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Nevanlinna algebras

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Abstract. The Nevanlinna algebras, $\mathcal{N}_\alpha^p$, of this paper are the L^p variants of classical weighted area Nevanlinna classes of analytic functions on $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. They are F-algebras, neither locally bounded nor locally convex, with a rich duality structure.

For $s = (\alpha+2)/p$, the algebra F_s of analytic functions $f : \mathbb{U} \to \mathbb{C}$ such that $(1-|z|)^s|f(z)| \to 0$ as $|z| \to 1$ is the Fréchet envelope of $\mathcal{N}_\alpha^p$. The corresponding algebra $\mathcal{N}_s^\infty$ of analytic $f : \mathbb{U} \to \mathbb{C}$ such that $\sup_{z\in\mathbb{U}}(1-|z|)^s|f(z)| < \infty$ is a complete metric space but fails to be a topological vector space. F_s is also the largest linear topological subspace of $\mathcal{N}_s^\infty$. F_s is even a nuclear power series space. $\mathcal{N}_\alpha^p$ and $\mathcal{N}_\beta^q$ generate the same Fréchet envelope iff $(\alpha+2)/p = (\beta+2)/q$; they can replace each other for quasi-Banach space-valued continuous multilinear mappings.

Results for composition operators between $\mathcal{N}_\alpha^p$'s can often be translated in a one-to-one fashion to corresponding ones on associated weighted Bergman spaces $\mathcal{A}_\alpha^p$. This follows from the fact that the invertible elements in each $\mathcal{N}_\alpha^p$ are precisely the exponentials of functions in $\mathcal{A}_\alpha^p$. Moreover, each $\mathcal{N}_\alpha^p$, $(\alpha+2)/p \le 1$, admits dense ideals.

$\mathcal{A}_\alpha^p$ embeds order boundedly into $\mathcal{A}_\beta^q$ iff $\mathcal{A}_\beta^q$ contains the Bloch type space $\mathcal{A}_{(\alpha+2)/p}^\infty$ iff $(\alpha+2)/p < (\beta+1)/q$. In particular, $\bigcup_{p>0}\mathcal{A}_\alpha^p$ and $\bigcap_{p>0}\mathcal{A}_\alpha^p$ do not depend on the particular choice of $\alpha > -1$. The first space is a nuclear space, a copy of the dual of the space of rapidly decreasing sequences; the second has properties much stronger than being a Schwartz space but fails to be nuclear.

Introduction. We discuss a class of "large" algebras of analytic functions on the open unit disk $\mathbb{U} \subset \mathbb{C}$. These algebras, denoted by $\mathcal{N}_\alpha^p$, are the L^p variants of the usual weighted area Nevanlinna classes, defined by the measures $d\sigma_\alpha(z) = (\alpha+1)(1-|z|^2)^{1/\alpha}\,d\sigma(z)$, $\alpha > -1$, σ being normalized area measure on $\mathbb{U}$. The $\mathcal{N}_\alpha^p$ are F-algebras, which we refer to as Nevanlinna

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algebras. They fail to be locally pseudoconvex and so are neither locally bounded nor locally convex. But each $\mathcal{N}_\alpha^p$ has a separating dual since point evaluations are continuous.

The bidual of $\mathcal{N}_\alpha^p$ turns out to be the "Fréchet envelope" of $\mathcal{N}_\alpha^p$, i.e., the completion of $\mathcal{N}_\alpha^p$ with respect to the finest locally convex topology which is coarser than the original one; see [21]. This Fréchet envelope can be looked at as a "$(p = \infty)$-case" of Nevanlinna algebras: it can be identified with the space F_s of analytic functions $f : U \to \mathbb{C}$ such that $\lim_{|z|\to 1}(1-|z|)^s|f(z)| = 0$, where $s = (\alpha+2)/p$. The corresponding space $\mathcal{N}_s^\infty$ of analytic functions $f : \mathbb{U} \to \mathbb{C}$ such that $\sup_{z\in\mathbb{U}}(1-|z|)^s|f(z)| < \infty$ is also an algebra. It carries a canonical complete metric such that addition is continuous but scalar multiplication is not. The rôle of F_s as the "locally convex hull" of $\mathcal{N}_\alpha^p$ is complemented by the property of being a "linear topological kernel": as in the case of the classical Smirnov and Nevanlinna classes, F_s is the largest linear subspace of $\mathcal{N}_s^\infty$ which is a topological vector space with respect to the induced metric. It is a locally convex space, even a specific nuclear power series space of finite type. $\mathcal{N}_\alpha^p$ and $\mathcal{N}_\beta^q$ generate the same Fréchet envelope if and only if $(\alpha+2)/p = (\beta+2)/q$, in which case they can be replaced by each other when dealing with quasi-Banach space valued continuous (multi-)linear mappings.

We also generalize several recent results on composition operators between the function spaces under consideration (cf. [5], [19], [21], [41]). We show that, for some important cases, results for composition operators between Nevanlinna algebras $\mathcal{N}_\alpha^p$ translate in a one-to-one fashion to corresponding ones between the associated weighted Bergman spaces $\mathcal{A}_\alpha^p$. This follows from the fact that the invertible elements in each $\mathcal{N}_\alpha^p$ are precisely the exponentials of functions in $\mathcal{A}_\alpha^p$. Another algebraic feature of Nevanlinna algebras $\mathcal{N}_\alpha^p$ is that they admit, at least for $(\alpha+2)/p \le 1$, dense (maximal) ideals and so fail to be Q-algebras.

Applying this to formal identities between the spaces under discussion, we deduce that $\mathcal{A}_\alpha^p$ embeds into $\mathcal{A}_\beta^q$ in an order bounded fashion iff $\mathcal{A}_\beta^q$ contains the Bloch type space $\mathcal{A}_{(\alpha+2)/p}^\infty$ (definitions below) iff $(\alpha+2)/p < (\beta+1)/q$. In particular, neither $\bigcup_{p>0}\mathcal{A}_\alpha^p$ nor $\bigcap_{p>0}\mathcal{A}_\alpha^p$ depends on the particular choice of $\alpha > -1$. The first one is a nuclear space, in fact a copy of the dual of the space of rapidly decreasing sequences, whereas the second fails to be nuclear, though it has much stronger properties than just being a Schwartz space.

Nevanlinna algebras. We denote by $\mathbb{U}$ the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane. The space $\mathcal{H}$ of all analytic functions $f : \mathbb{U} \to \mathbb{C}$ is a Fréchet space with respect to the topology of local uniform convergence (= uniform convergence on compact subsets of $\mathbb{U}$). We will be interested in certain "big" subspaces of $\mathcal{H}$.

Let σ be normalized area measure on $\mathbb{U}$; so $d\sigma(x+iy) = \pi^{-1}dxdy$. For each $\alpha > -1$, a (Borel) probability measure on $\mathbb{U}$ is given by

$$d\sigma_\alpha(z) := (\alpha+1)(1-|z|^2)^\alpha d\sigma(z).$$

For $0 < p < \infty$, we denote by $\|\cdot\|_{\alpha,p}$ the usual norm (p-norm if $p < 1$) on the Lebesgue space $L^p(\sigma_\alpha)$. The corresponding *weighted Bergman space* $\mathcal{A}^p_\alpha$ consists of all $f \in \mathcal{H}$ which define an element of $L^p(\sigma_\alpha)$; briefly, $\mathcal{A}^p_\alpha := \mathcal{H} \cap L^p(\sigma_\alpha)$. With respect to the induced (p-) norm $\|\cdot\|_{\alpha,p}$, this is a Banach space (a p-Banach space when $0 < p < 1$).

Normalized Lebesgue measure on the unit circle $\mathbb{T} = \partial\mathbb{U}$ will be denoted by m; so $dm = dt/(2\pi)$. The *Hardy space* H^p consists of all $f \in \mathcal{H}$ such that $\sup_{r<1} \int_{\mathbb{T}} |f(\cdot)|^p \, dm$ is finite ($0 < p < \infty$). H^p is a (p-) Banach space, the (p-) norm being given by the pth root of the above expression. It is well known that, for $f \in H^p$, $\widetilde{f}(\zeta) = \lim_{r\to 1-} f(r\zeta)$ exists for m-almost all $\zeta \in \mathbb{T}$, and that $f \mapsto \widetilde{f}$ allows us to consider H^p as a closed subspace of $L^p(m)$.

The H^2-functions $w \mapsto k(z,w) := (1-\bar{z}w)^{-1}$, $z \in U$, satisfy $(k(z,\cdot) \mid f) = f(z)$ for all f in H^2 [here $(\cdot|\cdot)$ is the scalar product of H^2]. The function k is the *reproducing kernel* of H^2. Each of the Hilbert spaces $\mathcal{A}^2_\alpha$, $\alpha > -1$, has a reproducing kernel as well, namely $k^{\alpha+2}$. Accordingly, we shall also write $\mathcal{A}^p_{-1}$ instead of H^p, and σ_{-1} instead of m.

Nevanlinna algebras are the "logarithmic analogues" of the above weighted Bergman spaces. Given $\alpha > -1$ and $0 < p < \infty$, let $L^p_{\log}(\sigma_\alpha)$ be the space of all (equivalence classes of) σ_α-measurable functions on $\mathbb{U}$ such that

$$|||f|||_{\alpha,p} := \Big(\int_{\mathbb{U}} [\log(1+|f|)]^p \, d\sigma_\alpha \Big)^{1/p}$$

is finite. This is an *F-space* (complete, metrizable topological vector space). A defining F-norm on $L^p_{\log}(\sigma_\alpha)$ is given by $|||\cdot|||_{\alpha,p}$ if $1 \le p < \infty$, and by $|||\cdot|||^p_{\alpha,p}$ if $0 < p < 1$. The canonical embedding of $L^p_{\log}(\sigma_\alpha)$ into the usual F-space $L^0(\sigma_\alpha)$ of all (equivalence classes of) Borel measurable functions is continuous. $[L^p_{\log}(\sigma_\alpha), |||\cdot|||_{\alpha,p}]$ is also an F-space. Since $\log(1+st) \le \log(1+s)+\log(1+t)$ for all $s,t \ge 0$, it is an algebra with respect to pointwise multiplication. This multiplication is (jointly) continuous: our F-space is an *F-algebra*. An F-space whose topology is locally convex will be called a *Fréchet space*. We refer to [20] for details on these and related concepts.

Our main interest is in the space

$$\mathcal{N}^p_\alpha := \mathcal{H} \cap L^p_{\log}(\sigma_\alpha) \qquad (\alpha > -1)$$

with the induced F-norm $|||\cdot|||_{\alpha,p}$. For the time being, we require $1 \le p < \infty$

since we need subharmonicity of $[\log(1+|f|)]^p$. It is readily verified that $\mathcal{N}_\alpha^p$ is a closed subalgebra of $L^p_{\log}(\sigma_\alpha)$ and so it is also an F-algebra. Polynomials are dense in $\mathcal{N}_\alpha^p$, so that $\mathcal{N}_\alpha^p$ can also be seen as the closure of the usual disk algebra in $L^p_{\log}(\sigma_\alpha)$.

Incorporation of $\alpha = -1$ requires some care. We want to say that $f \in \mathcal{H}$ belongs to $\mathcal{N}_{-1}^p$ if $\sup_{r\to 1-}(\int_{\mathbb{T}} \log(1+|f(r\zeta)|)^p\, dm(\zeta))^{1/p}$ is finite and that this expression equals the F-norm of f as an element of $L^p_{\log}(dm)$. No problems occur if $1 < p < \infty$; see M. Stoll [39]. If $p = 1$, then the natural choice for $\mathcal{N}_{-1}^1$ is the *Smirnov class* rather than the usual Nevanlinna class. Recall (e.g. from P. Duren [12]) that $f \in \mathcal{H}$ belongs to the *Nevanlinna class* $\mathcal{N}$ if $\sup_{r<1} \int_{\mathbb{T}} \log(1+|f(r\zeta)|)\, dm(\zeta)$ is finite. In this case, boundary values $\widetilde{f}(\zeta) = \lim_{r\to 1-} |f(r\zeta)|$ exist for m-almost all $\zeta \in \mathbb{T}$; they define an element $\widetilde{f}$ of $L^1_{\log}(m)$. The function f belongs to the Smirnov class $\mathcal{N}^+$ if the F-norm of $\widetilde{f}$ in $L^1_{\log}(m)$ equals $\sup_{r<1} \int_{\mathbb{T}} \log(1+|f(r\zeta)|)\, dm(\zeta)$. It is well known that $\mathcal{N}^+$ is properly contained in $\mathcal{N}$; in fact, $\mathcal{N}$ fails to be a topological vector space in the canonical metric, whereas $\mathcal{N}^+$ is the largest subspace of $\mathcal{N}$ which is a topological vector space with respect to this metric; see J. H. Shapiro and A. L. Shields [36].

For $1 < p < \infty$, the $\mathcal{N}_{-1}^p$ are the *Hardy–Orlicz spaces* of [39]. The $\mathcal{N}_\alpha^1$, $\alpha > -1$, are known as *area Nevanlinna classes*. We call a *Nevanlinna algebra* each of the $\mathcal{N}_\alpha^p$'s, $\alpha \geq -1$, $1 \leq p < \infty$; we also say that $\mathcal{N}_\alpha^p$ is the Nevanlinna algebra which corresponds to $\mathcal{A}_\alpha^p$.

Later on, we will briefly indicate how our methods can be used to extend the definition of $\mathcal{N}_\alpha^p$ to $0 < p < 1$, essentially preserving what is discussed in this paper for $p \geq 1$.

The case $p = \infty$**.** An important rôle will be played by L^∞ versions of the above spaces. H^∞ is the usual Banach space of bounded analytic functions $\mathbb{U} \to \mathbb{C}$. Given $s > 0$, we denote by $\mathcal{A}_s^\infty$ the "Bloch type space" consisting of all $f \in \mathcal{H}$ such that

$$\|f\|_s^\infty := \sup_{z\in\mathbb{U}} (1-|z|^2)^s |f(z)| < \infty.$$

With $\|\cdot\|_s^\infty$ as a norm, this is a Banach space. The functions f in $\mathcal{A}_s^\infty$ which satisfy $\lim_{|z|\to 1-}(1-|z|^2)^s|f(z)| = 0$ form a closed subspace of $\mathcal{A}_s^\infty$ which we denote by $\mathcal{A}_{s,0}^\infty$. Again we will be interested in the "logarithmic analogues" of these spaces. $\mathcal{N}_s^\infty$ is the space of all $f \in \mathcal{H}$ such that $\sup_{z\in\mathbb{U}} (1-|z|^2)^s \log(1+|f(z)|) < \infty$. The subspace $\mathcal{N}_{s,0}^\infty$ consists of all $f \in \mathcal{N}_s^\infty$ such that even $\lim_{|z|\to 1-}(1-|z|^2)^s \log(1+|f(z)|) = 0$.

Let $M(f,r) := \max_{|z|=r} |f(z)|$ $(0 \leq r < 1)$ be the usual maximal function of $f \in \mathcal{H}$. Clearly, f belongs to $\mathcal{A}_s^\infty$ iff $\sup_{r<1}(1-r)^s M(f,r) < \infty$. Analogously for $\mathcal{A}_{s,0}^\infty$, $\mathcal{N}_s^\infty$, and $\mathcal{N}_{s,0}^\infty$.

It is readily seen that if we put

$$|||f|||_{s,\infty} := \sup_{z\in\mathbb{U}} (1-|z|^2)^s \log(1+|f(z)|)$$

then $(f,g) \mapsto |||f-g|||_{s,\infty}$ defines a complete, translation invariant metric on $\mathcal{N}_s^\infty$. But $|||\cdot|||_{s,\infty}$ fails to be an F-norm; there are functions $f \in \mathcal{N}_s^\infty$ such that $(|||n^{-1}f|||_{s,\infty})_{n\in\mathbb{N}}$ does not converge to zero. For example, $f(z) = \exp((1-z)^{-s})$ is in $\mathcal{N}_s^\infty \setminus \mathcal{N}_{s,0}^\infty$, and if we choose $0 < r_n < 1$ such that $\exp((1-r_n)^{-s}) = n^2$, then $|||n^{-1}f|||_{s,\infty} \geq (1-r_n)^s \log(n+1) \geq 1/2$ for all $n \in \mathbb{N}$.

Inside $\mathcal{N}_s^\infty$, the space $\mathcal{N}_{s,0}^\infty$ takes the rôle of the Smirnov class $\mathcal{N}^+$ inside the Nevanlinna class $\mathcal{N}$:

Theorem 1. *$\mathcal{N}_{s,0}^\infty$ is the largest linear subspace of $\mathcal{N}_s^\infty$ which is a topological vector space in the induced topology*; *it is actually an F-space.*

Proof. As is readily seen, $G := \{f \in \mathcal{N}_s^\infty : \lim_{t\to 0} |||t\cdot f|||_{s,\infty} = 0\}$ is the largest linear subspace of $\mathcal{N}_s^\infty$ which is a topological vector space with respect to the induced topology, and that it is complete (compare L. Drewnowski [11], M. Nawrocki [29]). So all we have to show is that G and $\mathcal{N}_{s,0}^\infty$ coincide.

Let $f \in \mathcal{N}_{s,0}^\infty$ and $\varepsilon > 0$ be given. Let $R \in (0,1)$ be such that $(1-|z|^2)^s \log(1+|f(z)|) < \varepsilon$ for all $R < |z| < 1$. Then $(1-|z|^2)^s \times \log(1+n^{-1}|f(z)|) < \varepsilon$ for all $R < |z| < 1$ and all $n \in \mathbb{N}$. By compactness of $R\cdot\overline{U}$, we can find $n_\varepsilon \in \mathbb{N}$ such that $(1-|z|^2)^s \log(1+n^{-1}|f(z)|) < \varepsilon$ for $|z| \leq R$ and $n \geq n_\varepsilon$. It follows that f is in G.

Suppose now that f belongs to G. Fix $\varepsilon > 0$ and choose an integer $N \geq 2$ such that $(1-r)^s \log(1+N^{-1}M_\infty(r,f)) < \varepsilon$ for all $0 < r < 1$. Then $(1-r)^s \log^+ M_\infty(r,f) < \varepsilon + (1-r)^s \log N$ for all r. If we choose $0 < R < 1$ so that $(1-R)^s \log N < \varepsilon$, then $(1-r)^s \log^+ M_\infty(r,f) < 2\varepsilon$ for $r \geq R$, hence $(1-r)^s \log(1+M_\infty(r,f)) \leq (1-r)^s[\log 2 + \log^+ M_\infty(r,f)] < \varepsilon$ (since $N \geq 2$). This shows that f is in $\mathcal{N}_{s,0}^\infty$. ∎

In order to conform with established notation, we write from now on F_s instead of $\mathcal{N}_{s,0}^\infty$. It was shown by M. Stoll [39] that F_s is even an F-algebra. He also obtained the next result (Theorem 2.2(a)⇔(c) in [39]); we state it as a lemma in a slightly modified fashion:

Lemma A. *An analytic function $f(z) = \sum_{n=0}^\infty a_n z^n$ on $\mathbb{U}$ belongs to F_s if and only if its Taylor coefficients a_n satisfy $\sum_{n=0}^\infty |a_n|^2 \exp(-n^{s/(s+1)}/k) < \infty$ for all $k \in \mathbb{N}$.*

For each $c > 0$, let $F_{s,c}$ be the space of all sequences $(a_n)_n \in \mathbb{C}^{\mathbb{N}_0}$ such that

$$q_c((a_n)_n) := \Big(\sum_{n=0}^\infty |a_n|^2 \exp(-cn^{s/(s+1)})\Big)^{1/2} < \infty.$$

$F_{s,c}$ is a Hilbert space with norm q_c. Via $(a_n)_n \mapsto \sum_{n=0}^{\infty} a_n z^n$, each $F_{s,c}$ can be considered as a linear subspace of $\mathcal{H}$.

If $0 < c < d$, then $F_{s,c}$ is a (dense) linear subspace of $F_{s,d}$ and the embedding is a nuclear (= trace class) operator. Consequently,

$$F_s = \bigcap_{c>0} F_{s,c} = \bigcap_{k\in\mathbb{N}} F_{s,1/k}$$

is a nuclear Fréchet space with respect to the corresponding projective limit topology, a so-called *nuclear power series space of finite type* ([20, Ch. 21]).

Note that we get the usual representation of $\mathcal{H}$ as a sequence space if we take $s = \infty$.

The arguments used to prove (c)⇒(a) of Theorem 2.2 in [39] yield

$$\mathcal{N}_s^{\infty} = \bigcup_{c>0} F_{s,c} = \bigcup_{k\in\mathbb{N}} F_{s,k}$$

(compare also [29, p. 170]). Consequently, $\mathcal{N}_s^{\infty}$ is a topological vector space with respect to the corresponding natural topological inductive limit topology; this topology is locally convex ([15, p. 164]) since we even deal with the inductive limit of a sequence of Hilbert spaces with nuclear linking mappings. With this topology, $\mathcal{N}_s^{\infty}$ is reflexive; it is in fact the strong dual of the "nuclear power series space of infinite type" $\bigcap_{k\in\mathbb{N}} F_{s,k}^*$. Here $F_{s,k}^*$ consists of all sequences $(a_n)_{n\in\mathbb{N}}$ such that $a_n = O(\exp(-kn^{s/(s+1)}))$, $k \in \mathbb{N}$. The existence of this topology, together with a general version of the closed graph theorem ([20, 5.5.4]), provides another proof of the fact that $|||\cdot|||_{s,\infty}$ cannot be an F-norm on $\mathcal{N}_s^{\infty}$.

It is easy to see that each $\mathcal{N}_{\alpha}^{p}$ embeds continuously into $F_{(\alpha+2)/p}$. We will see below that relations between these algebras are in fact much tighter.

But first we take a look at some algebraic properties of Nevanlinna algebras.

On the algebras $\mathcal{N}_{\alpha}^{p}$. By a result of N. Mochizuki [28], each of the algebras $\mathcal{N}_{-1}^{p}$, $1 < p < \infty$, admits dense (maximal) ideals [ideals are assumed to be non-trivial]. By a different argument this can be extended to the algebras $\mathcal{N}_{\alpha}^{p}$ and $F_{(\alpha+2)/p}$ for suitably chosen α and p.

PROPOSITION 1. *Let $\alpha > -1$ and $1 \le p < \infty$ be such that $(\alpha+2)/p \le 1$. Then the F-algebras $\mathcal{N}_{\alpha}^{p}$ and $F_{(\alpha+2)/p}$ contain dense ideals.*

Proof. Put $s = (\alpha+2)/p$. Since $s \le 1$, $f(z) := (1-z)^{-s}$ has positive real parts and so $\varphi := e^{-f}$ belongs to H^{∞}. As before (Theorem 1), $1/\varphi$ is a member of $\mathcal{N}_s^{\infty}$ but not of $F_s = \mathcal{N}_{s,0}^{\infty}$ and, a fortiori, not of $\mathcal{N}_{\alpha}^{p}$.

We use the well-known fact that f is in H^r for any $0 < r < 1/s$ and apply a result of J. Bourgain [4, Proposition 1.6]. Accordingly, there are a constant $C = C(r)$ and, for each $\lambda > 0$, functions $g, h \in H^r$ such that

$$f = g + h, \quad |g|, |h| \le C|f|, \quad |g| \le C\lambda,$$
$$\Big(\int_{\mathbb{T}} |h|^r\, dm\Big)^{1/r} \le C\Big(\int_{\{|f|>\lambda\}} |f|^r\, dm\Big)^{1/r}.$$

Fix $\varepsilon > 0$ and choose λ so large that the right hand side in the last expression is $\le \varepsilon$. Note that $\gamma := \exp(g)$ belongs to H^∞. Let X be $\mathcal{N}^p_\alpha$ or $F_{(\alpha+2)/p}$. Since H^r embeds canonically into X we have $\|\varphi\gamma - 1\|_X \le \|h\|_{\alpha,p} \le \|h\|_{H^r} \le \varepsilon$. So φ admits an "approximative inverse" in X ([28]), and so the ideal generated by φ is dense in X. The ideal is proper since $1/\varphi$ does not belong to X. ■

A modification of the preceding argument yields another proof of Mochizuki's result mentioned above (the case $\alpha = -1$, $1 < p < \infty$). We do not know whether the proposition is also valid if $(\alpha + 2)/p > 1$.

If a commutative F-algebra admits no (proper) dense ideals then it is obviously a *Q-algebra*: its invertible elements form an open subset. So the algebras $\mathcal{N}^p_\alpha$ and $F_{(\alpha+2)/p}$ provide examples for non-Q-algebras, the latter being locally convex and even nuclear.

It is well known that the invertible elements in the Smirnov class $\mathcal{N}^1_{-1} = \mathcal{N}^+$ are just the outer functions. By [39], the invertible elements in $\mathcal{N}^p_{-1}$, $1 < p < \infty$, are precisely the exponentials of functions in H^p. This carries over to Nevanlinna algebras:

Proposition 2. *Suppose that $\alpha > -1$ and $1 \le p < \infty$. Then $f \in \mathcal{N}^p_\alpha$ is invertible if and only if $f = e^g$ for some $g \in \mathcal{A}^p_\alpha$.*

Proof. It is clear that if g is in $\mathcal{A}^p_\alpha$, then e^g is invertible in $\mathcal{N}^p_\alpha$. Let conversely $f \in \mathcal{N}^p_\alpha$ be invertible. Then f does not vanish, hence $f = e^g$ for some $g \in \mathcal{H}$. Writing $g = u + iv$ with $\mathbb{R}$-valued harmonic functions u and v we get

$$\infty > \|\!|f|\!\|^p_{\alpha,p} = \int_{\mathbb{U}} [\log(1 + e^u)]^p\, d\sigma_\alpha \ge \int_{\{u\ge 0\}} |u|^p\, d\sigma_\alpha.$$

Similarly, $\infty > \int_{\mathbb{U}} [\log(1 + 1/|f|)]^p\, d\sigma_\alpha \ge \int_{\{u\le 0\}} |u|^p\, d\sigma_\alpha$, hence $u \in L^p(\sigma_\alpha)$. Since $\alpha > -1$ this implies that v also belongs to $L^p(\sigma_\alpha)$ [27], and so g is a member of $\mathcal{A}^p_\alpha$. ■

However, we do not have a characterization of the invertible elements in F_s.

By results of J. W. Roberts and M. Stoll [32] and N. Mochizuki [28], the closed maximal ideals in $\mathcal{N}^p_{-1}$ $(1 \le p < \infty)$ are precisely the kernels of the point evaluations $f \mapsto f(z)$, $z \in \mathbb{U}$. This extends to the algebras F_s (see

[27, Theorem 2]), but we do not know if such a result is also true for the algebras $\mathcal{N}_\alpha^p$ for $\alpha > -1$ and $1 \le p < \infty$.

Multipliers. We start by extending results which are known for scalar-valued function spaces [21], and in the vector-valued case for $\alpha = -1, 0$ (cf. [14]). The common strategy employed in these papers originates in the work of N. Yanagihara [42] who had settled the case of the scalar-valued Smirnov class in 1973. The desired close ties between $\mathcal{N}_\alpha^p$ and $F_{(\alpha+2)/p}$ are derived from a characterization of multipliers from these spaces into "small" ones, and even into "small" spaces of analytic functions with values in a quasi-Banach space. The announced extension is obtained by combining such results with a construction which is based on a lemma due to E. Beller [3] (see Lemma C below).

Let X be a quasi-Banach space. Following N. Kalton [24] we say that a function $f : \mathbb{U} \to X$ is *analytic* if there is a sequence (x_n) in X such that

$$(*) \qquad f(z) = \sum_{n=0}^{\infty} x_n z^n \quad \text{for all } z \in \mathbb{U}.$$

Here $x_n = f^{(n)}(0)/n!$, as usual. By the Aoki–Rolewicz theorem, every quasi-Banach space can be renormed to be an r-Banach space for some $0 < r \le 1$. This simplifies working with series like $(*)$. The analogue of Cauchy's theorem, however, may fail if X cannot be given an equivalent norm, and non-analytic, differentiable functions $\mathbb{U} \to X$ may exist ([1], [40]).

The analytic functions $\mathbb{U} \to X$ form a linear space, $\mathcal{H}(X)$, which is an F-space with respect to local uniform convergence. For each $0 < p < \infty$, let $\mathcal{A}_\alpha^p(X)$ consist of all $f \in \mathcal{H}(X)$ such that $\|f(\cdot)\|$ is in $L^p(\sigma_\alpha)$.

More function spaces will be needed. The first one is $\mathbb{A}(X)$, the X-valued analogue of the disk algebra; it consists of all $f \in \mathcal{H}(X)$ admitting a continuous extension $\overline{\mathbb{U}} \to X$. The second is the space $\mathbb{W}_q(X)$, $0 < q \le 1$, consisting of all $f \in \mathcal{H}(X)$ which have a representation $(*)$ such that $\sum_{n=0}^{\infty} \|x_n\|^q < \infty$, $\|\cdot\|$ being a defining r-norm on X. In a canonical fashion, $\mathcal{A}_\alpha^p(X)$ is an s-Banach space for $s = p \wedge r$, $\mathbb{A}(X)$ is an r-Banach space with respect to uniform convergence on $\overline{\mathbb{U}}$, and $\mathbb{W}_q(X)$ is an s-Banach space for $s = q \wedge r$. If F is any of the spaces $\mathcal{A}_\alpha^p$, $\mathbb{A}$, $\mathbb{W}_q$, then $\|\cdot\|_{F(X)}$ will denote the canonical quasinorm on $F(X)$.

Suppose in addition that E is any of the F-spaces of analytic scalar-valued functions considered here. Given a quasi-Banach space X, let us denote by $[\![E, F(X)]\!]$ the collection of all sequences $(x_n)_{n=0}^{\infty}$ in X such that $\sum_{n=0}^{\infty} a_n z^n x_n$ belongs to $F(X)$ whenever $\sum_{n=0}^{\infty} a_n z^n$ belongs to E. This is a linear space; its elements are called (coefficient) *multipliers* from E into $F(X)$. By the Closed Graph Theorem, each $(x_n) \in [\![E, F(X)]\!]$ defines a continuous linear operator $\Lambda : E \to F(X) : \sum_{n=0}^{\infty} a_n z^n \mapsto \sum_{n=0}^{\infty} a^n z^n x_n$.

The following theorem is of fundamental importance. Fix $-1 \le \alpha < \infty$ and $1 \le p < \infty$.

THEOREM 2. *For every sequence* (x_n) *in a quasi-Banach space* X *the following statements are equivalent*:

(i) $(x_n) \in [\![\mathcal{N}_\alpha^p, \mathbb{A}(X)]\!]$.
(ii) $(x_n) \in [\![\mathcal{N}_\alpha^p, \mathbb{W}_q(X)]\!]$ *for some* (*and then every*) $0 < q \le 1$.
(iii) *There are constants* $C, c > 0$ *such that*

$$\|x_n\| \le C \exp[-cn^{(\alpha+2)/(\alpha+2+p)}].$$

If X *is even a Banach space*, *the following statements can be added*:

(iv) $(x_n) \in [\![\mathcal{N}_\alpha^p, \mathcal{A}_\beta^q(X)]\!]$ *for some* $\beta \ge -1$ *and some* $0 < q \le \infty$.
(v) $(x_n) \in [\![\mathcal{N}_\alpha^p, \mathcal{A}_\beta^q(X)]\!]$ *for all* $\beta \ge -1$ *and all* $0 < q \le \infty$.

This was obtained by C. M. Eoff [14] for $\alpha = -1, 0$. For scalar-valued functions, the theorem was proved in [21].

Results of this kind are derived from precise information about the Taylor coefficients of the members of our function spaces. Such information is provided by the following series of lemmas. The first one is taken from [17, Lemma B].

LEMMA B. *Let* $\alpha > -1$ *and* $1 \le p < \infty$. *If* $f(z) = \sum_{n=0}^\infty a_n z^n$ *belongs to* $\mathcal{N}_\alpha^p$, *then*

$$a_n = O(\exp[o(n^{(\alpha+2)/(\alpha+2+p)})]) \quad (n = 0, 1, \ldots).$$

The next result is due to E. Beller [3, Lemma 2].

LEMMA C. *Let* $a > 0$ *and* $r_0 > 0$ *be given. Define* $f(z) = \sum_{n=0}^\infty a_n z^n$ *by* $f(z) = \exp[r/(1-z)^a]$ *for* $0 < r \le r_0$. *Then there exist constants* K *and* C, *depending only on* a *and* r_0, *such that*

$$a_n \ge K \exp[Cn^{a/(a+1)}] \quad \textit{for all } n \in \mathbb{N}_0.$$

Note that the a_n's are indeed positive.

For Banach space valued functions, we may state:

LEMMA D. *Let* X *be a Banach space. Given* $\beta \ge -1$ *and* $0 < q < \infty$, *there exists a constant* $C = C(X, \beta, q)$ *such that if* $g(z) = \sum_{n=0}^\infty x_n z^n$ *belongs to* $\mathcal{A}_\beta^q(X)$, *then*

$$\|x_n\| \le C\|g\|_{\beta,q} n^{(\beta+2)/q} \quad \textit{for all } n \in \mathbb{N}.$$

This is well known in the scalar case (see e.g. Lemma B.5 in W. Smith [38]). The Banach space case can be obtained either by working with the weak topology of X, or else by repeating the proof with absolute values replaced by norms.

No analogue of Lemma D seems to be available for functions with values in a quasi-Banach space. However, thanks to the following result of N. Kalton [24, Theorem 6.1], a partial generalization, sufficient for our purposes, does exist.

Lemma E. *Let X be an r-Banach space. Then there are constants $\gamma > 0$ and $C = C(X) > 0$ such that if $g = \sum_{n=0}^{\infty} x_n z^n$ belongs to $\mathbb{A}(X)$ then*

$$\|x_m\| \le C m^{\gamma} \|g\|_{\mathbb{A}(X)} \quad \textit{for all } m \in \mathbb{N}.$$

Theorem 2 can now be proved essentially by adjusting the arguments given in [21]. For the sake of completeness, we include the details.

Proof of Theorem 2. For convenience, we confine ourselves to the case $\alpha > -1$. Up to natural (notational) changes, the same arguments work for $\alpha = -1$.

(i)⇒(iii). Consider the function

$$f_{(s)}(z) = \exp\left[\frac{c(1-s)^{(\alpha+2)/p}}{(1-sz)^{2(\alpha+2)/p}}\right] - 1 \quad \text{where } 0 < s < 1 \text{ and } c > 0.$$

By [43, Lemma 4.2.2] there is a constant $K_0 = K_0(\alpha)$ such that

$$\begin{aligned} \|\!|f_{(s)}\|\!|_{\alpha,p}^p &= \int_{\mathbb{U}} [\log(1 + |f_{(s)}|)]^p \, d\sigma_\alpha \le c^p \int_{\mathbb{U}} \frac{(1-s)^{\alpha+2}}{|1-sz|^{2(\alpha+2)}} \, d\sigma_\alpha(z) \\ &= c^p(\alpha+1)(1-s)^{\alpha+2} \int_{\mathbb{U}} \frac{(1-|z|^2)^\alpha}{|1-sz|^{2\alpha+4}} \, d\sigma(z) \le c^p K_0^p. \end{aligned}$$

Let $\Lambda : \mathcal{N}_\alpha^p \to \mathbb{A}(X)$ be the multiplier induced by (x_n). By continuity, there is a $C > 0$ such that $\|\Lambda(g)\|_{\mathbb{A}(X)} \le 1$ whenever $g \in \mathcal{N}_\alpha^p$ satisfies $\|\!|g\|\!|_{\alpha,p} \le C$. Hence, if we choose $c \le C/K_0$ in the definition of $f_{(s)}$, then $\|\Lambda(f_{(s)})\|_{\mathbb{A}(X)} \le 1$ for all $0 < s < 1$.

Put now $R := c(1-s)^{(\alpha+2)/p}$ and consider the function

$$f(z) = \exp\left[\frac{R}{(1-z)^{2(\alpha+2)/p}}\right] - 1.$$

Clearly, $f_{(s)}(z) = f(sz)$, so that if we write $f_{(s)}(z) = \sum_{n=0}^{\infty} a_{n,s} z^n$ and $f(z) = \sum_{n=0}^{\infty} c_n z^n$, then $a_{n,s} = s^n c_n$ for all $n \in \mathbb{N}_0$. We know from Lemma C that the $a_{n,s}$ are positive and that

$$a_{n,s} \ge C_1 s^n \exp[L(1-s)^{(\alpha+2)/(2\alpha+4+p)} n^{(2\alpha+4)/(2\alpha+4+p)}]$$

where $L = c^{p/(2\alpha+4+p)}(p/(2\alpha+4))^{(2\alpha+4)/(2\alpha+4+p)}$ and C_1 is a constant.

Choose $0 < b, d < 1$ such that $d < b < (L/2) d^{(\alpha+2)/(2\alpha+4+p)}$, and then $N_0 \in \mathbb{N}$ so large that $b \le (3/4) N_0^\theta$ where $\theta = p/(\alpha+2+p)$. For $N \ge N_0$ and s satisfying $dN^{-\theta} \le 1 - s \le bN^{-\theta}$ we get (using once more the fact that $1-t > e^{-2t}$ for $0 < t \le 3/4$)

$$\begin{aligned} a_{N,s} &\ge C_1(1-bN^{-\theta})^N \exp[L(dN^{-\theta})^{(\alpha+2)/(2\alpha+4+p)} N^{(2\alpha+4)/(2\alpha+4+p)}] \\ &\ge C_1 \exp[-2bN^{1-\theta} + L^{(\alpha+2)/(2\alpha+4+p)} N^{(\alpha+2)/(\alpha+2+p)}] \\ &= C_1 \exp[C_2 N^{(\alpha+2)/(\alpha+2+p)}]; \end{aligned}$$

here $C_2 := Ld^{(\alpha+2)/(2\alpha+4+p)} - 2b$ (>0).

Lemma E provides us with constants $C_3, \lambda > 0$ such that $\|a_{n,s}x_n\| \le C_3 n^\lambda$ for all n and s. It follows that

$$\|x_N\| \le \frac{C_3}{C_1} N^\lambda \exp[-C_2 N^{(\alpha+2)/(\alpha+2+p)}] \quad \text{for } N \ge N_0,$$

whence our claim.

(iii)$\Rightarrow$(ii). Assume that $f(z) = \sum_{n=0}^\infty a_n z^n$ is in $\mathcal{N}_\alpha^p$. By Lemma B, there are a constant C' and a null sequence of positive numbers b_n such that $|a_n| \le C' \exp[b_n n^{(\alpha+2)/(\alpha+2+p)}]$ for all n. It follows that

$$\sum_{n=0}^\infty \|a_n x_n\|^q \le CC' \sum_{n=0}^\infty \exp[q(b_n - c)n^{(\alpha+2)/(\alpha+2+p)}] < \infty.$$

Since (ii)$\Rightarrow$(i) is trivial, we are done. ■

The Banach space parts of Theorem 2 can be obtained similarly, using Lemma D instead of Lemma E. Alternatively, duality can be applied to derive them directly from the results on scalar-valued function spaces in [21].

Duality. Now we are in a position to clarify the relations between spaces $\mathcal{N}_\alpha^p$ and $F_{(\alpha+2)/p}$ ($\alpha \ge -1$, $1 \le p < \infty$). Recall that $F_{(\alpha+2)/p}$ is $\bigcap_{k\in\mathbb{N}} F_{\alpha,1/k}$ where $F_{\alpha,1/k}$ consists of all sequences (a_n) in $\mathbb{C}$ such that $q_k((a_n)) = \sum_{n=0}^\infty |a_n|^2 \exp[-(1/k)n^{-(\alpha+2)/(\alpha+2+p)}] < \infty$ $(k \in \mathbb{N})$. It follows that

$$(\circ) \qquad F^*_{(\alpha+2)/p} = \bigcup_{k\in\mathbb{N}} \Big\{(a_n) : \sum_{n=0}^\infty |a_n|^2 \exp[(1/k)n^{(\alpha+2)/(\alpha+2+p)}] < \infty\Big\}.$$

It was shown in [21] that Theorem 2 (for scalar-valued functions) yields:

Corollary 1. *If $\alpha \ge -1$ and $1 \le p < \infty$, then*:

(a) *$\mathcal{N}_\alpha^p$ embeds continuously and densely into $F_{(\alpha+2)/p}$.*

(b) *The duals of $\mathcal{N}_\alpha^p$ and of $F_{(\alpha+2)/p}$ coincide and can be identified with the space of all $g \in \mathcal{H}$ whose Taylor coefficients b_n satisfy $b_n = O(\exp[-c\, n^{(\alpha+2)/(\alpha+2+p)}])$ for some $c > 0$. The action of g on $f(z) = \sum_{n=0}^\infty a_n z^n$ in $\mathcal{N}_\alpha^p$ is given by*

$$\langle g, f\rangle = \lim_{r\to 1-} \int_0^{2\pi} g(re^{it}) f(re^{-it}) \frac{dt}{2\pi} \quad \Big(= \sum_n a_n b_n\Big).$$

(c) $F_{(\alpha+2)/p}$ *induces on* $\mathcal{N}_\alpha^p$ *the Mackey topology* $\mu(\mathcal{N}_\alpha^p, (\mathcal{N}_\alpha^p)^*)$.
(d) $F_{(\alpha+2)/p}$ *is the Fréchet envelope of* $\mathcal{N}_\alpha^p$.
(e) $F_{(\alpha+2)/p}$ *is the bidual of* $\mathcal{N}_\alpha^p$.

Note that (d) complements the result that $F_{(\alpha+2)/p}$ is the "linear topological kernel" of $\mathcal{N}_{(\alpha+2)/p}$ (Theorem 1).

A few remarks are in order. It is clear that $\mathcal{N}_\alpha^p$ is barrelled (see [20], 11.1) under the Mackey topology $\mu(\mathcal{N}_\alpha^p, (\mathcal{N}_\alpha^p)^*)$, which therefore coincides with the strong topology $\beta(\mathcal{N}_\alpha^p, (\mathcal{N}_\alpha^p)^*)$. Moreover, identifying $(F_{(\alpha+2)/p})^*$ and $(\mathcal{N}_\alpha^p)^*$, we refer in (e) to the strong topology $\beta((\mathcal{N}_\alpha^p)^*, F_{(\alpha+2)/p})$. It coincides with $\beta((\mathcal{N}_\alpha^p)^*, E)$ for every dense linear topological subspace $E \subset F_{(\alpha+2)/p}$ ([20], 9.4.2), e.g. for $E = [\mathcal{N}_\alpha^p, \mu(\mathcal{N}_\alpha^p, (\mathcal{N}_\alpha^p)^*)]$. Being nuclear, $F_{(\alpha+2)/p}$ is reflexive: the strong dual of $[(\mathcal{N}_\alpha^p)^*, \beta((\mathcal{N}_\alpha^p)^*, F_{(\alpha+2)/p})]$ is $F_{(\alpha+2)/p}$. If τ denotes the original topology of the F-space $\mathcal{N}_\alpha^p$ then, by metrizability, subsets of $\mathcal{N}_\alpha^p$ must exist which are bounded for $\mu(\mathcal{N}_\alpha^p, (\mathcal{N}_\alpha^p)^*)$ and unbounded for τ. We do not have a satisfactory characterization of τ-bounded subsets of $\mathcal{N}_\alpha^p$, and we do not know what the dual of $[(\mathcal{N}_\alpha^p)^*, \gamma]$ is when γ is the topology of uniform convergence on τ-bounded subsets of $\mathcal{N}_\alpha^p$.

It follows from Corollary 1 that $\mathcal{N}_\alpha^p$ cannot be locally bounded: otherwise $F_{(\alpha+2)/p}$ would be a nuclear Banach space and so finite-dimensional. $\mathcal{N}_\alpha^p$ cannot be locally convex either; there are in fact various possibilities to see that $\mathcal{N}_\alpha^p$ is a proper subset of $F_{(\alpha+2)/p}$. We shall provide a "soft" argument below (Corollary 6).

Again by Corollary 1, if X is a Banach space (or just a complete locally convex space), then every continuous linear map $\mathcal{N}_\alpha^p \to X$ admits a unique extension to a continuous linear map $F_{(\alpha+2)/p} \to X$. Theorem 2 yields more:

THEOREM 3. *Let* $\alpha_1, \ldots, \alpha_n \geq -1$ *and* $1 \leq p_1, \ldots, p_n < \infty$ *be given* $(n \in \mathbb{N})$, *and let* X *be a quasi-Banach space. Then every continuous n-linear map* $b : \mathcal{N}_{\alpha_1}^{p_1} \times \ldots \times \mathcal{N}_{\alpha_n}^{p_n} \to X$ *has a (unique) extension to a continuous n-linear map* $\widetilde{b} : F_{(\alpha_1+2)/p_1} \times \ldots \times F_{(\alpha_n+2)/p_n} \to X$.

This relies on the following lemma which generalizes Propositions 4.5 and 4.6 in C. M. Eoff's paper [14]. The proof is entirely similar and is therefore omitted.

LEMMA F. *Let* X *be a quasi-Banach space and* $u : \mathcal{N}_\alpha^p \to X$ *a linear map, where* $\alpha \geq -1$ *and* $1 \leq p < \infty$. *For* u *to be continuous it is necessary and sufficient that* $(uz^n)_n$ *belongs to* $[\![\mathcal{N}_\alpha^p, \mathbb{A}(X)]\!]$ *and that* $u(f) = \sum_{n=0}^\infty a_n u(z^n)$ *converges in* X *for each* $f(z) = \sum_{n=0}^\infty a_n z^n$ *in* $\mathcal{N}_\alpha^p$.

Here z^n is used to denote the monomial $z \mapsto z^n$.

Proof of Theorem 3. We start by looking at the case $n = 1$. Let $u : \mathcal{N}_\alpha^p \to X$ be linear and continuous. By Lemma F, the $x_n := u(z^n)$

form a sequence in $[\![\mathcal{N}_\alpha^p, \mathbb{A}(X)]\!]$. By Theorem 2, there are constants C and c such that $\|x_n\| \le C\exp[-cn^{(\alpha+2)/(\alpha+2+p)}]$ for all n. We conclude that, for appropriately chosen $C' > 0$ and $k \in \mathbb{N}$,

$$\begin{aligned}\|uf\|^r &\le \sum_{n=0}^\infty |a_n|^r \|x_n\|^r \\ &\le C\sum_{n=0}^\infty |a_n|^r \exp(-crn^{(\alpha+2)/(\alpha+2+p)}) \le C' q_k((a_n)_n)^r\end{aligned}$$

whenever $f(z) = \sum_{n=0}^\infty a_n z^n$ is in $\mathcal{N}_\alpha^p$. Therefore u extends uniquely to a continuous linear map $\widetilde{u} : F_{(\alpha+2)/p} \to X$.

Note that this implies that if (U_n) is a 0-basis in $\mathcal{N}_\alpha^p$ and $0 < q \le 1$ then the q-convex hulls of U_n form a 0-basis for $\mu(\mathcal{N}_\alpha^p, (\mathcal{N}_\alpha^p)^*)$.

Using this, we can settle the case $n \ge 1$ by induction. For notational convenience, we just treat the bilinear case.

Let us write $\mathcal{N}_j = \mathcal{N}_{\alpha_j}^{p_j}$ and $F_j = F_{(\alpha_j+2)/p_j}$, $j = 1, 2$. Let $b : \mathcal{N}_1 \times \mathcal{N}_2 \to X$ be bilinear and continuous, X a q-Banach space. Let U_1, U_2 be 0-neighbourhoods in $\mathcal{N}_1$ resp. $\mathcal{N}_2$ such that $b(U_1 \times U_2) \subset B_X$. By the above remark, the q-convex hull $\widehat{U}_2$ of U_2 is a 0-neighbourhood for the Mackey topology, μ_2, of $\mathcal{N}_2$. By q-convexity of B_X we have $b(U_1 \times \widehat{U}_2) \subset B_X$, so that b is continuous as a map $\mathcal{N}_1 \times [\mathcal{N}_2, \mu_2] \to X$.

Now μ_2 is the topology which $\mathcal{N}_2$ inherits from F_2. As we know, $F_2 = \bigcap_{k\in\mathbb{N}} G_k$ with Banach spaces G_k each of which contains F_2 densely and whose norms, $\|\cdot\|_k$, generate the topology of F_2. Accordingly, there is a $k \in \mathbb{N}$ such that $\widehat{U}_2$ contains some multiple, V_k say, of $\mathcal{N}_2 \cap B_{G_k}$. From $b(U_1 \times V_k) \subset B_X$ we conclude that $u_b : \mathcal{N}_1 \to \mathcal{L}([\mathcal{N}_2, \|\cdot\|_k], X) : f_1 \mapsto (f_2 \mapsto b(f_1, f_2))$ is well defined, linear and continuous if we endow $\mathcal{L}([\mathcal{N}_2, \|\cdot\|_k], X)$ with its canonical q-norm. Since $\mathcal{N}_2$ is dense in G_k, the latter space can be identified with the q-Banach space $\mathcal{L}(G_k, X)$, and so u_b appears as a continuous linear map $\mathcal{N}_1 \to \mathcal{L}(G_k, X)$. By the first part, it has a unique continuous linear extension $\widetilde{u}_b : F_1 \to \mathcal{L}(G_k, X)$. The restriction of $F_1 \times G_k \to X : (f_1, f_2) \mapsto \widetilde{u}_b(f_1)(f_2)$ to $F_1 \times F_2$ is the continuous bilinear extension of β we are looking for. ∎

Let b be as in Theorem 3. Nuclearity of the $F_{(\alpha_j+2)/p_j}$ implies that b has the form

$$b(f_1, \dots, f_n) = \sum_{k=1}^\infty \lambda_k \langle g_k^{(1)}, f_1\rangle \dots \langle g_k^{(n)}, f_n\rangle z_k \qquad (f_1 \in \mathcal{N}_{\alpha_1}^{p_1}, \dots, f_n \in \mathcal{N}_{\alpha_n}^{p_n}),$$

where each of the sequences $(g_k^{(j)})_k$ is equicontinuous in $(\mathcal{N}_{\alpha_j}^{p_j})^*$ $(1 \le j \le n)$, $(z_k)_k$ is bounded in X, and $(\lambda_k)_k$ is taken from ℓ_r. It follows that $\widetilde{b}$ can be extended further to a continuous n-linear map $E_1 \times \dots \times E_n \to X$

provided each E_j is a Hausdorff locally convex space containing $F_{(\alpha_j+2)/p_j}$ as a subspace. Analogously for polynomials.

Taking $n = 1$, we obtain a slight improvement of what was already used in the proof of Theorem 3:

COROLLARY 2. (a) *Let* (U_n) *be a* 0*-basis in* $\mathcal{N}_\alpha^p$ *and let* (s_n) *be any sequence in* $(0,1]$. *For each* $n \in \mathbb{N}$, *let* V_n *be the* s_n*-convex hull of* U_n. *Then* $(V_n)_n$ *is a* 0*-basis for the Mackey topology* $\mu(\mathcal{N}_\alpha^p, (\mathcal{N}_\alpha^p)^*)$.

(b) *The space* $\mathcal{N}_\alpha^p$ *fails to be "locally pseudoconvex"*: *it cannot be represented as a projective limit of quasi-Banach spaces.*

It follows that if a locally pseudoconvex space is isomorphic to a quotient of $\mathcal{N}_\alpha^p$, then it is locally convex and nuclear. In particular, no infinite-dimensional quasi-Banach space can be isomorphic to a quotient of $\mathcal{N}_\alpha^p$. On the other hand, we do not know if there is an infinite-dimensional quasi-Banach space which is isomorphic to a subspace of $\mathcal{N}_\alpha^p$.

As we shall see later, there is, however, "enough room" to embed power series spaces F_s continuously into $\mathcal{N}_\alpha^p$, for a precise range of parameters s (Corollary 10).

As another consequence of Corollary 1 we may state:

COROLLARY 3. (a) *The algebras* $\mathcal{N}_\alpha^p$ *and* $\mathcal{N}_\beta^q$ *have the same Fréchet envelope if and only if* $(\alpha+2)/p = (\beta+2)/q$.

(b) *If* $(\alpha+2)/p = (\beta+2)/q$, *then every continuous linear map from* $\mathcal{N}_\alpha^p$ *to a complete locally pseudoconvex space* X *gives rise to a unique continuous linear map* $\mathcal{N}_\beta^q \to X$, *and conversely. Accordingly*, *the corresponding spaces of continuous linear mappings can be identified*:

$$\mathcal{L}(\mathcal{N}_\alpha^p, X) \cong \mathcal{L}(F_{(\alpha+2)/p}, X) \cong \mathcal{L}(\mathcal{N}_\beta^q, X).$$

More generally, if $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \geq -1$ and $1 \leq p_1, \ldots, p_n, q_1, \ldots, q_n < \infty$ ($n \in \mathbb{N}$) are such that $(\alpha_k+2)/p_k = (\beta_k+2)/q_k$, $1 \leq k \leq n$, then Theorem 3 leads to the natural identification of the corresponding spaces of continuous n-linear mappings:

$$\mathcal{L}_n(\mathcal{N}_{\alpha_1}^{p_1} \times \ldots \times \mathcal{N}_{\alpha_n}^{p_n}; X) \cong \mathcal{L}_n(\mathcal{N}_{\beta_1}^{q_1} \times \ldots \times \mathcal{N}_{\beta_n}^{q_n}; X).$$

The case $p < 1$. The results obtained so far can be used to introduce Nevanlinna algebras with analogous properties also for $p < 1$. We have seen that $\mathcal{N}_\alpha^p$ and $\mathcal{N}_{\alpha'}^1$ generate the same dual whenever $\alpha, \alpha' \geq -1$, $1 \leq p < \infty$ and $\alpha' + 2 = (\alpha+2)/p$. If $0 < p < 1$, then we *define*, with the same choice of α',

$$\mathcal{N}_\alpha^p := L_{\log}^p(\sigma_\alpha) \cap \mathcal{N}_{\alpha'}^1.$$

An F-norm $\|\!| \cdot \|\!|_{\alpha,p}^\bullet$ is obtained on $\mathcal{N}_\alpha^p$ by

$$\|\!| f \|\!|_{\alpha,p}^\bullet := \max\{\|\!| f \|\!|_{\alpha,p}^p, \|\!| f \|\!|_{\alpha',1}\}.$$

Plainly, $\mathcal{N}_\alpha^p$ is an F-algebra with respect to this F-norm and embeds continuously into $\mathcal{N}_{\alpha'}^1$. Using this and Lemma B, we get:

- *if* $f = \sum_{n=0}^{\infty} a_n z^n$ *belongs to* $\mathcal{N}_\alpha^p$ *then* $a_n = O(\exp[o(n^{(\alpha+2)/(\alpha+2+p)})])$.

Moreover, Lemmas C and E allow us to extend Theorems 2 and 3 in a straightforward manner. In particular, if $\alpha, \beta \geq -1$ and $0 < p, q < \infty$, then

- $\mathcal{N}_\alpha^p$ embeds continuously and densely into $F_{(\alpha+2)/p}$;
- the duals of $\mathcal{N}_\alpha^p$ and of $F_{(\alpha+2)/p}$ coincide;
- $F_{(\alpha+2)/p}$ induces on $\mathcal{N}_\alpha^p$ the Mackey topology $\mu(\mathcal{N}_\alpha^p, (\mathcal{N}_\alpha^p)^*)$;
- $F_{(\alpha+2)/p}$ is the Fréchet envelope of $\mathcal{N}_\alpha^p$;
- $\mathcal{N}_\alpha^p$ fails to be locally pseudoconvex;
- $\mathcal{N}_\alpha^p$ and $\mathcal{N}_\beta^q$ have the same Fréchet envelope if and only if $(\alpha+2)/p = (\beta+2)/q$.

The description of the dual of $\mathcal{N}_\alpha^p$ for $0 < p < 1$ is as in (b) of Corollary 1, and the resulting corollaries remain valid in the extended setting. Details will be contained in [16].

In this note, however, we keep to the case $p \geq 1$.

Composition operators. Let X be $\mathcal{A}_\alpha^p$ or $\mathcal{N}_\alpha^p$, α, p being arbitrary. It is well known that every analytic map $\varphi : \mathbb{U} \to \mathbb{U}$ gives rise to a continuous linear map $C_\varphi : X \to X : f \mapsto f \circ \varphi$, the *composition operator* induced by φ. By standard arguments (see [16]), if X is an algebra $\mathcal{N}_\alpha^p$, then composition operators are just the algebra homomorphisms $X \to X$.

We are going to look at composition operators which operate between $\mathcal{N}_\alpha^p$ and $\mathcal{N}_\beta^q$ (resp. $\mathcal{A}_\alpha^p$ and $\mathcal{A}_\beta^q$) for different (α, p) and (β, q).

In what follows, *arcs* will be subsets of $\partial\mathbb{U}$ of the form $I = \{z \in \partial\mathbb{U} : \theta_1 \leq \arg z < \theta_2\}$ ($\theta_1, \theta_2 \in [0, 2\pi)$, $\theta_1 < \theta_2$). Normalized length of an arc I is $|I| = \int_I dm(z)$. The set

$$S(I) := \{z \in \mathbb{U} : 1 - |I| \leq |z| < 1,\ z/|z| \in I\}$$

is the *Carleson box* based on I. An *s-Carleson measure*, $s > 0$, on $\mathbb{U}$ is a Borel measure μ such that $\mu(S(I)) = O(|I|^s)$ for all arcs I. We say that μ is a *compact s-Carleson measure* if even $\mu(S(I)) = o(|I|^s)$ for all arcs I.

To incorporate the "Hardy case" we need to work on $\overline{\mathbb{U}}$. Here we use the same definitions, with $\mathbb{U}$ replaced by $\overline{\mathbb{U}}$. So Carleson boxes are now subsets of $\overline{\mathbb{U}}$, and Carleson measures are Borel measures on $\overline{\mathbb{U}}$.

Let $\varphi : \mathbb{U} \to \mathbb{U}$ be analytic. $\sigma_{\beta,\varphi}$ will be the image measure $\sigma_\beta \circ \varphi^{-1}$. If $\beta > -1$ then $\sigma_{\beta,\varphi}$ is a Borel measure on $\mathbb{U}$. If $\beta = -1$, then $\sigma_\beta = m$ is normalized Lebesgue measure on $\mathbb{T}$, and $\sigma_{\beta,\varphi}$ is a Borel measure on $\overline{\mathbb{U}}$.

The parallelism between the spaces $\mathcal{N}_\alpha^p$ and $\mathcal{A}_\alpha^p$ is emphasized by the following

Theorem 4. *Let $\varphi : \mathbb{U} \to \mathbb{U}$ be analytic, and let $\alpha, \beta \geq -1$ and $1 \leq p \leq q < \infty$.*

(a) *C_φ exists as a bounded operator $\mathcal{N}^p_\alpha \to \mathcal{N}^q_\beta$ if and only if $\sigma_{\beta,\varphi}$ is a $q(\alpha+2)/p$-Carleson measure.*

(b) *C_φ defines a compact operator $\mathcal{N}^p_\alpha \to \mathcal{N}^q_\beta$ if and only if $\sigma_{\beta,\varphi}$ is a compact $q(\alpha+2)/p$-Carleson measure.*

This is a generalization of a result obtained in [23]. For convenience, we include the essential parts of the argument.

Proof. We only treat the case $\beta > -1$; the modifications needed for the case $\beta = -1$ are left to the reader. So our measure $\sigma_{\beta,\varphi}$ lives on $\varphi(\mathbb{U})$ and, for every $f \in \mathcal{N}^q_\beta$, we have $\int_\mathbb{U} [\log(1+|f\circ\varphi|)]^q \, d\sigma_\beta = \int_\mathbb{U} [\log(1+|f|)]^q \, d\sigma_{\beta,\varphi}$.

(a) Suppose that $C_\varphi : \mathcal{N}^p_\alpha \to \mathcal{N}^q_\beta$ is bounded. Then there is a $\varrho > 0$ such that $\|\!|C_\varphi f\|\!|_{\beta,q} \leq 1$ for all $f \in \mathcal{N}^p_\lambda$ satisfying $\|\!|f\|\!|_{\alpha,p} \leq \varrho$. To show that $\sigma_{\beta,\varphi}$ is a $q(\alpha+2)/p$-Carleson measure, fix $\theta \in [0, 2\pi)$, $h \in (0,1]$, and consider the function

$$f_w(z) = \exp\left[c\left(\frac{1-|w|^2}{(1-\overline{w}z)^2}\right)^{(\alpha+2)/p}\right] - 1,$$

where $w = (1-h)e^{i\theta}$. The constant c will be fixed below. By [43, Lemma 4.2.2], there is a constant $K = K(\alpha)$ such that, for each $w \in \mathbb{U}$,

$$\begin{aligned}\|\!|f_w\|\!|_{\alpha,p} &= \Big(\int_\mathbb{U} [\log(1+|f_w|)]^p \, d\sigma_\alpha\Big)^{1/p} \leq c\bigg(\int_\mathbb{U} \Big(\frac{1-|w|^2}{|1-\overline{w}z|^2}\Big)^{\alpha+2} d\sigma_\alpha(z)\bigg)^{1/p} \\ &\leq c\bigg(\int_\mathbb{U} \frac{(1-|z|^2)^\alpha}{|1-\overline{w}z|^{2\alpha+4}} \, d\sigma(z)\bigg)^{1/p} \leq cK.\end{aligned}$$

If we choose $c := \varrho/K$, then $\|\!|f_w\|\!|_{\alpha,p} \leq \varrho$ and so $\|\!|C_\varphi f_w\|\!|_{\beta,q} \leq 1$.

Look at arcs I of sufficiently small length h. There is a constant c_0 such that if z is in $S(I)$, then $|1-\overline{w}z|^{-4(\alpha+2)} \geq c_0 \, h^{-4(\alpha+2)}$ and $\mathrm{Re}[(1-w\overline{z})^{2(\alpha+2)}] \geq c_0 \, h^{2(\alpha+2)}$. It follows that

$$\begin{aligned}\log(1+|f_w(z)|) &= \log\left|\exp\left[c\frac{(1-|w|^2)^{\alpha+2}(1-w\overline{z})^{2(\alpha+2)}}{|1-\overline{w}z|^{4(\alpha+2)}}\right]\right|^{1/p} \\ &= \left(\frac{c(1-|w|^2)^{\alpha+2}\,\mathrm{Re}[(1-w\overline{z})^{2(\alpha+2)}]}{|1-\overline{w}z|^{4(\alpha+2)}}\right)^{1/p} \\ &\geq \frac{cc_0^2\,(1-|w|^2)^{(\alpha+2)/p}}{h^{(\alpha+2)/p}}\end{aligned}$$

and so, with an appropriate constant C,

$$1 \geq \|C_\varphi f_w\|_{\beta,q} \geq \Big(\int_{S(I)} [\log 1 + |f_w|]^q \, d\sigma_{\beta,\varphi} \Big)^{1/q} \geq \frac{C}{h^{(\alpha+2)/p}} \sigma_{\beta,\varphi}(S(I))^{1/q}.$$

We have shown that $\sigma_{\beta,\varphi}$ is a $q(\alpha+2)/p$-Carleson measure.

Suppose conversely that $\sigma_{\beta,\varphi}$ is a $q(\alpha+2)/p$-Carleson measure. To prove that C_φ maps $\mathcal{N}^p_\alpha$ boundedly into $\mathcal{N}^q_\beta$, we divide $\mathbb{U}$ into "dyadic boxes", that is, we consider Carleson boxes based on the members the family of all arcs of the form

$$I_{n,k} := \Big\{ z \in \partial\mathbb{U} : \frac{2\pi k}{2^n} \leq \arg z < \frac{2\pi(k+1)}{2^n} \Big\}, \qquad k = 0, 1, \ldots, 2^n - 1, \quad n = 0, 1, \ldots$$

Let $H_{n,k}$ be the "inner half" of $S(I_{n,k})$:

$$H_{n,k} := \{ z \in S(I_{n,k}) : 1 - |I_{n,k}| \leq |z| < 1 - |I_{n,k}|/2 \}.$$

Notice that these sets are pairwise disjoint and cover $\mathbb{U}$. For each (n,k), let $a_{n,k}$ be the "center" of $H_{n,k}$ in the sense that $|a_{n,k}|$ bisects the interval of absolute values of points in $H_{n,k}$ and $\arg a_{n,k}$ bisects the interval of their arguments. Note that $|I_{n,k}| \asymp 1 - |a_{n,k}|$. Let $f \in \mathcal{N}^p_\alpha$ be given. Let $a^*_{n,k}$ be a point in $\overline{H}_{n,k}$ where $|f|$ attains its maximum. The disk $U_{n,k}$ with radius $2^{-(n+1)}$ and center at $a^*_{n,k}$ interesects at most four of the sets $H_{n,k}$. By subharmonicity of $[\log(1+|f|)]^p$ and since $p \leq q$,

$$\begin{aligned}
\|C_\varphi f\|^q_{\beta,q} &= \int_{\mathbb{U}} [\log(1+|f|)]^q \, d\sigma_{\beta,\varphi} = \sum_{n,k} \int_{H_{n,k}} [\log(1+|f|)]^q \, d\sigma_{\beta,\varphi} \\
&\leq \sum_{n,k} \sup_{w \in H_{n,k}} [\log(1+|f(w)|)]^q \, \sigma_{\beta,\varphi}(H_{n,k}) \\
&\leq C \sum_{n,k} [\log(1+|f(a^*_{n,k})|)]^q (1-|a_{n,k}|^2)^{q(\alpha+2)/p} \\
&\leq C \Big(\sum_{n,k} [\log(1+|f(a^*_{n,k})|)]^p (1-|a_{n,k}|^2)^{\alpha+2} \Big)^{q/p} \\
&\leq C \Big(\sum_{n,k} \int_{\mathbb{U}_{n,k}} [\log(1+|f(z)|)]^p (1-|z|^2)^\alpha \, d\sigma(z) \Big)^{q/p} \\
&\leq C \Big(\sum_{n,k} \int_{H_{n,k}} [\log(1+|f(z)|)]^p \, d\sigma_\alpha(z) \Big)^{q/p} = C \|f\|^q_{\alpha,p};
\end{aligned}$$

here C denotes a constant which may change from line to line.

(b) is just the "little o" version of (a) and has a similar proof which is based on the fact that a sequence in $\mathcal{N}^p_\alpha$ converges to zero iff it is bounded and converges to zero locally uniformly. We skip the details. ∎

It is not clear to what extent the conditions on p and q can be relaxed. Note that if $p > q$ and $\beta \geq \alpha$ then C_φ always acts boundedly from $\mathcal{N}^p_\alpha$ to $\mathcal{N}^q_\beta$: it is bounded from $\mathcal{N}^p_\alpha$ to itself and the canonical embedding $\mathcal{N}^p_\alpha \hookrightarrow \mathcal{N}^p_\beta$ is continuous.

If $\alpha, \beta > -1$, then $C_\varphi : \mathcal{N}^p_\alpha \to \mathcal{N}^q_\beta$ is even compact in such a case. In fact, combining Theorem 4 with [10, Proposition 6] and [23, Theorems 1.2 and 1.3], we may first of all state:

Corollary 4. *Let $\varphi : \mathbb{U} \to \mathbb{U}$ be analytic, and let $\alpha, \beta \geq -1$ and $1 \leq p \leq q < \infty$. Then C_φ maps $\mathcal{N}^p_\alpha$ boundedly (resp. compactly) into $\mathcal{N}^q_\beta$ iff it maps $\mathcal{A}^p_\alpha$ boundedly (resp. compactly) into $\mathcal{A}^q_\beta$.*

To get from this the above compactness statement, we need to know that, by atomic decomposition, $\mathcal{A}^r_\alpha$ is isomorphic to ℓ^r (compare with [6] or [27]) and that, by Pitt's theorem ([26, p. 208]), any operator $\ell^q \to \ell^p$ is compact whenever $p < q$.

We also have an immediate reduction to a plain Hilbert space setting:

Corollary 5. *Suppose that $p \leq q(\alpha + 2)$ and define α' by $\alpha' + 2 = q(\alpha + 2)/p$. For C_φ to exist as a bounded (resp. compact) operator $\mathcal{N}^p_\alpha \to \mathcal{N}^q_\alpha$, it is necessary and sufficient that C_φ maps the Hilbert space $\mathcal{A}^2_{\alpha'}$ boundedly (resp. compactly) into the Hilbert space $\mathcal{A}^2_\beta$.*

It is not hard to see that if that if $\alpha, \beta \geq -1$ and $1 \leq p \leq q < \infty$, then $\mathcal{N}^p_\alpha \subset \mathcal{N}^q_\beta$ (continuously) if and only if $(\beta + 2)/q \geq (\alpha + 2)/p$ and that the embedding is compact if and only if $(\beta + 2)/q > (\alpha + 2)/p$. Formally, this appears as a special case of the situation discussed here since formal identities can be considered as composition operators induced by the identity map of $\mathbb{U}$. Note that $\mathcal{N}^p_\alpha$ and $\mathcal{N}^q_\beta$ can be replaced with $\mathcal{A}^p_\alpha$ and $\mathcal{A}^q_\beta$, respectively.

As a consequence, if $\alpha, \beta \geq -1$, $1 \leq p \leq q < \infty$ and $\mathcal{N}^p_\alpha = \mathcal{N}^q_\beta$, then $\alpha = \beta$ and $p = q$. Actually, no restriction on p and q is needed.

Corollary 6. *Let $\alpha, \beta \geq -1$ and $1 \leq p, q < \infty$. Then $\mathcal{N}^p_\alpha$ and $\mathcal{N}^q_\beta$ coincide only when $\alpha = \beta$ and $p = q$.*

Proof. If $\mathcal{N}^p_\alpha = \mathcal{N}^q_\beta$, then $\mathcal{A}^p_\alpha = \mathcal{A}^q_\beta$ by Proposition 2, and so the statement follows. ∎

This leads to the promised "soft" proof that $\mathcal{N}^p_\alpha$ cannot be locally convex. In fact, otherwise we would have $\mathcal{N}^p_\alpha = F_{(\alpha+2)/p}$, hence $\mathcal{N}^p_\alpha \subset \mathcal{N}^q_\beta \subset F_{(\alpha+2)/p}$ and so $\mathcal{N}^p_\alpha = \mathcal{N}^q_\beta$ if we choose β, q such that $(\alpha + 2)/p = (\beta + 2)/q$ and $p < q$; this is a contradiction with Corollary 6.

Order boundedness. Let X be a quasi-Banach space and Y a subspace of a quasi-Banach lattice L. An operator $u : X \to Y$ is called *order bounded* if there is an element $g \geq 0$ in L such that $|uf| \leq g$ for all f in B_X, the unit ball of X. It is thus required that u maps B_X into an order interval of L. The choice of L is part of the definition; every Banach space operator $u : X \to Y$ becomes order bounded when Y is considered as a subspace of the Banach lattice $C(K)$ of continuous functions on an appropriate compact Hausdorff space K.

On the other hand, if I is an order interval in a quasi-Banach lattice L, then the span of I, with the order inherited from L and normed by the gauge functional of I, is a Banach lattice. It is in fact an abstract M-space and so, by Kakutani's theorem, isometrically isomorphic to some Banach lattice $C(K)$. Consequently, an order bounded operator $u : X \to Y$, considered as an operator with values in the given lattice L, admits a factorization $X \to C(K) \overset{j}{\hookrightarrow} L$, j being the canonical map.

Within the setting of Banach lattices and Banach spaces, there are close relations with absolutely summing operators; see e.g. [7, Chs. 11 & 16]. We just mention that, for arbitrary measures μ and ν, an operator $L^2(\mu) \to L^2(\nu)$ is order bounded if and only if it is a Hilbert–Schmidt operator. Order bounded operators with values e.g. in $\mathcal{A}^q_\beta$ ($q \ge 1$) are known to be q-integral. They are q-nuclear if the dual of the domain space has the Radon–Nikodym property. Details can be found in [8] and [7].

Order boundedness is generalized to operators from $\mathcal{N}^p_\alpha$ into $\mathcal{A}^q_\beta$ or $\mathcal{N}^q_\beta$ by requiring that every 0-neighbourhood in $\mathcal{N}^p_\alpha$ is mapped into an order interval in $L^q(\sigma_\beta)$, resp. $L^q_{\log}(\sigma_\beta)$.

Order boundedness of composition operators between weighted Bergman spaces has been investigated in [10], for example. It was shown there that, given $\alpha, \beta \ge -1$, $p, q \ge 1$ and $\varphi : \mathbb{U} \to \mathbb{U}$ analytic, C_φ exists as an order bounded operator $\mathcal{A}^p_\alpha \to \mathcal{A}^q_\beta$ if and only if $(1 - |\varphi|^2)^{-q(\alpha+2)/p}$ is σ_β-integrable. The same condition characterizes order boundedness of composition operators between Nevanlinna algebras:

THEOREM 5. *Let $\varphi : \mathbb{U} \to \mathbb{U}$ be analytic, and let $\alpha, \beta \ge -1$ and $p, q \ge 1$. Then C_φ exists as an order bounded operator $\mathcal{N}^p_\alpha \to \mathcal{N}^q_\beta$ if and only if $(1 - |\varphi|^2)^{-1}$ belongs to $L^{q(\alpha+2)/p}(\sigma_\beta)$.*

Proof. Suppose first that $C_\varphi : \mathcal{N}^p_\alpha \to \mathcal{N}^q_\beta$ is order bounded: for each $s > 0$ there exists a $g_s \in L^q_{\log}(\sigma_\beta)$ such that $|f(\varphi(z))| \le g_s(z)|$ a.e. on $\mathbb{U}$ for all $f \in \mathcal{N}^p_\alpha$ satisfying $\|f\|_{\alpha,p} \le s$. Given $c > 0$ and $w \in \mathbb{U}$, we define

$$F_w(z) := \exp\left[c\left(\frac{1 - |\varphi(w)|^2}{(1 - \overline{\varphi(w)}z)^2}\right)^{(\alpha+2)/p}\right] - 1 \quad (z \in \mathbb{U});$$

so $F_w := f_{\varphi(w)}$ in the notation used in the proof of Theorem 4. Accordingly, there is a constant $K = K(\alpha) > 0$ such that $\|F_w\|_{\alpha,p} \le cK$. We put $c = s/K$

so as to have $\|F_w\|_{\alpha,p} \le s$, hence

$$g_s(z) \ge |F_w(\varphi(z))| = \left|\exp\left[c\left(\frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^2}\right)^{(\alpha+2)/p}\right]-1\right|.$$

In particular, $g_s(w) \ge \exp[c((1-|\varphi(w)|^2)^{-(\alpha+2)/p}]-1$, so that

$$[\log(1+|g_s(w)|)]^q \ge c((1-|\varphi(w)|^2)^{-q(\alpha+2)/p}.$$

The statement follows.

The converse is also straightforward. Assume that $(1-|\varphi|^2)^{-q(\alpha+2)/p}$ is σ_β-integrable. A standard subharmonicity argument provides us with a constant $M_0 = M_0(\alpha)$ such that $[\log(1+|f(w)|)]^p \le M_0 s/(1-|w|^2)^{\alpha+2}$ whenever $\|f\|_{\alpha,p} \le s^{1/p}$; it follows that

$$[\log(1+|f(\varphi(z))|]^q \le \frac{M_0 s}{(1-|\varphi(z)|^2)^{q(\alpha+2)/p}}.$$

The function to the right is in $L^1(\sigma_\beta)$ and provides an order bound for the functions f under consideration. Since $s>0$ was arbitrary, $C_\varphi : \mathcal{N}_\alpha^p \to \mathcal{N}_\beta^q$ is order bounded. ∎

In addition to the situation for bounded and compact operators, we may state:

Corollary 7. *Suppose that $p \le q(\alpha+2)$. Then C_φ is order bounded as an operator $\mathcal{N}_\alpha^p \to \mathcal{N}_\beta^q$ iff it acts as a Hilbert–Schmidt operator from $\mathcal{A}_{\alpha'}^2$ into $\mathcal{A}_\beta^2$, where $\alpha'+2 = q(\alpha+2)/p$.*

In the setting of Banach spaces and Banach lattices, every order bounded operator admits a factorization through some $C(K)$-space. As for composition operators, we can be more precise. We start by a result on spaces $\mathcal{A}_{s,0}^\infty$ and $\mathcal{A}_s^\infty$, $s>0$. We will use the fact that $\mathcal{A}_s^\infty$ is canonically isomorphic to the bidual of $\mathcal{A}_{s,0}^\infty$ (this is true even in a more general context; see L. A. Rubel and A. L. Shields [34]). In fact, $\mathcal{A}_{s,0}^\infty$ is isomorphic to the sequence space c_0.

Theorem 6. *Let $s>0$, $\beta \ge -1$, $0<q<\infty$ and an analytic function $\varphi : \mathbb{U} \to \mathbb{U}$ be given. Then the composition operator $C_\varphi : \mathcal{A}_{s,0}^\infty \to \mathcal{A}_\beta^q$ is order bounded whenever it is defined. In this case, $(1-|\varphi|)^{-s}$ belongs to $L^q(\sigma_\beta)$.*

Proof. We confine ourselves to $\beta > -1$. The case $\beta = -1$ is similar; see [18, Theorem 3].

We prove order boundedness of $C_\varphi : \mathcal{A}_s^\infty \to \mathcal{A}_\beta^q$ first under the assumption $0 < s \le 1$, $q \ge 1$. By weak compactness, C_φ maps $\mathcal{A}_s^\infty$ into $\mathcal{A}_\beta^q$ if and only if $C_\varphi(\mathcal{A}_{s,0}^\infty) \subset \mathcal{A}_\beta^q$.

We apply a lacunary series argument (compare [2, Theorem 16] and [18, Theorem 4]). For each (non-dyadic) $t \in [0,1]$, define $f_t(z) := \sum_{n=0}^\infty r_n(t) 2^{ns} z^{2^n}$ where (r_n) is the sequence of Rademacher functions. The

f_t's form a bounded family in $\mathcal{A}_s^\infty$; see [22, p. 435] (it is claimed there that $\|f_t\|_{s,\infty}$ does not depend on t, but this seems to be too optimistic). Our hypothesis implies that $K := \sup_t \|C_\varphi f_t\|_{\beta,q}$ exists. By Fubini's theorem and Khinchin's inequality there is a $C > 0$ such that

$$K^q \ge \int_0^1 \|C_\varphi f_t\|_{\beta,q}^q \, dt = \int_{\mathbb{U}}\int_0^1 \Big|\sum_{n=0}^\infty r_n(t) 2^{ns} \varphi(z)^{2^n}\Big|^q dt \, d\sigma_\beta(z)$$

$$\ge C \int_{\mathbb{U}} \Big(\sum_{n=0}^\infty 2^{2ns} |\varphi(z)|^{2^{n+1}}\Big)^{q/2} d\sigma_\beta(z).$$

Put $I_n = \{k \in \mathbb{N} : 2^n \le k+1 < 2^{n+1}\}$, $n \in \mathbb{N}_0$. Since $\sum_{k\in I_n}(k+1) \le 2^{2n+1}$ we get, for $0 < r < 1$,

$$\begin{aligned}
&\sum_{n=0}^\infty 2^{2ns} r^{2^{n+1}} \\
&\quad \ge 2^{-s} \sum_{n=0}^\infty \Big[\sum_{k\in I_n}(k+1)\Big]^s (r^2)^{2^n} \ge 2^{-s}\Big[\sum_{n=0}^\infty \sum_{k\in I_n} (k+1)(r^{2/s})^{k+1}\Big]^s \\
&\quad = 2^{-s}\Big[\sum_{n=0}^\infty (n+1)(r^{2/s})^{n+1}\Big]^s \ge 2^{-s}\Big[\sum_{n=0}^\infty (n+1)(r^{4/s})^n - (1-r^{2/s})\Big]^s \\
&\quad \ge 2^{-s}\Big[\sum_{n=0}^\infty (n+1)(r^{4/s})^n - 1\Big]^s \ge 2^{-s}\Big[\frac{1}{(1-r^{4/s})^{2s}} - 1\Big].
\end{aligned}$$

Hence $\int_{\mathbb{U}}[(1-|\varphi|^{4/s})^{-2s} - 1]^{q/2} \, d\sigma_\beta \le C$ for some $C > 0$. It follows that $(1-|\varphi|^{4/s})^{-sq}$ and so $(1-|\varphi|^2)^{-sq}$ are σ_β-integrable.

The cases where $s > 1$ and/or q is arbitrary can now be settled by the same reduction procedure. Let C_φ be as in the theorem. Then C_φ maps $\mathcal{A}_{s/2}^\infty$ into A_β^{2q}. In fact, $f \in B_{\mathcal{A}_{s/2}^\infty}$ implies $f^2 \in B_{\mathcal{A}_s^\infty}$, hence

$$\|f\circ\varphi\|_{\beta,2q}^{2q} = \|f^2\circ\varphi\|_{\beta,q}^q \le K\,(\sup_{z\in\mathbb{U}}(1-|z|^2)^s|f^2(z)|)^q \le K,$$

where $K = \|C_\varphi : \mathcal{A}_s^\infty \to A_\beta^q\|^q$. Iterate to see that C_φ maps each $\mathcal{A}_{s/2^k}^\infty$ (boundedly) into $\mathcal{A}_\beta^{2^k q}$, $k \in \mathbb{N}$. Choose k so that $s/2^k \le 1$ and $2^k q \ge 1$. Then $C_\varphi : \mathcal{A}_{s/2^k}^\infty \to \mathcal{A}_\beta^{2^k q}$ is order bounded by the first part of the proof, and again we can conclude that $(1-|\varphi|^2)^{-s} \in L^q(\sigma_\beta)$. ∎

By the last step, if $0 < q, s, t < \infty$ and $\beta \ge -1$ are given and if C_φ maps $\mathcal{A}_s^\infty$ into $\mathcal{A}_\beta^q$ then it maps $\mathcal{A}_{st}^\infty$ into $\mathcal{A}_\beta^{q/t}$. These operators are order bounded; moreover, $\mathcal{A}_s^\infty$ and $\mathcal{A}_{st}^\infty$ can be replaced by $\mathcal{A}_{s,0}^\infty$ and $\mathcal{A}_{st,0}^\infty$, respectively.

It is readily seen that $\mathcal{A}^p_\alpha$ embeds (boundedly) into $\mathcal{A}^\infty_{(\alpha+2)/p,0}$. Therefore we can now state:

COROLLARY 8. *A composition operator* $C_\varphi : \mathcal{A}^p_\alpha \to \mathcal{A}^q_\beta$ *is order bounded if and only if* C_φ *even maps* $\mathcal{A}^\infty_{(\alpha+2)/p,0}$ (*equivalently*, $\mathcal{A}^\infty_{(\alpha+2)/p}$) *into* $\mathcal{A}^q_\beta$.

Compare with [22] and [11]. Again there is a corresponding result for composition operators between Nevanlinna algebras. Recall that $F_{(\alpha+2)/p}$ is the Nevanlinna algebra corresponding to $\mathcal{A}^\infty_{(\alpha+2)/p,0}$ (Lemma A).

COROLLARY 9. *Let* $\varphi : \mathbb{U} \to \mathbb{U}$ *be analytic*, $\alpha, \beta \geq -1$, $1 \leq p, q < \infty$ ($q > 1$ *if* $\beta = -1$).

(a) $C_\varphi : F_{(\alpha+2)/p} \to \mathcal{N}^q_\beta$ *is order bounded whenever it is defined.*

(b) C_φ *maps* $\mathcal{N}^p_\alpha$ *order boundedly into* $\mathcal{N}^q_\beta$ *iff it even maps* $F_{(\alpha+2)/p}$ *into* $\mathcal{N}^q_\beta$.

Proof. (a) Write $s = (\alpha+2)/p$ and look at any $f \in \mathcal{A}^\infty_{s,0}$. Then $\exp(\pm f) \in F_s$ and so $C_\varphi(\exp(\pm f)) \in \mathcal{N}^q_\beta$, by assumption. Since $C_\varphi(\exp(\pm f(z))) = \exp(\pm(C_\varphi f)(z))$ for all $z \in \mathbb{U}$, we see that we are dealing with invertible elements in $\mathcal{N}^q_\beta$. It follows from Proposition 2 that $C_\varphi f$ belongs to $\mathcal{A}^q_\beta$.

The resulting operator $C_\varphi : \mathcal{A}^\infty_{s,0} \to \mathcal{A}^q_\beta$ is order bounded by Theorem 6, and we have $1/(1-|\varphi|^2)^s \in L^q(\sigma_\beta)$. The functions $\exp c/(1-|\varphi|^2)^s \in L^q_{\log}(\sigma_\beta)$, $c > 0$, provide order bounds for the $C_\varphi f$'s, $f \in F_s$: as an operator $F_s \to \mathcal{N}^q_\beta$, C_φ is order bounded.

(b) is a straightforward consequence of the preceding results. ∎

Concluding remarks. Passing again to the special case of embeddings, we may state:

COROLLARY 10. *Let* $\alpha \geq -1$, $\beta > -1$ *and* $0 < p, q < \infty$. *The following statements are equivalent*:

(i) $(\beta+1)/q > (\alpha+2)/p$.

(ii) $\mathcal{A}^p_\alpha \hookrightarrow \mathcal{A}^q_\beta$ *order boundedly.*

(iii) $\mathcal{A}^\infty_{(\alpha+2)/p,0} \hookrightarrow \mathcal{A}^q_\beta$ (*order boundedly*).

Analogously for embeddings $\mathcal{N}^p_\alpha \hookrightarrow \mathcal{N}^q_\beta$ *and* $F_{(\alpha+2)/p} \hookrightarrow \mathcal{N}^q_\beta$ $(1 \leq p, q < \infty)$.

It was shown in [33, Theorem 6.7] that if $s < 1$ then $F_s \hookrightarrow \mathcal{N}^1_0$. By Corollary 10, the converse is true as well. This answers a corresponding question raised in [33, Remark 7].

There is no chance to extend Corollary 10 to $\beta = -1$. In fact, suppose that F_c embeds into $\mathcal{N}^q_{-1}$ for some $0 < c < 1$ and $1 \leq q < \infty$. Put $p = 1/c$. Then $p \geq q$, $\mathcal{N}^p_{-1}$ embeds order boundedly into $\mathcal{N}^q_{-1}$, and so H^p embeds order boundedly into H^q. Being a q-summing operator with domain a reflexive space, the embedding would be compact ([7, Ch. 5]), and this is impossible.

Clearly, the Nevanlinna class $\mathcal{N}$ embeds (continuously) into $\mathcal{N}^q_\beta$ for any $\beta > -1$ and $p \geq 1$. It is an easy consequence of results of J. S. Choa, H. O. Kim and J. H. Shapiro [5] and T. Domenig [9] that this embedding is compact if $q = 1$; see [23]. We can now strengthen these results as follows:

COROLLARY 11. (a) *If* $\beta > -1$ *and* $(\beta + 2)/q > 1$, *then* $\mathcal{N}$ *embeds compactly into* $\mathcal{N}^q_\beta$. *Moreover*, *every analytic function* $\varphi : \mathbb{U} \to \mathbb{U}$ *gives rise to a compact composition operator* $C_\varphi : \mathcal{N} \to \mathcal{N}^q_\beta$.

(b) *If* $\beta > -1$ *and* $(\beta + 1)/q > 1$, *then* $\mathcal{N}$ *embeds order boundedly into* $\mathcal{N}^q_\beta$. *Moreover*, *every analytic function* $\varphi : \mathbb{U} \to \mathbb{U}$ *gives rise to an order bounded composition operator* $C_\varphi : \mathcal{N} \to \mathcal{N}^q_\beta$.

Here compactness and order boundedness of operators with domain $\mathcal{N}$ are defined in the same way as for operators on $\mathcal{N}^p_\alpha$; compare [5] and [19].

By Corollary 10,

$$\mathcal{A} := \bigcap_{s>0} \mathcal{A}^\infty_{s,0} = \bigcap_{s>0} \mathcal{A}^\infty_s = \bigcap_{p>0} \mathcal{A}^p_\alpha \quad \text{and} \quad \mathcal{F} := \bigcap_{s>0} F_s = \bigcap_{s>0} \mathcal{N}^\infty_s = \bigcap_{p>0} \mathcal{N}^p_\alpha$$

for every $\alpha > -1$. In a natural fashion, $\mathcal{F}$ is a nuclear Fréchet space, and $\mathcal{A}$ is a Fréchet–Schwartz space (Pitt's theorem). As a consequence, Corollary 10 does not extend to $\alpha = -1$: $\bigcap_{p<\infty} H^p$ cannot be a Schwartz space since otherwise embeddings like $H^\infty \hookrightarrow H^2$ would be compact. By Theorem 1, $\bigcap_{p<\infty} \mathcal{N}^p_{-1} \subset \mathcal{F}$ is a proper inclusion as well.

We claim that $\mathcal{A}$ fails to be nuclear. To see this, we employ the ideals $\mathcal{S}_r$ of Banach space operators whose approximation numbers belong to ℓ^r, $0 < r < \infty$. For details we refer to A. Pietsch [30], Chapter 2. Assume that $\mathcal{A}$ is nuclear. Then, by a known multiplication theorem ([30, 2.3.13]), the standard embedding $\mathbb{W} \hookrightarrow \mathcal{A}^2_\alpha$, for example, is in each $\mathcal{S}_r$ since it admits a factorization through $\mathcal{A}$. Here $\mathbb{W}$ is the Wiener algebra, as before. We identify, in the canonical way, $\mathbb{W}$ with ℓ^1 and $\mathcal{A}^2_\alpha$ with ℓ_2. Then the above embedding is just the diagonal operator $D : \ell^1 \to \ell^2$ given by $(d_n)^\infty_{n=0}$ where $d_n = \Gamma(\alpha+1)^{1/2}\Gamma(n+1)^{1/2}\Gamma(\alpha+n+2)^{-1/2}$ is the norm of the nth monomial z^n in $\mathcal{A}^2_\alpha$. By [30, 2.9.12], D belongs to $\mathcal{S}_r(\ell^1, \ell^2)$ iff $(d_n)_n$ belongs to ℓ^r. It follows that $\{0 < r < \infty : D \in \mathcal{S}_r(\ell^1, \ell^2)\}$ is bounded away from zero, a contradiction.

On the other hand, we lack information about the topological structure of the F-space $\bigcap_{p<\infty} \mathcal{N}^p_{-1}$.

Similarly we see that, independently of $\alpha > -1$,

$$\mathcal{A}^{-\infty} := \bigcup_{p>0} \mathcal{A}^p_\alpha = \bigcup_{s>0} \mathcal{A}^\infty_s = \bigcup_{s>0} \mathcal{A}^\infty_{s,0}.$$

$\mathcal{A}^{-\infty}$ is an algebra and was introduced by B. Korenblum [25]. Selecting α_n's in $(-1,\infty)$ such that $\alpha_{n+1} > \alpha_n + 1$ for each n, we obtain an inductive limit representation $\mathcal{A}^{-\infty} = \bigcup_n \mathcal{A}^2_{\alpha_n}$ of Hilbert spaces. By Corollary 10, the linking mappings are Hilbert–Schmidt, and it is well known that therefore the (linear) inductive limit topology on $\mathcal{A}^{-\infty}$ is nuclear (compare K. Floret and J. Wloka [15, p. 163]). In our case, this can also be seen more directly. In fact, it is readily verified that an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ belongs to $\mathcal{A}^{-\infty}$ iff $(a_n n^{-k})_n$ is bounded for some $k \in \mathbb{N}$. Accordingly, the dual of $\mathcal{A}^{-\infty}$ consists of all analytic functions $g(z) = \sum_{n=0}^{\infty} b_n z^n$ such that all $(n^k b_n)_n$ are bounded (see [25, p. 189]) and so can be identified with the classical space $\mathbf{s}$ of rapidly decreasing sequences. $\mathbf{s}$ is the "universal" nuclear space in the sense that every nuclear space is isomorphic to a subspace of a product of copies of $\mathbf{s}$. It follows that $\mathcal{A}^{-\infty}$ itself is just another model of the dual of $\mathbf{s}$.

It also follows from Corollary 10 that if $(\beta+2)/(\beta+1) < q'/q$ and if a composition operator C_φ maps $\mathcal{A}^p_\alpha$ into $\mathcal{A}^{q'}_\beta$, then it is order bounded as an operator $\mathcal{A}^p_\alpha \to \mathcal{A}^q_\beta$. Again, this does not apply directly to $\beta = -1$, but in that case a much better result was obtained by H. Hunziker [17]: a composition operator C_φ maps H^p order boundedly into H^q if $C_\varphi(H^p) \subset H^{\lambda q}$ holds for some $\lambda > 1 + p/q$; compare also [15, Theorem 5.5]. It was shown by R. Riedl [31] that this cannot be improved: if φ maps $\mathbb{U}$ conformally onto a polygonal domain inside $\mathbb{U}$ and if γ is the greatest interior angle at a point of contact of the polygon and $\mathbb{T}$, then C_φ maps H^1 into $H^{\pi/\gamma}$. This is best possible: $C_\varphi : H^1 \to H^{\pi/\gamma}$ cannot be compact.

We do not have a sharp result like this for weighted Bergman spaces. We can only complement the above statements by the following simple

PROPOSITION 3. *Suppose that $p \le q$ and that C_φ maps $\mathcal{A}^p_\alpha$ into $\mathcal{A}^{\lambda q}_\beta$ for some $\lambda > 1 + 2p/(q(\alpha+2))$. Then C_φ is order bounded as an operator $\mathcal{A}^p_\alpha \to \mathcal{A}^q_\beta$.*

Proof. Put $\gamma := q(\alpha+2)/p$. We want to show that $(1-|z|)^{-\gamma}$ is $\sigma_{\beta,\varphi}$-integrable whenever $\sigma_{\beta,\varphi}$ is a $\lambda\gamma$-Carleson measure.

Let $C > 0$ be such that $\sigma_{\beta,\varphi}(S(I)) \le C|I|^{\alpha\gamma}$ for every arc $I \subset \mathbb{T}$. For each $n \in \mathbb{N}$, put $A_n := \{z \in \mathbb{U} : |z| > 1 - 1/n\}$ and $B_n := A_n \setminus A_{n+1}$. Partition $\mathbb{T}$ into n disjoint arcs I_k of equal length. Then A_n is the disjoint union of the Carleson boxes $S(I_k)$, whence

$$\sigma_{\beta,\varphi}(A_n) = \sum_{k=1}^{n} \sigma_{\beta,\varphi}(S(I_k)) \le \frac{C}{n^{\lambda\gamma-1}}.$$

It follows that

$$\int \frac{1}{(1-|z|)^\gamma}\,d\sigma_{\beta,\varphi}(z) = \sum_{n=1}^{\infty}\int_{B_n} \frac{1}{(1-|z|)^\gamma}\,d\sigma_{\beta,\varphi}(z) \le \sum_{n=1}^{\infty}(n+1)^\gamma\sigma_{\beta,\varphi}(A_n) \le C'\sum_{n=1}^{\infty} n^{\lambda\gamma-2},$$

where $C' = C2^\gamma$. The sum converges since $\lambda\gamma - 2 > \gamma \ge 1$.

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