STUDIA MATHEMATICA 187 (3) (2008)

Minimality properties of Tsirelson type spaces

by

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Dedicated to the memory of Lior Tzafriri

Abstract. We study minimality properties of partly modified mixed Tsirelson spaces. A Banach space with a normalized basis (e_k) is said to be *subsequentially minimal* if for every normalized block basis (x_k) of (e_k), there is a further block basis (y_k) of (x_k) such that (y_k) is equivalent to a subsequence of (e_k). Sufficient conditions are given for a partly modified mixed Tsirelson space to be subsequentially minimal, and connections with Bourgain's ℓ^1-index are established. It is also shown that a large class of mixed Tsirelson spaces fails to be subsequentially minimal in a strong sense.

The class of mixed Tsirelson spaces plays an important role in the structure theory of Banach spaces and has been well investigated (e.g., [2, 3, 5, 17, 20, 21]). In this paper, we will study aspects of the subspace structure of mixed Tsirelson spaces and (partly) modified mixed Tsirelson spaces (see definitions below). We are particularly interested in properties connected with minimality. An infinite-dimensional Banach space X is *minimal* if every infinite-dimensional subspace has a further subspace isomorphic to X. The work of Gowers [15] had motivated some recent studies on minimality (e.g., [11, 12, 22]).

A Banach space X with a normalized basis (e_k) is said to be *subsequentially minimal* if for every normalized block basis (x_k) of (e_k), there is a further block (y_k) of (x_k) such that (y_k) is equivalent to a subsequence of (e_k). It is well known that the Tsirelson space $T[(\mathcal{S}_1, 1/2)]$ has the prop-

2000 *Mathematics Subject Classification*: Primary 46B20; Secondary 46B45.

Key words and phrases: subsequential minimality, partly modified mixed Tsirelson spaces, Bourgain's ℓ^1-index.

The first and second authors are pleased to acknowledge the support they received to attend the Workshop in Linear Analysis at Texas A & M University in August 2005, where the research reported in this paper was begun.

Research of the second author was partially supported by AcRF project no. R-146-000-086-112.

erty that every normalized block basis of its standard basis is equivalent to a subsequence of (e_k) (see [8]). In particular, it is subsequentially minimal. In [18, Theorem 9], it was shown that if a nonincreasing null sequence (θ_n) in $(0,1)$ is regular ($\theta_{m+n} \geq \theta_m\theta_n$) and satisfies

$$(\dagger) \qquad \lim_{m\to\infty} \limsup_{n\to\infty} \frac{\theta_{m+n}}{\theta_n} > 0,$$

then the space $T[(\mathcal{S}_n,\theta_n)_{n=1}^{\infty}]$ is subsequentially minimal if and only if every block subspace of $T[(\mathcal{S}_n,\theta_n)_{n=1}^{\infty}]$ admits an ℓ^1-$\mathcal{S}_\omega$-spreading model, if and only if every block subspace of $T[(\mathcal{S}_n,\theta_n)_{n=1}^{\infty}]$ has Bourgain ℓ^1-index greater than ω^ω. In particular, if $\sup_n \theta_n^{1/n} = 1$, then the mixed Tsirelson space $T[(\mathcal{S}_n,\theta_n)_{n=1}^{\infty}]$ is subsequentially minimal [20].

This paper is divided into two parts. In the first part, we investigate the analogs of the results quoted above in the context of partly modified mixed Tsirelson spaces. In this connection, it is worth pointing out that a subsequentially minimal partly modified mixed Tsirelson space is quasi-minimal in the sense of Gowers [15]. Since these spaces are strongly asymptotic ℓ^1, by [10] they do not contain minimal subspaces, and therefore they are strictly quasi-minimal. The only typical known example of a strictly quasi-minimal space was the Tsirelson space. While the unit vector basis of the Tsirelson space has the *block property* (every normalized block basis is equivalent to a subsequence of the unit vector basis) [8], among our examples of strictly quasi-minimal spaces there are cases which do not have this property. The subsequentially minimal mixed Tsirelson spaces, mentioned above, are also quasi-minimal, but it is not known if they are strictly quasi-minimal (see the remarks in [10]). In the second part of the paper, we give a general sufficient condition for an (unmodified) mixed Tsirelson space to fail to be subsequentially minimal in a strong sense.

In a recent article [13], Ferenczi and Rosendal undertook a deep analysis of minimality and proved several dichotomy results. Generally speaking, their results show that in every (infinite-dimensional) Banach space, one can find a further (infinite-dimensional closed) subspace that has either some form of minimality or a related form of "tightness". They also refined Gower's classification of Banach spaces. The subsequentially minimal partly modified mixed Tsirelson spaces considered in §2 below all belong to class 5c in the Ferenczi–Rosendal classification. (The fact that they are tight with constants follows from Proposition 4.2 of [13].) This puts them in the same class as Tsirelson's space T. Every normalized block basis in T has the block property. On the other hand, if (θ_n) is a regular sequence with $\sup_n \theta_n^{1/n} = 1$, then our results show that $T[(\mathcal{S}_n,\sigma_n,\theta_n)_{n=1}^{\infty}]$ is saturated with block sequences that have the block property and also saturated with block sequences that fail to have the block property. (The same can be said for the space $X_{M(1),u}$ constructed in [3].) Thus it is unclear if there is a further dichotomy within class 5c that can distinguish T from these other spaces. Let us also mention that a dichotomy between subsequential minimality on the one hand and a property called "tight by range" on the other was established in [13, Theorem 1.3]. The property of being "tight by range" is stronger than what we term "strongly non-subsequentially minimal" in §4 below.

1. Preliminaries. Denote by $\mathbb{N}$ the set of natural numbers. For any infinite subset M of $\mathbb{N}$, let $[M]$ and $[M]^{<\infty}$ be the sets of all infinite and all finite subsets of M respectively. These are subspaces of the power set of $\mathbb{N}$, which is identified with $2^{\mathbb{N}}$ and endowed with the topology of pointwise convergence. A subset $\mathcal{F}$ of $[\mathbb{N}]^{<\infty}$ is said to be *hereditary* if $G \in \mathcal{F}$ whenever $G \subseteq F$ and $F \in \mathcal{F}$. It is *spreading* if for all strictly increasing sequences $(m_i)_{i=1}^k$ and $(n_i)_{i=1}^k$, $(n_i)_{i=1}^k \in \mathcal{F}$ if $(m_i)_{i=1}^k \in \mathcal{F}$ and $m_i \le n_i$ for all i. We also call $(n_i)_{i=1}^k$ a *spreading* of $(m_i)_{i=1}^k$. A *regular* family is a subset of $[\mathbb{N}]^{<\infty}$ that is hereditary, spreading and compact (as a subspace of $2^{\mathbb{N}}$). If I and J are nonempty finite subsets of $\mathbb{N}$, we write $I < J$ to mean $\max I < \min J$. We also allow that $\emptyset < I$ and $I < \emptyset$. For a singleton $\{n\}$, $\{n\} < J$ is abbreviated to $n < J$. If $\mathcal{F}, \mathcal{G} \subseteq [\mathbb{N}]^{<\infty}$, let

$$\mathcal{F}[\mathcal{G}] = \Big\{ \bigcup_{i=1}^k G_i : G_i \in \mathcal{G},\, G_1 < \cdots < G_k,\, (\min G_i)_{i=1}^k \in \mathcal{F} \Big\},$$
$$(\mathcal{F}, \mathcal{G}) = \{F \cup G : F < G,\, F \in \mathcal{F},\, G \in \mathcal{G}\}.$$

Inductively, set $(\mathcal{F})^1 = \mathcal{F}$ and $(\mathcal{F})^{n+1} = (\mathcal{F}, (\mathcal{F})^n)$ for all $n \in \mathbb{N}$. It is clear that $\mathcal{F}[\mathcal{G}]$ and $(\mathcal{F}, \mathcal{G})$ are regular if both $\mathcal{F}$ and $\mathcal{G}$ are. A class of regular families that has played a central role is the class of generalized Schreier families [1].

Let $\mathcal{S}_0$ consist of all singleton subsets of $\mathbb{N}$ together with the empty set. Then define $\mathcal{S}_1$ to be the collection of all $A \in [\mathbb{N}]^{<\infty}$ such that $|A| \le \min A$ together with the empty set, where $|A|$ denotes the cardinality of the set A. If $\mathcal{S}_\alpha$ has been defined for some countable ordinal α, set $\mathcal{S}_{\alpha+1} = \mathcal{S}_1[\mathcal{S}_\alpha]$. For a countable limit ordinal α, specify a sequence (α_n) that strictly increases to α. Then define

$$\mathcal{S}_\alpha = \{F : F \in \mathcal{S}_{\alpha_n} \text{ for some } n \le \min F\} \cup \{\emptyset\}.$$

Given a nonempty compact family $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$, let $\mathcal{F}^{(0)} = \mathcal{F}$ and $\mathcal{F}^{(1)}$ be the set of all limit points of $\mathcal{F}$. Continue inductively to define $\mathcal{F}^{(\alpha+1)} = (\mathcal{F}^{(\alpha)})^{(1)}$ for all ordinals α and $\mathcal{F}^{(\alpha)} = \bigcap_{\beta<\alpha} \mathcal{F}^{(\beta)}$ for all limit ordinals α. The *index* $\iota(\mathcal{F})$ is the smallest α such that $\mathcal{F}^{(\alpha+1)} = \emptyset$. Since $[\mathbb{N}]^{<\infty}$ is countable, $\iota(\mathcal{F}) < \omega_1$ for any compact family $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$. It is well known that $\iota(\mathcal{S}_\alpha) = \omega^\alpha$ for all $\alpha < \omega_1$ [1, Proposition 4.10].

A sequence (x_n) in a normed space said to *dominate* a sequence (y_n) in a possibly different space if there is a finite constant K such that $\|\sum a_n y_n\| \leq K\|\sum a_n x_n\|$ for all $(a_n) \in c_{00}$. If two sequences dominate each other, then they are *equivalent*, and we write $(x_n) \sim (y_n)$. If (e_n) is a basic sequence and $F \subseteq \mathbb{N}$, then $[(e_n)_{n\in F}]$ denotes the closed linear span of $\{e_n : n \in F\}$. If (e_n) is a normalized basis of X, then $(x_n) \prec (e_n)$ or $(x_n) \prec X$ will indicate that (x_n) is a normalized block basis of (e_n). We say that Y is a *block subspace* of X, written $Y \prec X$, if X has a basis (x_n) and $Y = [(y_n)_{n\in\mathbb{N}}]$ for some $(y_n) \prec (x_n)$. A normalized sequence (x_n) is said to be an *ℓ^1-$\mathcal{S}_\beta$-spreading model with constant* K if $\|\sum_{n\in F} a_n x_n\| \geq K^{-1}\sum_{n\in F}|a_n|$ whenever $F \in \mathcal{S}_\beta$.

Partly modified mixed Tsirelson spaces. Let (θ_n) be a null sequence in the interval $(0,1)$ and $\sigma_n \in \{U, M\}$ for every n. We say that a family $(E_i)_{i=1}^k$ of subsets of $\mathbb{N}$ is $(\mathcal{S}_n,\sigma_n)$-*adapted* if $(\min E_i)_{i=1}^k \in \mathcal{S}_n$ and

$$\begin{cases} E_i \cap E_j = \emptyset, & 1 \leq i \neq j \leq k \quad \text{if } \sigma_n = M,\\ E_i < E_{i+1}, & 1 \leq i < k \quad\quad\;\; \text{if } \sigma_n = U.\end{cases}$$

An $(\mathcal{S}_n,\sigma_n)$-adapted family $(E_i)_{i=1}^k$ is said to be $\mathcal{S}_n$-*admissible* (respectively $\mathcal{S}_n$-*allowable*) if $\sigma_n = U$ (respectively $\sigma_n = M$). Define the space $X = T[(\mathcal{S}_n,\sigma_n,\theta_n)_{n=1}^\infty]$ to be the completion of c_{00} under the implicitly defined norm

$$\|x\| = \max\Big\{\|x\|_{c_0}, \sup_n \theta_n \sup \sum_i \|E_i x\|\Big\}, \tag{1}$$

where the last supremum is taken over all $(\mathcal{S}_n,\sigma_n)$-adapted families (E_i). If $\sigma_n = U$ for all n (respectively $\sigma_n = M$ for all n), then X is a mixed Tsirelson space (respectively modified mixed Tsirelson space). If $\sigma_{p_0} = M$ for at least one p_0, we call X a *partly modified mixed Tsirelson space.*

Norming trees. Equation (1) can be viewed as an iterative prescription for computing the norm. The procedure may be summarized in terms of norming trees (see [7]), from which the existence and uniqueness of a norm satisfying (1) also follows. An $((\mathcal{S}_n,\sigma_n)_n$-$)$*adapted tree* $\mathcal{T}$ is a finite collection (E_i^m), $0 \leq m \leq r$, $1 \leq i \leq k(m)$, of elements in $[\mathbb{N}]^{<\infty}$ with the following properties:

(1) $k(0) = 1$,
(2) every E_i^{m+1} is a subset of some E_j^m,
(3) for each j and m, the collection $\{E_i^{m+1} : E_i^{m+1} \subseteq E_j^m\}$ is $(\mathcal{S}_k,\sigma_k)$-adapted for some k.

The set E_1^0 is called the *root* of the adapted tree. The elements E_i^m are called *nodes* of the tree. If $E_i^n \subseteq E_j^m$ and $n > m$, we say that E_i^n is a *descendant* of E_j^m and E_j^m is an *ancestor* of E_i^n. If, in the above notation, $n = m+1$, then E_i^n is said to be an *immediate successor* of E_j^m, and

E_j^m the *immediate predecessor* or *parent* of E_i^n. Nodes with no descendants are called *terminal nodes* or *leaves* of the tree. The collection of all leaves of $\mathcal{T}$ is denoted by $\mathcal{L}(\mathcal{T})$. Assign *tags* to the individual nodes inductively as follows. Let $t(E_1^0) = 1$. If $t(E_i^m)$ has been defined and the collection (E_j^{m+1}) of all immediate successors of E_i^m is $(\mathcal{S}_k, \sigma_k)$-adapted, then define $t(E_j^{m+1}) = \theta_k t(E_i^m)$ for all immediate successors E_j^{m+1} of E_i^m. If $x \in c_{00}$ and $\mathcal{T}$ is an adapted tree, let $\mathcal{T}x = \sum t(E)\|Ex\|_{c_0}$, where the sum is taken over all leaves in $\mathcal{T}$. It follows from the implicit description (1) of the norm in X that $\|x\| = \max \mathcal{T}x$, with the maximum taken over the set of all adapted trees. Let us also point out that if $\mathcal{E}$ is a collection of pairwise disjoint nodes of an adapted tree $\mathcal{T}$ so that $E \subseteq \bigcup \mathcal{E}$ for every leaf E of $\mathcal{T}$ and $x \in c_{00}$, then $\mathcal{T}x \le \sum_{F\in\mathcal{E}} t(F)\|Fx\|$. Given a node $E \in \mathcal{T}$ with tag $t(E) = \prod_{i=1}^m \theta_{n_i}$, define $\mathrm{ord}_{\mathcal{T}}(E) = \sum_{i=1}^m n_i$. When there is no confusion, we write $\mathrm{ord}(E)$ instead of $\mathrm{ord}_{\mathcal{T}}(E)$.

Let $\mathcal{T}$ be an adapted tree. A node $E \in \mathcal{T}$ is said to be a *sibling* of $F \in \mathcal{T}$ if they have the same parent. If (z_i) is a block sequence, we say that E *begins at* z_k if $E \cap \operatorname{supp} z_k \ne \emptyset$ and $E \cap \operatorname{supp} z_j = \emptyset$ for all $j < k$. To say that E *begins before* z_k means that E begins at z_j for some $j < k$ and we denote this condition by $E \lhd z_k$.

ℓ^1-Trees and Bourgain's ℓ^1-index. A *tree* in a Banach space B is a subset $\mathcal{T}$ of $\bigcup_{n=1}^\infty B^n$ so that $(x_1, \ldots, x_n) \in \mathcal{T}$ whenever $(x_1, \ldots, x_n, x_{n+1}) \in \mathcal{T}$. Elements of the tree are called *nodes*. The tree is *well-founded* if there is no infinite sequence (x_n) so that $(x_1, \ldots, x_m) \in \mathcal{T}$ for all m. If B has a basis, then a tree $\mathcal{T}$ is said to be a *block tree* (with respect to the basis) if every node is a block basis of the given basis. For any well-founded tree $\mathcal{T}$, its *derived tree* is the tree $\mathcal{D}^{(1)}(\mathcal{T})$ consisting of all nodes $(x_1, \ldots, x_n)$ such that $(x_1, \ldots, x_n, x) \in \mathcal{T}$ for some x. Inductively, set $\mathcal{D}^{(\alpha+1)}(\mathcal{T}) = \mathcal{D}^{(1)}(\mathcal{D}^{(\alpha)}(\mathcal{T}))$ for all ordinals α and $\mathcal{D}^{(\alpha)}(\mathcal{T}) = \bigcap_{\beta<\alpha} \mathcal{D}^{(\beta)}(\mathcal{T})$ for all limit ordinals α. The *order* of a tree $\mathcal{T}$ is the smallest ordinal $\mathrm{o}(\mathcal{T}) = \alpha$ such that $\mathcal{D}^{(\alpha)}(\mathcal{T}) = \emptyset$.

Definition. Given a finite constant $K \ge 1$, an *ℓ^1-K-tree* in a Banach space B is a tree in B so that every node $(x_1, \ldots, x_n)$ is a normalized sequence such that $\|\sum a_k x_k\| \ge K^{-1} \sum |a_k|$ for all (a_k). If B has a basis, an *ℓ^1-K-block tree* is a block tree that is also an ℓ^1-K-tree. Suppose that B does not contain ℓ^1, and let $I(B, K) = \sup \mathrm{o}(\mathcal{T})$, where the sup is taken over the set of all ℓ^1-K-trees in X. The *Bourgain ℓ^1-index* of B is defined to be $I(B) = \sup_{K<\infty} I(B, K)$. The *block ℓ^1-index* $I_{\mathrm{b}}(B)$ is defined analogously using block trees if B has a basis. In [16, Lemmas 5.7 and 5.11], it was shown that $I_{\mathrm{b}}(B) \ne I_{\mathrm{b}}(B, K)$ and $I(B) \ne I(B, K)$ for every K. In particular, $I_{\mathrm{b}}(B), I(B)$ are limit ordinals. It was also shown [16, Corollary 5.13] that $I(B) = I_{\mathrm{b}}(B)$ when both are defined and are $\ge \omega^\omega$ each.

2. Sufficient conditions for subsequential minimality. The purpose of the present section is to give sufficient conditions for a partly modified mixed Tsirelson space to be subsequentially minimal. Prior experience with mixed Tsirelson spaces [18] suggests that there may be some connection with the Bourgain ℓ^1-index. This indeed turns out to be the case but the proof requires a different approach.

The main result of the section is the following theorem. The smallest integer greater than or equal to $a \in \mathbb{R}$ is denoted by $\lceil a \rceil$. *For the rest of the section, X will denote a partly modified mixed Tsirelson space.*

THEOREM 1. *Let X be a partly modified mixed Tsirelson space. If $Y \prec X$ and $I(Y) > \omega^\omega$, then there exists $(x_n) \prec Y$ such that $(x_n) \sim (e_{p_n})$, where $p_n = \min \operatorname{supp} x_n$. Consequently, X is subsequentially minimal if $I(Y) > \omega^\omega$ for all $Y \prec X$.*

Let us remark that by Theorem 1 and Proposition 14, any partly modified mixed Tsirelson space X such that $I(Y) > \omega^\omega$ for all block subspaces Y of X is saturated with subspaces with subsequentially minimal bases. Thus X is *sequentially minimal* in the terminology of [13]. Before proceeding with the proof of the theorem, let us draw the following corollary.

COROLLARY 2. *Suppose that there exists $\varepsilon > 0$ such that*

$$\sup\{n/m : \theta_n \geq \varepsilon^m\} = \infty.$$

Then X is subsequentially minimal. This holds in particular if $\sup \theta_n^{1/n} = 1$.

Proof. Clearly, for any $n \in \mathbb{N}$ and any $Y \prec X$, every normalized block sequence in Y is an ℓ^1-$\mathcal{S}_n$-spreading model with constant θ_n^{-1}. By [16], if Y contains an ℓ^1-$\mathcal{S}_{2n}$-spreading model with constant K, then it contains an ℓ^1-$\mathcal{S}_n$-spreading model with constant $\sqrt{K}$. With the assumption of the corollary, for any $k \in \mathbb{N}$, there are m, n so that $n/m \geq 2k$ and $\theta_n \geq \varepsilon^m$. Choose i and j so that $2^i \leq m < 2^{i+1}$ and $2^j \leq n < 2^{j+1}$. Then any $Y \prec X$ contains an ℓ^1-$\mathcal{S}_{2^j}$-spreading model with constant θ_n^{-1}, and hence, by the remark above, an ℓ^1-$\mathcal{S}_{2^{j-i}}$-spreading model with constant $\theta_n^{-1/2^i}$. Since $\theta_n^{-1/2^i} \leq \varepsilon^{-2}$ and $2^{j-i} \geq k$, Y has an ℓ^1-$\mathcal{S}_k$-spreading model with constant ε^{-2} for all k. Hence there is an ℓ^1-ε^{-2}-tree on Y of order ω^ω. Thus $I_{\mathrm{b}}(Y, \varepsilon^{-2}) \geq \omega^\omega$ and so $I(Y) = I_{\mathrm{b}}(Y) > I_{\mathrm{b}}(Y, \varepsilon^{-2}) \geq \omega^\omega$. The desired result now follows from Theorem 1.

Finally, assume that $\sup \theta_n^{1/n} = 1$. Given $0 < \varepsilon < 1$ and $k \in \mathbb{N}$, there exists $n > k$ such that $\theta_n^{1/n} > \varepsilon^{1/k}$. Set $m = \lceil n/k \rceil \geq 2$. Then $\theta_n \geq \varepsilon^m$ and $n/m \geq k(1 - 1/m) \geq k/2$. ∎

The proof of Theorem 1 is in two stages. First we show that from any block subspace of X with a high ℓ^1-index a "slow-growing" block sequence may be extracted (see property $(*)$ defined below). In the second part, we show that this block sequence is equivalent to a subsequence of the unit vector basis (e_k).

DEFINITION. Let $Y = [(y_k)]$ be a block subspace of X. We say that Y has *property* $(*)$ if there exists a constant $C < \infty$ such that for all $n \in \mathbb{N}$, there exists a normalized vector $x \in Y_n = \operatorname{span}\{(y_k)_{k=n}^{\infty}\}$ such that $\sum \|E_i x\| \le C$ whenever (E_i) is $\mathcal{S}_n$-allowable.

First we recall a lemma.

LEMMA 3 ([17, Proposition 14]). *Let $\mathcal{T}$ be a well-founded block tree in a Banach space B with a basis. Define*

$$\mathcal{H} = \{(\max\operatorname{supp} x_j)_{j=1}^{r} : (x_j)_{j=1}^{r} \in \mathcal{T}\},$$
$$\mathcal{G} = \{G : G \textit{ is a spreading of a subset of some } H \in \mathcal{H}\}.$$

Then $\mathcal{G}$ is hereditary and spreading. If $\mathcal{G}$ is compact, then $\iota(\mathcal{G}) \ge \mathrm{o}(\mathcal{T})$.

LEMMA 4. *If $I(Y) > \omega^{\omega}$ then Y has property $(*)$.*

Proof. There exists $K < \infty$ such that $I_{\mathrm{b}}(Y, K) \ge \omega^{\omega}$. Let $\mathcal{T}$ be an ℓ^1-K-block tree in Y such that $\mathrm{o}(\mathcal{T}) \ge \omega^{\omega}$. Given $n \ge p_0$, consider the tree $\widehat{\mathcal{T}}$ consisting of all nodes of the form $(x_j)_{j=n}^{r}$ for some $(x_j)_{j=1}^{r} \in \mathcal{T}$, $r \ge n$. Then $\widehat{\mathcal{T}}$ is an ℓ^1-K-block tree in Y_n such that $\mathrm{o}(\widehat{\mathcal{T}}) \ge \omega^{\omega}$. Define

$$\mathcal{H} = \{(\max\operatorname{supp} x_j)_{j=n}^{r} : (x_j)_{j=n}^{r} \in \widehat{\mathcal{T}}\},$$
$$\mathcal{G} = \{G : G \text{ is a spreading of a subset of some } H \in \mathcal{H}\}.$$

By Lemma 3, $\mathcal{G}$ is hereditary and spreading, and either $\mathcal{G}$ is non-compact or it is compact with $\iota(\mathcal{G}) \ge \mathrm{o}(\widehat{\mathcal{T}}) \ge \omega^{\omega} > \omega^{n+1}$. By [14, Theorem 1.1], there exists $M \in [\mathbb{N}]$ such that

$$\mathcal{S}_{n+1} \cap [M]^{<\infty} \subseteq \mathcal{G}.$$

Now [21, Proposition 3.6] gives a finite set $G \in \mathcal{S}_{n+1} \cap [M]^{<\infty}$ and a sequence $(a_p)_{p \in G}$ of positive numbers such that $\sum a_p = 1$ and $\sum_{p \in F} a_p < (\theta_{p_0})^P$, where $P = \lceil n/p_0 \rceil$, whenever $F \subseteq G$ and $F \in \mathcal{S}_n$. By definition, there exist a node $(x_j)_{j=n}^{r} \in \widehat{\mathcal{T}}$ and a subset J of the integer interval $[n, r]$ such that G is a spreading of $(\max\operatorname{supp} x_j)_{j \in J}$. Denote by u the unique order preserving bijection from J onto G and consider the vector $y = \sum_{j \in J} a_{u(j)} x_j$. Since $(x_j)_{j=n}^{r}$ is a normalized ℓ^1-K-block sequence in Y_n and $\sum a_{u(j)} = 1$, $y \in Y_n$ and $\|y\| \ge 1/K$.

Let (E_i) be $\mathcal{S}_n$-allowable. Let $J_1 = \{j \in J : \text{some } E_i \text{ begins at } x_j\}$ and $J_2 = J \setminus J_1$. Note that $\mathcal{S}_n \subseteq \mathcal{S}_{p_0 P} = [\mathcal{S}_{p_0}]^P$. Thus for each j, $(\theta_{p_0})^P \sum_i \|E_i x_j\| \le \|x_j\| = 1$. Also, since $\{u(j) : j \in J_1\} \in \mathcal{S}_n$, $\sum_{j \in J_1} a_{u(j)} < (\theta_{p_0})^P$. Hence

$$(2)\qquad \sum_i \Big\| E_i \sum_{j \in J_1} a_{u(j)} x_j \Big\| \le \sum_{j \in J_1} a_{u(j)} \sum_i \|E_i x_j\| \le \sum_{j \in J_1} a_{u(j)} \frac{1}{(\theta_{p_0})^P} < 1.$$

On the other hand, the collection $\{E_i \cap \operatorname{supp} x_j : E_i \lhd x_j\}$ of pairwise disjoint sets is $\mathcal{S}_1$-allowable and thus $\mathcal{S}_{p_0}$-allowable. Therefore,

$$(3)\quad \sum_i \Big\| E_i \sum_{j\in J_2} a_{u(j)} x_j \Big\| \le \sum_{j\in J_2} a_{u(j)} \sum_i \|E_i x_j\| = \sum_{j\in J_2} a_{u(j)} \sum_{E_i \lhd x_j} \|E_i x_j\|$$
$$= \sum_{j\in J_2} a_{u(j)} \sum_{E_i\lhd x_j} \|(E_i \cap \operatorname{supp} x_j) x_j\| \le \sum_{j\in J_2} a_{u(j)} \frac{1}{\theta_{p_0}}.$$

Combining inequalities (2) and (3) gives

$$\sum \|E_i y\| = \sum \Big\| E_i \sum_{j\in J} a_{u(j)} x_j \Big\|$$
$$\le \sum \Big\| E_i \sum_{j\in J_1} a_{u(j)} x_j \Big\| + \sum \Big\| E_i \sum_{j\in J_2} a_{u(j)} x_j \Big\| \le 1 + \frac{1}{\theta_{p_0}}.$$

It is clear that the normalized element $x = y/\|y\|$ satisfies the statement of the lemma with the constant $C = (1 + 1/\theta_{p_0})K$. ∎

We record the quantitative restatement of Lemma 4 for future reference.

LEMMA 5. *Let $\mathcal{T}$ be an ℓ^1-K-block tree on a block subspace Y of X of order $\mathrm{o}(\mathcal{T}) \ge \omega^\omega$. Then for all $n \in \mathbb{N}$, there is a normalized vector x in the span of a node of $\mathcal{T}$ such that $\sum \|E_i x\| \le K(1 + \theta_{p_0}^{-1})$ whenever (E_i) is $\mathcal{S}_n$-allowable.*

For each $n \in \mathbb{N}$, define

$$\xi_n = \sup\{\theta_{m_1} \cdots \theta_{m_j} : m_1 + \cdots + m_j > n\}.$$

Then (ξ_n) is a null sequence. Assume that Y has property $(*)$. Taking $n_0 = 0$, choose a normalized, finitely supported vector x_1 so that $\sum \|E_s x_1\| \le C$ whenever (E_s) is $\mathcal{S}_{n_0}$-allowable. Since (ξ_n) is a null sequence, there exists $n_1 > n_0$ so that $\xi_{n_1} \|x_1\|_{\ell^1} \le 1/2$. Let $q_1 = \max \operatorname{supp} x_1$. We can choose a normalized vector $x_2 \in \operatorname{span}\{(e_k)_{k=2q_1}^\infty\}$ so that $\sum \|E_s x_1\| \le C$ whenever (E_s) is $\mathcal{S}_{n_1}$-allowable. Continuing inductively, we obtain $(x_k) \prec Y$ and a strictly increasing sequence (n_k), $n_0 = 1$, so that for each k,

(α) $\sum \|E_s x_k\| \le C$ whenever (E_s) is $\mathcal{S}_{n_{k-1}}$-allowable,

(β) $\xi_{n_k} \|x_k\|_{\ell^1} \le 1/2^k$,

(γ) $2q_k \le p_{k+1}$ for all k, where $p_k = \min \operatorname{supp} x_k$ and $q_k = \max \operatorname{supp} x_k$.

Let $(b_k)_{k=1}^N \in c_{00}^+$ and set $x = \sum_{k=1}^N b_k x_k$.

LEMMA 6. *Let $\mathcal{T}$ be an adapted tree. If $\mathcal{E}$ is a collection of pairwise disjoint nodes of $\mathcal{T}$ such that $\operatorname{ord}(E) \le m$ for all $E \in \mathcal{E}$, then $\mathcal{E}$ is $\mathcal{S}_m$-allowable.*

Proof. Note that if $\mathcal{T}$ is an adapted tree, then it is an allowable tree with nodes of the same orders. The conclusion follows from [19, Lemma 3]. ∎

LEMMA 7. *Given any adapted tree $\mathcal{T}$, there exists an adapted tree $\mathcal{T}'$ such that*

(a) *if $E \in \mathcal{T}'$ and $E \cap \operatorname{supp} x_k \neq \emptyset$, then $\operatorname{ord}(E) \le n_k$,*
(b) $\mathcal{T}x \le \mathcal{T}'x + \sum_k b_k/2^k$.

Proof. Given an adapted tree $\mathcal{T}$ and $F \subseteq \mathbb{N}$, define

$$\mathcal{T}_F = \{E \cap F : E \in \mathcal{T},\ E \cap F \neq \emptyset\}.$$

Clearly, $\mathcal{T}_F$ is an adapted tree. For all $k = 2, \ldots, N$, define a set F_k by

$$F_k^c = \bigcup\{E \cap \operatorname{supp} x_k : E \in \mathcal{T},\ \operatorname{ord}(E) > n_k\}.$$

Then

$$\mathcal{T}x_k = \sum_{E\in\mathcal{L}(\mathcal{T})} t(E)\|Ex_k\| = \sum_{\substack{E\in\mathcal{L}(\mathcal{T})\\ \operatorname{ord}(E)\le n_k}} t(E)\|Ex_k\| + \sum_{\substack{E\in\mathcal{L}(\mathcal{T})\\ \operatorname{ord}(E)> n_k}} t(E)\|Ex_k\|$$

$$\le \mathcal{T}_{F_k}x_k + \xi_{n_k}\|x_k\|_{\ell^1} \le \mathcal{T}_{F_k}x_k + 1/2^k.$$

Let $\mathcal{T}' = \mathcal{T}_{F_2\cap F_3\cap\cdots\cap F_N}$. Note that $\mathcal{T}'$ satisfies (a) and $\mathcal{T}'x_k = \mathcal{T}_{F_k}x_k$ if $2 \le k \le N$. Hence

$$\mathcal{T}x = \sum b_k\mathcal{T}x_k \le b_1\mathcal{T}x_1 + \sum_{k=2}^{N} b_k\Big(\mathcal{T}_{F_k}x_k + \frac{1}{2^k}\Big)$$

$$= \sum b_k\mathcal{T}'x_k + \sum\frac{b_k}{2^k} = \mathcal{T}'x + \sum\frac{b_k}{2^k}. \ \blacksquare$$

Define $\mathcal{E}_k = \{E \in \mathcal{T}' : E$ begins at x_k and has a sibling that begins before $x_k\}$.

LEMMA 8. $\sum_{E\in\mathcal{E}_k}\|Ex_k\| \le C$ *for all $k = 2, \ldots, N$.*

Proof. Note that if $E \in \mathcal{E}_k$, then E has a sibling E' that begins before x_k. Hence $\operatorname{ord}(E) = \operatorname{ord}(E') \le n_{k-1}$ by property (a) of Lemma 7. By Lemma 6, $\mathcal{E}_k$ is $\mathcal{S}_{n_{k-1}}$-allowable. The conclusion follows from condition (α). ∎

Proof of Theorem 1. As (e_k) is a 1-unconditional basis of X, it is enough to consider nonnegative coefficients. As above, consider $(b_k)_{k=1}^N \in c_{00}^+$ and set $x = \sum_{k=1}^N b_kx_k$, $y = \sum_{k=1}^N b_ke_{p_k}$. It is easy to see that $\|y\| \le \|x\|$. We will show that $\|x\| \le (2+C)\|y\|$, where C is the constant in condition (α). Given an adapted tree $\mathcal{T}$, we obtain an adapted tree $\mathcal{T}'$ as in Lemma 7. We may further assume that every node $E \in \mathcal{T}'\setminus\mathcal{L}(\mathcal{T}')$ is the union of its immediate successors, that $E \subseteq \bigcup_k \operatorname{supp} x_k$ for every $E \in \mathcal{T}'$ and that, upon relabeling if necessary, the root of $\mathcal{T}'$ begins at x_1. With these assumptions, every node $E \in \mathcal{L}(\mathcal{T}')$ that intersects $\operatorname{supp} x_k$, $k \ge 2$, is a descendant of some node in $\mathcal{E}_k$. For each $k \ge 2$, choose $E_k \in \mathcal{E}_k$ such that $t(E_k) = \max\{t(E) : E \in \mathcal{E}_k\}$. By

Lemma 8, for $k \geq 2$,

$$\mathcal{T}'x_k = \sum_{E\in\mathcal{E}_k} t(E)\|Ex_k\| \leq t(E_k) \sum_{E\in\mathcal{E}_k} \|Ex_k\| \leq t(E_k)C.$$

Therefore,

$$\begin{aligned}\mathcal{T}x &\leq \mathcal{T}(b_1x_1) + \sum_{k=2}^{N} b_k\mathcal{T}'x_k + \sum_{k=2}^{N} \frac{b_k}{2^k} \leq b_1 + C\sum_{k=2}^{N} b_k t(E_k) + \sum_{k=2}^{N} \frac{b_k}{2^k} \\ &\leq C\sum_{k=2}^{N} t(E_k)b_k + 2\|(b_k)\|_{c_0} \leq C\sum_{k=2}^{N} t(E_k)b_k + 2\|y\|.\end{aligned}$$

To complete the proof, it suffices to appeal to Proposition 9 below to see that $\sum_{k=2}^{N} t(E_k)b_k \leq \|y\|$. ■

Remark. The proof above shows that if (x_k) is a (possibly finite) normalized block sequence in X satisfying conditions (α)–(γ) for some (n_k), then (x_k) is $(2+C)$-equivalent to (e_{p_k}).

Proposition 9. *There is an $(\mathcal{S}_n, \sigma_n)_{n=1}^{\infty}$-adapted tree $\mathcal{T}''$ such that*

$$\mathcal{T}''y \geq \sum_{k=1}^{N} b_k t(E_k).$$

In particular, $\sum_{k=1}^{N} b_k t(E_k) \leq \|y\|$.

Proof. The tree $\mathcal{T}''$ is constructed by replacing each node E in $\mathcal{T}'$ with one or two nodes, which we now proceed to describe. For each $E \in \mathcal{T}'$, define $G_E = \{p_j : E_j \subsetneq E\}$. If $E \in \mathcal{T}'$ and $E \neq E_k$ for any k, substitute G_E for E. If $E = E_k$ for some k, put two nodes, namely $\{p_k\}$ and G_E, in place of E. The resulting collection of nodes is denoted by $\mathcal{T}''$. Note that since the root of $\mathcal{T}'$ begins at x_1, it cannot be equal to E_k for any k. Thus the root of $\mathcal{T}'$ is replaced with a single node. If $E \in \mathcal{T}'$ has immediate successors $(F_i)_{i=1}^{s}$ which form an $(\mathcal{S}_n, \sigma_n)$-adapted family, then in the process, the F_i's are replaced with sets from the collection $(G_{F_i})_{i=1}^{s} \cup P$, where $P = \{\{p_k\} : F_i = E_k \text{ for some } i\}$, whereas G_E is one of the substitutes for E (perhaps the only one). Claim 1 below shows that $\mathcal{T}''$ remains a tree; Claims 2 and 3 show that $(G_{F_i})_{i=1}^{s} \cup P$ is an $(\mathcal{S}_n, \sigma_n)$-adapted family if $(F_i)_{i=1}^{s}$ is $(\mathcal{S}_n, \sigma_n)$-adapted.

Claim 1. *$(G_{F_i})_{i=1}^{s} \cup P$ is a family of pairwise disjoint subsets of G_E.*

By definition, $\{p_k\} \subseteq G_E$ for any $\{p_k\} \in P$. Let us show that $G_{F_i} \subseteq G_E$. Indeed, if $p_j \in G_{F_i}$, then $E_j \subsetneq F_i \subseteq E$. Thus $p_j \in G_E$.

Now, if $i \neq i'$, then $F_i \cap F_{i'} = \emptyset$. By definition, G_{F_i} is disjoint from $G_{F_{i'}}$. If $F_i = E_k$ for some i and k, then for any i' (including i itself), $E_k \subsetneq F_{i'}$ cannot hold. Therefore, $\{p_k\}$ and $G_{F_{i'}}$ are disjoint for all i'. Since obviously any two sets in P are disjoint, the claim is established.

Claim 2. *If* $(F_i)_{i=1}^s$ *consists of successive sets, then so does* $(G_{F_i})_{i=1}^s \cup P$.

First we show that if $F_i < F_{i'}$, then $G_{F_i} < G_{F_{i'}}$. Let $p_j \in G_{F_i}$ and $p_{j'} \in G_{F_{i'}}$. Then $E_j \subsetneq F_i$ and $E_{j'} \subsetneq F_{i'}$. Since E_j begins at x_j, $E_{j'}$ begins at $x_{j'}$ and $F_i < F_{i'}$, it follows that $j < j'$ and hence $p_j < p_{j'}$. This shows that $G_{F_i} < G_{F_{i'}}$.

Next, if $F_i < F_{i'} = E_k$ for some i, i' and k, then we claim that $G_{F_i} < \{p_k\} < G_{F_{i'}}$. To see the first inequality, pick a point $p_j \in G_{F_i}$. Then $E_j \subsetneq F_i$. In particular, $E_j < F_{i'} = E_k$. Since E_j begins at x_j and E_k begins at x_k, we deduce that $j < k$ and thus $p_j < p_k$. Hence $G_{F_i} < \{p_k\}$. Similarly, if $p_j \in G_{F_{i'}}$, then $E_j \subsetneq F_{i'} = E_k$. Since E_j begins at x_j and E_k begins at x_k, we deduce that $k < j$. This shows that $\{p_k\} < G_{F_{i'}}$.

Let $\widehat{P} = \{p_k : \{p_k\} \in P\}$.

Claim 3. $(\min G_{F_i})_{i=1}^s \cup \widehat{P} \in \mathcal{S}_n$.

The proof of this claim requires several short lemmas.

Lemma 10. *For any* $E \in \mathcal{T}'$, $\min G_E \geq 2 \min E$.

Proof. Suppose that $p_j \in G_E$. Then $E_j \subsetneq E$. Since E_j has a sibling that begins before x_j, E begins before x_j. This implies that

$$2 \min E \leq 2q_{j-1} \leq p_j \quad \text{by } (\gamma). \ \blacksquare$$

Lemma 11. $\widehat{P}$ *is a spreading of a subset of* $(\min F_i)_{i=1}^s$ *and* $p_k \geq 2 \min F_1$ *for all* $p_k \in \widehat{P}$.

Proof. We may assume that $\min F_1 < \cdots < \min F_s$. For each k, let $H_k = \{\min F_i : \min F_i \in \operatorname{supp} x_k\}$. List the k's such that $H_k \neq \emptyset$ in increasing order as $k_1 < \cdots < k_r$. Since every F_i begins at or after x_{k_1}, $E_{k_1} \neq F_i$ for any i. Therefore, $\widehat{P} \subseteq (p_{k_l})_{l=2}^r$. For each $2 \leq l \leq r$, choose i_{l-1} such that $\min F_{i_{l-1}} \in H_{k_{l-1}}$. Then $(p_{k_l})_{l=2}^r$ is a spreading of $(\min F_{i_{l-1}})_{l=2}^r$. Also note that $p_k \geq p_{k_2} \geq 2q_{k_1} \geq 2 \min F_1$ for all $p_k \in \widehat{P}$. $\blacksquare$

It follows from Lemmas 10 and 11 that $(\min G_{F_i})_{i=1}^s \cup \widehat{P}$ can be written as $\bigcup_{j\in B} A_j$, where $B = \{2 \min F_1\} \cup (\min F_i)_{i=2}^s$, $\min A_j \geq j$, and $|A_j| \leq 2$ for all $j \in B$.

Lemma 12. *Suppose that* $n \in \mathbb{N}$, $L \in \mathcal{S}_n$ *and* B *is a spreading of* L *such that* $\min B \geq 2 \min L$. *If* $|A_j| \leq 2$ *and* $\min A_j \geq j$ *for all* $j \in B$, *then* $\bigcup_{j\in B} A_j \in \mathcal{S}_n$.

Proof. It is easy to see that we may assume $A_j < A_{j'}$ if $j < j'$. Write $L = \bigcup_{k=1}^p L_k$, where $L_1 < \cdots < L_p$ are in $\mathcal{S}_{n-1}$ and $p \leq \min L_1$. Then $B = \bigcup_{k=1}^p B_k$, where each B_k is a spreading of L_k and $B_1 < \cdots < B_p$. Denoting by $\mathcal{A}_2$ the collection of subsets of $\mathbb{N}$ having at most two elements, we appeal to [17, Remark on p. 312] to deduce that

$$\bigcup_{j\in B_k} A_j \in \mathcal{S}_{n-1}[\mathcal{A}_2] \subseteq (\mathcal{S}_{n-1})^2.$$

Hence $\bigcup_{j\in B} A_j = \bigcup_{k=1}^{2p} C_i$, where $C_1 < \cdots < C_{2p}$ are in $\mathcal{S}_{n-1}$. Since $2p \leq 2\min L_1 \leq \min B \leq \min C_1$, the conclusion of the lemma follows. ∎

Completion of proof of Proposition 9. It follows from the claims and lemmas above that the nodes of $\mathcal{T}''$ form an $(S_n, \sigma_n)_{n=1}^{\infty}$-adapted tree, where the tag of any node in $\mathcal{T}''$ is the same as the tag of the node in $\mathcal{T}'$ for which it is a substitute. Moreover, it follows from Claim 1 that all nodes in P are terminal. Therefore,

$$\mathcal{T}''y \geq \sum_{\{p_k\}\in P} t(\{p_k\})b_k = \sum_{k=2}^{N} t(E_k)b_k. \ ∎$$

Recall that a Banach space Z is said to be *minimal* if every infinite-dimensional subspace of Z has a further subspace isomorphic to Z. This definition is due to Rosenthal. In [15], Gowers introduced the more general notion of quasi-minimal spaces. Two Banach spaces are said to be *totally incomparable* if they do not have isomorphic infinite-dimensional subspaces. A Banach space is said to be *quasi-minimal* if it does not contain a pair of totally incomparable infinite-dimensional closed subspaces. Using Theorem 1, Corollary 2 and Proposition 14 below, we obtain

COROLLARY 13. *Let $X = T[(\mathcal{S}_n, \sigma_n, \theta_n)_{n=1}^{\infty}]$ be a partly modified mixed Tsirelson space so that $I(Y) > \omega^{\omega}$ for every block subspace Y of X. Then X is quasi-minimal. This holds if there exists $\varepsilon > 0$ such that* $\sup\{n/m : \theta_n \geq \varepsilon^m\} = \infty$, *and in particular if* $\sup \theta_n^{1/n} = 1$.

PROPOSITION 14. *Let (p_k) and (q_k) be subsequences of $\mathbb{N}$ so that $p_k \leq q_k < 2q_k \leq p_{k+1}$ for all k. Then the sequences (e_{p_k}) and (e_{q_k}) are 2-equivalent in any partly modified mixed Tsirelson space $X = T[(\mathcal{S}_n, \sigma_n, \theta_n)_{n=1}^{\infty}]$.*

Proof. Define a sequence of norms on X follows. Let $\|x\|_0 = \|x\|_{c_0}$ and

$$\|x\|_{i+1} = \max\Big\{\|x\|_0, \sup_n \sup \theta_n \sum_m \|E_m x\|_i\Big\},$$

where the final supremum is taken over all $(\mathcal{S}_n, \sigma_n)$-adapted families (E_m). It is clear that $\|x\| = \lim \|x\|_i$ for all $x \in X$. For any finite subset E of (q_k), let the *shift* of E be the set $s(E) = \{p_k : q_k \in E\}$. We claim that for any i, any $(a_k) \in c_{00}$ and any $E \subseteq (q_k)$, there exist $p_j \in s(E)$ and $F \subseteq s(E)$ such that $p_j < F$ and

$$\Big\|E\sum a_k e_{q_k}\Big\|_i \leq |a_j| + \Big\|F\sum a_k e_{p_k}\Big\|_i. \tag{4}$$

Once the claim is proved, it follows easily that $\|\sum a_k e_{q_k}\| \le 2\|\sum a_k e_{p_k}\|$. Since each $\mathcal{S}_n$ is spreading, we clearly have $\|\sum a_k e_{p_k}\| \le \|\sum a_k e_{q_k}\|$, and the proof of the proposition would be complete. We now prove the claim (4) by induction on i.

The case $i = 0$ is trivial. Suppose that the claim holds for some i. We may assume that

$$\Big\|E\sum a_k e_{q_k}\Big\|_{i+1} = \theta_n \sum_{m=1}^{d} \Big\|E_m \sum a_k e_{q_k}\Big\|_i,$$

where $(E_m)_{m=1}^d$ is an $(\mathcal{S}_n, \sigma_n)$-adapted family of subsets of E, arranged so that $(\min E_m)_{m=1}^d$ is an increasing sequence. By induction, for each m, there are $p_{j_m} \in s(E_m)$ and $F_m \subseteq s(E_m)$ such that $p_{j_m} < F_m$ and

$$\Big\|E_m \sum a_k e_{q_k}\Big\|_i \le |a_{j_m}| + \Big\|F_m \sum a_k e_{p_k}\Big\|_i.$$

Observe that for every m, $2\min E_m \le 2q_{j_m} < p_{j_m+1} \le \min F_m$. Also, for $m \ge 2$, $2\min E_{m-1} \le \min s(E_m) \le p_{j_m}$. Let m_0 be such that $p_{j_{m_0}}$ is the minimum of the sequence $(p_{j_m})_{m=1}^d$. Then $(p_{j_m})_{m\ne m_0} \cup (\min F_m)_{m=1}^d$ may be written as $\bigcup_{j\in B} A_j$, where B is a spreading of $(\min E_m)_{m=1}^d$ such that $\min B \ge 2\min E_1$, $|A_j| \le 2$ and $A_j \ge j$ for all $j \in B$. By Lemma 12, $(p_{j_m})_{m\ne m_0} \cup (\min F_m)_{m=1}^d \in \mathcal{S}_n$. Clearly, $\{\{p_{j_m}\} : m \ne m_0\} \cup \{F_m : 1 \le m \le d\}$ is a pairwise disjoint family that is successive if $(E_m)_{m=1}^d$ is. Thus, this family is $(\mathcal{S}_n, \sigma_n)$-adapted. We may then conclude that

$$\begin{aligned}\Big\|E\sum a_k e_{q_k}\Big\|_{i+1} &= \theta_n \sum_{m=1}^{d} \Big\|E_m \sum a_k e_{q_k}\Big\|_i \\ &\le \theta_n |a_{j_{m_0}}| + \theta_n \Big(\sum_{m\ne m_0} |a_{j_m}| + \sum_{m=1}^{d} \Big\|F_m \sum a_k e_{p_k}\Big\|_i \Big) \\ &\le |a_{j_{m_0}}| + \Big\|F \sum a_k e_{p_k}\Big\|_{i+1},\end{aligned}$$

where $F = \{p_{j_m} : m \ne m_0\} \cup \bigcup_{m=1}^d F_m \subseteq s(E)$ and $F > p_{j_{m_0}} \in s(E)$. ∎

If $X = T[(\mathcal{S}_n, \sigma_n, \theta_n)_{n=1}^\infty]$ is a partly modified mixed Tsirelson space where $\sigma_{p_0} = M$, then it is clear that every disjointly supported sequence $(x_k)_{k=1}^n$ in $[(e_k)_{k=n}^\infty]$ is $\theta_{p_0}^{-1}$-equivalent to the unit vector basis of $\ell^1(n)$. Such spaces are called *strongly asymptotic ℓ^1 spaces*. In [10], it was proved that every minimal, strongly asymptotic Banach space with a basis is isomorphic to a subspace of ℓ^1. Since partly modified spaces are reflexive (this may be proved using the arguments of [3]; alternatively, it follows from the computation of the ℓ^1-index below (Theorem 17)), we deduce that no partly modified mixed Tsirelson space contains a minimal subspace. Hence the class of partly modified mixed Tsirelson spaces X such that $I(Y) > \omega^\omega$ for every subspace Y of X provides examples of strictly quasi-minimal Banach spaces in the sense of Gowers [15]. By [9], Tsirelson's space T does not contain a minimal subspace. Also, T has the *block property* (see [8]): every normalized block basis is equivalent to a subsequence of the unit vector basis. It follows easily that T is strictly quasi-minimal. However, among the strictly quasi-minimal partly modified mixed Tsirelson spaces are spaces that fail the block property. Indeed, it can be deduced from the arguments in §2 of [3] that the space $X_{M(1),u}$ constructed there is one such example. We may also obtain further examples using the arguments in the present paper. Recall that a *regular sequence* (θ_n) is a nonincreasing null sequence in $(0,1)$ so that $\theta_{m+n} \geq \theta_m\theta_n$ for all m, n. By [21, Lemma 4.13], $\theta = \lim_n \theta_n^{1/n}$ exists and is equal to $\sup \theta_n^{1/n}$.

Proposition 15. *Let (θ_n) be a regular sequence so that $\sup \theta_n^{1/n} = 1$. Then $X = \mathcal{T}[(\mathcal{S}_n, \sigma_n, \theta_n)_{n=1}^\infty]$ is saturated with block subspaces that fail to have the block property.*

Proof. If $\lim \theta_n^{1/n} = 1$, then the hypothesis of Theorem 23 below is fulfilled. In fact, assume that there exists m such that

$$\limsup_n \inf_{n_1+\cdots+n_s \geq n} \frac{\theta_{m+n}}{\theta_{n_1}\cdots\theta_{n_s}} < \frac{1}{2}.$$

Since (θ_n) is regular, $\theta_{n_1}\cdots\theta_{n_s} \leq \theta_n$ if $n_1 + \cdots + n_s \geq n$. So we have

$$\limsup_n \frac{\theta_{m+n}}{\theta_n} < \frac{1}{2}.$$

Pick n_0 so that $\theta_{m+n}/\theta_n < 1/2$ for $n \geq n_0$. For all $k \in \mathbb{N}$,

$$\theta_{km+n_0} \leq \frac{1}{2}\,\theta_{(k-1)m+n_0} \leq \cdots \leq \left(\frac{1}{2}\right)^k \theta_{m+n_0}.$$

Thus

$$\limsup_k \theta_{km+n_0}^{1/km+n_0} \leq \frac{1}{2^{1/m}},$$

a contradiction. By Theorem 23, $X = T[(\mathcal{S}_n, \sigma_n, \theta_n)_{n=1}^\infty]$ contains ℓ^1-$\mathcal{S}_m$-spreading models with uniform constant, say K.

Claim. *For any $\varepsilon > 0$, there exists n_0 so that*

$$\|x\| \leq \varepsilon\|x\|_{\ell^1} + \|x\|_{\mathcal{S}_{n_0}}$$

for all $x \in X$, where $\|x\|_{\mathcal{S}_{n_0}} = \sup_{E\in\mathcal{S}_{n_0}} \|Ex\|_{\ell^1}$.

Let $\mathcal{T}$ be an adapted tree so that $\|x\| = \sum t(E)\|Ex\|_{c_0}$, where the sum is taken over all leaves of $\mathcal{T}$. It is clear that there exists n_0 so that any leaf

$E \in \mathcal{T}$ with $t(E) > \varepsilon$ satisfies $\operatorname{ord}(E) \le n_0$. By Lemma 6, the set of all leaves E with $t(E) > \varepsilon$ is $\mathcal{S}_{n_0}$-allowable. Thus

$$\sum\nolimits' t(E)\|Ex\|_{c_0} \le \|x\|_{\mathcal{S}_{n_0}},$$

where $\sum'$ is taken over all leaves in $\mathcal{T}$ with $t(E) > \varepsilon$. Let $\sum''$ be the sum over the remaining leaves. Then

$$\|x\| \le \sum\nolimits'' t(E)\|Ex\|_{c_0} + \sum\nolimits' t(E)\|Ex\|_{c_0} \le \varepsilon\|x\|_{\ell^1} + \|x\|_{\mathcal{S}_{n_0}},$$

as desired.

Next, we show that there is a normalized block basis (y_n) in X and a subsequence (e_{p_n}) of the unit vector basis of X with $y_n < e_{p_n} < y_{n+1}$ for all n so that (y_n) is not equivalent to (e_{p_n}). Indeed, if this were not true, there would be a uniform constant M such that any two such sequences are M-equivalent. Let K be the constant chosen before the Claim and let n_0 be obtained from the Claim corresponding to $\varepsilon = 1/2KM$. Take any $m > n_0$. There is a normalized block basis (y_n) in X that is an ℓ^1-$\mathcal{S}_m$-spreading model with constant K and a subsequence (e_{p_n}) of the unit vector basis of X such that $y_n < e_{p_n} < y_{n+1}$ for all n. For any finite sequence (a_n) supported on a set in $\mathcal{S}_m$,

$$\begin{aligned}\sum |a_n| \le K\Big\|\sum a_n y_n\Big\| &\le KM\Big\|\sum a_n e_{p_n}\Big\| \\ &\le \frac{1}{2}\sum |a_n| + KM\Big\|\sum a_n e_{p_n}\Big\|_{\mathcal{S}_{n_0}}.\end{aligned}$$

Hence $\|(a_n)\|_{\mathcal{S}_m} \le 2KM\|\sum a_n e_{p_n}\|_{\mathcal{S}_{n_0}}$ for any sequence (a_n). It is easy to see that this does not hold since $m > n_0$.

If X has the block property, then (y_n) chosen above is equivalent to some subsequence (e_{p_n}) of the unit vector basis. From Proposition 14, we obtain a subsequence (y_{n_k}) and $r_1 < r_2 < \cdots$ such that $y_{n_k} < e_{r_k} < y_{n_{k+1}}$. Since (y_{n_k}) is still an ℓ^1-$\mathcal{S}_m$-spreading model with constant K, the same contradiction ensues.

The argument above may be carried out in any subspace $[(e_{q_n})]$ of X, with (e_{q_n}) a subsequence of the unit vector basis. Since X is subsequentially minimal by Corollary 2, it follows that X is saturated with block subspaces that fail to have the block property and also saturated with normalized block basic sequences (z_n) such that (z_{2n-1}) is not equivalent to (z_{2n}). ■

3. The Bourgain ℓ^1-index. In this section, we develop the techniques of §2 further to investigate the Bourgain ℓ^1-index of partly modified mixed Tsirelson spaces. In the first part of the section, we show that $I(X)$ does not exceed $\omega^{\omega\cdot 2}$. In the second part, we pinpoint the value of $I(X)$ in certain cases in terms of the sequence of coefficients (θ_n).

In the following proposition, we will require the concepts of block subtrees, minimal trees $\mathcal{T}_\alpha$ and replacement trees $\mathcal{T}(\alpha,\beta)$ defined, constructed and developed in [16]. We refer the reader to that paper for details. The proof below is comparable to that of [16, Lemma 4.2]. When two trees $\mathcal{T}$ and $\mathcal{T}'$ are isomorphic, we write $\mathcal{T} \simeq \mathcal{T}'$. Given two finite sequences $\vec{x} = (x_1,\dots,x_m)$ and $\vec{y} = (y_1,\dots,y_n)$, let $\vec{x} \sqcup \vec{y} = (x_1,\dots,x_m,y_1,\dots,y_n)$. We say that a normalized vector x has *property* $(*)$ for the couple $(n,C) \in \mathbb{N}\times\mathbb{R}^+$ if $\sum \|E_i x\| \le C$ whenever (E_i) is $\mathcal{S}_n$-allowable.

PROPOSITION 16. *If $\mathcal{T}$ is an ℓ^1-K-block tree of order $\mathrm{o}(\mathcal{T}) \ge \omega^\omega\cdot\alpha$, then for any $n_0 \in \mathbb{N}$ and any positive sequence (ε_i), there exists a block subtree $\mathcal{T}'$ of $\mathcal{T}$, isomorphic to $\mathcal{T}_\alpha$, such that every node $(x_1,\dots,x_d) \in \mathcal{T}'$ satisfies*:

(1) *there exist $n_1 < \dots < n_{d-1}$, with $n_1 > n_0$, such that each x_i has property $(*)$ for the couple (n_{i-1},C), where $C = (1+\theta_{p_0}^{-1})K$,*
(2) *$\xi_{n_i}\|x_i\|_{\ell^1} \le \varepsilon_i$ for $1 \le i < d$,*
(3) *$2\max\operatorname{supp} x_i \le \min\operatorname{supp} x_{i+1}$ if $1 \le i < d$.*

Proof. The proof is by induction on α. The case $\alpha = 1$ follows from Lemma 5. Suppose that $\mathcal{T}$ is an ℓ^1-K-block tree of order $\mathrm{o}(\mathcal{T}) \ge \omega^\omega\cdot(\alpha+1)$. According to [16, Lemma 3.7], upon replacing $\mathcal{T}$ by a subtree if necessary, we may assume that $\mathcal{T}$ is isomorphic to the "replacement tree" $\mathcal{T}\,(\alpha+1,\omega^\omega)$. From the definition of $\mathcal{T}(\alpha+1,\omega^\omega)$, we see that $(\mathcal{T}(\alpha+1,\omega^\omega))^{(\omega^\omega\cdot\alpha)}$ is the minimal tree $\mathcal{T}_{\omega^\omega}$. Applying the case $\alpha = 1$ to $\mathcal{T}^{(\omega^\omega\cdot\alpha)} \simeq \mathcal{T}_{\omega^\omega}$, we obtain a normalized block y of a node $\vec{x} = (x_1,\dots,x_m)$ in $\mathcal{T}^{(\omega^\omega\cdot\alpha)}$ such that y has property $(*)$ for the couple (n_0,C). Choose $n_1 > n_0$ such that $\xi_{n_1}\|y\|_{\ell^1} \le \varepsilon_1$. Without loss of generality, we may assume that $\vec{x}$ is a terminal node in $\mathcal{T}^{(\omega^\omega\cdot\alpha)}$. By the construction of $\mathcal{T}(\alpha+1,\omega^\omega)$, the subtree $\mathcal{T}_{\vec{x}}$ of $\mathcal{T}$ consisting of all nodes $\vec{z} > \vec{x}$ is isomorphic to $\mathcal{T}(\alpha,\omega^\omega)$ and hence has order $\omega^\omega\cdot\alpha$. Consider the "restricted subtree" $\mathcal{R}(\mathcal{T}_{\vec{x}})$ [16, Definition 4.1] consisting of all $(w_j,\dots,w_k)$ where $\vec{x}\sqcup(w_1,\dots,w_k) \in \mathcal{T}_{\vec{x}}$ and j is the smallest integer such that $\min\operatorname{supp} w_j \ge 2\max\operatorname{supp} x_m$. Then $\mathcal{R}(\mathcal{T}_{\vec{x}})$ is an ℓ^1-K-block tree of order $\omega^\omega\cdot\alpha$. Apply the inductive hypothesis to $\mathcal{R}(\mathcal{T}_{\vec{x}})$ with the parameters n_1 and (ε_{i+1}) to obtain a block subtree $\mathcal{T}''$ of $\mathcal{R}(\mathcal{T}_{\vec{x}})$. Define $\mathcal{T}' = \{(y)\sqcup\vec{w} : \vec{w} \in \mathcal{T}''\}$. It is easy to check that $\mathcal{T}'$ satisfies the desired conclusion (for the ordinal $\alpha+1$).

Suppose that $\mathcal{T}$ is an ℓ^1-K-block tree of order $\mathrm{o}(\mathcal{T}) \ge \omega^\omega\cdot\alpha$, where α is a limit ordinal. Let (α_n) be a sequence of ordinals strictly increasing to α. Then $\mathcal{T}$ contains pairwise disjoint subtrees $\mathcal{T}_n$ with $\mathrm{o}(\mathcal{T}_n) \ge \omega^\omega\cdot\alpha_n$ for all n. For each n, apply the inductive hypothesis to obtain a block subtree $\mathcal{T}_n'$ of $\mathcal{T}_n$. The block subtree $\mathcal{T}' = \bigcup \mathcal{T}_n'$ of $\mathcal{T}$ satisfies the conclusion of the proposition. ∎

If (ε_i) is chosen to be $(1/2^i)$, then from the remark following the proof of Theorem 1, we see that every node $(x_1, \dots, x_d) \in \mathcal{T}'$ is $(2+C)$-equivalent to (e_{p_i}), where $p_i = \min \operatorname{supp} x_i$. For $y \in c_{00}$, let $\|y\|_{\mathcal{S}_p} = \sup_{E \in \mathcal{S}_p} \|Ey\|_{\ell^1}$.

THEOREM 17. *The Bourgain ℓ^1-index of $X = T[(\mathcal{S}_n, \sigma_n, \theta_n)_{n=1}^\infty]$ is $I(X) \leq \omega^{\omega \cdot 2}$.*

Proof. If $I(X) > \omega^{\omega \cdot 2}$, then by [16, Corollary 5.13], there exists an ℓ^1-K-block tree $\mathcal{T}$ with $\mathrm{o}(\mathcal{T}) \geq \omega^{\omega \cdot 2}$ for some $K > 0$. Let n be chosen so that $\xi_n < 1/2K(2+C)$. By Proposition 16, we obtain an ℓ^1-K-block tree $\mathcal{T}'$ of $\mathcal{T}$ with $\mathrm{o}(\mathcal{T}') = \omega^{n+1}$ such that every node $(x_1, \dots, x_d)$ in $\mathcal{T}'$ is $(2+C)$-equivalent to (e_{p_i}). Define

$$\mathcal{H} = \{(p_j)_{j=n}^r : (x_j)_{j=n}^r \in \mathcal{T}'\},$$
$$\mathcal{G} = \{G : G \text{ is a spreading of a subset of some } H \in \mathcal{H}\}.$$

By Lemma 3, $\mathcal{G}$ is hereditary and spreading, and either $\mathcal{G}$ is noncompact, or it is compact with $\iota(\mathcal{G}) \geq \mathrm{o}(\mathcal{T}') \geq \omega^{n+1} > \omega^n$. By [14, Theorem 1.1], there exists $M \in [\mathbb{N}]$ such that $\mathcal{S}_n \cap [M]^{<\infty} \subseteq \mathcal{G}$. As in the proof of Lemma 4, we obtain a node $(x_j)_{j=n}^r \in \mathcal{T}'$, $J \subseteq [n, r]$, an order preserving map u from J onto a spreading of $(p_j)_{j \in J}$ and a sequence $(a_{u(j)})_{j \in J}$ of positive numbers such that $\sum_{j \in J} a_{u(j)} = 1$ and $\sum_{j \in A} a_{u(j)} < \xi_n$ whenever $\{u(j) : j \in A\} \in \mathcal{S}_{n-1}$. Let $y = \sum_{j \in J} a_{u(j)} x_j$. Since $(x_j)_j$ is a normalized ℓ^1-K-block sequence, $\|y\| \geq 1/K$. On the other hand,

$$\begin{aligned} \|y\| &= \Big\| \sum_{j \in J} a_{u(j)} x_j \Big\| \leq (2+C) \Big\| \sum_{j \in J} a_{u(j)} e_{p_i} \Big\| \\ &\leq (2+C) \Big(\Big\| \sum_{j \in J} a_{u(j)} e_{p_j} \Big\|_{\mathcal{S}_{n-1}} + \xi_n \|(a_{u(j)})\|_{\ell^1} \Big) \leq 2(2+C)\xi_n, \end{aligned}$$

contradicting the choice of n. ∎

In the second half of the section, we obtain an estimate on the norms of vectors spanned by normalized block sequences in X (Proposition 21), from which the value of the Bourgain ℓ^1-index $I(X)$ may be deduced. *For the remainder of the section, assume that (x_k) is a normalized block sequence in $X = T[\mathcal{S}_n, \sigma_n, \theta_n)_{n=1}^\infty]$, $(a_k) \in c_{00}$ and $q_k = \max \operatorname{supp} x_k$.* Set $x = \sum a_k x_k$. Recall the assumption that $\sigma_{p_0} = M$ for some p_0. Given a node E in an adapted tree $\mathcal{T}$, we say that it is a *long node* (with respect to x) if $E \cap \operatorname{supp} x_k \neq \emptyset$ for more than one k. Otherwise, we term the node *short*.

LEMMA 18. *For any N, there exists an adapted tree $\mathcal{T}$ such that all long nodes $E \in \mathcal{T}$ satisfy $t(E) > \theta_N$ and*

$$\|x\| \leq \mathcal{T}x + \frac{\theta_N}{\theta_{p_0}} \|(a_k)\|_{\ell^1}.$$

Proof. Choose an adapted tree $\mathcal{T}'$ such that $\|x\| = \mathcal{T}'x$. Let $\mathcal{E}$ be the collection of minimal elements in the set of long nodes E with $t(E) \le \theta_N$. For each $E \in \mathcal{E}$, let k_E be the smallest k such that $\operatorname{supp} x_k \cap E \neq \emptyset$, and let $F_E = \operatorname{supp} x_{k_E} \cap E$. For each k, the nonempty sets in the collection $\{(E \setminus F_E) \cap \operatorname{supp} x_k\}$ are $\mathcal{S}_1$-allowable and hence $\mathcal{S}_{p_0}$-allowable. Thus,

$$\sum_{E\in\mathcal{E}} t(E)\|(E \setminus F_E)x_k\| \le \theta_N \sum_{E\in\mathcal{E}} \|(E \setminus F_E)x_k\| \le \frac{\theta_N}{\theta_{p_0}}.$$

Then

$$\sum_{E\in\mathcal{E}} t(E)\|(E \setminus F_E)x\| \le \frac{\theta_N}{\theta_{p_0}}\,\|(a_k)\|_{\ell^1}.$$

Let $\mathcal{T}$ be the tree obtained from $\mathcal{T}'$ by changing all nodes $G \in \mathcal{T}'$, $G \subseteq E$ for some $E \in \mathcal{E}$ to $G \cap F_E$. Then $\mathcal{T}$ is an adapted tree such that every long node H in $\mathcal{T}$ satisfies $t(H) > \theta_N$. Moreover,

$$\|x\| = \mathcal{T}'x \le \mathcal{T}x + \sum_{E\in\mathcal{E}} t(E)\|(E \setminus F_E)x\| \le \mathcal{T}x + \frac{\theta_N}{\theta_{p_0}}\,\|(a_k)\|_{\ell^1}. \quad \blacksquare$$

Fix N and let $\mathcal{T}$ be the tree given by Lemma 18. For any $\varepsilon > 0$, let $k(\varepsilon) = \max\{n_1 + \cdots + n_j : \theta_{n_1}\cdots\theta_{n_j} > \varepsilon\}$. Let $\mathcal{E}$ denote the set of all minimal short nodes in $\mathcal{T}$.

LEMMA 19. *If* $\mathcal{E}_1 = \{E \in \mathcal{E} : E$ *has a long sibling*$\}$, *then*

$$\sum_{E\in\mathcal{E}_1} t(E)\|Ex\| \le \Big\|\sum a_k e_{q_k}\Big\|_{\mathcal{S}_{k(\theta_N)}}.$$

Proof. If $E \in \mathcal{E}_1$, then $t(E) > \theta_N$ and hence $\operatorname{ord}(E) \le k(\theta_N)$. Hence by Lemma 6, $\mathcal{E}_1$ is $\mathcal{S}_{k(\theta_N)}$-allowable. Since each $E \in \mathcal{E}_1$ is a short node, it follows that the set $Q_0 = \{q_k : \operatorname{supp} x_k \cap E \neq \emptyset$ for some $E \in \mathcal{E}_1\} \in \mathcal{S}_{k(\theta_N)}$. Thus

$$\sum_{E\in\mathcal{E}_1} t(E)\|Ex\| \le \sum_{q_k\in Q_0} |a_k| \le \Big\|\sum a_k e_{q_k}\Big\|_{\mathcal{S}_{k(\theta_N)}}. \quad \blacksquare$$

For $m, n \in \mathbb{N}$, define

$$\eta_{m,n} = \inf \frac{\theta_{m+n}}{\theta_{n_1}\cdots\theta_{n_s}},$$

where the infimum is taken over all $n_1, \ldots, n_s$ such that $n_1 + \cdots + n_s \ge n$, with the additional requirement that $\sigma_{n_1} = \cdots = \sigma_{n_s} = M$ if $\sigma_{m+n} = M$. Obviously, $\eta_{m,n}$ majorizes the quantity

$$\inf_{n_1+\cdots+n_s\ge n} \frac{\theta_{m+n}}{\theta_{n_1}\cdots\theta_{n_s}},$$

which occurs in Theorem 23 below.

LEMMA 20. *Suppose that*
$$\inf_m \limsup_n \eta_{m,n} = 0.$$
For any $\varepsilon > 0$, *there exist* m *and* n_0 *such that*
$$\sum_{E\in\mathcal{E}\setminus\mathcal{E}_1} t(E)\|Ex\| \le \varepsilon\|(a_k)\|_{\ell^1} + 2\Big\|\sum a_k e_{q_k}\Big\|_{\mathcal{S}_{k(\theta_N)+n_0+m}}.$$

Proof. Choose m and n_0 such that $\eta_{m,n} < \varepsilon$ if $n \ge n_0$. Let $\mathcal{D} = (D_i)$ be the set of all parents of nodes in $\mathcal{E}\setminus\mathcal{E}_1$. In particular, each D_i is a long node and hence $t(D_i) > \theta_N$. It follows that $\operatorname{ord}(D_i) \le k(\theta_N)$. Also, the nodes in $\mathcal{D}$ are pairwise disjoint since no $E \in \mathcal{E}\setminus\mathcal{E}_1$ can have a long sibling. For each i, there exists some n_i such that $\mathcal{F}_i = \{E \in \mathcal{E}\setminus\mathcal{E}_1 : E \subseteq D_i\}$ is $(\mathcal{S}_{n_i}, \sigma_{n_i})$-adapted. Let $I = \{i : n_i \le n_0+m\}$. Then $\operatorname{ord}(E) = \operatorname{ord}(D_i)+n_i \le k(\theta_N)+n_0+m$ for all $E \in \bigcup_{i\in I}\mathcal{F}_i$. By Lemma 6, $\bigcup_{i\in I}\mathcal{F}_i$ is an $\mathcal{S}_{k(\theta_N)+n_0+m}$-allowable collection of short nodes. It follows that
$$Q_0 = \Big\{q_k : \operatorname{supp} x_k \cap E \ne \emptyset \text{ for some } E \in \bigcup_{i\in I}\mathcal{F}_i\Big\} \in \mathcal{S}_{k(\theta_N)+n_0+m}.$$
Therefore,
$$\sum_{E\in\bigcup_{i\in I}\mathcal{F}_i} t(E)\|Ex\| \le \sum_{q_k\in Q_0} |a_k| \le \Big\|\sum a_k e_{q_k}\Big\|_{\mathcal{S}_{k(\theta_N)+n_0+m}}. \tag{5}$$
Now consider those $i \notin I$. Let $\mathcal{F}_{ik} = \{E \in \mathcal{F}_i : E \subseteq \operatorname{supp} x_k\}$. For each k, let
$$I_k = \{i \notin I : \{\min E : E \in \mathcal{F}_{ik}\} \in \mathcal{S}_{n_i-m}\},$$
$$I'_k = \{i \notin I : \{\min E : E \in \mathcal{F}_{ik}\} \notin \mathcal{S}_{n_i-m}\}.$$
Suppose that $i \in I_k$. Choose $n_1^{(i)}, \dots, n_s^{(i)}$ such that $n_1^{(i)}+\dots+n_s^{(i)} \ge n_i - m$,
$$\frac{\theta_{m+n_i-m}}{\theta_{n_1^{(i)}}\cdots\theta_{n_s^{(i)}}} < \varepsilon$$
and $\sigma_{n_1^{(i)}} = \dots = \sigma_{n_s^{(i)}} = M$ if $\sigma_{n_i} = M$. This is possible since $i \notin I$ implies that $n_i - m \ge n_0$ and hence $\eta_{m,n_i-m} < \varepsilon$.

If $\sigma_{n_i} = U$, then the sets in $\mathcal{F}_i$ and hence $\mathcal{F}_{ik}$ are successive. Since $\{\min E : E \in \mathcal{F}_{ik}\} \in \mathcal{S}_{n_i-m}$, $\mathcal{F}_{ik}$ is $\mathcal{S}_{n_i-m}$-admissible and hence $\mathcal{S}_{n_1^{(i)}+\dots+n_s^{(i)}}$-admissible. Then
$$\sum_{E\in\mathcal{F}_{ik}} \theta_{n_1^{(i)}}\cdots\theta_{n_s^{(i)}}\|Ex_k\| = \sum_{E\in\mathcal{F}_{ik}} \theta_{n_1^{(i)}}\cdots\theta_{n_s^{(i)}}\|ED_ix_k\| \le \|D_ix_k\|. \tag{6}$$
If $\sigma_{n_i} = M$, then $\mathcal{F}_{ik}$ is $\mathcal{S}_{n_i-m}$-allowable and hence $\mathcal{S}_{n_1^{(i)}+\dots+n_s^{(i)}}$-allowable. Since $\sigma_{n_1^{(i)}} = \dots = \sigma_{n_s^{(i)}} = M$, we obtain the same inequality as in (6).

From inequality (6),

$$\begin{aligned}\sum_{i\in I_k}\sum_{E\in\mathcal{F}_{ik}} t(E)\|Ex_k\| &= \sum_{i\in I_k} t(D_i)\theta_{n_i}\sum_{E\in\mathcal{F}_{ik}}\|Ex_k\| \\ &\le \varepsilon\sum_{i\in I_k} t(D_i)\theta_{n_1^{(i)}}\cdots\theta_{n_s^{(i)}}\sum_{E\in\mathcal{F}_{ik}}\|Ex_k\| \\ &\le \varepsilon\sum_{i\in I_k} t(D_i)\|D_i x_k\| \le \varepsilon.\end{aligned}$$

Therefore,

$$\sum_{\{(i,k)\,:\,i\in I_k\}}\sum_{E\in\mathcal{F}_{ik}} t(E)\|Ex\| \le \varepsilon||(a_k)||_{\ell^1}. \tag{7}$$

For each $i \notin I$, set $J_i = \{k : i \in I'_k\}$. Then $\{\min E : E \in \mathcal{F}_{ik}\} \notin \mathcal{S}_{n_i - m}$ for each $k \in J_i$ but $\bigcup_k\{\min E : E \in \mathcal{F}_{ik}\} = \{\min E : E \in \mathcal{F}_i\} \in \mathcal{S}_{n_i}$. By [17, Lemma 2], $(\min\bigcup_{E\in\mathcal{F}_{ik}} E)_{k\in J_i} \in \mathcal{S}_m$. Now $\mathrm{ord}(D_i) \le k(\theta_N)$ for all i and $\mathcal{D}$ consists of pairwise disjoint sets. Thus by Lemma 6, $\mathcal{D}$ is $\mathcal{S}_{k(\theta_N)}$-allowable. Therefore, $\{q_k : k \in \bigcup_{i\notin I} J_i\} \in \mathcal{S}_{k(\theta_N)+m}$. It follows that

$$\sum_{\{(i,k)\,:\,i\in I'_k\}}\sum_{E\in\mathcal{F}_{ik}} t(E)\|Ex\| \le \sum_{k\in\bigcup_{i\notin I} J_i} |a_k| \le \Big\|\sum a_k e_{q_k}\Big\|_{\mathcal{S}_{k(\theta_N)+m}}. \tag{8}$$

Combining (5), (7) and (8) yields

$$\sum_{E\in\mathcal{E}\setminus\mathcal{E}_1} t(E)\|Ex\| = \varepsilon\|(a_k)\|_{\ell^1} + 2\Big\|\sum a_k e_{q_k}\Big\|_{\mathcal{S}_{k(\theta_N)+n_0+m}}. \ \blacksquare$$

From Lemmas 18–20 we have

Proposition 21. *Suppose that* $\inf_m \limsup_n \eta_{m,n} = 0$. *Then given any* $\varepsilon > 0$ *and* N, *there exist* m *and* n_0 *such that*

$$\|x\| \le \left(\varepsilon + \frac{\theta_N}{\theta_{p_0}}\right)\|(a_k)\|_{\ell^1} + 3\Big\|\sum a_k e_{q_k}\Big\|_{\mathcal{S}_{k(\theta_N)+n_0+m}}.$$

Theorem 22. *If* $\inf_m \limsup_n \eta_{m,n} = 0$, *then* $I(Y) = \omega^\omega$ *for all* $Y \prec X$.

Proof. Since Y contains ℓ^1-$\mathcal{S}_n$-spreading models with constant θ_n^{-1} for all n and all $Y \prec X$, it is clear that $I(Y) \ge \omega^\omega$. Suppose that $I(Y) > \omega^\omega$ for some $Y \prec X$. Then $I(X) > \omega^\omega$. There exist $K > 1$ and an ℓ^1-K-block tree $\mathcal{T}$ such that $\mathrm{o}(\mathcal{T}) > \omega^\omega$. Let $\mathcal{H}(\mathcal{T}) = \{(\max\operatorname{supp} x_j)_{j=1}^r : (x_j)_{j=1}^r \in \mathcal{T}\}$ and $\mathcal{G} = \{G : G$ is a spreading of a subset of some $H \in \mathcal{H}\}$. Then $\iota(\mathcal{G}) \ge \mathrm{o}(\mathcal{T}) > \omega^\omega$. Choose ε and N such that $\varepsilon + \theta_N/\theta_{p_0} < 1/2K$ and let $r = k(\theta_N) + n_0 + m$, where n_0, m are such that $\eta_{m,n} < \varepsilon$ if $n \ge n_0$. By [14, Theorem 1.1], there exists $M \in [\mathbb{N}]$ such that $\mathcal{S}_\omega \cap [M]^{<\infty} \subseteq \mathcal{G}$. Hence, it follows from [21, Proposition 3.6] that there exist $G = (t_i) \in \mathcal{G}$ and $(a_i) \in c_{00}^+$ such that $\sum a_i = 1$ and $\|\sum a_i e_{t_i}\|_{\mathcal{S}_r} < 1/6K$.

By definition, there exists a normalized ℓ^1-K-block sequence $(x_i)_{i=1}^k$ in X such that (t_i) is a spreading of $(q_i) = (\max\operatorname{supp} x_i)$. By Proposition 21,

$$\begin{aligned}\frac{1}{K} \le \Big\|\sum a_i x_i\Big\| &\le \frac{1}{2K}\|(a_i)\|_{\ell^1} + 3\Big\|\sum a_i e_{q_i}\Big\|_{\mathcal{S}_r} \\ &\le \frac{1}{2K} + 3\Big\|\sum a_i e_{t_i}\Big\|_{\mathcal{S}_r} < \frac{1}{K},\end{aligned}$$

a contradiction. ∎

THEOREM 23. *If*

$$\inf_m \limsup_{n\to\infty} \inf_{n_1+\cdots+n_s\ge n} \frac{\theta_{m+n}}{\theta_{n_1}\cdots\theta_{n_s}} > 0,$$

then X contains ℓ^1-$\mathcal{S}_m$-spreading models with a uniform constant. In particular, $I(X) = \omega^{\omega\cdot 2}$.

If X satisfies the hypothesis of Theorem 23 and is subsequentially minimal, as happens for instance if (θ_n) is regular and $\sup\theta_n^{1/n} = 1$, then $I(Y) = \omega^{\omega\cdot 2}$ for all infinite-dimensional closed subspaces Y of X. This follows from the fact that the following construction can be carried out on any subsequence of the unit vector basis.

LEMMA 24. *For any $n \in \mathbb{N}$, $\varepsilon > 0$ and $L \in [\mathbb{N}]$, there exists $x \in c_{00}$ such that*

$$\|x\|_{\ell^1} = 1/\theta_n, \quad \operatorname{supp} x \in \mathcal{S}_{N+1} \cap [L]^{<\infty} \quad \textit{and} \quad \|x\|_X \le 1 + 1/\varepsilon,$$

where $N = \max\{n_1 + \cdots + n_s : \varepsilon\theta_{n_1}\cdots\theta_{n_s} > \theta_n\}$. (*We take* $\max\emptyset = 0$.)

Proof. According to [21, Proposition 3.6], there exists $x \in c_{00}$ such that $\|x\|_{\ell^1} = 1/\theta_n$, $\operatorname{supp} x \in \mathcal{S}_{N+1} \cap [L]^{<\infty}$ and $\|x\|_{\mathcal{S}_N} \le 1$. If $\mathcal{T}$ is an adapted tree, then

$$\begin{aligned}\mathcal{T}x &= \sum_{\substack{E\in\mathcal{L}(\mathcal{T})\\ \varepsilon t(E)\le\theta_n}} t(E)\|Ex\|_{c_0} + \sum_{\substack{E\in\mathcal{L}(\mathcal{T})\\ \varepsilon t(E)>\theta_n}} t(E)\|Ex\|_{c_0} \\ &\le \frac{\theta_n}{\varepsilon}\|x\|_{\ell^1} + \sum_{\substack{E\in\mathcal{L}(\mathcal{T})\\ \varepsilon t(E)>\theta_n}} \|Ex\|_{c_0}.\end{aligned}$$

But $\varepsilon t(E) > \theta_N$ implies that $\operatorname{ord}(E) \le N$. It follows from Lemma 6 that $\{E \in \mathcal{L}(\mathcal{T}) : \varepsilon t(E) > \theta_n\}$ is $\mathcal{S}_N$-allowable. Then $\mathcal{T}x \le 1/\varepsilon + \|x\|_{\mathcal{S}_N} \le 1/\varepsilon + 1$. ∎

Proof of Theorem 23. Let $\varepsilon > 0$ be such that

$$\inf_m \limsup_{n\to\infty} \inf_{n_1+\cdots+n_s\ge n-m} \frac{\theta_n}{\theta_{n_1}\cdots\theta_{n_s}} > \varepsilon.$$

Given any m, choose $n > m$ such that

$$\inf_{n_1+\cdots+n_s \geq n-m} \frac{\theta_n}{\theta_{n_1}\cdots\theta_{n_s}} > \varepsilon.$$

Then $N = \max\{n_1+\cdots+n_s : \varepsilon\theta_{n_1}\cdots\theta_{n_s} > \theta_n\} < n-m$. Choose a block sequence (x_k) such that $\|x_k\|_{\ell^1} = 1/\theta_n$, $\operatorname{supp} x_k \in \mathcal{S}_{N+1}$ and $\|x_k\|_X \leq 1+1/\varepsilon$ for all k. Let $p_k = \min\operatorname{supp} x_k$ for all k. If $F \in \mathcal{S}_m$, then $(p_k)_{k\in F} \in \mathcal{S}_m$ and hence $\bigcup_{k\in F}\operatorname{supp} x_k \in \mathcal{S}_{m+N+1} \subseteq \mathcal{S}_n$. Thus for any $(a_k) \in c_{00}$,

$$\Big\|\sum_{k\in F} a_k x_k\Big\| \geq \theta_n \Big\|\sum_{k\in F} a_k x_k\Big\|_{\ell^1} = \sum_{k\in F} |a_k|.$$

This shows that $(x_k/\|x_k\|)$ is an ℓ^1-$\mathcal{S}_m$-spreading model with constant $1+1/\varepsilon$.

Let K be a fixed constant such that for each m, there is a normalized block sequence $(x_i^m)_{i=1}^\infty$ that is an ℓ^1-$\mathcal{S}_m$-spreading model with constant K. If $\mathcal{F}$ is a regular family, consider the tree $\mathcal{T}(\mathcal{F})$ in X consisting of all sequences of the form $(x_i^{m_1})_{i\in I_1} \cup \cdots \cup (x_i^{m_r})_{i\in I_r}$ with $I_k \in \mathcal{S}_{m_k}$, $1 \leq k \leq r$, $i_{k+1} > \max_{i\in I_k}\max\operatorname{supp} x_i^{m_k}$ for all $i_{k+1} \in I_{k+1}$, $1 \leq k < r$, and $(\min I_1,\ldots,\min I_r) \in \mathcal{F}$. If $(x_i^{m_1})_{i\in I_1} \cup \cdots \cup (x_i^{m_r})_{i\in I_r} \in \mathcal{T}(\mathcal{F}^{(1)})$, take $i_0 = \max_{i\in I_r}\max\operatorname{supp} x_i^{m_r}$. There exists j_0 such that $(\min I_1,\ldots,\min I_r, j_0) \in \mathcal{F}$. Then $(x_i^{m_1})_{i\in I_1} \cup \cdots \cup (x_i^{m_r})_{i\in I_r} \cup (x_i^m)_{i\in I} \in \mathcal{T}(\mathcal{F})$ provided $I \in \mathcal{S}_m$ and $I > \max\{i_0, j_0\}$. It follows easily that $\mathcal{T}(\mathcal{F}^{(1)}) \subseteq \mathcal{T}(\mathcal{F})^{(\omega^\omega)}$. Carrying on inductively, one deduces that $\mathrm{o}(\mathcal{T}(\mathcal{S}_n)) \geq \omega^\omega\cdot\omega^n$ for all n. Finally, note that if $(x_i^{m_1})_{i\in I_1} \cup \cdots \cup (x_i^{m_r})_{i\in I_r} \in \mathcal{T}(\mathcal{S}_n)$, then for all scalars (a_i^m),

$$\Big\|\sum_{k=1}^r\sum_{i\in I_k} a_i^{m_k}x_i^{m_k}\Big\| \geq \theta_n\sum_{k=1}^r\Big\|\sum_{i\in I_k} a_i^{m_k}x_i^{m_k}\Big\| \geq \frac{\theta_n}{K}\sum_{k=1}^r\sum_{i\in I_k}|a_i^{m_k}|.$$

Hence $\mathcal{T}(\mathcal{S}_n)$ is an ℓ^1-$K\theta_n^{-1}$-tree in X of order at least $\omega^{\omega+n}$. Thus $I(X) \geq \omega^{\omega\cdot 2}$. The reverse inequality holds by Theorem 17. ∎

Problem. If there are n_1, n_2 so that $\sigma_{n_1} = M$ and $\sigma_{n_2} = U$, then Theorems 22 and 23 leave the value of $I(X)$ undetermined when

$$\inf_m \limsup_{n\to\infty} \eta_{m,n} > 0 \quad \text{and} \quad \inf_m \limsup_{n\to\infty} \inf_{n_1+\cdots+n_s\geq n} \frac{\theta_{m+n}}{\theta_{n_1}\cdots\theta_{n_s}} = 0.$$

The gap is bridged if X is either boundedly modified or (completely) modified.

Corollary 25. *Suppose that there exists N such that $\sigma_n = U$ for all $n > N$, or that $\sigma_n = M$ for all n. Then*

(1) *$I(X) = \omega^\omega$ if*

$$\inf_m \limsup_{n\to\infty} \inf_{n_1+\cdots+n_s\geq n} \frac{\theta_{m+n}}{\theta_{n_1}\cdots\theta_{n_s}} = 0,$$

(2) *$I(X) = \omega^{\omega\cdot 2}$ if*

$$\inf_m \limsup_{n\to\infty} \inf_{n_1+\cdots+n_s\ge n} \frac{\theta_{m+n}}{\theta_{n_1}\cdots\theta_{n_s}} > 0.$$

In this case X has ℓ^1-$\mathcal{S}_m$-spreading models with a uniform constant.

4. Mixed Tsirelson spaces that are strongly non-subsequentially minimal. In the final section, it is shown that a large class of (unmodified) mixed Tsirelson spaces fails to be subsequentially minimal in a strong sense. We consider a mixed Tsirelson space $X = T[(\mathcal{S}_n,\theta_n)_{n=1}^\infty] = T[(\mathcal{S}_n,\sigma_n,\theta_n)_{n=1}^\infty]$, where $\sigma_n = U$ for all n. In this case, we may assume without loss of generality that (θ_n) is a regular sequence, i.e., (θ_n) is a nonincreasing null sequence in $(0,1)$ such that $\theta_{m+n} \ge \theta_n\theta_m$ for all $m,n\in\mathbb{N}$. By [21, Lemma 4.13], $\theta = \lim_n \theta_n^{1/n}$ exists and is equal to $\sup \theta_n^{1/n}$. Also, we let $\varphi_n = \theta_n/\theta^n$.

Definition. We say that a Banach space X with a normalized basis (e_k) is *strongly non-subsequentially minimal* if for every normalized block basis (x_k) of (e_k), there exists $(y_k) \prec (x_k)$ such that for all $(z_k)\prec(y_k)$, (z_k) is not equivalent to any subsequence of (e_k).

The main result of this section is Theorem 35, where it is shown that X is strongly non-subsequentially minimal if $\theta<1$ and $0<\inf\varphi_n\le\sup\varphi_n<1$.

Proposition 26 ([19, Proposition 21]). *If $\theta<1$ and $\inf\varphi_n>0$, then (θ_n) satisfies*

($\neg\dagger$) $\lim_m \limsup_n \dfrac{\theta_{m+n}}{\theta_n} = 0$ *and*

($\ddagger$) *There exists $F:\mathbb{N}\to\mathbb{R}$ with $\lim_{n\to\infty}F(n)=0$ such that for all $R,t\in\mathbb{N}$ and any arithmetic progression $(s_i)_{i=1}^R$ in $\mathbb{N}$,*

$$\max_{1\le i\le R}\frac{\theta_{s_i+t}}{\theta_{s_i}} \le F(R)\sum_{i=1}^R \frac{\theta_{s_i+t}}{\theta_{s_i}}.$$

The main tool in our investigation is a construction of certain "layered repeated averages" that can be carried out under the assumptions ($\neg\dagger$) and ($\ddagger$). The basic units of the construction are the *repeated averages* due to Argyros, Mercourakis and Tsarpalias [6], which we recall here. An $\mathcal{S}_0$-repeated average is a vector e_k for some $k\in\mathbb{N}$. For any $p\in\mathbb{N}$, an $\mathcal{S}_p$-repeated average is a vector of the form $(1/k)\sum_{i=1}^k x_i$, where $x_1<\cdots<x_k$ are $\mathcal{S}_{p-1}$-repeated averages and $k=\min\operatorname{supp} x_1$. Observe that any $\mathcal{S}_p$-repeated average x is a convex combination of $\{e_k : k\in\operatorname{supp} x\}$ such that $\|x\|_\infty \le (\min\operatorname{supp} x)^{-1}$ and $\operatorname{supp} x\in\mathcal{S}_p$.

Construction of layered repeated averages. Assume that ($\neg\dagger$) and ($\ddagger$) hold. Given $N\in\mathbb{N}$ and $V\in[\mathbb{N}]$, choose sequences $(p_k)_{k=1}^N$ and $(L_k)_{k=1}^N$ in $\mathbb{N}$, $L_k\ge 2$, that satisfy the following conditions:

(A) if $0 \le M \le N-2$ and $n \ge p_N$, then

$$\frac{\theta_{p_{M+1}+n}}{\theta_n} \le \frac{\theta_1}{24N^2} \prod_{i=1}^{M} \theta_{L_i p_i},$$

(the vacuous product $\prod_{i=1}^{0} \theta_{L_i p_i}$ is taken to be 1),

(B) if $0 < M \le N-2$, then $p_{M+1} > \sum_{i=1}^{M} L_i p_i$,

(C) if $0 < M \le N-2$, then

$$F(L_{M+1}) \le \frac{\theta_1}{144N^2} \prod_{i=1}^{M} \theta_{L_i p_i}.$$

If $k \in \mathbb{N}$ and $1 \le M \le N$, define $r_M(k)$ to be the integer in $\{1, \dots, L_M\}$ such that $L_M \,|\, (k - r_M(k))$. We can construct sequences of vectors $\mathbf{x}^0, \dots, \mathbf{x}^N$ with the following properties ((e_k) is the unit vector basis of $X = T[(\mathcal{S}_n, \theta_n)_{n=1}^{\infty}]$):

($\boldsymbol{\alpha}$) $\mathbf{x}^0$ is a subsequence of $(e_k)_{k \in V}$.

($\boldsymbol{\beta}$) Say $\mathbf{x}^M = (x_j^M)$ and $m_j = \min \operatorname{supp} x_j^M$. Then there is a sequence (I_k^{M+1}) of integer intervals such that

$$I_k^{M+1} < I_{k+1}^{M+1}, \qquad \bigcup_{k=1}^{\infty} I_k^{M+1} = \mathbb{N}$$

and each vector $x_k^{M+1} \in \mathbf{x}^{M+1}$ is of the form

$$x_k^{M+1} = \sum_{j \in I_k^{M+1}} a_j x_j^M,$$

where $\theta_{r_{M+1}(k) p_{M+1}} \sum_{j \in I_k^{M+1}} a_j e_{m_j}$ is an $\mathcal{S}_{r_{M+1}(k) p_{M+1}}$-repeated average. Moreover, the sequence $(a_j)_{j=1}^{\infty}$ is decreasing.

Each x_k^{M+1} is made up of components of diverse complexities. We analyze it by decomposing it into components of "pure forms" in the following manner. We adhere to the notation in **($\boldsymbol{\beta}$)**.

"Pure forms". Given $1 \le r_i \le L_i$, $1 \le M \le N-1$, write

$$x_k^{M+1}(r_M) = \sum_{\substack{j \in I_k^{M+1} \\ r_M(j) = r_M}} a_j x_j^M.$$

For $1 \le s < M$, define

$$x_k^{M+1}(r_s, \dots, r_M) = \sum_{\substack{j \in I_k^{M+1} \\ r_M(j) = r_M}} a_j x_j^M(r_s, \dots, r_{M-1}).$$

If $1 \le s \le M$, it is clear that $x_k^{M+1} = \sum x_k^{M+1}(r_s, \ldots, r_M)$, where the sum is taken over all possible values of $r_s, \ldots, r_M$.

Given $r_1, \ldots, r_N$, write $p(r_1, \ldots, r_j) = \sum_{i=1}^{j} r_i p_i$, $1 \le j \le N$. Set

$$\Phi_k^N = \frac{\theta_1}{2} \sum_{r_1,\ldots,r_{N-1}} \theta_{p(r_1,\ldots,r_N(k))} \theta^{-1}_{r_N(k)p_N} \prod_{i=1}^{N-1} \theta^{-1}_{r_i p_i} L_i^{-1}.$$

If $p \ge N$, define

$$\Theta_p = \Theta_p(N) = \max\Big\{\prod_{i=1}^{N} \theta_{l_i} : l_i \in \mathbb{N}, \sum_{i=1}^{N} l_i = p\Big\}.$$

The following estimates are crucial for subsequent computations. From here on, we fix a k satisfying

$$k \ge 42N^2 \prod_{i=1}^{N} L_i \theta^{-1}_{L_i p_i}. \tag{9}$$

Proposition 27 ([19, Theorem 20; see also the remark following the proof of the theorem]).

$$\|x_k^N\| \le \bigg(\frac{2}{N} + 4\theta_1^{-1} \sup_{r_1,\ldots,r_{N-1}} \frac{\Theta_{p(r_1,\ldots,r_N(k))}}{\theta_{p(r_1,\ldots,r_N(k))}}\bigg) \Phi_k^N.$$

Proposition 28 ([19, Corollary 9]).

$$\|x_k^N(r_1, \ldots, r_{N-1})\|_{\ell^1} \ge \frac{1}{2}\, \theta^{-1}_{r_N(k)p_N} \prod_{i=1}^{N-1} \theta^{-1}_{r_i p_i} L_i^{-1}.$$

For all $m \in \mathbb{N}$, $z \in c_{00}$, define

$$\|z\|_m = \theta_m \sup\Big\{\sum \|E_l z\| : (E_l) \text{ is } \mathcal{S}_m\text{-admissible}\Big\}.$$

Proposition 29. *Suppose that $x = x_k^N = \sum_{i=1}^{l} b_i e_{m_i}$, (z_i) is a normalized block basis of (e_k) with* $\min\operatorname{supp} z_i = m_i$, *$q = \sum_{j=1}^{N} L_j p_j$, and there exists $K < \infty$ such that $\|z_i\|_s \ge 1/K$ for all $1 \le s \le q$, $1 \le i \le l$. Let $z = \sum_{i=1}^{l} b_i z_i$. Then*

$$\|x\| \le \bigg(\frac{2}{N} + 4\theta_1^{-1} \sup_{r_1,\ldots,r_{N-1}} \frac{\Theta_{p(r_1,\ldots,r_N(k))}}{\theta_{p(r_1,\ldots,r_N(k))}}\bigg) K\theta_1 \|z\|.$$

Proof. According to Proposition 27, it suffices to show that $\|z\| \ge (\theta_1 K)^{-1} \Phi_k^N$. For each $1 \le i \le l$, let $(r_1, \ldots, r_{N-1})$ be the unique $(N-1)$-tuple such that $m_i \in \operatorname{supp} x_k^N(r_1, \ldots, r_{N-1})$. Since $\|z_i\|_t \ge 1/K$ for $t = q - p(r_1, \ldots, r_{N-1})$, there exists an $\mathcal{S}_t$-admissible family $\mathcal{G}_i$ such that $G \subseteq$

$\operatorname{supp} z_i$ for all $G \in \mathcal{G}_i$ and

$$\|z_i\|_t = \theta_t \sum_{G\in\mathcal{G}_i} \|Gz_i\| \geq \frac{1}{K}. \tag{10}$$

We estimate the norm of z by means of a particular tree $\mathcal{T}$. If $0 \leq n \leq N$ and $\operatorname{supp} x_j^{N-n} \subseteq \operatorname{supp} x_k^N$, let

$$E_j^n = \bigcup\{\operatorname{supp} z_i : m_i \in \operatorname{supp} x_j^{N-n}\}, \qquad \mathcal{E}^n = \{E_j^n : \operatorname{supp} x_j^{N-n} \subseteq \operatorname{supp} x_k^N\}.$$

By **(β)** in the construction of $\mathbf{x}^{N-n}$, E_s^n is an $\mathcal{S}_{r_{N-n}(s)p_{N-n}}$-admissible union of the sets $\{E_j^{n+1} : \operatorname{supp} x_j^{N-n-1} \subseteq \operatorname{supp} x_k^N\}$. Hence $\bigcup_{n=1}^N \mathcal{E}^n$ is an admissible tree so that

$$\operatorname{ord}(E_j^{n+1}) = \operatorname{ord}(E_s^n) + r_{N-n}(s)p_{N-n} \quad \text{if } E_j^{n+1} \subseteq E_s^n. \tag{11}$$

Note that $\operatorname{supp} x_j^0$ is a singleton $\{m_i\}$ for some i and hence $E_j^N = \operatorname{supp} z_i$. It follows from (11) that $\operatorname{ord}(E_j^N) = p(r_1, \ldots, r_{N-1}) + r_N(k)p_N$, where $(r_1, \ldots, r_{N-1})$ is the unique $(N-1)$-tuple determined by m_i. Set $\mathcal{E}^{N+1} = \bigcup_{i=1}^l \mathcal{G}_i$. Since $\mathcal{G}_i$ is an $\mathcal{S}_{q-p(r_1,\ldots,r_{N-1})}$-admissible family with $\bigcup_{G\in\mathcal{G}_i} G \subseteq \operatorname{supp} z_i = E_j^N \in \mathcal{E}^N$, $\mathcal{T} = \bigcup_{n=0}^{N+1} \mathcal{E}^n$ is an admissible tree such that $\operatorname{ord}(G) = q + r_N(k)p_N$ for each of the leaves G of $\mathcal{T}$. By Lemma 6, $\bigcup_{i=1}^l \mathcal{G}_i$ is $\mathcal{S}_{r_N(k)p_N+q}$-admissible. Therefore,

$$\begin{aligned}
\|z\| &\geq \theta_{q+r_N(k)p_N} \sum_{i=1}^{l} b_i \sum_{G\in\mathcal{G}_i} \|Gz_i\| \\
&\geq \theta_{q+r_N(k)p_N} \sum_{i=1}^{l} b_i (K\theta_{q-p(r_1,\ldots,r_{N-1})})^{-1} \qquad \text{(by (10))} \\
&= \theta_{q+r_N(k)p_N} \sum_{r_1,\ldots,r_{N-1}} (K\theta_{q-p(r_1,\ldots,r_{N-1})})^{-1} \|x(r_1, \ldots, r_{N-1})\|_{\ell^1}.
\end{aligned}$$

By the regularity of (θ_n), $\theta_{p(r_1,\ldots,r_{N-1},r_N(k))}\theta_{q-p(r_1,\ldots,r_{N-1})} \leq \theta_{q+r_N(k)p_N}$. Applying Proposition 28 to the above gives

$$\begin{aligned}
\|z\| &\geq \frac{\theta_{q+r_N(k)p_N}}{K} \sum_{r_1,\ldots,r_{N-1}} \frac{\theta_{p(r_1,\ldots,,r_N(k))}}{\theta_{q+r_N(k)p_N}} \left(\frac{1}{2}\,\theta^{-1}_{r_N(k)p_N} \prod_{i=1}^{N-1} \theta^{-1}_{r_ip_i} L_i^{-1}\right) \\
&= \frac{1}{K}\left(\frac{1}{2} \sum_{r_1,\ldots,r_{N-1}} \theta_{p(r_1,\ldots,r_N(k))}\theta^{-1}_{r_N(k)p_N} \prod_{i=1}^{N-1} \theta^{-1}_{r_ip_i} L_i^{-1}\right) = (\theta_1 K)^{-1}\Phi_k^N. \ \blacksquare
\end{aligned}$$

We need a few preparatory results in order to exploit the estimate established in Proposition 29.

LEMMA 30. *If* $(x_k) \prec (e_k)$, $\varepsilon > 0$ *and* $p \in \mathbb{N}$, *then there exists* $y \in \operatorname{span}(x_k)$, $\|y\| = 1$, *such that* $\|y\|_{\mathcal{S}_p} < \varepsilon$.

Proof. Assume the contrary. There exist $\varepsilon > 0$ and $p \in \mathbb{N}$ such that for all $y \in \operatorname{span}\{(x_k)\}$, $\|y\|_{\mathcal{S}_p} \geq \varepsilon\|y\|$. On the other hand, $\|y\| \geq \theta_p \|y\|_{\mathcal{S}_p}$. Hence $\|\cdot\|$ and $\|\cdot\|_{\mathcal{S}_p}$ are equivalent on $\operatorname{span}\{(x_k)\}$. However, the Schreier space $\mathcal{S}_p$ is c_0-saturated. It follows that $[(x_k)]$ and thus X contains a copy of c_0, contradicting the reflexivity of X. ∎

LEMMA 31. *If $(z_k) \prec (y_k) \prec (e_k)$ and $\|y_k\|_{\mathcal{S}_{k-1}} \leq 1/2^{k+2}$ for all k, then*

$$\|z_k\|_{\mathcal{S}_{k-1}} \leq \frac{1}{2^{k+1}} \quad \text{for all } k.$$

Proof. Write $z_k = \sum_{j\in J_k} a_j y_j$. Note that $|a_j| \leq \|z_k\| = 1$ for all $j \in J_k$. Therefore,

$$\begin{aligned}\|z_k\|_{\mathcal{S}_{k-1}} &\leq \sum_{j\in J_k} \|y_j\|_{\mathcal{S}_{k-1}} \\ &\leq \sum_{j\in J_k} \|y_j\|_{\mathcal{S}_{j-1}} \quad (\text{since } k \leq \min J_k \leq j) \\ &\leq \sum_{j\in J_k} \frac{1}{2^{j+2}} \leq \frac{1}{2^{k+1}}. \ \blacksquare\end{aligned}$$

LEMMA 32. *Assume that $\theta < 1$ and $\inf_n \varphi_n > 0$. If $(z_k) \prec (e_k)$ and*

$$\|z_k\|_{\mathcal{S}_{k-1}} \leq \frac{1}{2^{k+1}} \quad \text{for all } k,$$

then there is a constant $K < \infty$ such that for all $z \in \operatorname{span}(z_k)_{k=n}^{\infty}$, we have $\|z\|_m \geq (1/2K)\|z\|$ for all $1 \leq m \leq n$.

Proof. First observe that

$$\frac{\theta_{m+n}}{\theta_m\theta_n} = \frac{\varphi_{m+n}}{\varphi_m\varphi_n} \leq \frac{1}{(\inf \varphi_n)^2} \quad \text{for all } m, n. \tag{12}$$

Let $K = 1/(\inf \varphi_n)^2$. Suppose that $z \in \operatorname{span}(z_k)_{k=n}^{\infty}$, $\|z\| = 1$ and $1 \leq m \leq n$. Choose an admissible tree $\mathcal{T}$ of z so that

$$\begin{aligned}1 = \|z\| = \mathcal{T}z &= \sum_{E\in\mathcal{L}(\mathcal{T})} t(E)\|Ez\|_{c_0} \\ &= \sum_{\substack{E\in\mathcal{L}(\mathcal{T}) \\ \operatorname{ord}(E)\leq m}} t(E)\|Ez\|_{c_0} + \sum_{\substack{E\in\mathcal{L}(\mathcal{T}) \\ \operatorname{ord}(E)>m}} t(E)\|Ez\|_{c_0}.\end{aligned}$$

Write $z = \sum_{k=n}^{\infty} a_k z_k$. Then $|a_k| \leq 1$ as $\|z\| = 1$. Note that according to Lemma 6, the collection $\{E \in \mathcal{L}(\mathcal{T}) : \operatorname{ord}(E) \leq m\}$ of leaves is $\mathcal{S}_m$-

admissible. Therefore,

$$\sum_{\substack{E\in\mathcal{L}(\mathcal{T})\\ \mathrm{ord}(E)\le m}} t(E)\|Ez\|_{c_0} \le \sum_{\substack{E\in\mathcal{L}(\mathcal{T})\\ \mathrm{ord}(E)\le m}} \|Ez\|_{c_0} \le \|z\|_{\mathcal{S}_m} \le \sum_{k=n}^{\infty}\|z_k\|_{\mathcal{S}_m}$$
$$\le \sum_{k=n}^{\infty}\|z_k\|_{\mathcal{S}_{k-1}} \le \sum_{k=n}^{\infty}\frac{1}{2^{k+1}} \le \frac{1}{2}.$$

Thus

$$\sum_{\substack{E\in\mathcal{L}(\mathcal{T})\\ \mathrm{ord}(E)> m}} t(E)\|Ez\|_{c_0} \ge \frac{1}{2}.$$

Let $\mathcal{E}$ be the collection of all nodes E in $\mathcal{T}$ that are minimal subject to the condition $\mathrm{ord}(E) > m$. Also, let $\mathcal{D}$ be the set of all immediate predecessors of nodes in $\mathcal{E}$. If $D \in \mathcal{D}$, let $\mathcal{E}(D)$ be the collection of its immediate successors. For each $E \in \mathcal{E}(D)$, $\mathrm{ord}(D) \le m < \mathrm{ord}(E)$. Therefore there exists an $\mathcal{S}_{m-\mathrm{ord}(D)}$-admissible collection $\mathcal{G}_D$ of subsets of D such that $\mathcal{E}(D) = \bigcup\{E \in \mathcal{E}(D) : E \subseteq G$ for some $G \in \mathcal{G}_D\}$ and $\{E \in \mathcal{E}(D) : E \subseteq G\}$ is $\mathcal{S}_{\mathrm{ord}(E)-m}$-admissible for each $G \in \mathcal{G}_D$. Now $\mathcal{G} = \bigcup_{D\in\mathcal{D}} \mathcal{G}_D$ is $\mathcal{S}_m$-admissible and $\theta_{\mathrm{ord}(E)} \ge t(E)$ by the regularity of (θ_n). Hence

$$\|z\|_m \ge \theta_m \sum_{G\in\mathcal{G}} \|Gz\| \ge \theta_m \sum_{G\in\mathcal{G}} \theta_{\mathrm{ord}(E)-m} \sum_{\substack{E\in\mathcal{E}\\ E\subseteq G}} \|Ez\|$$
$$\ge \sum_{G\in\mathcal{G}} \frac{\theta_{\mathrm{ord}(E)}}{K} \sum_{\substack{E\in\mathcal{E}\\ E\subseteq G}} \|Ez\| \quad \text{(by (12) and the definition of } K\text{)}$$
$$\ge \frac{1}{K}\sum_{E\in\mathcal{E}} t(E)\|Ez\| \ge \frac{1}{K} \sum_{\substack{E\in\mathcal{L}(\mathcal{T})\\ \mathrm{ord}(E)> m}} t(E)\|Ez\| \ge \frac{1}{2K}. \ \blacksquare$$

We shall show that, for appropriate (θ_n), if $(z_k) \prec (e_k)$ satisfies the conclusion of Lemma 31, then it is not equivalent to a subsequence of (e_k).

LEMMA 33. *If* $0 < \inf_n \varphi_n \le \sup_n \varphi_n < 1$, *then*

$$\lim_N \sup_{p\ge N} \frac{\Theta_p(N)}{\theta_p} = 0.$$

Proof. Let $\varepsilon > 0$. Choose N such that $d^N/c < \varepsilon$, where $0 < c = \inf_n \varphi_n \le \sup_n \varphi_n = d < 1$. Let $p \in \mathbb{N}$. If $(\ell_i)_{i=1}^N$ is a sequence of positive integers such that $\sum_{i=1}^N \ell_i = p$, then

$$\prod_{i=1}^N \theta_{\ell_i} = \theta^p \prod_{i=1}^N \varphi_{\ell_i} \le \theta^p d^N \quad \text{and} \quad \theta_p = \varphi_p\theta^p \ge c\theta^p.$$

Thus
$$\sup_{p\ge N}\frac{\Theta_p(N)}{\theta_p}\le\frac{d^N}{c}<\varepsilon. \ \blacksquare$$

PROPOSITION 34. *If (z_k) is a normalized block basis that is equivalent to a subsequence of (e_k), then there is a subsequence (z_{k_j}) of (z_k) such that (z_{k_j}) is equivalent to (e_{m_j}), where $m_j=\min\operatorname{supp} z_{k_j}$.*

Proof. By [17, Proposition 9], two subsequences (e_{n_i}) and (e_{l_i}) of (e_k) are equivalent whenever $\max\{n_i,l_i\}<\min\{n_{i+1},l_{i+1}\}$ for all i. If (z_k) is equivalent to a subsequence of (e_k), then there is a subsequence (z_{k_j}) of (z_k) that is equivalent to a subsequence (e_{n_j}) of (e_k) with
$$\max\{\min\operatorname{supp} z_{k_j},n_j\}<\min\{\min\operatorname{supp} z_{k_{j+1}},n_{j+1}\}\quad\text{ for all } j.$$
Thus $\max\{n_j,m_j\}<\min\{n_{j+1},m_{j+1}\}$ and hence (e_{n_j}) is equivalent to (e_{m_j}). Consequently, (z_{k_j}) is equivalent to (e_{m_j}). ∎

We are now ready to prove the main result of this section.

THEOREM 35. *If $0<\inf_n\varphi_n\le\sup_n\varphi_n<1$, then X is strongly non-subsequentially minimal.*

Proof. Let (x_k) be a normalized block basis of (e_k). By Lemma 30, there exists $(y_k)\prec(x_k)$ such that $\|y_k\|_{\mathcal{S}_{k-1}}\le 1/2^{k+2}$ for all k. Suppose that there exists $(z_k)\prec(y_k)$ that is equivalent to a subsequence of (e_k). Applying Proposition 34, we may assume that (z_k) is equivalent to (e_{m_k}), where $m_k=\min\operatorname{supp} z_k$. Pick $\varepsilon>0$ so that
$$\varepsilon\Big\|\sum b_kz_k\Big\|\le\Big\|\sum b_ke_{m_k}\Big\|\quad\text{ for all } (b_k)\in c_{00}.$$
By a combination of Lemmas 31 and 32 there is a constant $K<\infty$ such that $\|z\|_s\ge(1/2K)\|z\|$ for all $z\in\operatorname{span}(z_k)_{k=n}^{\infty}$, $1\le s\le n$. Use Lemma 33 to choose N such that
$$\frac{2}{N}+4\theta_1^{-1}\sup_p\frac{\Theta_p(N)}{\theta_p}<\frac{\varepsilon}{2K\theta_1}\quad\text{ if } p\ge N.$$
With the chosen N and $V=(m_i)_{i=q}^{\infty}$ construct the layered repeated average vector $x=x_k^N=\sum_{i=q}^{l}b_ie_{m_i}$ with k satisfying inequality (9). Let $z=\sum_{i=q}^{l}b_iz_i$. (Recall that $q=\sum_{j=1}^{N}L_jp_j$, where $(p_j)_{j=1}^N$ and $(L_j)_{j=1}^N$ are chosen to satisfy conditions (A)–(C) once N is determined.) According to Proposition 29,
$$\|x\|\le\bigg(\frac{2}{N}+4\theta_1^{-1}\sup_{r_1,\dots,r_{N-1}}\frac{\Theta_{p(r_1,\dots,r_N(k))}}{\theta_{p(r_1,\dots,r_N(k))}}\bigg)2K\theta_1\|z\|<\varepsilon\|z\|,$$
contrary to the choice of ε. ∎

The following example shows that the condition $\sup_n\varphi_n<1$ is not necessary for the conclusion of the theorem to hold.

Example 36. If $\theta < 1$, then there exists a regular sequence (θ_n) with $\sup_n \theta_n^{1/n} = \theta$ and $\lim_n \varphi_n = 1$ such that X is strongly non-subsequentially minimal.

Proof. Suppose that $0 < \theta < 1$. In [19, Example 23], a regular sequence (θ_n) is constructed such that $\sup_n \theta_n^{1/n} = \theta$, $\lim_n \varphi_n = 1$ and, for all $N \in \mathbb{N}$, there are sequences $(p_k)_{k=1}^N$ and $(L_k)_{k=1}^N$ satisfying conditions (A)–(C) and

$$\lim_{N\to\infty} \sup_{r_1,\dots,r_{N-1}} \frac{\Theta_{p(r_1,\dots,r_N(k))}}{\theta_{p(r_1,\dots,r_N(k))}} = 0. \tag{13}$$

Following the arguments in Theorem 35 with Lemma 33 replaced by (13) shows that X is strongly non-subsequentially minimal. ∎

In view of Proposition 14, any subsequentially minimal partly modified mixed Tsirelson space is quasi-minimal. However, the existence of strongly non-subsequentially minimal mixed Tsirelson spaces prompts the following question.

Question. Does every (partly modified) mixed Tsirelson space $T[(\mathcal{S}_n, \theta_n)_{n=1}^\infty]$ (or $T[(\mathcal{S}_n, \sigma_n, \theta_n)_{n=1}^\infty]$) contain a quasi-minimal subspace?

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Received February 2, 2007
Revised version April 30, 2008 (6101)