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Some Banach spaces of Dirichlet series

by

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Abstract. The Hardy spaces of Dirichlet series, denoted by $\mathcal{H}^p$ ($p \geq 1$), have been studied by Hedenmalm et al. (1997) when $p = 2$ and by Bayart (2002) in the general case. In this paper we study some L^p-generalizations of spaces of Dirichlet series, particularly two families of Bergman spaces, denoted $\mathcal{A}^p$ and $\mathcal{B}^p$. Each could appear as a "natural" way to generalize the classical case of the unit disk. We recover classical properties of spaces of analytic functions: boundedness of point evaluation, embeddings between these spaces and "Littlewood–Paley" formulas when $p = 2$. Surprisingly, it appears that the two spaces have a different behavior relative to the Hardy spaces and that these behaviors are different from the usual way the Hardy spaces $H^p(\mathbb{D})$ embed into Bergman spaces on the unit disk.

1. Introduction

1.1. Background and notation. In [14], the authors defined the Hardy space $\mathcal{H}^2$ of Dirichlet series with square-summable coefficients. It is a space of analytic functions on $\mathbb{C}_{1/2} := \{s \in \mathbb{C} : \Re(s) > 1/2\}$ and this domain is maximal. This space is isometrically isomorphic to the Hardy space $H^2(\mathbb{T}^\infty)$ (see [9] for the definition of $H^2(\mathbb{T}^\infty)$, and we refer to [14] for results on $\mathcal{H}^2$).

F. Bayart [5] introduced the more general class of Hardy spaces of Dirichlet series $\mathcal{H}^p$ ($1 \leq p < \infty$). We shall recall the definitions below.

In another direction, McCarthy [21] defined some weighted Hilbert spaces, for instance, the spaces of Dirichlet series whose coefficients satisfy $\sum |a_n|^2(\log(n))^\alpha < \infty$. We shall recover these spaces (and their properties) as a special case of our spaces $\mathcal{A}^p_\mu$, with $p = 2$ and a suitable measure μ.

[21] is the starting point of much recent research on spaces of Dirichlet series: for instance in [23], [24] and [25], some local properties of these spaces are studied, and in [2], [3], [4], [5], [6], [18], [27] and [28] some results about composition operators on these spaces are obtained.

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The study of Dirichlet series may appear more complicated than the study of power series. For instance, there is a first important difference: all the notions of radius of convergence coincide for Taylor series, while Dirichlet series have several abscissas of convergence. The two most standard ones are the abscissa σ_{c} of simple convergence and the abscissa σ_{a} of absolute convergence (see [26], [31]).

Let f be a Dirichlet series of the form

$$f(s)=\sum_{n=1}^{\infty} a_n n^{-s}. \tag{1}$$

We shall need another two abscissas:

$$\sigma_{\mathrm{u}}(f)=\inf\{a : \text{the series (1) is uniformly convergent for } \Re(s)>a\},$$

the *abscissa of uniform convergence* of f, and

$$\sigma_{\mathrm{b}}(f)=\inf\{a : \text{the function } f \text{ has an analytic, bounded extension for } \Re(s)>a\},$$

the *abscissa of boundedness of* f. Actually, these two abscissas coincide: for all Dirichlet series f, one has $\sigma_{\mathrm{b}}(f)=\sigma_{\mathrm{u}}(f)$ (see [7]). This result due to Bohr is really important for the study of $\mathcal{H}^\infty$, the algebra of bounded Dirichlet series on the right half-plane $\mathbb{C}_+$ (see [20]), which also turns out to be the space of multipliers of $\mathcal{H}^2$. We shall denote by $\|\cdot\|_\infty$ the norm on this space:

$$\|f\|_\infty := \sup_{\Re(s)>0} |f(s)|.$$

Let us now recall Bohr's point of view on Dirichlet series: Let $n\ge 2$ be an integer; it can be (uniquely) written as a product of prime numbers, $n=p_1^{\alpha_1}\dots p_k^{\alpha_k}$ where $\alpha_j\ge 1$, and where $p_1=2$, $p_2=3$. If s is a complex number and $z=(p_1^{-s},p_2^{-s},\dots)$, then by (1),

$$f(s)=\sum_{n=1}^{\infty} a_n (p_1^{-s})^{\alpha_1}\dots(p_k^{-s})^{\alpha_k}=\sum_{n=1}^{\infty} a_n z_1^{\alpha_1}\dots z_k^{\alpha_k}.$$

So we can consider a Dirichlet series as a Fourier series on the infinite-dimensional polytorus $\mathbb{T}^\infty$. We shall denote this Fourier series by $D(f)$:

$$D(f)(z_1,z_2,\dots)=\sum_{\substack{n\ge 1\\ n=p_1^{\alpha_1}\cdots p_k^{\alpha_k}}} a_n z_1^{\alpha_1}\dots z_k^{\alpha_k}.$$

This correspondence is not just formal. For instance, let $\mathbb{P}$ be the set of prime numbers. Bohr proved the following result.

Theorem ([7]). *Let f be a Dirichlet series of the form* (1). *Then*

$$\sum_{p\in\mathbb{P}} |a_p| \le \|f\|_\infty.$$

The infinite-dimensional polytorus $\mathbb{T}^\infty$ can be identified with the group of complex-valued characters χ on the positive integers, which satisfy the following properties:

$$\begin{cases} |\chi(n)| = 1 & \forall n \geq 1, \\ \chi(nm) = \chi(n)\chi(m) & \forall n, m \geq 1. \end{cases}$$

To obtain this identification for $\chi = (\chi_1, \chi_2, \dots) \in \mathbb{T}^\infty$, it suffices to define χ on the prime numbers by $\chi(p_i) = \chi_i$ and use multiplicativity.

We shall denote by m the normalized Haar measure on $\mathbb{T}^\infty$.

Now, let us recall how one can define the Hardy spaces of Dirichlet series $\mathcal{H}^p$. We fix $p \geq 1$. The space $H^p(\mathbb{T}^\infty)$ is the closure of the set of analytic polynomials with respect to the norm of $L^p(\mathbb{T}^\infty, m)$. Let f be a Dirichlet polynomial; from Bohr's point of view, $D(f)$ is an analytic polynomial on $\mathbb{T}^\infty$. By definition $\|f\|_{\mathcal{H}^p} := \|D(f)\|_{H^p(\mathbb{T}^\infty)}$. The space $\mathcal{H}^p$ is defined to be the completion of the Dirichlet polynomials with respect to this norm. Consequently, $\mathcal{H}^p$ and $H^p(\mathbb{T}^\infty)$ are isometrically isomorphic. We already mentioned the case $p = \infty$, nevertheless, it is easy to adapt the previous description to the case $p = \infty$. When $p = 2$, $\mathcal{H}^2$ is just the space of Dirichlet series of the form (1) which satisfy

$$\sum_{n=1}^{\infty} |a_n|^2 < \infty.$$

Let $\mathcal{D}$ be the space of functions which admit a representation by a convergent Dirichet series on some half-plane. When a function f belongs to $\mathcal{D}$ and $\sigma > 0$, we define $f_\sigma \in \mathcal{D}$ to be the translate of f by σ, i.e. $f_\sigma(s) := f(\sigma + s)$. We then define a map from $\mathcal{D}$ to $\mathcal{D}$ by $T_\sigma(f) = f_\sigma$.

For $\theta \in \mathbb{R}$, $\mathbb{C}_\theta$ is the half-plane $\{s \in \mathbb{C} : \Re(s) > \theta\}$.

We shall denote by $\mathcal{P}$ the space of Dirichlet polynomials, that is, the vector space spanned by the functions $\mathrm{e}_n(z) = n^{-z}$, where $n \geq 1$. Finally, for $p \geq 1$, we write p' for its conjugate exponent: $1/p + 1/p' = 1$.

1.2. Organization of the paper. In the present paper, we introduce two classes of Bergman spaces of Dirichlet series. We give some properties of these spaces, and estimate the growth of point evaluation of functions belonging to these spaces. Finally, we compare them to the Hardy spaces of Dirichlet series: here some curious phenomena appear.

Definition 1. Let $p \geq 1$, P be a Dirichlet polynomial and μ be a probability measure on $(0, \infty)$ such that 0 belongs to the support $\operatorname{supp}(\mu)$ of μ. We define

$$\|P\|_{\mathcal{A}^p_\mu} = \Big(\int_0^\infty \|P_\sigma\|^p_{\mathcal{H}^p} \, d\mu(\sigma) \Big)^{1/p}.$$

The space $\mathcal{A}^p_\mu$ is the completion of $\mathcal{P}$ with respect to this norm. When $\mu(\sigma) = 2e^{-2\sigma}\, d\sigma$, we denote this space simply by $\mathcal{A}^p$. More generally, fix $\alpha > -1$ and consider the probability measure μ_α defined by

$$d\mu_\alpha(\sigma) = \frac{2^{\alpha+1}}{\Gamma(\alpha+1)}\sigma^\alpha \exp(-2\sigma)\, d\sigma.$$

The space $\mathcal{A}^p_{\mu_\alpha}$ will be denoted simply $\mathcal{A}^p_\alpha$.

Definition 2. On the infinite-dimensional polydisk $\mathbb{D}^\infty$, we consider the measure $A = \lambda \otimes \lambda \otimes \cdots$ where λ is the normalized Lebesgue measure on the unit disk $\mathbb{D}$. For $p \geq 1$, the space $B^p(\mathbb{D}^\infty)$ is the closure of the set of analytic polynomials with respect to the norm of $L^p(\mathbb{D}^\infty, A)$. Let f be a Dirichlet polynomial, and set $\|f\|_{\mathcal{B}^p} := \|D(f)\|_{B^p(\mathbb{D}^\infty)}$. The space $\mathcal{B}^p$ is defined to be the closure of $\mathcal{P}$ with respect to this norm.

In Section 2, we prove that point evaluation is bounded on the spaces $\mathcal{A}^p_\mu$ for any $s \in \mathbb{C}_{1/2}$. More precisely, let δ_s be the operator of point evaluation at $s \in \mathbb{C}_{1/2}$, which is *a priori* defined for Dirichlet polynomials (or convergent Dirichlet series). We prove that the operator extends to a bounded operator which we still denote by δ_s, and we show that there exists a constant c_p such that for every $s \in \mathbb{C}_{1/2}$,

$$\|\delta_s\|_{(\mathcal{A}^p)^*} \leq c_p \left(\frac{\Re(s)}{2\Re(s)-1}\right)^{2/p}.$$

It turns out that the classical ideas to prove the boundedness of point evaluation do not apply here and we have to find some new ideas. In particular, these ideas also apply to the classical (one variable) Bergman (or Dirichlet) spaces (although they do not give the best constants in that case). We also show that the identity from $\mathcal{H}^2$ to $\mathcal{A}^p$ is not bounded when $p > 2$ (but is compact when $p = 2$): this is probably the most surprising result of the paper because it completely differs from the classical result that the classical Hardy space of the unit disk, $H^2(\mathbb{D})$, embeds into the Bergman space $\mathcal{A}^4 = \mathcal{B}^4$ (of the unit disk). Finally, we obtain a Littlewood–Paley formula for the Hilbert spaces $\mathcal{A}^2_\mu$.

In Section 3, we prove that the point evaluation at any $s \in \mathbb{C}_{1/2}$ is bounded on $\mathcal{B}^p$ and

$$\|\delta_s\|_{(\mathcal{B}^p)^*} = \zeta(2\Re(s))^{2/p}.$$

By a hypercontractivity result, we find that the injection from $\mathcal{H}^p$ to $\mathcal{B}^{2p}$ is bounded. This phenomenon is similar to what happens in the classical framework of Hardy/Bergman spaces in one variable. Nevertheless, concerning compactness, we have the following curiosity: the injection from $\mathcal{H}^p$ to $\mathcal{B}^p$ is not compact. We also obtain a Littlewood–Paley formula for $\mathcal{B}^2$.

2. The Bergman spaces $\mathcal{A}^p_\mu$

2.1. Hilbert spaces of Dirichlet series with weighted ℓ^2 norm. First, we recall some facts from [21]. We have changed the definition in order to include the constant functions in these spaces, which seems to us more natural.

Let $w = (w_n)_{n\geq 1}$ be a sequence of positive numbers. The space $\mathcal{A}^2_w$ is defined by

$$\mathcal{A}^2_w := \Big\{ f \in \mathcal{D} : f(s) = \sum_{n=1}^{\infty} a_n n^{-s},\ \|f\| = \Big(\sum_{n=1}^{\infty} |a_n|^2 w_n \Big)^{1/2} < \infty \Big\}.$$

Of course, if $w \equiv 1$, then $\mathcal{A}^2_w$ is just the classical Hardy space $\mathcal{H}^2$. In order to obtain good properties for these spaces, we need to impose some conditions on the weights. This is motivated by the ideas and results of [17].

Definition ([21]). Let μ be a probability measure on $(0,\infty)$ such that $0 \in \operatorname{supp}(\mu)$. For $n \geq 1$ we define

$$w_n := \int_0^\infty n^{-2\sigma}\, d\mu(\sigma).$$

In this case, we say that the space $\mathcal{A}^2_\mu := \mathcal{A}^2_w$ is a (hilbertian) *Bergman-like space* and that w is a *Bergman weight.*

Example. When $\mu = \delta_0$, the Dirac mass at 0, we have $w_n = 1$ and we get the Hardy space $\mathcal{H}^2$. In the opposite situation, when $\mu(\{0\}) = 0$, it is easy to see that the sequence w converges to 0.

In the case $\mu = \mu_\alpha$, where $\alpha > -1$, we have $w_n = (\log(n)+1)^{-1-\alpha}$ for $n \geq 1$ and the associated space is $\mathcal{A}^2_\alpha$. For $\alpha = 0$, we recover the space $\mathcal{A}^2$ and we notice that the limit (degenerate) case $\alpha = -1$ corresponds to $\mathcal{H}^2$.

McCarthy [21] proved that these spaces are spaces of analytic functions on $\mathbb{C}_{1/2}$. This is a consequence of the following lemma:

Lemma ([21]). *Let w be a Bergman weight. Then w is non-increasing and it decreases more slowly than any negative power of n, that is,*

$$\forall \varepsilon > 0,\ \exists c > 0,\ \forall n \geq 1, \qquad w_n > c n^{-\varepsilon}.$$

In addition, $\mathbb{C}_{1/2}$ is a maximal domain. Indeed, let ζ be the Riemann zeta function ([31]). Then for every $\varepsilon > 0$ and every Bergman weight w,

$$\zeta(1/2 + s + \varepsilon) = \sum_{n=1}^{\infty} \frac{1}{n^{1/2+\varepsilon+s}} \in \mathcal{A}^2_w.$$

But this Dirichlet series has a pole at $1/2 - \varepsilon$.

2.2. Point evaluation on $\mathcal{A}^p_\mu$. First we can easily compute the norm of point evaluation in the case of the Hilbert spaces $\mathcal{A}^2_\mu$. In this case point evaluation is bounded on $\mathbb{C}_{1/2}$, it is optimal (in the sense that point evaluation cannot be defined on $\mathbb{C}_a$ with $a < 1/2$) and the reproducing kernel at $s \in \mathbb{C}_{1/2}$ is

$$K_\mu(s, w) := \sum_{n=1}^{\infty} \frac{n^{-w-\overline{s}}}{w_n}$$

and

$$\|\delta_s\|_{(\mathcal{A}^2_\mu)^*} \leq \left(\sum_{n=1}^{\infty} \frac{n^{-2\sigma}}{w_n}\right)^{1/2} \qquad \text{for every } s = \sigma + it \in \mathbb{C}_{1/2}.$$

In the general case, the next theorem provides us with a majorization which gives the right order of growth when the abscissa is close to the critical value 1/2. Actually we are going to distinguish the behaviour according to the valuation of the function (i.e. the least v such that $a_n = 0$ for every $n < v$), so we shall need some estimates depending on whether the constant coefficient vanishes or not. It could be interesting to work with (truncated) functions with higher valuation; however, in the present paper, we shall only concentrate on the cases $v = 0$ and $v = 1$, because these are the only cases needed here.

Definition 3. Let $\mathcal{H}^p_\infty$ be the subspace of $\mathcal{H}^p$ of functions whose valuation is at least 1, i.e. the space of Dirichlet series whose constant coefficient a_1 is zero (remember that a_1 is actually the value at infinity, and this explains our notation).

Let $\mathcal{A}^p_{\mu,\infty}$ be the subspace of $\mathcal{A}^p_\mu$ of functions whose constant coefficient vanishes. In the particular case of the measure μ_α, we write $\mathcal{A}^p_{\alpha,\infty}$. Finally, when $\alpha = 0$, we simply use the natural notation $\mathcal{A}^p_\infty$.

On the spaces $\mathcal{H}^p$ (resp. $\mathcal{H}^p_\infty$), we define $\Delta_p(s)$ (resp. $\Delta_{p,\infty}(s)$) as the norm of the evaluation at $s \in \mathbb{C}_{1/2}$. We recall that $\Delta_p(s) = \zeta(2\Re(s))^{1/p}$ by [5].

Theorem 1. *Let $p \geq 1$ and μ be a probability measure on $(0, \infty)$ such that $0 \in \operatorname{supp}(\mu)$. Then the point evaluation at any $s \in \mathbb{C}_{1/2}$ is bounded on $\mathcal{P} \cap \mathcal{A}^p_\mu$ (resp. on $\mathcal{P} \cap \mathcal{A}^p_{\mu,\infty}$). Hence it extends to a bounded operator on $\mathcal{A}^p_\mu$ (resp. on $\mathcal{A}^p_{\mu,\infty}$) whose norm satisfies*

(i) $$\|\delta_s\|_{(\mathcal{A}^p_\mu)^*} \leq \inf_{\eta \in (0, \Re(s)-1/2)} \frac{\|\Delta_p(\Re(s) - \bullet)\|_{L^{p'}([0,\Re(s)-1/2-\eta],\, d\mu)}}{\mu([0, \Re(s) - 1/2 - \eta])}.$$

(ii) $$\|\delta_s\|_{(\mathcal{A}^p_{\mu,\infty})^*} \leq \inf_{\eta \in (0, \Re(s)-1/2)} \frac{\|\Delta_{p,\infty}(\Re(s) - \bullet)\|_{L^{p'}([0,\Re(s)-1/2-\eta],\, d\mu)}}{\mu([0, \Re(s) - 1/2 - \eta])}.$$

Proof. We prove only (i) since the proof of (ii) is the same. Fix η in $(0, \Re(s) - 1/2)$. We can assume that $s = \sigma \in (1/2, \infty)$ thanks to the vertical translation invariance of the norm on $\mathcal{A}^p_\mu$. Let P be a Dirichlet polynomial. We have

$$P(\sigma) = P_\varepsilon(\sigma - \varepsilon) \quad \text{for any } \varepsilon \in (0, \sigma - 1/2).$$

We know that point evaluation is bounded on $\mathcal{H}^p$:

$$|P(\sigma)| \le \Delta_p(\sigma - \varepsilon)\|P_\varepsilon\|_{\mathcal{H}^p} \quad \text{for any } \varepsilon \in (0, \sigma - 1/2).$$

By integration on $(0, \sigma - 1/2 - \eta)$ we obtain

$$\mu([0, \sigma - 1/2 - \eta])|P(\sigma)| \le \int_0^{\sigma-1/2-\eta} \Delta_p(\sigma - \varepsilon)\|P_\varepsilon\|_{\mathcal{H}^p}\, d\mu(\varepsilon).$$

Then, by Hölder's inequality,

$$\mu([0, \sigma - 1/2 - \eta])|P(\sigma)| \le \|P\|_{\mathcal{A}^p_\mu} \cdot \|\Delta_p(\sigma - \bullet)\|_{L^{p'}([0,\sigma-1/2-\eta],d\mu)}.$$

Since $\eta \in (0, \Re(s) - 1/2)$ is arbitrary, the result follows. ∎

Corollary 1. *Let $p \ge 1$ and $\alpha > -1$.*

(i) *The point evaluation at any $s \in \mathbb{C}_{1/2}$ is bounded on $\mathcal{A}^p_\alpha$ and there exists a positive constant $c_{p,\alpha}$ such that for every $s \in \mathbb{C}_{1/2}$ we have*

$$\|\delta_s\|_{(\mathcal{A}^p_\alpha)^*} \le c_{p,\alpha}\left(\frac{\Re(s)}{2\Re(s) - 1}\right)^{(2+\alpha)/p}.$$

(ii) *The point evaluation at any $s \in \mathbb{C}_{1/2}$ is bounded on $\mathcal{A}^p_{\alpha,\infty}$ and there exists a positive constant $c'_{p,\alpha}$ such that for every $s \in \mathbb{C}_{1/2}$ we have*

$$\|\delta_s\|_{(\mathcal{A}^p_{\alpha,\infty})^*} \le \frac{c'_{p,\alpha}}{(2\Re(s) - 1)^{(2+\alpha)/p}}.$$

Proof. We shall use the fact that $\zeta(x) \le \frac{x}{x-1}$ for every $x > 1$. Moreover, $A \lesssim B$ means that there exists for some constant c depending on p and α only such that $A \le cB$.

Fix $s = \sigma \in (1/2, \infty)$ and $\eta \in (0, \sigma - 1/2)$. In our framework, there exists some constant C_α depending on α only, such that, for every $A > 0$,

$$\mu_\alpha([0, A]) \ge C_\alpha \min(1, A^{\alpha+1}).$$

Let us prove (i). We first consider the case $p = 1$. We choose $\eta = (\sigma - 1/2)/2$. Since

$$\sup_{\varepsilon\in[0,(\sigma-1/2)/2]} |\zeta(2\sigma - 2\varepsilon)| = \zeta(\sigma + 1/2) \le \frac{2\sigma + 1}{2\sigma - 1},$$

the conclusion follows from the preceding theorem.

Now assume that $p>1$ and $p\neq 2$ (we already know the exact norm of the evaluation in the case $p=2$). We have

$$\int_0^{\sigma-1/2-\eta}\zeta(2\sigma-2\varepsilon)^{p'/p}\,d\mu_\alpha(\varepsilon)\lesssim(2\sigma)^{p'/p}\int_0^{\sigma-1/2-\eta}\frac{\varepsilon^\alpha}{(2\sigma-2\varepsilon-1)^{p'/p}}e^{-2\varepsilon}\,d\varepsilon.$$

We split our discussion into two cases, according to whether $p>2$ or $2>p>1$.

First assume that $p>2$. We have $p'/p<1$, hence the last integral converges for $\eta=0$ and is majorized by

$$\int_0^{\sigma-1/2}\frac{\varepsilon^\alpha}{(2\sigma-2\varepsilon-1)^{p'/p}}\,d\varepsilon=\frac{1}{(2\sigma-1)^{p'/p}}\int_0^{\sigma-1/2}\frac{\varepsilon^\alpha}{(1-2\varepsilon/(2\sigma-1))^{p'/p}}\,d\varepsilon$$
$$=\frac{(2\sigma-1)^{\alpha+1}}{2^{\alpha+1}(2\sigma-1)^{p'/p}}\int_0^1 t^\alpha(1-t)^{-p'/p}\,dt\quad\text{with}\quad t=\frac{2\varepsilon}{2\sigma-1}.$$

Finally, we obtain

$$\int_0^{\sigma-1/2-\eta}\zeta(2\sigma-2\varepsilon)^{p'/p}\,d\mu_\alpha(\varepsilon)\lesssim(2\sigma)^{p'/p}\frac{B(\alpha+1,1-p'/p)}{(2\sigma-1)^{p'/p-\alpha-1}}$$

where B is the classical Beta function ([11]). Moreover, with the choice $\eta=0$ in Theorem 1, we get

$$\|\delta_\sigma\|_{(\mathcal{A}^p_\alpha)^*}\lesssim\left(\frac{B(\alpha+1,1-p'/p)}{(2\sigma-1)^{p'/p-\alpha-1}}\right)^{1/p'}\frac{(2\sigma)^{1/p}}{\min(1,(\sigma-1/2)^{\alpha+1})}.$$

This estimate is good when σ is bounded (and more precisely when σ is close to $1/2$). We have to consider the asymptotic behavior. So, assume $\sigma\geq 1$; the above integral

$$\int_0^{\sigma-1/2}\zeta(2\sigma-2\varepsilon)^{p'/p}\,d\mu_\alpha(\varepsilon)$$

is then majorized by

$$\int_0^{\sigma-1}\sup_{x\geq 2}|\zeta(x)|^{p'/p}\,d\mu_\alpha(\varepsilon)+\int_{\sigma-1}^{\sigma-1/2}\zeta(2\sigma-2\varepsilon)^{p'/p}\,d\mu_\alpha(\varepsilon).$$

The first integral is bounded uniformly in σ and the second is majorized by

$$\int_{\sigma-1}^{\sigma-1/2}\frac{\varepsilon^\alpha(2\sigma)^{p'/p}}{(2\sigma-2\varepsilon-1)^{p'/p}}e^{-2\varepsilon}\,d\varepsilon\lesssim\sigma^{\alpha+p'/p}e^{-2\sigma}\int_0^1\frac{1}{u^{p'/p}}\,du\lesssim 1.$$

This proves that the norm of the evaluation is uniformly bounded when $\sigma>1$. Collecting everything proves (i) when $p>2$.

Now for $1 < p < 2$, we have $p'/p > 1$ and we cannot choose $\eta = 0$ because the integral is not convergent. But in fact, it suffices to choose the middle point $\eta = (\sigma - 1/2)/2$. We conclude in the same way.

Let us prove (ii). Obviously $\|\delta_s\|_{(\mathcal{A}^p_{\alpha,\infty})^*} \le \|\delta_s\|_{(\mathcal{A}^p_\alpha)^*}$, hence the conclusion follows from (i) when the real part of s is bounded by 1.

It suffices to consider the behavior when $\sigma > 1$ and the result will follow from the (asymptotic) behavior of $\Delta_{p,\infty}$:

$$\Delta_{p,\infty}(s) \le \frac{1}{\Re(s) - 1}.$$

Indeed, for every $f \in \mathcal{P} \cap \mathcal{H}^p_\infty \subset \mathcal{H}^1_\infty$, we have, for any $s \in \mathbb{C}_1$,

$$f(s) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \overline{\widetilde{\zeta}(s+it)}\, f(it)\, dt,$$

where $\widetilde{\zeta}(z) = \sum_{n\ge 2} n^{-z}$. Hence

$$|f(s)| \le \|\widetilde{\zeta}_\sigma\|_{\mathcal{H}^\infty} \|f\|_{\mathcal{H}^1} \le \frac{1}{\sigma - 1} \|f\|_{\mathcal{H}^p}.$$

Now, the rest of the proof follows the lines of the proof of (i), so we leave the details to the reader. ∎

Remarks. (i) Let us clarify why the estimate of the norm is optimal in many cases: The behavior of $\|\delta_s\|_{(\mathcal{A}^p_\alpha)^*}$ around the critical line $\sigma = 1/2$ cannot be a power of $\Re(s)/(2\Re(s) - 1)$ better than $(2+\alpha)/p$. Indeed, let $\sigma > 1/2$; we would like to consider the function $\zeta_\sigma^{2/p}$. Let us mention that we can define the function ζ^q (where $q > 0$) through the Euler product:

$$\zeta^q(z) = \prod_{p\in\mathbb{P}} \left[\frac{1}{1 - p^{-z}}\right]^q.$$

Actually we first work with F being a partial sum of $(\zeta_\sigma)^{2/p}$ and we obtain

$$|F(\sigma)|^p \le \|\delta_\sigma\|^p_{(\mathcal{A}^p_\alpha)^*} \|F\|^p_{\mathcal{A}^p_\alpha} \lesssim \|\delta_\sigma\|^p_{(\mathcal{A}^p_\alpha)^*} \int_0^\infty \|F_\varepsilon\|^p_{\mathcal{H}^p} \varepsilon^\alpha \exp(-2\varepsilon)\, d\varepsilon$$

because F is a Dirichlet polynomial. Now if we assume that $p > 1$, we know (see [1]) that $(\mathrm{e}_n)_{n\ge 1}$ is a Schauder basis for $\mathcal{H}^p$. Hence there exists $c_p > 0$ such that

$$|F(\sigma)|^p \lesssim c_p \|\delta_\sigma\|^p_{(\mathcal{A}^p_\alpha)^*} \int_0^\infty \|\zeta^{2/p}_{\sigma+\varepsilon}\|^p_{\mathcal{H}^p} \varepsilon^\alpha \exp(-2\varepsilon)\, d\varepsilon.$$

But

$$\|\zeta^{2/p}_{\sigma+\varepsilon}\|^p_{\mathcal{H}^p} = \|\zeta_{\sigma+\varepsilon}\|^2_{\mathcal{H}^2} = \zeta(2\sigma + 2\varepsilon)$$

and we get (since F was an arbitrary partial sum of $\zeta_\sigma^{2/p}$)

$$|\zeta(2\sigma)|^2 \lesssim \|\delta_\sigma\|^p_{(\mathcal{A}^p_\alpha)^*} \sum_{n\geq 1} \frac{n^{-2\sigma}}{(1+\ln(n))^{\alpha+1}}.$$

When $\alpha < 0$, we get $|\zeta(2\sigma)|^2 \lesssim \|\delta_\sigma\|^p_{(\mathcal{A}^p_\alpha)^*}(2\sigma-1)^\alpha$, hence

$$\frac{1}{(2\sigma-1)^{2+\alpha}} \lesssim \|\delta_\sigma\|^p_{(\mathcal{A}^p_\alpha)^*},$$

which proves our claim, in a strong way: the majorization in (i) of Corollary 1 is actually also (up to a constant) a minorization.

When $\alpha \geq 0$, we have

$$\frac{1}{(2\sigma-1)^{2+\alpha}|\log(2\sigma-1)|} \lesssim \|\delta_\sigma\|^p_{(\mathcal{A}^p_\alpha)^*},$$

which proves that we cannot get a better exponent than $(2+\alpha)/p$ in Corollary 1(i).

(ii) Let $\sigma > 1/2$ and $\mu = \mu_\alpha$. We already know that the reproducing kernel at σ is defined by

$$\forall w \in \mathbb{C}_{1/2}, \qquad K_{\mu_\alpha}(\sigma, w) = \sum_{n=1}^{\infty} (1+\log(n))^{\alpha+1} n^{-\sigma-w}.$$

Then

$$K_{\mu_\alpha}(\sigma,\sigma) \leq \|\delta_\sigma\|_{(\mathcal{A}^2_\alpha)^*} \|K_{\mu_\alpha}(\sigma, \bullet)\|_{\mathcal{A}^2_\alpha}$$

and by the property of the reproducing kernel,

$$K_{\mu_\alpha}(\sigma,\sigma)^{1/2} \leq \|\delta_\sigma\|_{(\mathcal{A}^2_\alpha)^*}.$$

The converse inequality is already known. Hence

$$\|\delta_\sigma\|_{(\mathcal{A}^2_\alpha)^*} = K_{\mu_\alpha}(\sigma,\sigma)^{1/2} = \left(\frac{\Gamma(2+\alpha)}{(2\sigma-1)^{2+\alpha}} + O(1)\right)^{1/2}$$

when σ goes to $1/2$ (see [25] for the second equality), and so our result is sharp when $p=2$.

(iii) With the same notation, we have

$$K_{\mu_\alpha}(\sigma,\sigma)^2 \leq \|\delta_\sigma\|_{(\mathcal{A}^1_\alpha)^*} \|K_{\mu_\alpha}(\sigma, \bullet)^2\|_{\mathcal{A}^1_\alpha} = \|\delta_\sigma\|_{(\mathcal{A}^1_\alpha)^*} \|K_{\mu_\alpha}(\sigma, \bullet)\|^2_{\mathcal{A}^2_\alpha},$$

and again by the property of the reproducing kernel, we obtain

$$K_{\mu_\alpha}(\sigma,\sigma) \leq \|\delta_\sigma\|_{(\mathcal{A}^1_\alpha)^*}.$$

We conclude as in (ii) and so the result is also sharp for $p=1$.

(iv) In (i), we have used the fact that $(\mathrm{e}_n)_{n\geq 1}$ is a Schauder basis for $\mathcal{H}^p$ when $p>1$. This result is also true for $\mathcal{A}^p_\mu$ when $p>1$: just use the result on $\mathcal{H}^p$, then make an integration and use the density of the Dirichlet polynomials. This remark is also true for the spaces $\mathcal{B}^p$.

Let us mention here that we are able to give a more precise majorization in the particular case of an even integer p: the constants are equal to 1. This follows immediately from a general method explained in the appendix at the end of the paper:

PROPOSITION 1. *Let p be an even integer, and μ be as in Theorem 1.*

(i) *For every $s \in \mathbb{C}_{1/2}$ we have*

$$\|\delta_s\|_{(\mathcal{A}^p_\mu)^*} \leq \|\delta_s\|^{2/p}_{(\mathcal{A}^2_\mu)^*}.$$

In particular,

(ii) *For every $s \in \mathbb{C}_{1/2}$ we have*

$$\|\delta_s\|_{(\mathcal{A}^p)^*} \leq \big((\zeta - \zeta')(2\Re(s))\big)^{1/p} \sim \frac{1}{(2\Re(s)-1)^{2/p}} \quad \textit{when } \Re(s) \to 1/2.$$

As soon as a Bergman-like space is defined, a Dirichlet-like space is naturally associated:

DEFINITION 4. Let $p \geq 1$ and μ be a probability measure on $(0,\infty)$. We define the Dirichlet space $\mathcal{D}^p_\mu$ as the space of Dirichlet series f such that

$$\|f\|^p_{\mathcal{D}^p_\mu} := |f(\infty)|^p + \|f'\|^p_{\mathcal{A}^p_\mu} < \infty.$$

Here $f(\infty)$ stands for $\lim_{\Re(s)\to\infty} f(s) = a_1$, where f has an expansion (1).

THEOREM 2. *Let $p \geq 1$ and μ be a probability measure on $(0,\infty)$. For any $s \in \mathbb{C}_{1/2}$, we have*

$$|f(s)| \leq 2^{1/p'} \max\Big(1, \int_{\Re(s)}^{\infty} \|\delta_t\|_{(\mathcal{A}^p_{\mu,\infty})^*}\, dt\Big) \times \|f\|_{\mathcal{D}^p_\mu}.$$

Proof. Without loss of generality we may assume that $s = \sigma \in (1/2,\infty)$. Now

$$\begin{aligned}|f(\sigma) - f(\infty)| &= \Big|\int_\sigma^\infty f'(t)\, dt\Big| \leq \int_\sigma^\infty |f'(t)|\, dt \\ &\leq \int_\sigma^\infty \|\delta_t\|_{(\mathcal{A}^p_{\mu,\infty})^*}\, dt \times \|f'\|_{\mathcal{A}^p_{\mu,\infty}}\end{aligned}$$

since the constant coefficient of f' vanishes, i.e. $f' \in \mathcal{A}^p_{\mu,\infty}$. So we get

$$\begin{aligned}|f(\sigma)| &\leq |f(\infty)| + \int_\sigma^\infty \|\delta_t\|_{(\mathcal{A}^p_{\mu,\infty})^*}\, dt \times \|f'\|_{\mathcal{A}^p_{\mu,\infty}} \\ &\leq \Big(1 + \Big(\int_\sigma^\infty \|\delta_t\|_{(\mathcal{A}^p_{\mu,\infty})^*}\, dt\Big)^{p'}\Big)^{1/p'} \times \big(|f(\infty)|^p + \|f'\|^p_{\mathcal{A}^p_{\mu,\infty}}\big)^{1/p}\end{aligned}$$

thanks to Hölder's inequality. Now it suffices to remark that

$$\Big(1+\Big(\int_{\sigma}^{\infty}\|\delta_t\|_{(\mathcal{A}^p_{\mu,\infty})^*}\,dt\Big)^{p'}\Big)^{1/p'}\le 2^{1/p'}\max\Big(1,\int_{\Re(s)}^{\infty}\|\delta_t\|_{(\mathcal{A}^p_{\mu,\infty})^*}\,dt\Big).\ \blacksquare$$

Corollary 2. *Let $\alpha>-1$ and $p\ge 1$. There exists $c_{p,\alpha}>0$ such that for every $s\in\mathbb{C}_{1/2}$, we have*

$$\|\delta_s\|_{\mathcal{D}^p_\alpha}\le\begin{cases} c_{p,\alpha}/(2\Re(s)-1)^{(2+\alpha)/p-1} & \text{if } \alpha\neq p-2,\\ c_{p,\alpha}\log(2\Re(s)-1) & \text{if } \alpha=p-2.\end{cases}$$

Proof. This follows from Theorem 2 and Corollary 1. ■

Let us make a digression. The proofs of Theorems 1 and 2 are based on the fact that we work with Bergman spaces with axial weights. Replacing axial weights on the half-plane by radial weights on the unit disk, the same idea can be adapted to classical Bergman and Dirichlet spaces on the unit disk $\mathbb{D}$. Let us describe here how we can easily estimate the norm of the evaluation on weighted spaces of analytic functions over the unit disc.

Let $\sigma:(0,1)\to(0,\infty)$ be a continuous function such that $\sigma\in L^1(0,1)$. We extend it on $\mathbb{D}$ by $\sigma(z)=\sigma(|z|)$. For $p\ge 1$, we consider the weighted Bergman space

$$A^p_\sigma:=H(\mathbb{D})\cap L^p(\mathbb{D},\sigma(|z|)\,d\lambda(z))$$

where $H(\mathbb{D})$ is the set of analytic functions on $\mathbb{D}$ and λ is the normalized Lebesgue measure on $\mathbb{D}$. This space is equipped with the norm

$$\|f\|_{A^p_\sigma}=\Big(\int_{\mathbb{D}}|f(z)|^p\sigma(z)\,d\lambda(z)\Big)^{1/p}.$$

We also consider the Dirichlet space $D^p_\sigma(\mathbb{D})$ of analytic functions on $\mathbb{D}$ whose derivative belongs to A^p_σ. This space is equipped with the norm

$$\|f\|_{D^p_\sigma}=\big(|f(0)|^p+\|f'\|^p_{A^p_\sigma}\big)^{1/p}.$$

We know that the point evaluation at $z\in\mathbb{D}$ is bounded on the (classical) Hardy spaces $H^p=H^p(\mathbb{D})$ (see [10]) and we have exactly

$$\|\delta_z\|_{(H^p)^*}=\frac{1}{(1-|z|^2)^{1/p}}.$$

Theorem 3. *Let $p\ge 1$ and $z\in\mathbb{D}$. The point evaluation at z is bounded on A^p_σ and we have*

$$\|\delta_z\|_{(A^p_\sigma)^*}\le\inf_{\eta\in(0,1-|z|)}\frac{\|r\mapsto(1-(|z|/r)^2)^{-1/p}\|_{L^{p'}([|z|+\eta,1],\sigma(r)dr)}}{S([|z|+\eta,1])}$$

where $S(I)=\int_I\sigma(r)\,dr$.

Example. When $\sigma \equiv 1$ (the classical Bergman space A^p), we recover

$$\|\delta_z\|_{(A^p)^*} \lesssim \frac{1}{(1-|z|^2)^{2/p}} \quad \text{for any } z \in \mathbb{D}.$$

Theorem 4. *Let $p \geq 1$ and $z \in \mathbb{D}$. We have*

$$\|\delta_z\|_{(D^p_\sigma)^*} \leq 2^{1/p'} \max\Big(1, \int_0^{|z|} \|\delta_r\|_{(A^p_\sigma)^*}\, dr\Big).$$

2.3. $\mathcal{A}^p_\mu$ as a space of Dirichlet series. The results of the preceding section allow us to define, for each $s \in \mathbb{C}_{1/2}$, the value of $f \in \mathcal{A}^p_\mu$ at s as $\delta_s(f)$. Of course, this coincides with the natural definition when f is a Dirichlet polynomial or when $f \in \mathcal{D} \cap \mathcal{A}^p_\mu$. Now we want more: we wish to check that we are actually working on spaces of Dirichlet series.

We first need the following tool.

Lemma 1. *Let $\varepsilon > 0$ and μ be a probability measure on $(0,\infty)$. Then*

$$T_\varepsilon : \mathcal{P} \cap \mathcal{A}^1_\mu \to \mathcal{A}^2_\mu, \qquad f \mapsto f_\varepsilon,$$

is bounded. It extends to a bounded operator (still denoted T_ε) from $\mathcal{A}^1_\mu$ to $\mathcal{A}^2_\mu$.

In the proof, we shall use the following sequence.

Definition 5. Let μ be a probability measure on $(0,\infty)$ with $0 \in \operatorname{supp}(\mu)$. For every integer $n \geq 1$ we define

$$\widetilde{w}_n := \int_0^\infty n^{-\sigma}\, d\mu(\sigma).$$

Proof of Lemma 1. We shall introduce three bounded operators. First we define $S_1 : \mathcal{P} \cap \mathcal{A}^1_\mu \to \mathcal{H}^1$ by

$$S_1\Big(\sum_{n=1}^\infty a_n \mathrm{e}_n\Big) := \sum_{n=1}^\infty a_n \widetilde{w}_n \mathrm{e}_n.$$

It is bounded because for any Dirichlet polynomial we have

$$\begin{aligned}
\Big\|\sum_{n=1}^N a_n \widetilde{w}_n \mathrm{e}_n\Big\|_{\mathcal{H}^1} &= \int_{\mathbb{T}^\infty} \Big|\sum_{n=1}^N a_n \widetilde{w}_n z_1^{\alpha_1} \ldots z_k^{\alpha_k}\Big|\, dm(z) \\
&= \int_{\mathbb{T}^\infty} \Big|\int_0^\infty \sum_{n=1}^N a_n n^{-\sigma} z_1^{\alpha_1} \ldots z_k^{\alpha_k}\, d\mu(\sigma)\Big|\, dm(z) \quad \text{by definition of } \widetilde{w}_n \\
&\leq \int_0^\infty \int_{\mathbb{T}^\infty} \Big|\sum_{n=1}^N a_n n^{-\sigma} z_1^{\alpha_1} \ldots z_k^{\alpha_k}\Big|\, dm(z)\, d\mu(\sigma) = \Big\|\sum_{n=1}^N a_n \mathrm{e}_n\Big\|_{\mathcal{A}^1_\mu}.
\end{aligned}$$

By density, this operator extends to a bounded operator (still denoted S_1).

Now we define $S_2 : \mathcal{H}^1 \to \mathcal{H}^2$ by

$$S_2\Big(\sum_{n=1}^{\infty} a_n \mathrm{e}_n\Big) := \sum_{n=1}^{\infty} \frac{a_n n^{-\varepsilon}}{\sqrt{\widetilde{w}_n}} \mathrm{e}_n.$$

It is bounded because $T_{\varepsilon/2} : \mathcal{H}^1 \to \mathcal{H}^2$ is bounded (see [5]) and because there exists $C > 0$ such that $\widetilde{w}_n > Cn^{-\varepsilon}$.

The third operator $S_3 : \mathcal{H}^2 \to \mathcal{A}^2_\mu$ is defined by

$$S_3\Big(\sum_{n=1}^{\infty} a_n \mathrm{e}_n\Big) := \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{\widetilde{w}_n}} \mathrm{e}_n.$$

It is bounded because $w_n \le \widetilde{w}_n$ for all $n \ge 1$.

Hence $S_3 \circ S_2 \circ S_1$ is bounded and clearly coincides with T_ε. ∎

Theorem 5. *The space $\mathcal{A}^p_\mu$ is a space of Dirichlet series: every $f \in \mathcal{A}^p_\mu$ belongs to $\mathcal{D}$ and $\sigma_{\mathrm{u}}(f) \le 1/2$.*

Proof. The result is obvious when $p \ge 2$ since then $\mathcal{A}^p_\mu \subset \mathcal{A}^2_\mu$. When $1 \le p < 2$, since $\mathcal{A}^p_\mu \subset \mathcal{A}^1_\mu$, we only have to prove the conclusion for $p = 1$. But this follows from the preceding lemma: Indeed, fix $f \in \mathcal{A}^1_\mu$, $\alpha > 1/2$ and $\varepsilon = \alpha - 1/2 > 0$. The function $T_\varepsilon(f)$ belongs to $\mathcal{A}^2_\mu$ so that for every $z \in \mathbb{C}_{1/2}$ we can write

$$T_\varepsilon(f)(z) = \sum_{n\ge 1} a_n^{(\varepsilon)}\, n^{-z}.$$

On the other hand, f is the limit of a sequence $(P_k)_{k\in\mathbb{N}}$ of Dirichlet polynomials in the space $\mathcal{A}^1_\mu$. The continuity of T_ε implies that $T_\varepsilon(f)$ is the limit of $(P_k(\varepsilon + \bullet))_{k\in\mathbb{N}}$ in the norm of $\mathcal{A}^2_\mu$. Invoking the continuity of the point evaluation both at $z + \varepsilon$ and at z, we get

$$f(z+\varepsilon) = \lim P_k(\varepsilon + z) = \lim T_\varepsilon(P_k)(z) = T_\varepsilon(f)(z) = \sum_{n\ge 1} a_n^{(\varepsilon)} n^{-z}.$$

In particular, for every $s \in \mathbb{C}_\alpha$, we have (with $z = s - \varepsilon \in \mathbb{C}_{1/2}$)

$$f(s) = \sum_{n\ge 1} (a_n^{(\varepsilon)} n^{\varepsilon}) n^{-s}.$$

Actually the coefficients do not depend on ε (by uniqueness of the Dirichlet expansion). Since $\alpha > 1/2$ is arbitrary, we get the conclusion. ∎

In view of the results of this section, we get the conclusion of Lemma 1.

It seems clear that $\mathcal{H}^p \subset \mathcal{A}^p_\mu$ for any $p \ge 1$ and any μ. Indeed, the following theorem makes this fact precise and shows that the way we may compute the norm remains valid for general functions of $\mathcal{A}^p_\mu$.

Theorem 6. *Let $p \ge 1$ and μ a probability measure whose support contains* 0.

(i) $\mathcal{H}^p \subset \mathcal{A}^p_\mu$ *and* $\|f\|_{\mathcal{A}^p_\mu} \le \|f\|_{\mathcal{H}^p}$ *for every* $f \in \mathcal{H}^p$.
(ii) *For every* $f \in \mathcal{H}^p$, *we have* $\|f\|_{\mathcal{A}^p_\mu} = (\int_0^\infty \|f_\sigma\|^p_{\mathcal{H}^p}\, d\mu)^{1/p}$.
(iii) *For every* $f \in \mathcal{A}^p_\mu$, *we have* $\|f\|_{\mathcal{A}^p_\mu} = \lim_{c\to 0^+} \|f_c\|_{\mathcal{A}^p_\mu}$.

Proof. For every Dirichlet polynomial f, we have $\|f\|_{\mathcal{A}^p_\mu} \le \|f\|_{\mathcal{H}^p}$, since μ is a probability measure and $\|f\|_{\mathcal{H}^p} = \sup_{c>0} \|f_c\|_{\mathcal{H}^p}$. Now (in the spirit of the proof of Theorem 5) a density argument, combined with the boundedness of point evaluation, easily yields the first assertion.

Now, let $f \in \mathcal{H}^p$ and $\varepsilon > 0$. There exists a Dirichlet polynomial P such that $\|f-P\|_{\mathcal{H}^p} < \varepsilon$. By the first assertion $\|f-P\|_{\mathcal{A}^p_\mu} < \varepsilon$ and so

$$\begin{aligned}\|f\|_{\mathcal{A}^p_\mu} &\le \varepsilon + \|P\|_{\mathcal{A}^p_\mu} = \varepsilon + \Big(\int_0^\infty \|P_\sigma\|^p_{\mathcal{H}^p}\, d\mu(\sigma)\Big)^{1/p} \\ &\le \varepsilon + \Big(\int_0^\infty \|P_\sigma - f_\sigma\|^p_{\mathcal{H}^p}\, d\mu(\sigma)\Big)^{1/p} + \Big(\int_0^\infty \|f_\sigma\|^p_{\mathcal{H}^p}\, d\mu(\sigma)\Big)^{1/p}.\end{aligned}$$

Now, T_σ is a contraction on $\mathcal{H}^p$ for every $\sigma > 0$ and so

$$\|f\|_{\mathcal{A}^p_\mu} \le 2\varepsilon + \Big(\int_0^\infty \|f_\sigma\|^p_{\mathcal{H}^p}\, d\mu(\sigma)\Big)^{1/p}.$$

In the same way we obtain a lower bound and finally the second assertion.

For the third assertion, we shall use the fact that T_c is a contraction on $\mathcal{A}^p_\mu$ for every $c > 0$. As in the first assertion, it suffices to check it on Dirichlet polynomials but in this case this is clear by definition of the norm for Dirichlet polynomials and the fact that T_c is a contraction on $\mathcal{H}^p$. Now let $f \in \mathcal{A}^p_\mu$ and $\varepsilon, c > 0$. There exists a Dirichlet polynomial P such that $\|f-P\|_{\mathcal{A}^p_\mu} < \varepsilon$. Then

$$\begin{aligned}\|f-f_c\|_{\mathcal{A}^p_\mu} &\le \|f-P\|_{\mathcal{A}^p_\mu} + \|P-P_c\|_{\mathcal{A}^p_\mu} + \|P_c - f_c\|_{\mathcal{A}^p_\mu} \\ &\le 2\|f-P\|_{\mathcal{A}^p_\mu} + \|P-P_c\|_{\mathcal{A}^p_\mu} \le 2\varepsilon + \|P-P_c\|_{\mathcal{A}^p_\mu}\end{aligned}$$

and by the Lebesgue dominated convergence theorem $\|P-P_c\|_{\mathcal{A}^p_\mu} \to 0$ as $c \to 0$, and so the result is proved. ∎

Remark. It follows from the preceding theorem that the norm of point evaluation on H^p is bounded by its norm on $\mathcal{A}^p_\mu$. Hence $\mathbb{C}_{1/2}$ is always the maximal domain where we can *a priori* define the Dirichlet series of $\mathcal{A}^p_\mu$.

2.4. Vertical limits and Littlewood–Paley formula. Let f be a Dirichlet series absolutely convergent in a half-plane. For any sequences $(\tau_n) \subset \mathbb{R}$, we can consider vertical translations of f,

$$(f_{\tau_n}(s)) := (f(s + i\tau_n)).$$

By Montel's theorem, this sequence is a normal family in the half-plane of absolute convergence of f and so there exists a subsequence convergent to some $\tilde{f}$. We say that $\tilde{f}$ is a *vertical limit* of f. We shall use the following result.

PROPOSITION ([14]). *Let f be a Dirichlet series of the form* (1), *absolutely convergent in a half-plane. The vertical limit functions of f are exactly the functions of the form*

$$f_\chi := \sum_{n=1}^{\infty} a_n \chi(n) \mathrm{e}_n \quad \textit{where } \chi \in \mathbb{T}^\infty.$$

In [14] it is shown that every element f in $\mathcal{H}^2$ admits vertical limit functions f_χ which converge m-almost everywhere on $\mathbb{C}_+$. We have the same result for the Bergman spaces $\mathcal{A}^p_\mu$. This is a consequence of the following theorem.

MEN'SHOV'S THEOREM ([22]). *Let $(\Omega, \mathcal{A}, \nu)$ be a probability space and (Φ_n) be an orthonormal sequence in $L^2(\Omega)$. Then*

$$\sum_{n=1}^{\infty} |c_n|^2 \log^2(n) < \infty \ \Rightarrow \ \sum_{n=1}^{\infty} c_n \Phi_n \textit{ converges } \nu\textit{-a.e.}$$

PROPOSITION 2. *Let $p \geq 1$, μ be a probability measure on $(0, \infty)$ and let $f \in \mathcal{A}^p_\mu$ or $\mathcal{D}^p_\mu$. For almost all $\chi \in \mathbb{T}^\infty$, f_χ converges on $\mathbb{C}_+$.*

REMARK. It suffices to give the proof in the case of the Bergman-like spaces: indeed, if f is in $\mathcal{D}^p_\mu$ then $f' \in \mathcal{A}^p_\mu$ and so f'_χ converge on $\mathbb{C}_+$ for almost every $\chi \in \mathbb{T}^\infty$, and the same holds for f.

Proof. First, we prove the result when $p = 2$. Let $f \in \mathcal{A}^2_\mu$ be of the form (1) and set $c_n := a_n n^{-\sigma - it}$ where $\sigma > 0$ and $t \in \mathbb{R}$. Clearly $(\chi(n))$ is an orthonormal family in $L^2(\mathbb{T}^\infty)$. We have

$$\sum_{n=2}^{\infty} |c_n|^2 \log^2(n) = \sum_{n=2}^{\infty} |a_n|^2 w_n \left(\frac{\log^2(n)}{n^{2\sigma} w_n} \right).$$

If w is a Bergman weight, we know that there exists a positive constant C such that $w_n > C n^{-\sigma}$ for all $n \geq 1$, so

$$\frac{\log^2(n)}{n^{2\sigma} w_n} \leq \frac{\log^2(n)}{C n^\sigma}.$$

In this case, the right hand side of this inequality is finite and by Men'shov's theorem, the proof is finished for $p = 2$.

Now we want to prove this result when $p \neq 2$. By inclusion between these spaces, it suffices to prove the result for $p = 1$.

Let $f \in \mathcal{A}^1_\mu$. By Lemma 1, $f_\varepsilon \in \mathcal{A}^2_\mu$ for every $\varepsilon > 0$. So

$$\text{for every } \varepsilon > 0, \text{ for almost all } \chi \in \mathbb{T}^\infty, (f_\varepsilon)_\chi \text{ converges on } \mathbb{C}_+.$$

Thus

$$\text{for every } n \geq 1, \text{ for almost all } \chi \in \mathbb{T}^\infty, \ (f_{1/n})_\chi \text{ converges on } \mathbb{C}_+.$$

Now we can invert the quantifiers:

$$\text{for almost all } \chi \in \mathbb{T}^\infty, \text{ for all } n \geq 1, \ (f_{1/n})_\chi \text{ converges on } \mathbb{C}_+.$$

Of course if $(f_{1/n})_\chi$ converges on $\mathbb{C}_+$ for every $n \geq 1$, then f_χ converges on $\mathbb{C}_+$, and so we obtain the result. ∎

Now, following some ideas from [17] in the case of the unit disk, we consider the case of weighted Bergman-like spaces when $d\mu(\sigma) = h(\sigma)d\sigma$ where $h \geq 0$, $\|h\|_{L^1(\mathbb{R}^+)} = 1$ and $0 \in \mathrm{supp}(h)$. Let w_h be the associated Bergman weight defined for $n \geq 1$ by

$$w_h(n) = \int_0^\infty n^{-2\sigma} h(\sigma)\, d\sigma.$$

For $\sigma > 0$, we define

$$\beta_h(\sigma) := \int_0^\sigma (\sigma - u) h(u)\, du = \int_0^\sigma \int_0^t h(u)\, du\, dt.$$

Remark. Observe that $\lim_{\sigma\to\infty} \beta_h(\sigma) n^{-2\sigma} = 0$ for every $n \geq 2$.

We can compute the first two derivatives of β_h:

$$\beta_h'(\sigma) = \int_0^\sigma h(u)\, du, \qquad \beta_h''(\sigma) = h(\sigma).$$

In order to obtain a Littlewood–Paley formula for $\mathcal{A}^2_\mu$, we need the following lemma.

Lemma ([6]). *Let η be a Borel probability measure on $\mathbb{R}$, and f of the form* (1). *Then*

$$\|f\|^2_{\mathcal{H}^2} = \int_{\mathbb{T}^\infty} \int_{\mathbb{R}} |f_\chi(it)|^2\, d\eta(t)\, dm(\chi).$$

Theorem 7 ("Littlewood–Paley formula"). *Let η be a Borel probability measure on $\mathbb{R}$. Then*

$$\|f\|^2_{\mathcal{A}_{w_h}} = |f(\infty)|^2 + 4 \int_{\mathbb{T}^\infty} \int_0^\infty \int_{\mathbb{R}} \beta_h(\sigma) |f_\chi'(\sigma + it)|^2\, d\eta(t)\, d\sigma\, dm(\chi).$$

Proof. Let $f \in \mathcal{A}^2_{w_h}$ be of the form (1). The previous lemma applied to f_σ where $\sigma > 0$ gives

$$\int_{\mathbb{T}^\infty} \int_{\mathbb{R}} |f_\chi'(\sigma + it)|^2\, d\eta(t)\, dm(\chi) = \sum_{n=2}^\infty |a_n|^2 n^{-2\sigma} \log^2(n) \qquad \forall \sigma > 0.$$

Now we multiply by $\beta_h(\sigma)$ and integrate over $\mathbb{R}_+$:

$$\int_0^\infty \int_{\mathbb{T}^\infty} \int_{\mathbb{R}} \beta_h(\sigma)|f'_\chi(\sigma+it)|^2\,d\eta(t)\,dm(\chi)\,d\sigma = \sum_{n=2}^\infty |a_n|^2 \log^2(n) \int_0^\infty n^{-2\sigma}\beta_h(\sigma)\,d\sigma.$$

Now, it suffices to prove that

$$w_h(n) = 4\log^2(n) \int_0^\infty n^{-2\sigma}\beta_h(\sigma)\,d\sigma.$$

But by definition, we have

$$w_h(n) = \int_0^\infty h(\sigma)n^{-2\sigma}\,d\sigma.$$

Integration by parts gives

$$\begin{aligned} w_h(n) &= \Big[\int_0^\sigma h(u)\,du \times n^{-2\sigma}\Big]_0^\infty + 2\log(n)\int_0^\infty\int_0^\sigma h(u)\,du \times n^{-2\sigma}\,d\sigma \\ &= \big[\beta'_h(\sigma)\times n^{-2\sigma}\big]_0^\infty + 2\log(n)\int_0^\infty \beta'_h(\sigma)n^{-2\sigma}\,d\sigma. \end{aligned}$$

But we know that $\beta'_h(\sigma)\to 0$ as $\sigma\to 0$ (because $h\in L^1(\mathbb{R}_+)$) and $\beta'_h(\sigma)\to \|h\|_1$ as $\sigma\to\infty$. So we have

$$w_h(n) = 2\log(n)\int_0^\infty \beta'_h(\sigma)\times n^{-2\sigma}\,d\sigma.$$

Using again integration by parts, we obtain

$$w_h(n) = 4\log^2(n)\int_0^\infty n^{-2\sigma}\beta_h(\sigma)\,d\sigma. \blacksquare$$

Example. Let $\alpha > -1$ and $f \in \mathcal{A}^2_\alpha$. We have $\beta_h(\sigma) \approx \sigma^{\alpha+2}$ when σ is small. Hence

$$\|f\|^2_{\mathcal{A}^2_\alpha} \approx |f(\infty)|^2 + \frac{2^{\alpha+3}}{\Gamma(\alpha+3)} \int_{\mathbb{T}^\infty}\int_0^\infty\int_{\mathbb{R}} \sigma^{\alpha+2}|f'_\chi(\sigma+it)|^2\,d\eta(t)\,d\sigma\,dm(\chi).$$

In the case of Dirichlet spaces, we obtain the following proposition.

Proposition 3. *Let $d\mu = hd\sigma$ be a probability measure. Then for $f \in \mathcal{D}^2_\mu$ we have*

$$\|f\|^2_{\mathcal{D}^2_\mu} = |f(\infty)|^2 + 4\int_{\mathbb{T}^\infty}\int_0^\infty\int_{\mathbb{R}} h(\sigma)|f'_\chi(\sigma+it)|^2\,d\eta(t)\,d\sigma\,dm(\chi).$$

Remark. These formulas are really useful to prove some criteria for compactness of composition operators (see [2]). We can also use them to compare $\mathcal{A}^2$ and $\mathcal{H}^2$ norms. For example, assume that $f \in \mathcal{A}^2_\infty$. Then for $x > 0$, we have

$$\|f\|^2_{\mathcal{A}^2} \geq 4 \int_0^x \int_{\mathbb{T}^\infty} \int_{\mathbb{R}} |f'_\chi(\sigma + it)|^2 \, d\eta(t) \, dm(\chi) \, \sigma^2 \, d\sigma.$$

But we know that

$$\int_{\mathbb{T}^\infty} \int_{\mathbb{R}} |f'_\chi(\sigma + it)|^2 \, d\eta(t) \, dm(\chi) = \|f_\sigma\|^2_{\mathcal{H}^2} \geq \|f_x\|^2_{\mathcal{H}^2}$$

when $\sigma \leq x$. Thus we have

$$\|f\|^2_{\mathcal{A}^2} \geq 4\|f_x\|^2_{\mathcal{H}^2} \int_0^x \sigma^2 \, d\sigma \quad \text{and so} \quad \|f_x\|_{\mathcal{H}^2} \leq \frac{\sqrt{3}\|f\|_{\mathcal{A}^2}}{2x^3} \quad \forall x > 0.$$

Obviously we can do the same with the spaces $\mathcal{A}^2_\mu$.

Corollary 3. *Let $\varepsilon > 0$. Then $T_\varepsilon(\mathcal{A}^2_\mu) \subset \mathcal{H}^2 \subset \mathcal{A}^2_\mu$.*

2.5. Comparison of $\mathcal{A}^p$ and $\mathcal{H}^p$. We already saw that $\mathcal{H}^p \subset \mathcal{A}^p$. The goal of this section is to prove the following theorem, which looks surprising, in view of the classical results on the unit disk.

Theorem 8. *Let $p > 2$. The identity from $\mathcal{H}^2$ to $\mathcal{A}^p$ is not bounded but the identity from $\mathcal{H}^2$ to $\mathcal{A}^2$ is compact.*

Corollary 4. *Let $p > 1$. The identity from $\mathcal{H}^1$ to $\mathcal{A}^p$ is not bounded.*

Proof of Corollary 4. If the identity from $\mathcal{H}^1$ to $\mathcal{A}^p$ were bounded then by using squares of Dirichlet polynomials the identity from $\mathcal{H}^2$ to $\mathcal{A}^p$ would be bounded, contrary to Theorem 8. ∎

We need the following lemma (we have not found any such formula in the literature).

Lemma 2. *For $n \geq 1$, we have*

$$\sum_{k=0}^\infty \binom{n+k}{n}^2 z^k = \frac{1}{(1-z)^{2n+1}} \sum_{k=0}^n \binom{n}{k}^2 z^k.$$

Proof. We give two proofs:

Proof 1. For $n = 1$, we easily check that

$$\sum_{k=0}^\infty \binom{k+1}{1}^2 z^k = \frac{1+z}{(1-z)^3}.$$

We can now prove the equality by induction just by noting that

$$\binom{n+k}{n} \frac{n+k+1}{n+1} = \binom{n+k+1}{n+1}.$$

Now it suffices to compute the second derivative of the equality for the rank n to obtain the equality for rank $n+1$; the computation is tedious and we leave it to the reader.

Proof 2. We also give a quick and elementary argument. Fix $z \in \mathbb{D}$. We want to estimate $S = \sum_{k=0}^{\infty} \binom{n+k}{k}^2 z^k$.

Since for every $w \in \mathbb{D}$, we have

$$\frac{1}{(1-w)^{n+1}} = \sum_{k=0}^{\infty} \binom{n+k}{k} w^k,$$

we observe that

$$S = \frac{1}{n!} G^{(n)}(1) \quad \text{where} \quad G(w) = \frac{w^n}{(1-zw)^{n+1}}.$$

Using now the Leibniz formula, we get

$$S = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k!} \cdot \frac{(n+k)! z^k}{n!(1-z)^{n+k+1}} = \frac{1}{(1-z)^{2n+1}} \tilde{S}$$

where

$$\tilde{S} = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} z^k (1-z)^{n-k},$$

which is the derivative of order n at $w = z$ of the function

$$w \mapsto \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} w^{n+k} (1-z)^{n-k} = \frac{w^n}{n!} (w+1-z)^n.$$

Hence with the help of the Leibniz formula once again, we obtain

$$\tilde{S} = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k!} z^k \cdot \frac{n!}{(n-k)!} (z+1-z)^{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2 z^k,$$

which gives the conclusion. ∎

After this work was completed, M. De La Salle communicated to us a third proof which relies on the computation of a residue.

Remark. By uniqueness, we obtain

$$\sum_{j=0}^{\min(m,2n+1)} (-1)^j \binom{2n+1}{j} \binom{n+m-j}{n}^2 = \binom{n}{m}^2 \qquad \forall n, m \geq 1.$$

Definition 6. Let $m \geq 1$ be an integer. We define d_m to be the following multiplicative function:

$$d_m(k) = \sum_{\substack{d_1 \ldots d_m = k \\ d_1, \ldots, d_m \geq 1}} 1.$$

REMARK. We denote by $*$ the Dirichlet convolution: $(a*b)_n = \sum_{kl=n} a_k b_l$. Now it is easy to see that d_m is multiplicative because $d_m = \mathbb{1} * \cdots * \mathbb{1}$ (m times) where $\mathbb{1}(n) = 1$ for every $n \geq 1$.

PROPOSITION 4. *Let $m \geq 1$ be an integer. There exists $\gamma_m > 0$ such that*

$$\sum_{n=1}^{\infty} d_m(n)^2 n^{-2\sigma} \sim \frac{\gamma_m}{(2\sigma-1)^{m^2}} \quad \textit{when } \sigma \to 1/2.$$

Proof. We know that d_m is multiplicative, so

$$\sum_{n=1}^{\infty} d_m(n)^2 n^{-2\sigma} = \prod_{p\in\mathbb{P}} \Big(\sum_{k\geq 0} d_m(p^k)^2 p^{-2\sigma k} \Big).$$

Now, we can compute each series in the product because

$$d_m(p^k) = \sum_{\substack{p^{\alpha_1}\cdots p^{\alpha_m} = p^k \\ \alpha_1,\dots,\alpha_m \geq 0}} 1 = \sum_{\substack{\alpha_1+\cdots+\alpha_m = k \\ \alpha_1,\dots,\alpha_k \geq 0}} 1 = \binom{m+k-1}{m-1}.$$

So, by Lemma 2, we have

$$\sum_{n=1}^{\infty} d_m(n)^2 n^{-2\sigma} = \prod_{p\in\mathbb{P}} \bigg(\frac{\sum_{k=0}^{m-1} \binom{m-1}{k}^2 (p^{-2\sigma})^k}{(1-(p^{-2\sigma})^k)^{2m-1}} \bigg).$$

Now we have

$$\bigg(\sum_{k=0}^{m-1} \binom{m-1}{k}^2 z^k \bigg) (1-z)^{(m-1)^2} = Q(z)$$

where $Q(0) = 1$ and $Q'(0) = 0$ because the coefficient of z is

$$\binom{m-1}{1}^2 - \binom{(m-1)^2}{1} = (m-1)^2 - (m-1)^2 = 0.$$

We find that $(Q(p^{-2\sigma}))_p \in \ell^1$ when $\sigma \geq 1/2$ and the infinite product $\prod_{p\in\mathbb{P}} Q(p^{-2\sigma})$ is convergent and has a positive limit when $\sigma \to 1/2$. Finally, we obtain

$$\sum_{n=1}^{\infty} d_m(n)^2 n^{-2\sigma} = \frac{\prod_{p\in\mathbb{P}} Q(p^{-2\sigma})}{\prod_{p\in\mathbb{P}} (1-p^{-2\sigma})^{(m-1)^2+2m-1}} = \frac{\prod_{p\in\mathbb{P}} Q(p^{-2\sigma})}{\prod_{p\in\mathbb{P}} (1-p^{-2\sigma})^{m^2}}.$$

And so when $\sigma \to 1/2$, we obtain

$$\sum_{n=1}^{\infty} d_m(n)^2 n^{-2\sigma} = \zeta(2\sigma)^{m^2} \Big(\prod_{p\in\mathbb{P}} Q(p^{-2\sigma}) \Big) \sim \frac{\gamma_m}{(2\sigma-1)^{m^2}}. \ \blacksquare$$

An immediate corollary is

Corollary 5. *Let $m \geq 1$ be an integer. There exists $c_m > 0$ such that*

$$\|\zeta^m(\sigma+\cdot)\|_{\mathcal{H}^2} \sim \frac{c_m}{(2\sigma-1)^{m^2/2}} \qquad \textit{when } \sigma \to 1/2.$$

Now we can prove Theorem 8.

Proof of Theorem 8. Assume that the injection from $\mathcal{H}^2$ to $\mathcal{A}^p$ is bounded. Then there exists $m \geq 1$ such that

$$2 < 2(m+1)/m < p.$$

The identity from $\mathcal{H}^2$ to $\mathcal{A}^{2(m+1)/m}$ is then also bounded: there exists $C > 0$ such that, for every $f \in \mathcal{H}^2$,

$$\|f\|_{\mathcal{A}^{2(m+1)/m}} \leq C\|f\|_{\mathcal{H}^2}.$$

We apply this inequality to the mth power of the reproducing kernels of $\mathcal{H}^2 : s \mapsto \zeta^m(\sigma+s)$ with $\sigma > 1/2$. Then

$$\|\zeta^m(\sigma+\bullet)\|_{\mathcal{A}^{2(m+1)/m}} \leq C\|\zeta^m(\sigma+\bullet)\|_{\mathcal{H}^2}$$

and thanks to the last corollary we know that

$$\|\zeta^m(\sigma+\bullet)\|_{\mathcal{H}^2} \sim \frac{c_m}{(2\sigma-1)^{m^2/2}} \qquad \text{when } \sigma \to 1/2.$$

Now the left hand side of the above inequality satisfies

$$\|\zeta^m(\sigma+\bullet)\|_{\mathcal{A}^{2(m+1)/m}} = \|\zeta^{m+1}(\sigma+\bullet)\|_{\mathcal{A}^2}^{\frac{m}{m+1}} = \left(\sum_{n=1}^{\infty} \frac{d_{m+1}(n)^2 n^{-2\sigma}}{\log(n)+1}\right)^{\frac{m}{2(m+1)}}.$$

By the previous proposition, we know that

$$\sum_{n=1}^{\infty} d_{m+1}(n)^2 n^{-2\sigma} \sim \frac{\gamma_{m+1}}{(2\sigma-1)^{(m+1)^2}} \qquad \text{when } \sigma \to 1/2.$$

So by integration, we obtain

$$\left(\sum_{n=1}^{\infty} \frac{d_{m+1}(n)^2 n^{-2\sigma}}{\log(n)+1}\right)^{\frac{m}{2(m+1)}} \sim \frac{\widetilde{\gamma}_m}{(2\sigma-1)^{\frac{m^2(m+2)}{2(m+1)}}}$$

for some $\widetilde{\gamma}_m > 0$.

Now using the inequality given by the boundedness of the identity, we obtain, for σ close to $1/2$,

$$1 \lesssim C(2\sigma-1)^{\frac{m^2(m+2)}{2(m+1)} - m^2/2} = (2\sigma-1)^{\frac{m^2}{2(m+1)}},$$

and this is obviously false.

To finish the proof we have to show that the injection from $\mathcal{H}^2$ to $\mathcal{A}^2$ is compact. It suffices to remark that this injection is a diagonal operator for the orthonormal canonical basis $(\mathrm{e}_n)_{n\geq 1}$ of $\mathcal{H}^2$: the eigenvalues, equal to $1/(\log(n)+1)$, tend to zero. ∎

2.6. Inequalities on coefficients. We shall give some inequalities between the $\mathcal{A}^p$ norm and some weighted ℓ^p norms of the coefficients of the functions. This follows in spirit the classical estimates on Bergman spaces (see [12, p. 81] for instance).

THEOREM 9. *Let $p \geq 1$ and μ be a probability measure on $(0,\infty)$ such that $0 \in \operatorname{supp}(\mu)$ and $(w_n)_{n\geq 1}$ the associated weight.*

(i) *If $1 \leq p \leq 2$ and $f = \sum_{n\geq 1} a_n \mathrm{e}_n \in \mathcal{A}^p_\mu$, then*
$$\|w_n^{1/p} a_n\|_{\ell^{p'}} \leq \|f\|_{\mathcal{A}^p_\mu}.$$

(ii) *If $p \geq 2$ and $\sum_{n\geq 1} w_n^{p'-1}|a_n|^{p'} < \infty$, then $f = \sum_{n\geq 1} a_n \mathrm{e}_n \in \mathcal{A}^p_\mu$ and*
$$\|f\|_{\mathcal{A}^p_\mu} \leq \Big(\sum_{n\geq 1} w_n^{p'-1}|a_n|^{p'}\Big)^{1/p'} = \|w_n^{1/p} a_n\|_{\ell^{p'}}.$$

An immediate corollary is

COROLLARY 6. *Let $p \geq 1$.*

(i) *If $1 \leq p \leq 2$ and $f = \sum_{n\geq 1} a_n \mathrm{e}_n \in \mathcal{A}^p$, then*
$$\Big\|\frac{a_n}{(1+\ln(n))^{1/p}}\Big\|_{\ell^{p'}} \leq \|f\|_{\mathcal{A}^p}.$$

(ii) *If $p \geq 2$ and $\sum_{n\geq 1} \frac{|a_n|^{p'}}{(1+\ln(n))^{p'-1}} < \infty$, then $f = \sum_{n\geq 1} a_n \mathrm{e}_n \in \mathcal{A}^p$ and*
$$\|f\|_{\mathcal{A}^p} \leq \Big(\sum_{n\geq 1} \frac{|a_n|^{p'}}{(1+\ln(n))^{p'-1}}\Big)^{1/p'} = \Big\|\frac{a_n}{(1+\ln(n))^{1/p}}\Big\|_{\ell^{p'}}.$$

Proof of Theorem 9. Let us detail the case $1 \leq p \leq 2$.

For every integer $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots \geq 1$ and $f \in L^p(\mathbb{R}^+ \times \mathbb{T}^\infty, d\mu \otimes dm)$, set
$$\tau_n(f) = \int_{\mathbb{R}^+\times\mathbb{T}^\infty} f(\sigma,z)\bar{z}^{(n)} n^{-\sigma}\, d\mu(\sigma) \otimes dm(z)$$
where $z^{(n)} = z_1^{\alpha_1} z_2^{\alpha_2} \ldots$.

To a Dirichlet polynomial $P(s) = \sum_{n\geq 1} a_n n^{-s}$, we can associate as usual $f(\sigma,z) = \sum_{n\geq 1} a_n n^{-\sigma} z_1^{\alpha_1} z_2^{\alpha_2} \ldots$. In that case $\tau_n(f) = w_n a_n$.

Set $Q(f) = (\tau_n(f))_{n\geq 1}$. Then Q defines a norm one operator from $L^1(\mathbb{R}^+ \times \mathbb{T}^\infty, d\mu \otimes dm)$ to $L^\infty(\omega)$ and from $L^2(\mathbb{R}^+ \times \mathbb{T}^\infty, d\mu \otimes dm)$ to $L^2(\omega)$, where $L^q(\omega)$ is the Lebesgue space on the positive integers with discrete measure whose mass at n is $1/w_n$. Indeed,
$$|\tau_n(f)| \leq \|f\|_1 \quad \text{and} \quad \sum_{n\geq 1} \frac{|\tau_n(f)|^2}{w_n} = \sum_{n\geq 1} |\langle f, b_n\rangle|^2$$

where $b_n(\sigma,z) = z^{(n)} n^{-\sigma}/\sqrt{w_n}$ is an orthonormal system in the Hilbert space $L^2(\mathbb{R}^+ \times \mathbb{T}^\infty, d\mu \otimes dm)$. So the Bessel inequality gives

$$\sum_{n\geq 1} \frac{|\tau_n(f)|^2}{w_n} \leq \|f\|_2^2.$$

Now by interpolation (apply the Riesz–Thorin theorem), Q is bounded from $L^p(\mathbb{R}^+ \times \mathbb{T}^\infty, d\mu \otimes dm)$ to $L^{p'}(\omega)$:

$$\Big(\sum_{n\geq 1} \frac{|\tau_n(f)|^{p'}}{w_n}\Big)^{1/p'} \leq \|f\|_p.$$

Writing this inequality in the particular case of f associated to a Dirichlet polynomial (as described at the beginning of the proof) yields the result.

The other case is obtained in the same way (it is even easier). ∎

3. The Bergman spaces $\mathcal{B}^p$

3.1. The Bergman spaces of the infinite polydisk. Recall that $A = \lambda \otimes \lambda \otimes \cdots$ where λ is the normalized Lebesgue measure on the unit disk $\mathbb{D}$. For $p \geq 1$, $B^p(\mathbb{D}^\infty)$ is the Bergman space of the infinite polydisk. It is defined as the closure in $L^p(\mathbb{D}^\infty, A)$ of the span of the analytic polynomials.

Remark. Let P be an analytic polynomial defined for $z = (z_1, z_2, \dots) \in \mathbb{D}^\infty$ by $P(z) := \sum_{n=1}^N a_n z_1^{\alpha_1} \dots z_k^{\alpha_k}$. Then

$$\|P\|_{B^2(\mathbb{D}^\infty)} = \Big(\sum_{n=1}^N \frac{|a_n|^2}{(\alpha_1+1)\dots(\alpha_k+1)}\Big)^{1/2}.$$

So clearly $H^2(\mathbb{T}^\infty) \subset B^2(\mathbb{D}^\infty)$. In fact this is also true for any $p \geq 1$: it suffices to apply this property several times in the case of the unit disk.

Recall that the *Bergman kernel* at $z, w \in \mathbb{D}$ is defined by

$$k(w,z) := \frac{1}{(1-\overline{w}z)^2}.$$

Definition 7. Let $z \in \mathbb{D}^\infty$ and $\zeta \in \mathbb{D}^\infty \cap \ell^2$. For $n \geq 1$, we define

$$K_n(\zeta,z) := \prod_{i=1}^n k(\zeta_i, z_i) \quad \text{and} \quad K(\zeta,z) := \prod_{i=1}^\infty k(\zeta_i, z_i).$$

Then K is well defined thanks to the condition on ζ and the fact that (K_n) converges pointwise to K.

Remark. We know that

$$\|k(\zeta_i, \bullet)\|_2^2 = k(\zeta_i,\zeta_i) = \frac{1}{(1-|\zeta_i|^2)^2}.$$

So $K(\zeta, \cdot) \in B^2(\mathbb{D}^\infty)$ and

$$\|K(\zeta, \bullet)\|^2_{B^2(\mathbb{D}^\infty)} = \prod_{i=1}^{\infty} \frac{1}{(1-|\zeta_i|^2)^2}.$$

Proposition 5. *Let P be an analytic polynomial on $\mathbb{D}^\infty$ and let $\zeta \in \mathbb{D}^\infty \cap \ell^2$. Then*

$$|P(\zeta)| \le \Big(\prod_{i=1}^{\infty} \frac{1}{1-|\zeta_i|^2}\Big) \|P\|_{B^2(\mathbb{D}^\infty)}.$$

Proof. By the reproducing kernel property of the classical Bergman space used several times, we obtain

$$P(\zeta_1, \ldots, \zeta_n) = \int\limits_{\mathbb{D}^n} P(z_1, \ldots, z_n)\overline{K_n(\zeta, z)}\, d\lambda(z_1)\ldots d\lambda(z_n).$$

The Cauchy–Schwarz inequality gives the result. ∎

With the help of the previous proposition, we can extend by density the evaluation defined on the analytic polynomials for $z \in \mathbb{D}^\infty \cap \ell^2$. For every $f \in B^2(\mathbb{D}^\infty)$, we denote this extension by $\tilde{f}(\zeta)$ and we have

$$|\tilde{f}(\zeta)| \le \Big(\prod_{i=1}^{\infty} \frac{1}{1-|\zeta_i|^2}\Big) \|f\|_{B^2(\mathbb{D}^\infty)}.$$

Moreover the norm of the evaluation is exactly $\prod_{i=1}^{\infty} \frac{1}{1-|\zeta_i|^2}$. Actually in [9], the authors proved (in a more general setting) that $\tilde{f}$ is holomorphic on $\mathbb{D}^\infty \cap \ell^2$. We shall need the following lemma.

Lemma ([9]). *Let $\zeta \in \mathbb{D}^\infty \cap \ell^2$, $N \ge 1$ and $a \in \mathbb{R}$, and set*

$$G_N(z) := \prod_{i=1}^{N} (1-\overline{\zeta_i} z_i)^a.$$

Then $\{G_N\}$ is a bounded martingale in $L^2(\mathbb{T}^\infty)$.

Remark. Each G_n belongs to $B^2(\mathbb{D}^\infty)$ and we have

$$\|G_n\|_{L^2(\mathbb{D}^\infty)} = \|G_n\|_{B^2(\mathbb{D}^\infty)} \le \|G_n\|_{H^2(\mathbb{T}^\infty)} = \|G_n\|_{L^2(\mathbb{T}^\infty)}.$$

So $\{G_n\}$ is a bounded martingale in $L^2(\mathbb{D}^\infty)$. By Doob's theorem we know that the product converges pointwise and in norm in $B^2(\mathbb{D}^\infty)$.

We need to recall some notation and results from [9].

Let U be a uniform algebra on a compact space X and μ be a measure on X. Then $H^p(\mu)$ is the closure of U in $L^p(\mu)$.

Proposition ([9]). *Let U be a uniform algebra on a compact space X, μ be a probability measure on X, and $y \in X$ such that the point evaluation* *at y extends continuously to $H^2(\mu)$. Assume that any real power of the reproducing kernel of this point evaluation, $x \mapsto K(x,y)$, belongs to $H^2(\mu)$. Then for $p \geq 1$, we have*

$$|\tilde{f}(y)|^p \leq K(y,y) \int |f(x)|^p \, d\mu(x)$$

for every function f in $H^p(\mu)$, and the norm of the point evaluation at y is exactly $K(y,y)^{1/p}$.

From the last remark and this proposition, we deduce that the point evaluation at $\zeta \in \mathbb{D}^\infty \cap \ell^2$ is bounded on $B^p(\mathbb{D}^\infty)$ and we have

$$|f(\zeta)|^p \leq \prod_{i=1}^{\infty} \frac{1}{(1-|\zeta_i|^2)^2} \|f\|^p_{B^p(\mathbb{D}^\infty)}.$$

Moreover $\tilde{f}$ is holomorphic on $\mathbb{D}^\infty \cap \ell^2$ thanks to [9].

3.2. Point evaluation on $\mathcal{B}^p$. In the following, R will denote the infinite product of the probability measures $2r_i dr_i$ on $[0,1]$.

Definition 8. Let $P \in \mathcal{P}$ be of the form $\sum_{n=1}^N a_n n^{-s}$. We define on $\mathcal{P}$ the norm

$$\|P\|_{\mathcal{B}^p} := \Big(\int\limits_{[0,1]^\infty} \Big\| \sum_{n=1}^N a_n r_1^{\alpha_1} \ldots r_k^{\alpha_k} \mathrm{e}_n \Big\|^p_{\mathcal{H}^p} \, dR \Big)^{1/p}.$$

Remark. The fact that this defines a norm follows from the next proposition.

Definition 9. Let $p \geq 1$. We denote by $\mathcal{B}^p$ the closure of $\mathcal{P}$ in the norm $\|\cdot\|_{\mathcal{B}^p}$; it is the *Bergman space of Dirichlet series.*

Remark. We denote by $d(n)$ the number of divisors of n. For f as in (1), one has

$$\|f\|_{\mathcal{B}^2} = \Big(\sum_{n=1}^{\infty} \frac{|a_n|^2}{d(n)} \Big)^{1/2}.$$

First we use Bohr's point of view to specify the link between $\mathcal{B}^p$ and $B^p(\mathbb{D}^\infty)$.

Proposition 6. *Let $p \geq 1$.*

(i) $\|P\|_{\mathcal{B}^p} = \|D(P)\|_{B^p}$ *for all $P \in \mathcal{P}$.*
(ii) $D : \mathcal{P} \to B^p(\mathbb{D}^\infty)$ *extends to an isometric isomorphism from $\mathcal{B}^p$ onto $B^p(\mathbb{D}^\infty)$.*

Proof. The first fact is clear. For the second one, recall that $\mathcal{B}^p$ is the closure of $\mathcal{P}$ and that $B^p(\mathbb{D}^\infty)$ is the closure of the set of analytic polynomials. ∎

THEOREM 10. *Let $p \geq 1$ and $f \in \mathcal{B}^p$. The abscissa of uniform convergence of f satisfies $\sigma_{\mathrm{u}}(f) \leq 1/2$. Moreover, when $\Re(w) > 1/2$, we have*

$$|f(w)| \leq \zeta(2\Re(w))^{2/p}\|f\|_{\mathcal{B}^p} \quad \text{and} \quad \|\delta_w\|_{(\mathcal{B}^p)^*} = \zeta(2\Re(w))^{2/p}.$$

In addition, there exists $f \in \mathcal{B}^p$ such that $\sigma_{\mathrm{b}}(f) = 1/2$.

Proof. Let $f \in \mathcal{B}^p$ and $s \in \mathbb{C}_{1/2}$. We define $z_s = (p_1^{-s}, p_2^{-s}, \ldots) \in \mathbb{D}^\infty \cap \ell^2$. We know that $D(f) \in B^p(\mathbb{D}^\infty)$ and so

$$|D(f)(z_s)|^p \leq \prod_{i=1}^{\infty} \frac{1}{(1-|p_i^{-s}|^2)^2}\|D(f)\|^p_{B^p(\mathbb{D}^\infty)}.$$

But thanks to the last proposition, $\|D(f)\|_{B^p(\mathbb{D}^\infty)} = \|f\|_{\mathcal{B}^p}$ and

$$\begin{aligned} D(f)(z_s) &= \sum_{\substack{n=p_1^{\alpha_1}\ldots p_k^{\alpha_k}\\ n\geq 1}} a_n (p_1^{-s})^{\alpha_1}\ldots(p_k^{-s})^{\alpha_k} \\ &= \sum_{\substack{n=p_1^{\alpha_1}\ldots p_k^{\alpha_k}\\ n\geq 1}} a_n (p_1^{\alpha_1}\ldots p_k^{\alpha_k})^{-s} = \sum_{n\geq 1} a_n n^{-s}. \end{aligned}$$

Then we have

$$|f(s)|^p \leq \prod_{i=1}^{\infty} \frac{1}{(1-p_i^{-2\Re(s)})^2}\|f\|_{\mathcal{B}^p} = \zeta(2\Re(s))^2\|f\|^p_{\mathcal{B}^p}.$$

So f admits a bounded extension on each smaller half-plane in $\mathbb{C}_{1/2}$. By Bohr's theorem we have $\sigma_{\mathrm{u}}(f) \leq 1/2$.

To prove that the norm of the evaluation is exactly $\zeta(2\Re(w))^{2/p}$, it suffices to use the corresponding result from [9] on $B^p(\mathbb{D}^\infty)$. ∎

3.3. Comparison of $\mathcal{B}^p$ and $\mathcal{H}^p$. In this section, we study the link between $\mathcal{B}^p$ and $\mathcal{H}^p$. This question is natural as soon as we keep in mind the behavior of the injection from $H^p(\mathbb{D})$ to $B^q(\mathbb{D})$ in the classical framework of one variable Hardy–Bergman spaces on the unit disk. We recall that $H^p \subset B^q$ if and only if $q \leq 2p$ and that this injection is compact if and only if $q < 2p$ (see [19] for recent results on the limit case $q = 2p$).

First, following ideas of [5], we obtain a hypercontractivity result between the spaces $\mathcal{B}^p$.

Let $1 \leq p \leq q < \infty$. For $f \in B^p(\mathbb{D}^\infty)$, $z \in \mathbb{D}^\infty$ and $k \geq 1$, we define $\hat{z}_k = (z_1, \ldots, z_{k-1}, z_{k+1}, \ldots)$. Let $f_{\hat{z}_k}(z_k) = f(z)$. Then

$$\int_{\mathbb{D}^\infty} \|f_{\hat{z}_k}\|^r_{L^r(\mathbb{D})}\, dm(\hat{z}_k) = \|f\|^r_{L^r(\mathbb{D}^\infty)}.$$

We consider a sequence of operators $S_k : B^p(\mathbb{D}) \to B^q(\mathbb{D})$ for $k \geq 1$ such that $S_k(1) = 1$. If P is an analytic polynomial on $\mathbb{D}^\infty$, we define

$$P^1 = P, \qquad P^{k+1} = S_k(P_{\hat{z}_k}(z_k)).$$

This is not in general a sequence of polynomials but if P depends on $z_1,\dots,z_n$ then so does each term of this sequence. So this sequence is stationary.

Proposition 7. *If $\prod_{k=1}^{\infty}\|S_k\|<\infty$ then $(P^k)_{k\ge1}$ converges to some $S(P)\in B^q(\mathbb{D}^\infty)$. In addition, S extends to a bounded operator from $B^p(\mathbb{D}^\infty)$ to $B^q(\mathbb{D}^\infty)$.*

Actually, if we consider a sequence of operators $S_k: H^p(\mathbb{D})\to B^q(\mathbb{D})$, we obtain the following similar result.

Proposition 8. *If $\prod_{k=1}^{\infty}\|S_k\|<\infty$ then $(P^k)_{k\ge1}$ converges to some $S(P)\in B^q(\mathbb{D}^\infty)$. In addition, S extends to a bounded operator from $H^p(\mathbb{T}^\infty)$ to $B^q(\mathbb{D}^\infty)$.*

We only give the proof of the second proposition.

Proof. It suffices to show that

$$\|P^{k+1}\|_{B^q(\mathbb{D}^\infty)}\le\Big(\prod_{i=1}^{k}\|S_i\|\Big)\|P\|_{H^p(\mathbb{T}^\infty)}.$$

One has

$$\begin{aligned}
\|P^{k+1}\|^q_{B^q(\mathbb{D}^\infty)}&=\int_{\mathbb{D}^\infty}|S_k(P^k_{\hat z_k}(z_k))|^q\,dA(z)\\
&=\int_{\mathbb{D}^\infty}\int_{\mathbb{D}}|S_k(P^k_{\hat z_k}(z_k))|^q\,d\lambda(\hat z_k)\,dA(z_k)\\
&=\int_{\mathbb{D}^\infty}\|S_k(P^k_{\hat z_k}(\cdot))\|^q_{B^q(\mathbb{D})}\,dA(\hat z_k)\\
&\le\|S_k\|^q\int_{\mathbb{D}^\infty}\|P^k_{\hat z_k}(\cdot)\|^q_{H^p(\mathbb{T})}\,dA(\hat z_k)\\
&=\|S_k\|^q\int_{\mathbb{D}^\infty}\Big(\int_{\mathbb{T}}|P^k_{\hat z_k}(\chi_k)|^p\,dm(\chi_k)\Big)^{q/p}dA(\hat z_k).
\end{aligned}$$

Since $q/p\ge1$, we get, by the integral triangular inequality,

$$\|P^{k+1}\|^q_{B^q(\mathbb{D}^\infty)}\le\|S_k\|^q\Big(\int_{\mathbb{T}}\Big(\int_{\mathbb{D}^\infty}|P^k_{\hat z_k}(\chi_k)|^q\,dm(\hat z_k)\Big)^{p/q}dm(\chi_k)\Big)^{q/p}.$$

By induction, we obtain the result. ∎

We shall give some applications of these propositions, but we first need other preliminaries, in the classical setting of the unit disk. In the following, for $q\ge1$, the space $B^q(\mathbb{D})$ (resp. $H^q(\mathbb{D})$) is the classical Bergman space (resp. the classical Hardy space).

Lemma 3. *The sequence* $\left(\frac{2}{n+2}\right)_{n\ge 0}$ *defines a multiplier from* $B^1(\mathbb{D})$ *to* $H^1(\mathbb{D})$ *with norm exactly* 1*: for every* $f(z)=\sum_{n\ge 1} a_n z^n \in B^1(\mathbb{D})$, *we have*

$$\Big\|\sum_{n\ge 1}\frac{2}{n+2}a_n z^n\Big\|_{H^1(\mathbb{D})} \le \|f\|_{B^1(\mathbb{D})}.$$

Proof. Let $r<1$ and $f\in B^1(\mathbb{D})$ of the form $f(z)=\sum_{n=0}^{\infty} a_n z^n$. Then if we denote by M the multiplier operator $B^1(\mathbb{D})\ni f\mapsto Mf(z)=\sum_{n\ge 1}\frac{2}{n+2}a_n z^n$, we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}|Mf(re^{i\theta})|\,d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\Big|\sum_{n=0}^{\infty}\frac{2}{n+2}a_n r^n e^{in\theta}\Big|\,d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi}\Big|\sum_{n=0}^{\infty}2\int_0^1 a_n\rho^{n+1}r^n e^{in\theta}\,d\rho\Big|\,d\theta\\ &\le \frac{1}{\pi}\int_0^{2\pi}\int_0^1\Big|\sum_{n=0}^{\infty}a_n\rho^n r^n e^{in\theta}\Big|\rho\,d\rho\,d\theta.\end{aligned}$$

Letting $r\to 1$, we obtain the result. ∎

Lemma 4. *Let* $r\le 2/3$. *Then* $\left(r^n\frac{n+2}{2\sqrt{n+1}}\right)_{n\ge 0}$ *is a multiplier from* $H^1(\mathbb{D})$ *to* $H^2(\mathbb{D})$ *with norm* 1.

Proof. We adapt a proof from [8]. Let $f\in H^1(\mathbb{D})$, with norm 1, be of the form

$$f(z)=\sum_{n=0}^{\infty}a_n z^n.$$

We consider a factorization $f=gh$ where g and h are in $H^2(\mathbb{D})$ and $|g|^2=|h|^2=1$. Denote by (b_n) and (c_n) the Fourier coefficients of g and h. Then

$$a_n=\sum_{k=0}^{n}b_k c_{n-k}.$$

We also know that

$$\sum_{n=0}^{\infty}|b_n|^2=\sum_{n=0}^{\infty}|c_n|^2=1.$$

We want to show that

$$\sum_{n=0}^{\infty}\Big|a_n r^n\frac{n+2}{2\sqrt{n+1}}\Big|^2\le 1.$$

We can assume that the coefficients b_n and c_n are all non-negative (at worst, the modulus of a_n becomes larger but we are looking for a sufficient condition for the inequality so this is not a problem).

So the last inequality is equivalent to

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} b_k c_{n-k} r^n d_n \frac{n+2}{2\sqrt{n+1}} \le 1$$

for all non-negative sequences (d_n) with ℓ^2 norm 1. This is equivalent to

$$\sum_{k,n=0}^{\infty} b_k c_n r^{n+k} d_{n+k} \frac{n+k+2}{2\sqrt{n+k+1}} \le 1.$$

We will get this inequality as soon as we show that

$$\sum_{k,n=0}^{\infty} \left(r^{n+k} \frac{n+k+2}{2\sqrt{n+k+1}} d_{n+k} \right)^2 \le 1.$$

But for any non-negative integer j, we only have $j+1$ ways to write j as the sum of two non-negative integers. So it suffices to prove that

$$\sum_{j=0}^{\infty} r^{2j} \frac{(j+2)^2}{4} d_j^2 \le 1.$$

By the definition of (d_n), this will follow from

$$r^{2j} \frac{(j+2)^2}{4} \le 1 \quad \text{for any } j \ge 0.$$

This latter inequality is clearly true for $j=0$ for any r, so we just have to compute

$$r_0 = \inf_{j\ge 1} \left(\frac{2}{j+2} \right)^{1/j}.$$

We can easily check that $x \mapsto \frac{\ln(2/(x+2))}{x}$ is increasing on $[1,\infty[$ so we obtain $r_0 = 2/3$. ■

The following is obvious and is just a rewriting of the norms.

Lemma 5. *The sequence* $(\sqrt{n+1})_{n\ge 0}$ *defines a multiplier from* $H^2(\mathbb{D})$ *to* $B^2(\mathbb{D})$ *with norm exactly* 1.

Now we can state a contractive type result on classical Bergman spaces. Here P_r denotes the blow-up operator $P_r(f)(z) = f(rz)$.

Theorem 11. *If* $r \le 2/3$, *then* $P_r : B^1(\mathbb{D}) \to B^2(\mathbb{D})$ *is bounded with norm* 1. *Conversely, if* $P_r : B^1(\mathbb{D}) \to B^2(\mathbb{D})$ *is bounded with norm* 1 *then* $r \le 1/\sqrt{2}$.

Proof. If $r \le 2/3$, it suffices to apply the previous three lemmas.

Conversely, assume that $P_r : B^1(\mathbb{D}) \to B^2(\mathbb{D})$ is bounded with norm 1. Let $a \in \mathbb{R}$. We have

$$\|1+arz\|_{B^2(\mathbb{D})}^2 = 1 + \frac{a^2 r^2}{2}, \qquad \|1+az\|_{B^1(\mathbb{D})} = 1 + \frac{a^2}{8} + o(a^2).$$

So we have

$$1+\frac{a^2r^2}{2}\le\left(1+\frac{a^2}{8}+o(a^2)\right)^2=1+\frac{a^2}{4}+o(a^2).$$

And thus $r^2\le 1/2$. ■

Now we have another consequence of the preceding results, which will be used in the next section, and is similar to Lemma 1.

PROPOSITION 9. *Let $\varepsilon>0$. Then $T_\varepsilon:\mathcal{B}^1\to\mathcal{B}^2$ is bounded.*

Proof. We consider the following sequence of operators (we keep the notation of the preceding theorem):

$$S_k:B^1(\mathbb{D})\to B^2(\mathbb{D}),\qquad f\mapsto P_{p_k^{-\varepsilon}}(f),$$

where P_r is the classical Poisson kernel. Indeed, if we apply Proposition 7 to this sequence of operators and to a Dirichlet series f of the form (1), we obtain

$$\begin{aligned}S(f)(s)&=\sum_{n=p_1^{\alpha_1}\dots p_k^{\alpha_k}\ge 1}a_n(p_1^{-s-\varepsilon})^{\alpha_1}\dots(p_k^{-s-\varepsilon})^{\alpha_k}\\&=\sum_{n=p_1^{\alpha_1}\dots p_k^{\alpha_k}\ge 1}a_n n^{-s-\varepsilon}=T_\varepsilon(f)(s).\end{aligned}$$

We know from the preceding theorem that $\|P_r\|_{B^1(\mathbb{D})\to B^2(\mathbb{D})}\le 1$ for r quite small and we obtain our result for T_ε because $p_k^{-\varepsilon}\to 0$ as $k\to\infty$, and so the infinite product of the norm is finite. ■

NOTATION. Let $p\ge 1$. We denote by $\mathcal{H}^p_{\mathbb{P}}$ (resp. $\mathcal{B}^p_{\mathbb{P}}$)) the following subspace of $\mathcal{H}^p$ (resp. $\mathcal{B}^p$):

$$\mathcal{H}^p_{\mathbb{P}}=\overline{\operatorname{span}(e_k:k\in\mathbb{P})}^{\mathcal{H}^p}\qquad(\text{resp. }\mathcal{B}^p_{\mathbb{P}}=\overline{\operatorname{span}(e_k:k\in\mathbb{P})}^{\mathcal{B}^p}).$$

THEOREM 12. *Let $p\ge 1$.*

(i) *The identity from $\mathcal{H}^p$ to $\mathcal{B}^{2p}$ is bounded with norm 1, but*
(ii) *the identity from $\mathcal{H}^p$ to $\mathcal{B}^p$ is not compact. Actually, it is not a strictly singular operator.*

Proof. (i) Recall [12] that the identity from $H^p(\mathbb{D})$ to $B^{2p}(\mathbb{D})$ is bounded with norm 1 so it suffices to use Proposition 8 to get the boundedness of our operator.

(ii) In [5], it is shown that $\mathcal{H}^p_{\mathbb{P}}=\mathcal{H}^2_{\mathbb{P}}$ and in the same way we find that $\mathcal{B}^p_{\mathbb{P}}=\mathcal{B}^2_{\mathbb{P}}$. But clearly $\mathcal{H}^2_{\mathbb{P}}=\mathcal{B}^2_{\mathbb{P}}$ so $\mathcal{H}^p$ and $\mathcal{B}^p$ have isomorphic infinite-dimensional closed subspaces and so the identity from $\mathcal{H}^p$ to $\mathcal{B}^p$ is not strictly singular. ■

REMARKS. (i) When $p = 1$, (i) has already been proved by Helson [16].

(ii) We can check easily that for every $n \neq m$ we have

$$\|\mathrm{e}_{p_n} - \mathrm{e}_{p_m}\|_{\mathcal{B}^p} \geq \|\mathrm{e}_{p_n}\|_{\mathcal{B}^p} = \left(\frac{2}{p+2}\right)^{1/p},$$

and hence we obtain another proof of non-compactness in Theorem 12(ii).

(iii) We mention that it is immediate (without invoking Theorem 12(ii)) that the identity from $\mathcal{H}^p$ to $\mathcal{B}^{2p}$ is not compact: indeed, if it were, by restriction to the variable $z_1 = 2^{-s}$, the identity from $H^p(\mathbb{D})$ to $B^{2p}(\mathbb{D})$ would be compact, which is not the case.

Actually, we can prove that $\mathcal{H}^2 \subset \mathcal{B}^4$ by a simple computation on the coefficients of the Dirichlet series. Let f be a Dirichlet series of the form (1). We want to show that $\|f\|_{\mathcal{B}^4} \leq \|f\|_{\mathcal{H}^2}$. We have

$$\|f\|_{\mathcal{B}^4}^4 = \|f^2\|_{\mathcal{B}^2}^2.$$

But $f^2(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ with $b_n = \sum_{d|n} a_d a_{n/d}$ for all $n \geq 1$. So

$$\|f^2\|_{\mathcal{B}^2}^2 = \sum_{n=1}^{\infty} \frac{|\sum_{d|n} a_d a_{n/d}|^2}{d(n)}.$$

Now we apply the Cauchy–Schwarz inequality using the fact that the sum contains exactly $d(n)$ terms to get

$$\|f^2\|_{\mathcal{B}^2}^2 \leq \sum_{n=1}^{\infty} \sum_{d|n} |a_d|^2\, |a_{n/d}|^2.$$

Exchanging the sums yields

$$\|f^2\|_{\mathcal{B}^2}^2 \leq \sum_{d=1}^{\infty} |a_d|^2 \sum_{d|n} |a_{n/d}|^2.$$

But if n is a multiple of d, then n/d is in $\mathbb{N}$, and

$$\sum_{d|n} |a_{n/d}|^2 = \|f\|_{\mathcal{H}^2}^2.$$

Finally, we get $\|f\|_{\mathcal{B}^4}^4 \leq \|f\|_{\mathcal{H}^2}^4$.

3.4. Generalized vertical limit functions

DEFINITION 10. Let $\chi \in \mathbb{D}^\infty$ and f be of the form (1). We denote by f_χ the Dirichlet series

$$f_\chi = \sum_{n=1}^{\infty} a_n \chi(n) \mathrm{e}_n.$$

In this part, we apply the same trick as in [13] and [5] to obtain another expression of the norm in $\mathcal{B}^p$, useful for the study of composition operators.

Let $\varphi_1(z) = \frac{1+z}{1-z}$ be the Cayley transform which maps $\mathbb{D}$ onto $\mathbb{C}_+$. We will say that a function f is in $H_i^p(\mathbb{C}_+)$ if $f \circ \varphi_1 \in H^p(\mathbb{D})$ (the classical Hardy space on the unit disk). In this case we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f \circ \varphi_1(e^{i\theta})|^p \, d\theta = \int_{\mathbb{R}} |f(it)|^p \, d\lambda_i(t)$$

where

$$d\lambda_i(t) = \frac{dt}{\pi(1+t^2)}.$$

Definition 11. Let $t \in \mathbb{R}$. We define the *Kronecker flow* T_t on $\mathbb{D}^\infty$ by

$$T_t(z_1, z_2, \dots) := (p_1^{-it} z_1, p_2^{-it} z_2, \dots).$$

Lemma 6. *Let $\chi \in \mathbb{D}^\infty$, $f \in \mathcal{B}^p$ and $t \in \mathbb{R}$. Set $g_\chi(it) := D(f)(T_t\chi)$. Then for any finite Borel measure w on $\mathbb{R}$, one has*

$$\int_{\mathbb{D}^\infty} \int_{\mathbb{R}} |g_\chi(it)|^p \, dw(t) \, dA(\chi) = w(\mathbb{R}) \|f\|_{\mathcal{B}^p}^p.$$

Proof. The Kronecker flow (T_t) is just a rotation on $\mathbb{D}^\infty$, so

$$\begin{aligned}\int_{\mathbb{D}^\infty} |g_\chi(it)|^p \, dA(\chi) &= \int_{\mathbb{D}^\infty} |D(f)(T_t\chi)|^p \, dA(\chi) \\ &= \int_{\mathbb{D}^\infty} |D(f)(\chi)|^p \, dA(\chi) = \|D(f)\|_{\mathcal{B}^p}^p.\end{aligned}$$

We conclude the proof using Fubini's theorem. ∎

Proposition 10. *Let $\chi \in \mathbb{D}^\infty$ and $f \in \mathcal{B}^p$. Then $g_\chi \in H_i^p(\mathbb{C}_+)$ and g_χ is an extension of f_χ onto $\mathbb{C}_+$.*

Proof. Thanks to the previous lemma, we already know that $g_\chi \in L^p(\lambda_i)$ for almost every $\chi \in \mathbb{D}^\infty$. So it suffices to show (we use here a characterization of the classical Hardy space)

$$\int_{-\infty}^{\infty} \left(\frac{1-it}{1+it}\right)^n g_\chi(it) \, d\lambda_i(t) = 0 \quad \text{for } n \geq 1.$$

We use the same ideas as in [13] and [5] but we have to adapt the proof because here we do not work with Fourier series on $\mathbb{T}^\infty$ but with functions in $B^p(\mathbb{D}^\infty)$. We fix $n \geq 1$ and define

$$G(\chi) := \int_{-\infty}^{\infty} \left(\frac{1-it}{1+it}\right)^n g_\chi(it) \, d\lambda_i(t).$$

Clearly $G \in L^p(\mathbb{D}^\infty)$ because

$$\int\limits_{\mathbb{D}^\infty} |G(\chi)|^p \, dA(\chi) = \int\limits_{\mathbb{D}^\infty} \Big| \int\limits_{-\infty}^{\infty} \Big(\frac{1-it}{1+it}\Big)^n g_\chi(it)\, d\lambda_i(t)\Big|^p \, dA(\chi)$$
$$\leq \int\limits_{\mathbb{D}^\infty} \int\limits_{-\infty}^{\infty} |g_\chi(it)|^p \, d\lambda_i(t)\, dA(\chi) = \|D(f)\|^p_{\mathcal{B}^p}$$

where the last inequality follows from the preceding lemma.

Actually, $G \in B^p(\mathbb{D}^\infty)$. It suffices to show that there exists a sequence of analytic polynomials which converges to G. Since $f \in B^p(\mathbb{D}^\infty)$, we have $D(f) \in B^p(\mathbb{D}^\infty)$ and there exists a sequence (P_k) of analytic polynomials such that

$$\|D(f) - P_k\|_{B^p(\mathbb{D}^\infty)} \xrightarrow[k\to\infty]{} 0.$$

Then we define the analytic polynomial

$$Q_k(\chi) := \int\limits_{-\infty}^{\infty} \Big(\frac{1-it}{1+it}\Big)^n P_k(T_t\chi)\, d\lambda_i(t)$$

and we claim that (Q_k) converges to G. Indeed,

$$\|G - Q_k\|^p_{B^p(\mathbb{D}^\infty)} = \int\limits_{\mathbb{D}^\infty} \Big| \int\limits_{-\infty}^{\infty} \Big(\frac{1-it}{1+it}\Big)^n (g_\chi(it) - P_{k,\chi}(it))\, d\lambda_i(t)\Big|^p \, dA(\chi).$$

We get, by Fubini's theorem,

$$\|G - Q_k\|^p_{B^p(\mathbb{D}^\infty)} \leq \int\limits_{-\infty}^{\infty} \|(D(f) - P_k)(T_t(\cdot))\|^p_{B^p(\mathbb{D}^\infty)}\, d\lambda_i(t);$$

but T_t is just a rotation, so

$$\|G - Q_k\|^p_{B^p(\mathbb{D}^\infty)} \leq \int\limits_{-\infty}^{\infty} \|D(f) - P_k\|^p_{B^p(\mathbb{D}^\infty)}\, d\lambda_i(t) = \|D(f) - P_k\|^p_{B^p(\mathbb{D}^\infty)},$$

which goes to zero as $k \to \infty$, and this proves our claim.

We claim now that G vanishes almost everywhere. Since $G \in B^p(\mathbb{D}^\infty)$, it suffices to prove that G is orthogonal to every monomial with positive index. Let $q \in \mathbb{N}$. We have

$$\int\limits_{\mathbb{D}^\infty} \overline{\chi}(q) G(\chi)\, dA(\chi) = \int\limits_{-\infty}^{\infty} \Big(\frac{1-it}{1+it}\Big)^n \int\limits_{\mathbb{D}^\infty} \overline{\chi}(q) g_\chi(it)\, dA(\chi)\, d\lambda_i(t).$$

Actually we have

$$\int\limits_{\mathbb{D}^\infty} \overline{\chi}(q) g_\chi(it)\, dA(\chi) = 0$$

because $g_\chi(it) = Df(T_t\chi) \in B^p(\mathbb{D}^\infty)$. This is clear for f being a polynomial, and by density this proves the claim.

The proof that g_χ is an extension of f_χ is the same as in the case of $\mathcal{H}^p$ (see [5]). ∎

Now we shall denote the extension by f_χ instead of g_χ. As in the case of $\mathcal{H}^p$ with $p \geq 1$, this extension is almost surely simple.

PROPOSITION 11. *Let $\chi \in \mathbb{D}^\infty$ and $f \in \mathcal{B}^p$ for $p \geq 1$. Then for almost every χ (relative to the measure A on $\mathbb{D}^\infty$), f_χ converges on $\mathbb{C}_+$.*

Proof. Let $f \in \mathcal{B}^2$ be of the form (1). We consider $L^2(\mathbb{D}^\infty, A)$ and the orthonormal sequence $\Phi_n(\chi) = \sqrt{d(n)}\,\chi(n)$. For $\sigma > 0$ and $t \in \mathbb{R}$, let $c_n := a_n n^{-\sigma - it}/\sqrt{d(n)}$. We observe that $(a_n/\sqrt{d(n)})_{n\geq 1} \in \ell^2$, and that $(n^{-\sigma}\log(n))_{n\geq 1} \in \ell^\infty$, hence

$$\sum_{n=1}^{\infty} |c_n|^2 \log^2(n) < \infty.$$

So Men'shov's theorem gives that $\sum c_n \Phi_n(\chi)$ converges for almost every χ. Therefore, we get the result when $p = 2$.

When $p \neq 2$, it suffices to prove the result for $p = 1$. As in the case of the spaces $\mathcal{A}^p$, the result follows from Proposition 9. ∎

Let $f \in \mathcal{B}^2$. We know that for almost all $\chi \in \mathbb{D}^\infty$, f_χ converges on $\mathbb{C}_+$ and so $g_\chi = f_\chi$. We obtain, for each probability measure w on $\mathbb{R}$,

$$\|f\|_{\mathcal{B}^2}^2 = \int_{\mathbb{R}} \int_{\mathbb{D}^\infty} |f_\chi(it)|^2 \, dA(\chi)\, dw(t).$$

THEOREM 13. *Let $f \in \mathcal{B}^2$ and w be a probability measure on $\mathbb{R}$. Then*

$$\|f\|_{\mathcal{B}^2}^2 = |f(\infty)|^2 + 4\int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{D}^\infty} \sigma |f_\chi(\sigma + it)|^2 \, dA(\chi)\, d\sigma\, dw(t).$$

Proof. For $\sigma > 0$, we have

$$\int_{\mathbb{D}^\infty} \int_{\mathbb{R}} |f_\chi'(\sigma + it)|^2 \, dw(t)\, dA(\chi) = \|f'\|_{\mathcal{B}^2}^2 = \sum_{n=2}^{\infty} \frac{|a_n|^2 n^{-2\sigma} \log^2(n)}{d(n)}.$$

We multiply by σ and it suffices to remark that

$$\int_0^\infty \sigma n^{-2\sigma} d\sigma = \frac{1}{4\log^2(n)} \cdot \ \blacksquare$$

3.5. Inequalities on coefficients of $\mathcal{B}^p$ functions. We shall give here some inequalities between the $\mathcal{B}^p$ norm and some weighted ℓ^p norms of the coefficients of the functions (as in Theorem 9).

THEOREM 14. *Let $p \geq 1$.*

(i) *If $1 \leq p \leq 2$ and $f = \sum_{n\geq 1} a_n \mathrm{e}_n \in \mathcal{B}^p$, then*

$$\left\| \frac{a_n}{d(n)^{1/p}} \right\|_{\ell^{p'}} \leq \|f\|_{\mathcal{B}^p}.$$

(ii) *If $p \geq 2$ and $\sum_{n\geq 1} \frac{|a_n|^{p'}}{d(n)^{p'-1}} < \infty$, then $f = \sum_{n\geq 1} a_n \mathrm{e}_n \in \mathcal{B}^p$ and*

$$\|f\|_{\mathcal{B}^p} \leq \left(\sum_{n\geq 1} \frac{|a_n|^{p'}}{d(n)^{p'-1}} \right)^{1/p'} = \left\| \frac{a_n}{d(n)^{1/p}} \right\|_{\ell^{p'}}.$$

Proof. We do not give the details since the proof follows the same ideas as the one of Theorem 9.

(i) For every integer $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots \geq 1$ and $f \in L^p(\mathbb{D}^\infty, dA)$, set

$$\tau_n(f) = \int_{\mathbb{D}^\infty} f(z)\bar{z}^{(n)}\, dA$$

where $\bar{z}^{(n)} = \bar{z}_1^{\alpha_1} \bar{z}_2^{\alpha_2} \ldots$

With a Dirichlet polynomial $P(s) = \sum_{n\geq 1} a_n n^{-s}$, we associate as usual $f(z) = D(P)(z) = \sum_{n\geq 1} a_n z_1^{\alpha_1} z_2^{\alpha_2} \ldots$. In that case $\tau_n(f) = a_n/d(n)$.

Then we consider $Q(f) = (\tau_n(f))_{n\geq 1}$. This defines norm one operators from $L^1(\mathbb{D}^\infty, dA)$ to $L^\infty(\omega)$ and from $L^2(\mathbb{D}^\infty, dA)$ to $L^2(\omega)$, with $\omega(n) = d(n)$. An interpolation argument gives the conclusion.

In the same spirit, we prove (ii). ∎

4. Comparison of $\mathcal{A}^p_\mu$ and $\mathcal{B}^p$. It is worth mentioning that $\mathcal{A}^p$ and $\mathcal{B}^p$ are not the same space. Actually, more generally, we have

THEOREM 15. *Let μ be a probability measure whose support contains* 0, *and let $\alpha > -1$ and $p, q \geq 1$.*

(1) $\mathcal{A}^p_\mu \not\subseteq \mathcal{B}^q$.
(2) $\mathcal{B}^q \not\subseteq \mathcal{A}^p_\alpha$.

In the following, $A \approx B$ means that there exist positive constants c, d depending on p (or q) such that $cA \leq B \leq dA$.

Proof. Our proof relies on some estimates of the norms of the e_n. We claim that:

(i) $\log \|\mathrm{e}_n\|_{\mathcal{A}^p_\alpha} \approx -\log(\log(n))$.
(ii) $\log \|\mathrm{e}_n\|_{\mathcal{B}^q} \approx \log(1/d(n))$.

Indeed, when $1 \leq p \leq 2$, on the one hand $\|\mathrm{e}_n\|_{\mathcal{A}^p_\alpha} \leq \|\mathrm{e}_n\|_{\mathcal{A}^2_\alpha}$. On the other hand, $\|\mathrm{e}_n\|_{\mathcal{A}^2_\alpha} \leq \|\mathrm{e}_n\|^{p/2}_{\mathcal{A}^p_\alpha} \|\mathrm{e}_n\|^{1-p/2}_\infty = \|\mathrm{e}_n\|^{p/2}_{\mathcal{A}^p_\alpha}$ so that

$$(\log(n+1))^{-(\alpha+1)/2} \geq \|\mathrm{e}_n\|_{\mathcal{A}^p_\alpha} \geq (\log(n+1))^{-(\alpha+1)/p}.$$

In the same way, when $p \geq 2$, we have

$$\log(n+1)^{-(\alpha+1)/2} \leq \|\mathrm{e}_n\|_{\mathcal{A}^p_\alpha} \leq \log(n+1)^{-(\alpha+1)/p}.$$

The same method gives $d(n)^{-1/q} \leq \|\mathrm{e}_n\|_{\mathcal{B}^q} \leq d(n)^{-1/2}$ when $q \leq 2$, and $d(n)^{-1/2} \leq \|\mathrm{e}_n\|_{\mathcal{B}^q} \leq d(n)^{-1/q}$ when $q \geq 2$. This proves (ii) and our claim.

Now the theorem follows easily:

Let us prove clause 1. Assume that $\mathcal{A}^p_\mu \subseteq \mathcal{B}^q$. Thanks to the closed graph theorem, the identity from $\mathcal{A}^p_\mu$ to $\mathcal{B}^q$ is bounded and there exists $C > 0$ such that $\|\mathrm{e}_{p_n}\|_{\mathcal{B}^q} \leq C\|\mathrm{e}_{p_n}\|_{\mathcal{A}^p_\mu}$ for every $n \geq 1$.

Moreover it is clear that $\lim_{n\to\infty} \|\mathrm{e}_{p_n}\|_{\mathcal{A}^p_\mu} = 0$.

But the previous claim implies that $\|\mathrm{e}_{p_n}\|_{\mathcal{B}^q} \approx 1$ and we get a contradiction.

Let us now prove the second point. Assuming the contrary, we would have some $C > 0$ such that for every $n \geq 1$,

$$\|\mathrm{e}_n\|_{\mathcal{A}^p_\alpha} \leq C\|\mathrm{e}_n\|_{\mathcal{B}^q}.$$

A fortiori

$$\log(d(n)) \lesssim \log(\log(n)).$$

But this contradicts the extremal order of $(\log(d(n)))_{\geq 1}$ (see [31, p. 82] for example):

$$\limsup_{n\to\infty} \frac{\log(d(n))\log(\log(n))}{\log(2)\log(n)} = 1. \ \blacksquare$$

5. Appendix: Around the norm of point evaluation. We wish to present here a principle of comparing (for different p) the norms of point evaluation. We shall work in a rather general framework of subspaces of functions of some L^p spaces. When one works on classical spaces of analytic functions (Hardy–Bergman spaces), this principle is useless, since one can essentially work with any power of a function. In the context of Dirichlet series, a major difficulty is that we have no way to consider f^α when α is not an integer (and $f \in \mathcal{D}$). The following method can be helpful and gives very precise results in some particular cases.

In this section, we consider some subspaces $X_p \subset L^p(\Omega, \nu)$ of functions on Ω, where ν is a probability measure on Ω and $p \geq 1$. We assume that there exists some algebra $\mathcal{P} \subset \bigcap_{p\geq 1} X_p$ which is dense in each X_p (think of the polynomials in many contexts).

We fix some $\omega \in \Omega$ and we assume that the point evaluation $X_p \ni f \mapsto f(\omega)$ is bounded with norm N_p.

Let us mention that most often, thanks to the theory of reproducing kernels, the value of N_2 is known (and easy to get).

We now list several simple observations which we have used in this paper.

Proposition 12. *With the previous notation:*

(i) *If* $1/p = 1/q_1 + 1/q_2$, *then* $N_p \geq N_{q_1} N_{q_2}$.
(ii) *If* $q \geq p \geq 1$, *then* $N_p \geq N_q$.
(iii) *Let* m *be an integer. Then* $N_{pm} \leq (N_p)^{1/m}$ *for every* $p \geq 1$. *In particular,* $N_{2m} \leq (N_2)^{1/m}$.

Proof. (i) Let $f, g \in \mathcal{P}$ with $\|f\|_{L^{q_1}} = 1$ and $\|g\|_{L^{q_2}} = 1$. The product fg still belongs to $\mathcal{P} \subset X_p$ and we have

$$N_p \geq N_p \|fg\|_{L^p} = N_p \|fg\|_{X^p} \geq |f(\omega)| \cdot |g(\omega)|.$$

Taking now the upper bound relative to f and to g yields the first assertion.

(ii) is trivial.

(iii) By an obvious induction, we have $N_p \geq N_{q_1} \ldots N_{q_r}$ as soon as $1/p = 1/q_1 + \cdots + 1/q_r$. In particular, since $1/p = 1/(pm) + \cdots + 1/(pm)$ (m times), we get $N_p \geq (N_{pm})^m$. ∎

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