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Almansi decompositions for polyharmonic, polyheat, and polywave functions

by

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Abstract. We construct Almansi decompositions for a class of differential operators, which include powers of the classical Laplace operator, heat operator, and wave operator. The decomposition is given in a constructive way.

1. Introduction. In 1899 Almansi stated the following classical decomposition theorem (Almansi expansion, see [1]):

ALMANSI'S THEOREM (cf. Aronszajn *et al.* [2]). *If f is polyharmonic of degree m in a star-like domain Ω with centre* 0, *then there exist unique functions $P_1 f, \dots, P_m f$ harmonic in Ω such that*

$$f(x) = P_1 f(x) + |x|^2 P_2 f(x) + \cdots + |x|^{2(m-1)} P_m f(x).$$

One can find important applications and generalizations of this result in the case of several complex variables in the monograph of Aronszajn *et al.* [2], e.g. concerning functions holomorphic in a neighbourhood of the origin in $\mathbb{C}^n$.

More recent generalizations of Almansi's Theorem, for instance for polyharmonic Bergman spaces, can be found in [9] and [4], and for kernels of iterates of weighted Laplace and Helmholtz operators in [11]. Also the case of other operators, like Dirac or Dunkl operators, was studied in [6], [10], and [8].

However, all these cases are limited to elliptic operators on domains star-like with respect to zero. In 1958 M. Nicolescu [7] obtained an Almansi decomposition for a class of operators depending on two variables which also includes the heat operator. As far as we know, no Almansi decomposition

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has been proved for the case of polywave functions. At first sight, the case of polyheat functions seems to be most difficult, due to the fact that the heat operator is not homogeneous; it turns out, however, that strangely enough the difficult case is the case of polyharmonic and polywave functions. While the decomposition of polyharmonic functions with respect to the unit ball is unique this is not the case for the half-space. Nevertheless, our proof will be constructive. We will give representation formulae both locally and, whenever possible, globally.

In Section 3 we will consider Almansi decompositions for the heat operator. In this case the decomposition turns out to be unique.

2. The case of polyharmonic and polywave functions. First consider the differential operator $\Box = \Delta_x + \lambda\frac{\partial^2}{\partial t^2}$, $\lambda \in \mathbb{C}\setminus\{0\}$ being a constant, where $\Delta_x = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$, $(x,t) \in \mathbb{R}^{n+1}$. We remark that for $\lambda = 1$ we have the $(n+1)$-dimensional Laplacian and for $\lambda = -1$ the wave operator.

We prove the following theorem:

THEOREM 2.1. *Let $D \subset \mathbb{R}^{n+1}$ be a domain and $f : D \to \mathbb{C}$ be a function such that $\Box^m f = 0$ for some $m \in \mathbb{N}$. Then for any $(x,t) \in D$ there exist functions $P_j f \in \ker\Box$, locally defined and unique up to a harmonic function in x, such that*

$$f(x,t) = P_1 f(x,t) + tP_2 f(x,t) + \cdots + t^{m-1} P_m f(x,t). \tag{1}$$

Conversely, if $P_1 f, \ldots, P_m f$ are functions in $\ker\Box$ then equation (1) defines a function $f \in \ker\Box^m$.

REMARK 2.1. The lack of uniqueness is in the nature of the problem, as the following simple example shows. If for $n = 1$ we take $f(x,t) = xt$ and $m = 2$ then we have the decomposition with $P_1 f(x,t) = xt$ and $P_2 f(x,t) = 0$ as well as with $P_1 f(x,t) = 0$ and $P_2 f(x,t) = x$.

For any given $(x,t) \in \mathbb{R}^{n+1}$, we let $(x,t) \in B \times (a,b)$ where B denotes the largest ball centred at x such that $\overline{B} \times [a,b] \subset D$. For the proof of the first part of Theorem 2.1 we need only prove (1) in $\overline{B} \times [a,b]$. For this reason we can assume that $f \in C^2(\overline{B} \times [a,b])$. We keep this assumption throughout this section.

To give explicit formulae for $P_j f$ we introduce the operator

$$\begin{aligned} Qf(x,t) &= \frac{1}{\lambda}\int_a^t f(x,s)\,ds + K\left(\frac{\partial f}{\partial t}(x,a)\right) \\ &= \frac{1}{\lambda}\int_a^t f(x,s)\,ds - \int_B \Phi(x-y)\frac{\partial f}{\partial t}(y,a)\,dy. \end{aligned}$$

Here,

$$Kg(x) = -\int_B \Phi(x-y)g(y)\,dy$$

stands for the Newtonian potential with

$$\Phi(x) = \begin{cases} \dfrac{\Gamma(n/2)}{(n-2)2\pi^{n/2}}\,|x|^{2-n}, & n > 2, \\ -\dfrac{1}{2\pi}\ln|x|, & n = 2, \end{cases}$$

being the fundamental solution of the Laplacian. Recall that K is a right inverse of the Laplacian, i.e. $\Delta_x Kg = g$ for any $g \in C^2(\overline{B})$.

Define

$$C_k = \frac{1}{2^k k!}$$

and let I be the identity operator. Then we get the following representation formulae for the operators P_j:

$$(2) \qquad \begin{cases} P_m = C_{m-1}Q^{m-1}\Box^{m-1}, \\ P_{m-1} = C_{m-2}Q^{m-2}\Box^{m-2}(I - C_{m-1}t^{m-1}Q^{m-1}\Box^{m-1}), \\ \quad\vdots \\ P_2 = C_1Q\Box(I - C_2t^2Q^2\Box^2)\cdots(I - C_{m-1}t^{m-1}Q^{m-1}\Box^{m-1}), \\ P_1 = (I - C_1Q\Box)(I - C_2t^2Q^2\Box^2)\cdots(I - C_{m-1}t^{m-1}Q^{m-1}\Box^{m-1}), \end{cases}$$

or, for $j = 2, \dots, m-1$,

$$P_j = C_{j-1}Q^{j-1}\Box^{j-1}(I - C_jt^jQ^j\Box^j)\cdots(I - C_{m-1}t^{m-1}Q^{m-1}\Box^{m-1}),$$

where t^k denotes the multiplication operator defined by $(t^kf)(x,t) = t^kf(x,t)$.

From these formulae we see that all P_j are linear operators.

If our operator $\Box$ is hypoelliptic, i.e. every distributional null solution $u \in L^2_{\mathrm{loc}}(\Omega)$ is C^∞ (see [12] or [5]), then we can consider Qf in the distributional sense. By replacing the operator K by the distributional convolution (cf. [3]) of $\frac{\partial f}{\partial t}(x,a)$ with the fundamental solution Φ we obtain global representation formulae. For instance, this will be the case if $\lambda \in \mathbb{R}$ and $\lambda > 0$. This leads to the following theorem.

THEOREM 2.2. *Let $D \subset \mathbb{R}^{n+1}$ be a convex domain with respect to the last variable t, let the operator $\Box$ be hypoelliptic and $f : D \to \mathbb{C}$ be a function such that $\Box^m f = 0$ for some $m \in \mathbb{N}$. Then for any $(x,t) \in D$ there exist functions $P_jf \in \ker\Box$, globally defined and unique up to a harmonic function in x, such that*

$$(3) \qquad f(x,t) = P_1f(x,t) + tP_2f(x,t) + \cdots + t^{m-1}P_mf(x,t).$$

Conversely, if $P_1 f, \dots, P_m f$ *are functions in* $\ker \Box$ *then equation* (3) *defines a function* $f \in \ker \Box^m$.

In the case of $\lambda = 1$ we obtain from Theorem 2.2 the Almansi decomposition for polyharmonic functions in the half-space:

PROPOSITION 2.1. *Let* $f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ *be a function such that* $\Delta^m f = 0$ *for some* $m \in \mathbb{N}$. *Then there exist harmonic functions* $P_j f \in \ker \Delta$, *uniquely determined up to a harmonic function in* x, *such that*

$$f(x,t) = P_1 f(x,t) + t P_2 f(x,t) + \dots + t^{m-1} P_m f(x,t) \tag{4}$$

for any $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$. *Conversely, if* $P_1 f, \dots, P_m f$ *are functions in* $\ker \Delta$ *then equation* (4) *defines a polyharmonic function* $f \in \ker \Delta^m$.

REMARK 2.2. In the case of $D = \Omega \times \mathbb{R}_+$ with Ω being a bounded, sufficiently smooth domain and $f \in C^1(\overline{\Omega} \times \mathbb{R}_+)$ we obtain global representation formulae for our operators P_j defined by (2) by defining

$$Qf(x,t) = \frac{1}{\lambda} \int_1^t f(x,s)\, ds - \int_\Omega \Phi(x-y) \frac{\partial f}{\partial s}(y,1)\, dy,$$

due to the fact that also in this case $\Delta_x K g = g$ for any $g \in C^\alpha(\overline{\Omega}), 0 < \alpha \leq 1$.

To prove Theorem 2.1 we need some lemmas.

LEMMA 2.1. *If* $\Box f(x,t) = 0$ *for all* $(x,t) \in \overline{B} \times [a,b]$, *then*

$$\Box Q f = 0, \qquad \lambda \frac{\partial}{\partial t} Q f = f. \tag{5}$$

Proof. The first property follows from

$$\begin{aligned} \Box Q f &= \Box\bigg(\frac{1}{\lambda} \int_a^t f(x,s)\, ds - K\bigg(\frac{\partial f}{\partial s}(x,a)\bigg)\bigg) \\ &= \lambda \frac{\partial^2}{\partial t^2}\bigg(\frac{1}{\lambda} \int_a^t f(x,s)\, ds\bigg) + \Delta\bigg(\frac{1}{\lambda} \int_a^t f(x,s)\, ds\bigg) - \Delta K\bigg(\frac{\partial f}{\partial s}(x,a)\bigg) \\ &= \frac{\partial}{\partial t} f(x,t) + \int_a^t -\frac{\partial^2 f}{\partial s^2}(x,s)\, ds - \frac{\partial f}{\partial s}(x,a) = 0. \end{aligned}$$

Here, we used the fact that $\Delta f = -\lambda \partial^2 f / \partial t^2$ for $f \in \ker \Box$.

The second property comes from

$$\lambda \frac{\partial}{\partial t} Q f(x,t) = \lambda \frac{\partial}{\partial t}\bigg(\frac{1}{\lambda} \int_a^t f(x,s)\, ds - K\bigg(\frac{\partial f}{\partial s}(x,a)\bigg)\bigg) = f(x,t),$$

because the second term in parentheses does not depend on t. ∎

LEMMA 2.2. *Let $k, q \in \mathbb{N}$ and $f \in \ker \Box$. Then there exist constants $C_{j,q}$ and functions $g_{j,s} \in \ker \Box$ such that*

$$\Box^q t^k f = \lambda^q \Big[2^q k(k-1)\cdots(k-q+1)t^{k-q} R^q f + \sum_{j=1}^{q} C_{j,q} k(k-1)\cdots(k-q-j-1)t^{k-q-j} g_{j,q} \Big],$$

where $R = \partial/\partial t$.

Proof. We will prove the lemma by induction on q. If $q = 1$, then

$$\begin{aligned}
\Box(t^k f) &= \Delta(t^k f) + \lambda \frac{\partial^2}{\partial t^2}(t^k f) \\
&= t^k \Delta f + \lambda\Big(\frac{\partial^2 t^k}{\partial t^2} f + 2\, \frac{\partial t^k}{\partial t}\, \frac{\partial f}{\partial t} + t^k\, \frac{\partial^2 f}{\partial t^2} \Big) \\
&= t^k \Box f + \lambda\Big(k(k-1)t^{k-2} f + 2kt^{k-1}\, \frac{\partial f}{\partial t} \Big) \\
&= \lambda(2kt^{k-1} Rf + k(k-1)t^{k-2} f)
\end{aligned}$$

for all $k \in \mathbb{N}$.

Now, assume that the statement holds for q. Then

$$\begin{aligned}
\Box^{q+1}(t^k f) &= \Box(\Box^q t^k f) \\
&= \lambda^q \Big(2^k k(k-1)\cdots(k-q+1)\Box(t^{k-q} R^q f) \\
&\qquad + \sum_{j=1}^{q} C_{j,q}(k-1)\cdots(k-q-j-1)\Box(t^{k-q-j} g_{j,q}) \Big).
\end{aligned}$$

Note that from the case $q = 1$ we have

$$\Box(t^{k-q} R^q f) = \lambda(2(k-q)t^{k-q-1} R^{q+1} f + (k-q)(k-q-1)t^{k-q-2} R^q f)$$

and

$$\begin{aligned}
&\Box(t^{k-q-j} g_{j,q}) \\
&\quad = \lambda(2(k-q-j)t^{k-q-j-1} R g_{j,q} + (k-q-j)(k-q-j-1)t^{k-q-j-2} g_{j,q}).
\end{aligned}$$

Here, we used the fact that $\Box R^q = R^q \Box$. The lemma follows immediately from the above identities. ∎

LEMMA 2.3. *Let $k \in \mathbb{N}$ and $\Box f(x,t) = 0$ for all $(x,t) \in \overline{B} \times [a,b]$. Then*

$$\Box^k t^k f = \lambda^k 2^k k! R^k f, \qquad \Box^k t^k\, \frac{1}{2^k k!}\, Q^k f = f.$$

Proof. We take $q = k$ in Lemma 2.2 to obtain the first identity. If we replace in it f by $Q^k f$ (cf. Lemma 2.1) then we have

$$\Box^k t^k\, \frac{1}{2^k k!}\, Q^k f = \lambda^k R^k Q^k f.$$

Notice that the operators R and Q do not commute. Nevertheless, we can prove that

$$R^k Q^k f = \frac{1}{\lambda^k} f$$

by induction on $k \in \mathbb{N}$. Indeed, if $k = 1$, then

$$RQf(x,t) = \frac{\partial}{\partial t}\left(\frac{1}{\lambda}\int_a^t f(x,s)\,ds - K\left(\frac{\partial f}{\partial s}(x,a)\right)\right) = \frac{1}{\lambda} f(x,t).$$

Assume that the statement is true for $k-1$. If f is in $\ker \Box$ then Lemma 2.1 shows that so is Qf. Therefore, by the inductive assumption we get

$$R^{k-1}Q^{k-1}(Qf) = \frac{1}{\lambda^{k-1}} Qf,$$

so that

$$R^k Q^k f = R(R^{k-1}Q^{k-1}(Qf)) = \frac{1}{\lambda^{k-1}} RQf = \frac{1}{\lambda^k} f.$$

The last step follows from the case $k = 1$. ∎

Let us now turn to the proof of the main theorem.

Proof of Theorem 2.1. As pointed out in the paragraph after Theorem 2.1, we need only prove decomposition (1) locally. Clearly, once we prove that (1) is unique, the local construction of P_j yields the global definition of P_j.

Set $H_m = \ker \Box^m$. We only need to show

$$H_m = H_{m-1} + t^{m-1}H_1.$$

Let us first prove the $\supset$ inclusion. It is clear that $H_m \supset H_{m-1}$. We prove that $H_m \supset t^{m-1}H_1$. In fact, for any $f \in H_1$, it follows from Lemma 2.2 that

$$\Box^m(t^{m-1}f) = 0.$$

To prove the other inclusion, let $f \in H_m$. We consider the decomposition

$$f = (I - C_{m-1}t^{m-1}Q^{m-1}\Box^{m-1})f + t^{m-1}(C_{m-1}Q^{m-1}\Box^{m-1}f) =: g + t^{m-1}h.$$

We have to verify that $g \in H_{m-1}$ and $h \in H_1$, i.e. $\Box^{m-1}g(x,t) = 0$ and $\Box h(x,t) = 0$ for all $(x,t) \in \overline{B} \times [a,b]$. Indeed,

$$\Box h = \Box(C_{m-1}Q^{m-1}\Box^{m-1}f) = C_{m-1}\Box Q^{m-1}(\Box^{m-1}f) = 0,$$

due to the fact that $\Box^{m-1}f \in H_1$ so that $Q(\Box^{m-1}f) \in \ker \Box$ (cf. Lemma 2.1). Repeating this procedure yields $Q^{m-1}(\Box^{m-1}f) \in \ker \Box$. Moreover, from Lemma 2.3, for $\Box^{m-1}f \in H_1$ we have

$$(C_{m-1}\Box^{m-1}t^{m-1}Q^{m-1})\Box^{m-1}f = \Box^{m-1}f$$

so that

$$\begin{aligned}\Box^{m-1}g &= \Box^{m-1}(I - C_{m-1}t^{m-1}Q^{m-1}\Box^{m-1})f\\ &= \Box^{m-1}f - (C_{m-1}\Box^{m-1}t^{m-1}Q^{m-1})\Box^{m-1}f\\ &= \Box^{m-1}f - \Box^{m-1}f = 0.\end{aligned}$$

Now, we are going to investigate the uniqueness of our decomposition

$$f = g + t^{m-1}h, \quad g \in H_{m-1},\ h \in H_1,$$

for any $f \in \ker\Box$. To this end we apply $Q^{m-1}\Box^{m-1}$ to both sides:

$$\begin{aligned}Q^{m-1}\Box^{m-1}f &= Q^{m-1}\Box^{m-1}g + Q^{m-1}\Box^{m-1}t^{m-1}h\\ &= Q^{m-1}\Box^{m-1}t^{m-1}h\\ &= \lambda^{m-1}C_{m-1}^{-1}Q^{m-1}R^{m-1}h.\end{aligned}$$

Therefore, the question is: what can we say about the operator QR?

Plugging the term Rv with $v \in C^2(\overline{B} \times \mathbb{R}_+)$ and $v \in \ker\Box$ into the definition of Q we obtain

$$\begin{aligned}QRv(x,t) &= \frac{1}{\lambda}\int_a^t \frac{\partial v}{\partial s}(x,s)\,ds + \int_B \Phi(x-y)\,\frac{\partial^2 v}{\partial t^2}(y,a)\,dy\\ &= \frac{1}{\lambda}\Big(v(x,t) - v(x,a) - \int_B \Phi(x-y)\Delta_x v(y,a)\,dy\Big)\\ &= \frac{1}{\lambda}\Big(v(x,t) - v(x,a) + v(x,a) - \int_{\partial B} \Phi(x-y)\,\frac{\partial v}{\partial n}(y,a)\,d\sigma_y\\ &\quad + \int_{\partial B} \frac{\partial \Phi(x-y)}{\partial n}\,v(y,a)\,d\sigma_y\Big),\end{aligned}$$

where $d\sigma$ denotes the surface measure. In the last identity we used the integral representation for C^2-functions in x. The last two terms define a harmonic function, so that we obtain

$$QRv(x,t) = \frac{1}{\lambda}\,v(x,t) + w(x)$$

with $\Delta_x w(x) = 0$. For $m = 2$ we obtain, for $f = g + th = g_1 + th_1$,

$$\begin{aligned}\Box f - \Box f &= \lambda C_1^{-1}QRh - \lambda C_1^{-1}QRh_1 = \lambda C_1^{-1}QR(h - h_1)\\ &= \lambda C_1^{-1}\Big(\frac{1}{\lambda}(h - h_1) + w(x)\Big),\end{aligned}$$

therefore,

$$\Delta_x(h(x,t) - h_1(x,t)) = 0$$

as well as

$$\Delta_x(g - g_1) = -t\Delta_x(h - h_1) = 0.$$

Thus our result follows by induction. ■

3. The case of polyheat functions. First consider the case of the operator $\Box = \Delta_x + \lambda \frac{\partial}{\partial t}$ with $\lambda \in \mathbb{C} \setminus \{0\}$ being a constant. For $\lambda = -1$ we have the heat operator.

THEOREM 3.1. *Let $D \subset \mathbb{R}^{n+1}$ be a domain and $f : D \to \mathbb{C}$ be a function such that $\Box^m f = 0$ for some $m \in \mathbb{N}$. Then there exist functions $P_j f \in \ker \Box$, uniquely determined, such that*

$$f(x,t) = P_1 f(x,t) + t P_2 f(x,t) + \cdots + t^{m-1} P_m f(x,t) \tag{6}$$

for any $(x,t) \in D$. Conversely, if $P_1 f, \ldots, P_m f$ are functions in $\ker \Box$ then equation (6) defines a function $f \in \ker \Box^m$.

Define $C_k = 1/\lambda^k k!$ and let I be the identity operator. Then we have the following representation formulae for the operators P_j:

$$\begin{cases} P_m = C_{m-1}\Box^{m-1}, \\ P_{m-1} = C_{m-2}\Box^{m-2}(I - C_{m-1}t^{m-1}\Box^{m-1}), \\ \quad\vdots \\ P_2 = C_1\Box(I - C_2 t^2\Box^2)\ldots(I - C_{m-1}t^{m-1}\Box^{m-1}), \\ P_1 = (I - C_1\Box)(I - C_2 t^2\Box^2)\ldots(I - C_{k-1}t^{k-1}\Box^{k-1}), \end{cases} \tag{7}$$

or, for $j = 2, \ldots, m-1$,

$$P_j = C_{j-1}\Box^{j-1}(I - C_j t^j \Box^j)\cdots(I - C_{m-1}t^{m-1}\Box^{m-1}),$$

where t^k again denotes the multiplication operator defined by $(t^k f)(x,t) = t^k f(x,t)$. Note that in this case there is no operator Q appearing in the formulae. Again, the operators P_j, $j = 1, \ldots, m$, are manifestly linear.

REMARK 3.1. Due to the nature of our operators P_j we see that if f is a polynomial in x and t then all the $P_j f$ are polynomials in x and t.

For the proof of Theorem 3.1 we again need some lemmas.

LEMMA 3.1. *If $f \in \ker \Box$ then*

$$\Box(t^k f) = \lambda k t^{k-1} f.$$

Proof. A straightforward calculation shows that

$$\begin{aligned} \Box(t^k f) &= \left(\Delta + \lambda\frac{\partial}{\partial t}\right)(t^k f) = t^k \Delta f + \lambda k t^{k-1} f + \lambda t^k \frac{\partial f}{\partial t} \\ &= t^k\left(\Delta + \lambda\frac{\partial}{\partial t}\right) f + \lambda k t^{k-1} f = \lambda k t^{k-1} f. \ \blacksquare \end{aligned}$$

LEMMA 3.2. *If $f \in \ker \Box$ then*

$$C_k \Box^k t^k f = f.$$

Proof. From Lemma 3.1 we obtain

$$\frac{1}{\lambda k}\,\Box t^k f = t^{k-1} f,$$

so that

$$C_k \Box^k t^k f = C_k \lambda k \Box^{k-1}\left(\frac{1}{\lambda k}\,\Box t^k f\right) = C_{k-1}\Box^{k-1} t^{k-1} f.$$

The statement now follows by induction. ∎

Now we are prepared for the proof of the theorem.

Proof of Theorem 3.1. Define $H_m = \ker \Box^m$. We only need to show

$$H_m = H_{m-1} + t^{m-1} H_1.$$

Let us first prove the $\supset$ inclusion. Since it is clear that $H_m \supset H_{m-1}$, we need only show $H_m \supset t^{m-1} H_1$. If $f \in H_1$, then Lemma 3.2 yields

$$\Box^m (t^{m-1} f) = \Box(\Box^{m-1}(t^{m-1} f)) = \Box\left(\frac{1}{C_{m-1}}\,f\right) = \frac{1}{C_{m-1}}\,\Box f = 0.$$

To prove the other inclusion, let $f \in H_m$ and consider the decomposition

$$f = (I - C_{m-1} t^{m-1} \Box^{m-1}) f + t^{m-1}(C_{m-1}\Box^{m-1} f).$$

Again applying Lemma 3.2 we obtain

$$\begin{aligned}\Box^{m-1}(I - C_{m-1} t^{m-1}\Box^{m-1}) f &= \Box^{m-1} f - (\Box^{m-1} C_{m-1} t^{m-1})\Box^{m-1} f \\ &= \Box^{m-1} f - \Box^{m-1} f = 0\end{aligned}$$

as well as

$$\Box(C_m \Box^{m-1} f) = 0.$$

As a result, for $f \in H_m$,

$$P_m f = C_{m-1}\Box^{m-1} f.$$

Now f can be written as

$$f = g + t^{k-1} P_m f, \qquad g \in H_{m-1},$$

with g given by $(I - C_{m-1} t^{m-1}\Box^{m-1}) f$. Therefore, by (descending) induction we get the desired expressions of P_j.

Let us now prove that for any $f \in \ker \Box$ the decomposition

$$f = g + t^{m-1} h, \quad g \in H_{m-1},\ h \in H_1,$$

is unique. To this end we again apply $\Box^{m-1}$ to both sides:

$$\Box^{m-1} f = \Box^{m-1} g + \Box^{m-1} t^{m-1} h = \Box^{m-1} t^{m-1} h = C_{m-1}^{-1} h.$$

Therefore,

$$h = C_{m-1}\Box^{m-1} f,$$

so that

$$g = f - t^{m-1} h = (I - C_{m-1} t^{m-1}\Box^{m-1}) f.$$

Thus uniqueness follows by induction. ∎

References

[1] E. Almansi, *Sull'integrazione dell'equazione differenziale* $\Delta^{2n}u = 0$, Ann. Mat. Pura Appl. (3) 2 (1899), 1–51.

[2] N. Aronszajn, T. M. Creese and L. J. Lipkin, *Polyharmonic Functions*, Oxford Univ. Press, New York, 1983.

[3] D. Bahuguna, *Introduction to distributions*, in: Topics in Sobolev Spaces and Applications, D. Bahuguna *et al.* (eds.), Alpha Science Internat., Pangbourne (UK), 2002.

[4] M. B. Balk, *Polyanalytic Functions and Their Generalizations*, Wiley, Chichester, 1996.

[5] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, Springer, Berlin, 2003.

[6] H. R. Malonek and G. B. Ren, *Almansi-type theorems in Clifford analysis*, Math. Methods Appl. Sci. 25 (2002), 1541–1552.

[7] N. Nicolescu, *Problème de l'analyticité par rapport à un opérateur linéaire*, Studia Math. 16 (1958), 353–363.

[8] D. Pascali, *Polyanalyticity and polyharmonicity*, Proc. Rom. Acad. Ser. A 5 (2004), 1–7.

[9] M. Pavlović, *Decompositions of* L^p *and Hardy spaces of polyharmonic functions*, J. Math. Anal. Appl. 216 (1997), 499–509.

[10] G. B. Ren, *Almansi decomposition for Dunkl operators*, Sci. China Ser. A 48 (2005), 333–342.

[11] G. B. Ren and H. R. Malonek, *Decomposition of kernels of iterated operators*, submitted.

[12] K. Yoshida, *Functional Analysis*, Springer, Berlin, 1980.

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