STUDIA MATHEMATICA 187 (3) (2008)

Local and global solutions of well-posed integrated Cauchy problems

by

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Abstract. We study the local well-posed integrated Cauchy problem

$$v'(t) = Av(t) + \frac{t^\alpha}{\Gamma(\alpha+1)}\, x, \qquad v(0) = 0, \qquad t \in [0,\kappa),$$

with $\kappa > 0$, $\alpha \geq 0$, and $x \in X$, where X is a Banach space and A a closed operator on X. We extend solutions increasing the regularity in α. The global case ($\kappa = \infty$) is also treated in detail. Growth of solutions is given in both cases.

1. Introduction. Let X be a Banach space, A a closed operator on X with domain $D(A)$ and $f : [0,\kappa) \to \mathbb{C}$ a continuous function, $f \in C([0,\kappa);\mathbb{C})$ ($0 < \kappa \leq \infty$). The evolution equations

$$(1) \qquad v'(t) = Av(t) + f(t)x, \qquad t \in [0,\kappa),$$

have a long history. Many ordinary and partial differential equations may be written in this form. Different ideas and techniques have been developed to deal with this problem.

Recently local convoluted semigroups have been deeply investigated to express the solution of this equation (see for example [10] and the references therein). α-Times integrated semigroups are examples of convoluted semigroups obtained for $f(t) = t^\alpha/\Gamma(\alpha+1)$ with $\alpha \in \mathbb{R}^+$ (in this case, equation (1) defines a local integrated Cauchy problem). α-Times integrated semigroups were introduced first for $\alpha \in \mathbb{N}$ ([1]) and later for $\alpha \in \mathbb{R}^+$ ([8]). In fact, to generalize n to α may or may not be complicated. Sometimes, it is straightforward (see Theorem 1); in other circumstances some integral expressions are needed (see equality (5)) or some estimates about special

2000 *Mathematics Subject Classification*: 47D62, 26A33.

Key words and phrases: abstract Cauchy problems, integrated semigroups, distribution semigroups.

Partially supported by the Spanish Project MTM2007-61446, MCYT, DGI-FEDER and the DGA proyect E-64.

functions (Example 1). The underlying background theory is the fractional calculus (see Section 3 and [18]).

In the second section, we consider solutions of well-posed integrated Cauchy problems. We show that they are in fact local α-times integrated semigroups or local mild α-times integrated existence families (Theorem 1). Moreover, we show that every local α-times integrated semigroup may be extended if one is ready to give up the regularity (Theorem 2). This interesting extension property appeared in [3] for local n-times integrated semigroups and in [22] for local C-semigroups.

On the other hand, Lions introduced in 1960 the so-called (vector-valued) distribution semigroups, in connection with Cauchy problems ([16]). Particular classes of distribution semigroups have since been considered; for example quasi-distribution semigroups were introduced and studied by Wang in [21]. A brief description of distribution semigroups as well as references on the subject can be found in [1] and [17].

Local α-times integrated semigroups are equivalent to quasi-distribution semigroups (see the third section). We show that quasi-distribution semigroups of fractional order are equivalent to global solutions of integrated Cauchy problems with fairly general growth. We present an approach which allows us to extend some known results ([2, Theorem 4.4], [14, Theorem 3.6], [21, Theorem 4.13]) quite significantly.

2. Extending solutions of well-posed local integrated Cauchy problems. Let X be a Banach space, $\mathcal{B}(X)$ the set of bounded linear operators on X, $(A, D(A))$ a closed linear operator on X, $x \in X$ and $\kappa > 0$. The local α-times integrated Cauchy problem

$$C_\alpha(\kappa) \equiv \begin{cases} v \in C([0,\kappa); D(A)) \cap C^{(1)}([0,\kappa); X), \\ v'(t) = Av(t) + \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}\, x, \qquad t \in [0,\kappa), \\ v(0) = 0, \end{cases}$$

has been studied in detail for $\alpha \in \mathbb{N}$ ([3], [21]) and later for $\alpha \in \mathbb{R}^+$ ([15]).

The Cauchy problem $C_\alpha(\kappa)$ is *well-posed* if for all $x \in X$ there exists a unique solution of $C_\alpha(\kappa)$. If $E(a,b) \subset \varrho(A)$ for some $a, b > 0$, where

$$E(a,b) := \{\lambda \in \mathbb{C};\ \Re\lambda \ge b,\ |\Im\lambda| \le e^{a\Re\lambda}\}$$

($\varrho(A)$ is the resolvent set of A), and

$$\|(\lambda - A)^{-1}\| \le M|\lambda|^{\alpha-1}, \qquad \lambda \in E(a,b),$$

for some $\alpha \in \mathbb{R}^+$, then the Cauchy problem $C_\beta(\kappa)$ is well-posed with $\beta > \alpha$ and $\kappa = a(\beta - \alpha)$ ([15, Theorem 2.2]). Using some ideas from special functions, one can also prove a converse result ([15, Theorem 2.1]).

Given $\alpha \in \mathbb{R}^+$, the solution $(v_x(t))_{t\in[0,\kappa)}$ of the well-posed abstract Cauchy problem $C_{\alpha+1}(\kappa)$ defines a family $(S_\alpha(t))_{t\in[0,\kappa)} \subset \mathcal{B}(X)$ by

$$S_\alpha(t)x := v_x'(t), \qquad x \in X,\, t \in [0,\kappa). \tag{2}$$

In fact, $(S_\alpha(t))_{t\in[0,\kappa)}$ is a nondegenerate local α-times integrated semigroup, i.e.,

$$S_\alpha(t)S_\alpha(s)x = \frac{1}{\Gamma(\alpha)}\Big(\int\limits_s^{t+s} - \int\limits_0^t (t+s-r)^{\alpha-1}S_\alpha(r)x\,dr\Big) \tag{3}$$

for $\kappa > t+s \geq t, s \geq 0$, and $x \in X$ (*nondegenerate* in the usual sense: if $S_\alpha(t)x = 0$ for all $t \in [0,\kappa)$ then $x = 0$); see [2], [15], [21]. It is straightforward to check that $(S_\beta(t))_{t\in[0,\kappa)}$ defined by

$$S_\beta(t)x = \frac{1}{\Gamma(\beta-\alpha)}\int\limits_0^t (t-s)^{\beta-\alpha-1}S_\alpha(s)x\,ds, \qquad x \in X,\, t \in [0,\kappa), \tag{4}$$

for $\beta > \alpha$ is a local β-times integrated semigroup. For a nondegenerate local α-times integrated semigroup we may define its *generator* in the following way. Let $D(A)$ be the set of all $x \in X$ for which there exists $y \in X$ such that

$$S_\alpha(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}\,x = \int\limits_0^t S_\alpha(s)y\,ds, \qquad t \in [0,\kappa);$$

and set $Ax := y$. It is easy to check that $(A, D(A))$ is a closed operator on X. Moreover $S_\alpha(t)x$ is differentiable in t for $t \in [0,\kappa)$ and $x \in X$ if and only if $S_\alpha(t)x \in D(A)$, and in this case

$$\frac{d}{dt}S_\alpha(t)x = AS_\alpha(t)x + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\,x, \qquad \kappa > t > 0.$$

In the case $\kappa = \infty$, the growth of $\|S_\alpha(t)\|$ as $t \to \infty$ can be faster than exponential (see for example [17, Example 1.2.5]). If $\|S_\alpha(t)\| \leq Ce^{\lambda_0 t}$ with C, $\lambda_0 \geq 0$, the condition (3) is equivalent (via Laplace transform) to

$$R(\lambda, A) := \lambda^\alpha \int\limits_0^\infty e^{-\lambda t}S_\alpha(t)\,dt, \qquad \Re\lambda > \lambda_0,$$

being a *pseudo-resolvent operator*, i.e., satisfying $R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A)$ for any $\Re\lambda, \Re\mu > \lambda_0$ ([8]). In the nondegenerate case, $\lambda \in \varrho(A)$ and $R(\lambda, A) = (\lambda - A)^{-1}$ for $\Re\lambda > \lambda_0$.

R. deLaubenfelds introduced the concept of mild n-times integrated existence family in [13]. Suppose A is a closed operator on X and $\alpha \geq 0$. A strongly continuous family $(W(t))_{t\in[0,\kappa)} \subset \mathcal{B}(X)$ is a *local mild α-times integrated existence family* for A if for any $x \in X$ and $t \in [0,\kappa)$, we have

$\int_0^t W(s)x\,ds \in D(A)$ and

$$A\int_0^t W(s)x\,ds = W(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}\,x$$

(see [21, Definition 2.3] for $\alpha \in \mathbb{N}$). The following theorem is well-known in the case $\alpha \in \mathbb{N}\cup\{0\}$ (see for example [3, Proposition 2.3] and [21, Theorem 2.4]). The special case $\alpha = 0$ appears in [3, Theorem 1.2]. The proof in the case $\alpha \in \mathbb{R}^+$ is similar to the case $\alpha \in \mathbb{N}$ and we omit it.

THEOREM 1. *Let $\alpha \geq 0$ and $0 < \kappa \leq \infty$. The following are equivalent.*

(i) *$C_{\alpha+1}(\kappa)$ is well-posed.*
(ii) *A generates a nondegenerate α-times integrated semigroup.*
(iii) *$C_{\alpha+1}(\kappa)$ is well-posed and there exists a local mild α-times integrated existence family for A.*

EXAMPLE 1. This example appears in [20] for the case $\alpha \in \mathbb{N}$ (see also [17, Example 1.2.6]). Let ℓ^2 be the Hilbert space of all square-summable sequences $x = (x_m)_{m=1}^\infty$ with the usual norm $\|x\| := (\sum_{m=1}^\infty |x_m|^2)^{1/2}$. Take $T > 0$ and define

$$a_m = \frac{m}{T} + i\biggl(\Bigl(\frac{e^m}{m}\Bigr)^2 - \Bigl(\frac{m}{T}\Bigr)^2\biggr)^{1/2}, \qquad m \in \mathbb{N},$$

where $i^2 = -1$. For any $\alpha \in \mathbb{R}^+$, let $(U_\alpha(t))_{t>0}$ be defined by

$$U_\alpha(t)x = \biggl(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{a_m s}x_m\,ds\biggr)_{m=1}^\infty$$

for $x \in D(U_\alpha(t))$, where $D(U_\alpha(t)) = \{x \in \ell^2;\, U_\alpha(t)x \in \ell^2\}$. Then $(U_\alpha(t))_{t\in[0,\alpha T)}$ is a local α-times integrated semigroup on ℓ^2:

We consider the case $\alpha \notin \mathbb{N}$. Then $0 < \alpha - [\alpha] < 1$ and

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,e^{a_m s}\,ds = \frac{e^{a_m t}}{a_m^{[\alpha]}}\int_0^t \frac{s^{\alpha-[\alpha]-1}}{\Gamma(\alpha-[\alpha])}\,e^{-a_m s}\,ds - \sum_{j=1}^{[\alpha]}\frac{t^{\alpha-j}}{\Gamma(\alpha+1-j)a_m^j}$$

for $t \geq 0$. Moreover,

$$\int_0^t \frac{s^{\alpha-[\alpha]-1}}{\Gamma(\alpha-[\alpha])}\,e^{-a_m s}\,ds = t^{\alpha-[\alpha]}\biggl(\frac{e^{-a_m t}}{a_m}\,O(1) + \frac{1}{a_m^{\alpha-[\alpha]}}\,O(1)\biggr), \qquad t \geq 0,$$

when $|a_m| \to \infty$ (see for example [15, Theorem 2.1]). Since $|a_m| = m^{-1}e^m$ and $|e^{a_m t}| = e^{mt/T}$, we find that

$$\|U_\alpha(t)\| = \sup_{m\in\mathbb{N}}\biggl|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{a_m s}\,ds\biggr| < \infty$$

if and only if $0 \le t < \alpha T$. It is easily shown that $(U_\alpha(t))_{t\in[0,\alpha T)}$ satisfies (3) and $t \mapsto U_\alpha(t)x$ is strongly continuous (for these ideas in the case $\alpha = n$ see [20]). Note that $(U_\alpha(t))_{t\in[0,\alpha T)}$ cannot be extended to $t \ge \alpha T$.

Now we prove that solutions in the local well-posed case may be extended. A loss of regularity appears in the same way as in the integer case [3, Theorem 4.1]. Note that the extension given in [3, formula (4.2)] for $\alpha \in \mathbb{N}$ is not possible in the case $\alpha \in \mathbb{R}^+$.

THEOREM 2. *Let $\kappa_0 > 0$ and $\alpha \in \mathbb{R}^+$. Suppose that $C_{\alpha+1}(\kappa_0)$ is well-posed. Then $C_{2\alpha+1}(2\kappa_0)$ is also well-posed. In particular, for all $\kappa' > 0$ there exists $\beta > 0$ such that $C_\beta(\kappa')$ is well-posed.*

Proof. Take $\kappa < \kappa_0$. We will prove that $C_{2\alpha+1}(2\kappa)$ is well-posed. By Theorem 1, there exists a nondegenerate local α-times integrated semigroup $(S_\alpha(t))_{t\in[0,\kappa_0)}$ generated by $(A, D(A))$. Then we define $(S_{2\alpha}(t))_{t\in[0,\kappa]}$ by (4) if $0 \le t \le \kappa$ and

$$S_{2\alpha}(t)x := S_\alpha(\kappa)S_\alpha(t-\kappa)x + \frac{1}{\Gamma(\alpha)}\int_0^\kappa (t-s)^{\alpha-1}S_\alpha(s)x\,ds + \frac{1}{\Gamma(\alpha)}\int_0^{t-\kappa} (t-s)^{\alpha-1}S_\alpha(s)x\,ds, \tag{5}$$

if $\kappa \le t \le 2\kappa$ and $x \in X$. It is clear $S_{2\alpha} : [0, 2\kappa] \to \mathcal{B}(X)$ is strongly continuous. To show that $C_{2\alpha+1}(2\kappa)$ is well-posed, we prove that $(S_{2\alpha}(t))_{t\in[0,2\kappa]}$ is a local mild 2α-times integrated existence family for A and apply Theorem 1.

If $0 \le t \le \kappa$, it is clear that

$$A\int_0^t S_{2\alpha}(s)x\,ds = S_{2\alpha}(t)x - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}x, \qquad x \in X.$$

Take $\kappa \le t \le 2\kappa$. Then

$$A\int_0^t S_{2\alpha}(s)x\,ds = S_{2\alpha}(\kappa)x - \frac{\kappa^{2\alpha}}{\Gamma(2\alpha+1)}x + A\int_\kappa^t S_{2\alpha}(s)x\,ds.$$

We use (5) and the Fubini theorem to obtain

$$\int_\kappa^t S_{2\alpha}(s)x\,ds = S_\alpha(\kappa)\int_0^{t-\kappa} S_\alpha(u)x\,du + \int_0^\kappa S_\alpha(r)x\int_\kappa^t \frac{(s-r)^{\alpha-1}}{\Gamma(\alpha)}\,ds\,dr + \int_0^{t-\kappa} S_\alpha(r)x\int_{r+\kappa}^t \frac{(s-r)^{\alpha-1}}{\Gamma(\alpha)}\,ds\,dr.$$

Note that

$$S_\alpha(\kappa)A\int_0^{t-\kappa} S_\alpha(u)x\,du = S_\alpha(\kappa)S_\alpha(t-\kappa)x - S_\alpha(\kappa)x\,\frac{(t-\kappa)^\alpha}{\Gamma(\alpha+1)}.$$

As

$$\int_\kappa^t \frac{(s-r)^{\alpha-1}}{\Gamma(\alpha)}\,ds = \frac{(t-r)^\alpha}{\Gamma(\alpha+1)} - \frac{(\kappa-r)^\alpha}{\Gamma(\alpha+1)},$$

we now check easily

$$A\int_0^\kappa S_\alpha(r)x\frac{(\kappa-r)^\alpha}{\Gamma(\alpha+1)}\,dr = S_{2\alpha}(\kappa)x - \frac{\kappa^{2\alpha}}{\Gamma(2\alpha+1)}x$$

and also we conclude that

$$A\int_0^\kappa S_\alpha(r)x\,\frac{(t-r)^\alpha}{\Gamma(\alpha+1)}\,dr = \int_0^\kappa S_\alpha(s)x\,\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds + S_\alpha(\kappa)x\,\frac{(t-\kappa)^\alpha}{\Gamma(\alpha+1)}$$
$$- \frac{\kappa^\alpha}{\Gamma(\alpha+1)}\,\frac{(t-\kappa)^\alpha}{\Gamma(\alpha+1)}x - \int_0^\kappa \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,\frac{s^\alpha}{\Gamma(\alpha+1)}\,ds\,x.$$

For $r > t-k$, we use similar ideas and

$$\int_{\kappa+r}^t \frac{(s-r)^{\alpha-1}}{\Gamma(\alpha)}\,ds = \frac{(t-r)^\alpha}{\Gamma(\alpha+1)} - \frac{\kappa^\alpha}{\Gamma(\alpha+1)}$$

to check easily that

$$A\int_0^{t-\kappa} S_\alpha(r)x\,\frac{\kappa^\alpha}{\Gamma(\alpha+1)}\,dr = \frac{\kappa^\alpha}{\Gamma(\alpha+1)}\,S_\alpha(t-\kappa)x - \frac{\kappa^\alpha}{\Gamma(\alpha+1)}\,\frac{(t-\kappa)^\alpha}{\Gamma(\alpha+1)}\,x,$$

and we obtain

$$A\int_0^{t-\kappa} S_\alpha(r)x\,\frac{(t-\kappa)^\alpha}{\Gamma(\alpha+1)}\,dr = \int_0^{t-\kappa} S_\alpha(s)x\,\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds + \frac{\kappa^\alpha}{\Gamma(\alpha+1)}\,S_\alpha(t-\kappa)x$$
$$- \int_0^{t-\kappa} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,\frac{s^\alpha}{\Gamma(\alpha+1)}\,ds - \frac{\kappa^\alpha}{\Gamma(\alpha+1)}\,\frac{(t-\kappa)^\alpha}{\Gamma(\alpha+1)}\,x.$$

Note that

$$\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\kappa^\alpha}{\Gamma(\alpha+1)}\,\frac{(t-\kappa)^\alpha}{\Gamma(\alpha+1)}$$
$$= \int_0^\kappa \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,\frac{s^\alpha}{\Gamma(\alpha+1)}\,ds + \int_0^{t-\kappa} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,\frac{s^\alpha}{\Gamma(\alpha+1)}\,ds$$

(see for example [12, Lemma 3.1]). To finish the proof we join together all summands to obtain

$$A\int_0^t S_{2\alpha}(s)x\,ds = S_\alpha(\kappa)S_\alpha(t-\kappa)x + \int_0^\kappa S_\alpha(s)x\,\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds$$
$$- \int_0^{t-\kappa} S_\alpha(s)x\,\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\,x = S_{2\alpha}(t)x - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\,x,$$

and this proves the claim. ∎

REMARK 3. In the case $\alpha = k$, we recover the extension given in [3, Theorem 4.1], in view of the identity

$$\frac{1}{\Gamma(\alpha)}\int_0^\kappa (t-s)^{\alpha-1}S_\alpha(s)x\,ds + \frac{1}{\Gamma(\alpha)}\int_0^{t-\kappa}(t-s)^{\alpha-1}S_\alpha(s)x\,ds$$
$$= \sum_{m=0}^{k-1}\frac{1}{m!}\,(\kappa^m S_{2k-m}(t-\kappa)x + (t-\kappa)^m S_{2k-m}(\kappa))x.$$

Due to uniqueness of the solutions and the proof of Theorem 2, the functional equation

$$S_{2\alpha}(t+s)x = S_\alpha(t)S_\alpha(s)x + \frac{1}{\Gamma(\alpha)}\Big(\Big(\int_0^t + \int_0^s\Big)\,(t+s-u)^{\alpha-1}S_\alpha(u)x\,du\Big)$$

holds for $t, s \in [0,\kappa)$ and $x \in X$.

Take $\alpha \ge 0$ and $\kappa \in \mathbb{R}^+ \cup \{\infty\}$. We denote by $\Omega_{\alpha,\kappa}$ the set of nondecreasing and continuous functions τ_α on $[0,\kappa)$ such that $\inf_{\kappa>t>0} t^{-\alpha}\tau_\alpha(t) > 0$ and there exists a constant $C_\alpha > 0$ with

$$\int_{[0,r]\cup[s,s+r]} t^{\alpha-1}\tau_\alpha(r+s-t)\,dt \le C_\alpha\tau_\alpha(r)\tau_\alpha(s), \qquad 0 \le r \le s \le s+r < \kappa.$$

If $\kappa' > \kappa$ then $\Omega_{\alpha,\kappa'} \subset \Omega_{\alpha,\kappa}$. The functions $\tau_\alpha(t) = t^\alpha$; $t^\beta(1+t)^\nu$ with $\beta \in [0,\alpha]$ and $\nu \ge \alpha - \beta$; and $t^\beta e^{\tau t}$ with $\tau > 0$ and $\beta \in [0,\alpha]$, belong to $\Omega_{\alpha,\infty}$. If $\tau_\alpha \in \Omega_{\alpha,\kappa}$ then $\tau_\nu \in \Omega_{\nu,\kappa}$, where $\tau_\nu(t) := t^{\nu-\alpha}\tau_\alpha(t)$ for $t \ge 0$ and $\nu \ge \alpha$. The subset of functions $\tau_\alpha(t) = t^\alpha w(t)$ (where w is a continuous and nondecreasing weight function on $[0,\kappa)$, $w(t+s) \le Cw(t)w(s)$ for $0 \le t, s \le t+s < \kappa$) is denoted by $\Omega^h_{\alpha,\kappa}$ (see [7] for more details).

We use the equality (2) and the proof of Theorem 2 to obtain the following corollary.

COROLLARY 4. *Take $\alpha \ge 0$ and $\tau_\alpha \in \Omega_{\alpha,\kappa}$. Let $(v_x(t))_{t\in[0,\kappa)}$ be the solution of the local well-posed integrated Cauchy problem $C_{\alpha+1}(\kappa)$ such that*

$$\|v_x'(t)\| \le C\tau_\alpha(t), \qquad t \in [0,\kappa).$$

Then the solution $(u_x(t))_{t\in[0,2\kappa)}$ *of the local integrated Cauchy problem* $C_{2\alpha+1}(2\kappa)$ *satisfies*

$$\|u_x'(t)\| \le \begin{cases} Ct^{\alpha}\tau_\alpha(t), & t\in[0,\kappa],\\ C\tau_\alpha(\kappa)(\kappa^{\alpha}+\tau_\alpha(t-\kappa)), & t\in[\kappa,2\kappa). \end{cases}$$

3. Quasi-distribution semigroups and global solutions of the abstract Cauchy problem. In this section we start considering quasi-distribution semigroups introduced in [21]. Let $\mathcal{D}_+$ be the class of $\mathcal{C}^\infty$ functions of compact support on $[0,\infty)$; $\mathcal{D}$ be the class of $\mathcal{C}^\infty$ functions of compact support on $\mathbb{R}$; and $\mathcal{D}_0$ be the subspace of those ϕ's of $\mathcal{D}$ with $\mathrm{supp}(\phi)\subset[0,\infty)$. Note that if $\phi,\psi\in\mathcal{D}_+$ then $\phi*\psi\in\mathcal{D}_+$ where

$$\phi*\psi(t)=\int_0^t \phi(t-s)\psi(s)\,ds,\qquad t\ge 0.$$

We consider the usual topology defined in $\mathcal{D}_+,\mathcal{D}$ and $\mathcal{D}_0$. A *quasi-distribution semigroup* (QDSG) on X is a continuous linear map $\mathcal{G}:\mathcal{D}_+\to\mathcal{B}(X)$ that satisfes:

(i) $\mathcal{G}(\phi*\psi)=\mathcal{G}(\phi)\mathcal{G}(\psi)$ for $\phi,\psi\in\mathcal{D}_+$,
(ii) $\bigcap\{\ker(\mathcal{G}(\phi));\ \phi\in\mathcal{D}_0\}=\{0\}$

(see [21, Definition 3.3]). Although Wang considered maps from $\mathcal{D}$ to $\mathcal{B}(X)$, both approaches are equivalent [21, Remark 3.4(ii)]. Quasi-distribution semigroups extend distribution semigroups in the sense of Lions ([21, Corollary 3.11]).

For a given QDSG $\mathcal{G}$, the operator A_1 is defined by

$$D(A_1)=\bigcup\{\mathrm{Im}(\mathcal{G}(\phi));\ \phi\in\mathcal{D}_+\},$$
$$A_1\mathcal{G}(\phi)(x):=-\mathcal{G}(\phi')(x)-\phi(0)x,\qquad x\in X,\ \phi\in\mathcal{D}_+.$$

It is not difficult to check that $(A_1,D(A_1))$ is well-defined and closable (see [21, Proposition 3.5]). The closure of A_1, denoted by A, is called the *generator* of the QDSG $\mathcal{G}$, and for any $\phi\in\mathcal{D}_+$, we have $\mathcal{G}(\phi)A\subseteq A\mathcal{G}(\phi)$ ([21, Proposition 3.7]). An alternative definition of a generator is given in [11, Definition 3.3]. The generator is defined by $A:=\mathcal{G}(-\delta')$ and it may be proved that $D(A)=\mathrm{span}\{\mathcal{G}(\mathcal{D}_+)X\}$, where span denotes the linear span (see more details in [11]).

It is well-known that QDSGs are equivalent to well-posed Cauchy problems $C_{\alpha+1}(\kappa)$.

Theorem 5 ([21, Theorems 2.4 and 3.8]). *Let A be a closed operator. Then the following are equivalent.*

(i) *There exist $\alpha\ge 0$ and $\kappa>0$ such that $C_{\alpha+1}(\kappa)$ is well-posed.*
(ii) *A generates a QDSG.*

A direct proof of (i)⇒(ii) involves Weyl fractional derivatives. For a function $f \in \mathcal{D}_+$ and $\alpha \in \mathbb{R}^+$, the *Weyl fractional integral* $W_+^{-\alpha} f$ of order α is defined by

$$W_+^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^\infty (s-t)^{\alpha-1} f(s)\, ds$$

for $t \geq 0$, and the *Weyl fractional derivative* $W_+^{\alpha} f$ of order α is defined by

$$W_+^{\alpha} f(t) := \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^\infty (s-t)^{n-\alpha-1} f(s)\, ds$$

with $n = [\alpha] + 1$ and $t \geq 0$. It can be seen that $W_+^{\alpha+\beta} = W_+^{\alpha}(W_+^{\beta})$ for any $\alpha, \beta \in \mathbb{R}$, where $W_+^0 = \mathrm{Id}$ is the identity operator ([19]). The Weyl fractional calculus can be applied to more functions than those belonging to $\mathcal{D}_+$ (see [19, p. 248]). In this sense, for example, let f and g be measurable functions on $[0, \infty)$ such that $W_+^{-\alpha} f$ exists and $g = W_+^{-\alpha} f$ a.e. Then we consider $W_+^{\alpha} g = f$ and we follow the same notation.

To show that (i)⇒(ii) let $(S_\alpha(t))_{t\in[0,\kappa)}$ be the nondegenerate α-times integrated semigroup which gives the solution of the problem $C_{\alpha+1}(\kappa)$ (see (2)) and let $\phi \in \mathcal{D}_+$ with $\mathrm{supp}(\phi) \subset [0, R]$. Then there exists $n \in \mathbb{N} \cup \{0\}$ such that $2^n\alpha > R$, and we define $\mathcal{G}(\phi)$ by

$$\mathcal{G}(\phi)x := \int_0^{2^n\alpha} W_+^{2^n\alpha} \phi(t) S_{2^n\alpha}(t)x\, dt, \qquad x \in X,$$

where $(S_{2^n\alpha}(t))_t$ is defined recursively from the extension given in Theorem 2. To prove that

$$\mathcal{G}(\phi * \psi) = \mathcal{G}(\phi)\mathcal{G}(\psi)$$

for $\phi, \psi \in \mathcal{D}_+$, see similar ideas in [18, Theorem 3.1]. We may conclude that $\mathcal{G}$ is a QDSG.

Now we consider the case $\kappa = \infty$. Although there are local α-times integrated semigroups which cannot be extended (see Example 1), differential operators in Euclidean spaces are important examples of global α-times integrated semigroups ([9]).

Let $\omega : [0, \infty) \to (0, \infty)$ be a weight function and $L^1(\mathbb{R}^+, \omega)$ the usual convolution Banach algebra of measurable functions on $[0, \infty)$ such that $\|f\|_\omega := \int_0^\infty |f(t)|\omega(t)\, dt < \infty$. The *Riesz functions* $(R_t^\theta)_{t\geq 0}$ are defined by

$$R_t^\theta(s) := \frac{(t-s)^\theta}{\Gamma(\theta+1)} \chi_{(0,t)}(s), \qquad s \geq 0,\ t > 0,$$

$R_0^\theta = 0$ and $\theta > -1$. We denote by $\mathrm{Mul}(\mathcal{A})$ the set of multipliers of a Banach algebra $\mathcal{A}$.

THEOREM 6 ([7, Propositions 1.4 and 1.5]). *Let* $\alpha \in \mathbb{R}^+$ *and* $\tau_\alpha \in \Omega_{\alpha,\infty}$. *The expression*

$$q_{\tau_\alpha}(f) := \frac{1}{\Gamma(\alpha+1)} \int_0^\infty \tau_\alpha(t) |W_+^\alpha f(t)|\, dt$$

defines a norm on $\mathcal{D}_+$. *Moreover*, $q_{\tau_\alpha}(f * g) \le C_\alpha q_{\tau_\alpha}(f) q_{\tau_\alpha}(g)$ *for* $f, g \in \mathcal{D}_+$, *and* $C_\alpha > 0$ *is independent of* f *and* g. *Denote by* $\mathcal{T}_+^{(\alpha)}(\tau_\alpha)$ *the Banach algebra obtained as the completion of* $\mathcal{D}_+$ *in the norm* q_{τ_α} *with* $\tau_\alpha \in \Omega_{\alpha,\infty}$. *Then we have the following continuous embeddings*:

(i) $\mathcal{T}_+^{(\alpha)}(\tau_\alpha) \hookrightarrow \mathcal{T}_+^{(\alpha)}(t^\alpha) \hookrightarrow L^1(\mathbb{R}^+)$.

(ii) *If* $\beta > \alpha \in \mathbb{R}^+$, *and* $\tau_\beta \in \Omega_{\beta,\infty}$ *is such that*

$$\frac{1}{\Gamma(\beta-\alpha)\Gamma(\alpha+1)} \int_0^t (t-s)^{\beta-\alpha-1} \tau_\alpha(s)\, ds \le \frac{1}{\Gamma(\beta+1)} \tau_\beta(t), \qquad t \ge 0,$$

then $\mathcal{T}_+^{(\beta)}(\tau_\beta) \hookrightarrow \mathcal{T}_+^{(\alpha)}(\tau_\alpha)$, *in particular* $\mathcal{T}_+^{(\beta)}(t^\beta) \hookrightarrow \mathcal{T}_+^{(\alpha)}(t^\alpha)$.

(iii) $R_t^{\nu-1} \in \mathcal{T}_+^{(\alpha)}(\tau_\alpha)$ *with* $t > 0$ *and* $\nu > \alpha$; $q_{\tau_\alpha}(R_t^{\nu-1}) \le C_{\nu,\alpha} t^{\nu-\alpha} \tau_\alpha(t)$ *for any* $t > 0$, *where* $C_{\nu,\alpha} > 0$ *is independent of* t

(iv) $R_t^{\alpha-1} \in \mathrm{Mul}(\mathcal{T}_+^{(\alpha)}(\tau_\alpha))$ *and* $\|R_t^{\alpha-1}\|_{\mathrm{Mul}(\mathcal{T}_+^{(\alpha)}(\tau_\alpha))} \le C\tau_\alpha(t)$ *with* $t > 0$.

In the case $\alpha = 0$, we identify $\mathcal{T}_+^{(0)}(\tau_0)$ and $L^1(\mathbb{R}^+, \tau_0)$. If $\alpha = n$ and $\tau_n(t) = t^n$ for any $t \ge 0$, the algebra $\mathcal{T}_+^{(n)}(t^n)$ is $\mathcal{T}_n^+$ as defined in [2] and considered in [4] and [5]. If $\alpha = n$ and $\tau_n(t) = e^{rt}$ ($r > 0$, $t \ge 0$), then the algebra $\mathcal{T}_+^{(n)}(e^{rt})$ is $\mathcal{D}_{r,n}^+$ defined as in [21].

If $\tau_\alpha \in \Omega_{\alpha,\infty}^h$ with $\alpha \ge 0$, the algebra $\mathcal{T}_+^{(\alpha)}(\tau_\alpha)$ has bounded approximate identities (take $\phi \in \mathcal{T}_+^{(\alpha)}(\tau_\alpha)$ such that $\int_0^\infty \phi(t)\, dt = 1$ and consider $(\phi_s = \frac{1}{s}\phi(\frac{\cdot}{s}))_{0<s<1}$). In general, the algebras $\mathcal{T}_+^{(\alpha)}(\tau_\alpha)$ do not have any bounded approximate identity ([7]).

DEFINITION 7. We say that a quasi-distribution semigroup $\mathcal{G} : \mathcal{D}_+ \to \mathcal{B}(X)$ is of *order* $\alpha \in \mathbb{R}^+$ and *growth* $\tau_\alpha \in \Omega_{\alpha,\infty}$ if $\mathcal{G}$ can be extended to a continuous algebra homomorphism from $\mathcal{T}_+^{(\alpha)}(\tau_\alpha)$ into $\mathcal{B}(X)$, i.e., $\mathcal{G} : \mathcal{T}_+^{(\alpha)}(\tau_\alpha) \to \mathcal{B}(X)$.

This definition contains [21, Definitions 4.3 and 4.7].

LEMMA 8. *Let* $\alpha \ge 0$, $\tau_\alpha \in \Omega_{\alpha,\infty}^h$, *and let* $(A, D(A))$ *be a closed and densely defined operator which generates a quasi-distribution semigroup* $\mathcal{G}$ *on* X *of order* $\alpha \in \mathbb{R}^+$ *and growth* τ_α. *Then* $\mathcal{G}(\mathcal{T}_+^{(\alpha)}(\tau_\alpha))X$ *is dense in* X.

Proof. As $\tau_\alpha(t) = t^\alpha \omega(t)$ for some continuous and nondecreasing weight $\omega : [0,\infty) \to [0,\infty)$ and $\omega(t) \le Ce^{\kappa t}$ ($t \ge 0$, $C, \kappa > 0$), we deduce that $(\kappa,\infty)\subset\varrho(A)$ and $\sup_{\lambda>\kappa+1}\|\lambda R(\lambda,A)\|<\infty$. By [1, Lemma 3.3.12], $\lim_{\lambda\to\infty}\lambda R(\lambda,A)x=x$ for all $x\in X$. Since $\mathcal{G}(e^{-\lambda(\cdot)})=R(\lambda,A)$ for $\lambda>\kappa$, we conclude that $\mathcal{G}(\mathcal{T}_+^{(\alpha)}(\tau_\alpha))X$ is dense in X. ∎

The following theorem extends results from [2], [14] and [18] given in terms of integrated semigroups.

Theorem 9. *Suppose that $(v_x(t))_{t\ge0}$ is the solution of $C_{\alpha+1}(\infty)$ such that*

$$\|v_x'(t)\|\le C\tau_\alpha(t)\|x\|,\qquad x\in X,$$

with $\tau_\alpha\in\Omega_{\alpha,\infty}$. Then there exists a quasi-distribution semigroup of order α and growth τ_α, $\mathcal{G}:\mathcal{T}_+^{(\alpha)}(\tau_\alpha)\to\mathcal{B}(X)$, given by

$$\mathcal{G}(f)x=\int_0^\infty W_+^\alpha f(t)v_x'(t)\,dt,\qquad x\in X,$$

with $f\in\mathcal{T}_+^{(\alpha)}(\tau_\alpha)$, generated by $(A,D(A))$.

Proof. By Theorem 5, $(A,D(A))$ is the generator of a quasi-distribution semigroup, $\tilde{\mathcal{G}}:\mathcal{D}_+\to\mathcal{B}(X)$. Since $\|v_x'(t)\|\le C\tau_\alpha(t)\|x\|$ for any $t\ge0$, the expression

$$\mathcal{G}(f)x:=\int_0^\infty W_+^\alpha f(t)v_x'(t)\,dt,\qquad f\in\mathcal{T}_+^{(\alpha)}(\tau_\alpha),\ x\in X,$$

defines a continuous linear homomorphism $\mathcal{G}:\mathcal{T}_+^{(\alpha)}(\tau_\alpha)\to\mathcal{B}(X)$. There are several ways to conclude that $\mathcal{G}_{|\mathcal{D}_+}=\tilde{\mathcal{G}}$. We do this by checking that $\mathcal{G}$ is a quasi-distribution semigroup generated by $(A,D(A))$ and $(v_x'(t))_{t\ge0}$ defines an α-times integrated semigroup (see similar proof in [18, Theorem 3.1]). ∎

Remark 10. Note that in this theorem we do not assume that A is a densely defined operator.

Example 2. The function $E_c:\mathbb{R}^+\to\mathbb{R}^+$ defined by $E_c(t):=e^{ct^2}$ $(t>0)$ does not belong to $\Omega_{1,\infty}$ for any $c>0$. We have

$$\lim_{t\to\infty}\frac{\int_t^{t+r}e^{cs^2}\,ds}{e^{ct^2}}=\lim_{t\to\infty}\frac{e^{c(t+r)^2}-e^{ct^2}}{2cte^{ct^2}}=\infty.$$

Thus there is no $C>0$ such that $\int_t^{t+r}e^{cs^2}\,ds\le Ce^{ct^2}e^{cr^2}$ for $0<r<t$. However, there are 1-integrated semigroups such that $\|S_1(t)\|=e^{t^2/4}$ $(t\ge0)$ ([17, Example 1.2.5]). In this case we cannot apply Theorem 9 with $\tau_1(t)=e^{t^2/4}$.

We now prove the first converse to Theorem 9. Some precedents of this result are [2, Theorem 4.4], [14, Theorem 3.6] and [21, Theorem 4.16]. They consider integer derivation and particular growths τ_α.

THEOREM 11. *Given* $\alpha \geq 0$, $\tau_\alpha \in \Omega_{\alpha,\infty}$, *and a quasi-distribution semigroup* $\mathcal{G} : \mathcal{T}_+^{(\alpha)}(\tau_\alpha) \to \mathcal{B}(X)$ *generated by* $(A, D(A))$. *Then for any* $\nu > \alpha$, *the abstract Cauchy problem* $C_{\nu+1}(\infty)$ *is well-posed and the solution* $(v_x(t))_{t\geq 0}$ *satisfies*

$$\|v'_x(t)\| \leq C_\nu t^{\nu-\alpha}\tau_\alpha(t)\|x\|, \qquad t \geq 0.$$

Proof. Take $\nu > \alpha$. The family $(R_t^{\nu-1})_{t\geq 0}$ of Riesz functions, is a ν-times integrated semigroup in $\mathcal{T}_+^{(\alpha)}(\tau_\alpha)$ and $q_{\tau_\alpha}(R_t^{\nu-1}) \leq C_{\nu,\alpha}t^{\nu-\alpha}\tau_\alpha(t)$ for $\nu > \alpha$, $t \geq 0$ (see Theorem 6(iii)). Put $S_\nu(t) := \mathcal{G}(R_t^{\nu-1})$ for any $t \geq 0$. It is clear that $(S_\nu(t))_{t\geq 0}$ satisfies (3) and is a ν-times integrated semigroup. From the continuity of $\mathcal{G}$ we have $\|S_\nu(t)\| \leq C_{\nu,\alpha}t^{\nu-\alpha}\tau_\alpha(t)$ for any $t \geq 0$. It is straightforward to check that $(S_\nu(t))_{t\geq 0}$ is generated by $(A, D(A))$. ■

By Theorem 6(iv), the Riesz function $R_t^{\alpha-1}$ is a multiplier of the algebra $\mathcal{T}_+^{(\alpha)}(\tau_\alpha)$ for every $t > 0$. If $\mathcal{T}_+^{(\alpha)}(\tau_\alpha)$ has a bounded approximate identity, we may calculate $(\mathcal{G}(R_t^{\alpha-1}))_{t\geq 0}$ to get the second converse to Theorem 9.

THEOREM 12. *Let* $\alpha \geq 0$, $\tau_\alpha \in \Omega^h_{\alpha,\infty}$, *and let* $(A, D(A))$ *be a closed and densely defined operator on* X. *The following conditions are equivalent.*

(i) *The abstract Cauchy problem* $C_{\alpha+1}(\infty)$ *is well-posed and the solution* $(v_x(t))_{t\geq 0}$ *satisfies*

$$\|v'_x(t)\| \leq C_\alpha \tau_\alpha(t)\|x\|, \qquad t \geq 0.$$

(ii) $(A, D(A))$ *generates a quasi-distribution semigroup* $\mathcal{G}$ *of order* $\alpha \in \mathbb{R}^+$ *and growth* τ_α.

Proof. Defining $\mathcal{G}$ as in Theorem 9, we obtain (i)⇒(ii).

(ii)⇒(i). As $\tau_\alpha \in \Omega^h_{\alpha,\infty}$, $\mathcal{T}_+^{(\alpha)}(\tau_\alpha)$ has a bounded approximate identity. Since $\mathcal{G}(\mathcal{T}_+^{(\alpha)}(\tau_\alpha))X$ is dense in X (see Lemma 8), we have $X = \mathcal{G}(\mathcal{T}_+^{(\alpha)}(\tau_\alpha))X$ by Cohen's factorization theorem. The map $\mathcal{G}$ extends to a Banach algebra homomorphism $\widetilde{\mathcal{G}} : \mathrm{Mul}(\mathcal{T}_+^{(\alpha)}(\tau_\alpha)) \to \mathcal{B}(X)$. Then, if $T \in \mathrm{Mul}(\mathcal{T}_+^{(\alpha)}(\tau_\alpha))$ and $x = \mathcal{G}(f)y$ ($f \in \mathcal{T}_+^{(\alpha)}(\tau_\alpha)$, $y \in X$) we define $\widetilde{\mathcal{G}}(T)x := \mathcal{G}(T(f))y$ (see [6, Proposition 5.2]). By Theorem 6(iv), $R_t^{\alpha-1} \in \mathrm{Mul}(\mathcal{T}_+^{(\alpha)}(\tau_\alpha))$ and we put $S_\alpha(t) := \widetilde{\mathcal{G}}(R_t^{\alpha-1})$ for all $t \geq 0$. The proof is finished in the similar way as the proof in Theorem 11. ■

Acknowledgments. The author wishes to thank the referee for pointing out some references, comments and advice, which have greatly improved the presentation of the paper.

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Received January 18, 2006
Revised version May 30, 2008 (5844)