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Topological classification of closed convex sets in Fréchet spaces

by

Taras Banakh (Kielce and Lviv) and Robert Cauty (Paris)

Abstract. We prove that each non-separable completely metrizable convex subset of a Fréchet space is homeomorphic to a Hilbert space. This resolves a more than 30 years old problem of infinite-dimensional topology. Combined with the topological classification of separable convex sets due to Klee, Dobrowolski and Toruńczyk, this result implies that each closed convex subset of a Fréchet space is homeomorphic to $[0,1]^n \times [0,1)^m \times \ell_2(\kappa)$ for some cardinals $0 \le n \le \omega$, $0 \le m \le 1$ and $\kappa \ge 0$.

The problem of topological classification of convex sets in linear metric spaces traces its history back to the founders of functional analysis, S. Banach and M. Fréchet. For separable closed convex sets in Fréchet spaces this problem was resolved by combined efforts of Klee [8] (see [3, III.7.1]), Dobrowolski and Toruńczyk [4], [5]:

Theorem 1 (Klee–Dobrowolski–Toruńczyk). *Each separable closed convex subset C of a Fréchet space is homeomorphic to $[0,1]^n \times [0,1)^m \times (0,1)^k$ for some cardinals $0 \le n, k \le \omega$ and $0 \le m \le 1$. In particular, C is homeomorphic to the separable Hilbert space ℓ_2 if and only if C is not locally compact.*

By a *Fréchet space* we mean a locally convex complete linear metric space. A *linear metric space* is a linear topological space endowed with an invariant metric that generates its topology. A topological space is called *completely metrizable* if its topology is generated by a complete metric.

In this paper we study the topological structure of non-separable (completely metrizable) convex sets in Fréchet spaces and prove the following theorem that answers problem LS10 in Geoghegan's list [7], repeated in [11] and [2].

Theorem 2. *Each non-separable completely metrizable convex subset of a Fréchet space is homeomorphic to a Hilbert space.*

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Theorems 1 and 2 imply the following topological classification of closed convex subsets in Fréchet spaces.

Theorem 3. *Each closed convex subset C of a Fréchet space is homeomorphic to $[0,1]^n \times [0,1)^m \times \ell_2(\kappa)$ for some cardinals $0 \le n \le \omega$, $0 \le m \le 1$ and $\kappa \ge 0$. In particular, C is homeomorphic to an infinite-dimensional Hilbert space if and only if C is not locally compact.*

Here $\ell_2(\kappa)$ stands for the Hilbert space that has an orthonormal basis of cardinality κ. The topology of infinite-dimensional Hilbert spaces was characterized by Toruńczyk [9], [10]. This characterization was used in the proof of the following criterion from [2] which is our main tool for the proof of Theorem 2.

Theorem 4 (Banakh–Zarichnyi). *A convex subset C of a linear metric space is homeomorphic to an infinite-dimensional Hilbert space if and only if C is a completely metrizable absolute retract with LFAP.*

A topological space X is defined to have the *locally finite approximation property* (briefly, LFAP) if for each open cover $\mathcal{U}$ of X there is a sequence of maps $f_n : X \to X$, $n \in \omega$, such that each f_n is $\mathcal{U}$-near to the identity $\mathrm{id}_X : X \to X$ and the family $(f_n(X))_{n\in\omega}$ is locally finite in X. The latter means that each $x \in X$ has a neighborhood $O(x) \subset X$ that meets only finitely many sets $f_n(X)$, $n \in \omega$.

Theorem 2 follows immediately from Theorem 4, the Borsuk–Dugundji Theorem [3, II.3.1] (saying that convex subsets of Fréchet spaces are absolute retracts) and the following theorem that will be proved in Section 3.

Theorem 5. *Each non-separable convex subset of a Fréchet space has LFAP.*

1. Separated approximation property. Theorem 5 establishing LFAP of non-separable convex sets will be proved with the help of the metric counterpart of LFAP, called SAP.

A metric space (X,d) is defined to have the *separated approximation property* (briefly, SAP) if for each $\varepsilon > 0$ there is a sequence of maps $f_n : X \to X$, $n \in \omega$, such that each f_n is ε-homotopic to id_X and the family $(f_n(X))_{n\in\omega}$ is *separated* in the sense that $\inf_{n\neq m} d(f_n(X), f_m(X)) > 0$.

Here for two non-empty subsets $A, B \subset X$ we put $d(A,B) = \inf\{d(a,b) : a \in A,\ b \in B\}$. Two maps $f, g : A \to X$ are called *ε-homotopic* if they can be linked by a homotopy $(h_t)_{t\in\mathbb{I}} : A \to X$ such that $h_0 = f$, $h_1 = g$ and $\operatorname{diam}\{h_t(a) : t \in \mathbb{I}\} \le \varepsilon$ for all $a \in A$. By $\mathbb{I}$ we denote the unit interval $[0,1]$.

The following lemma is proved by analogy with Lemma 1 of [5] and Lemma 5.2 of [2].

Lemma 1. *Each metric space with SAP satisfies LFAP.*

Proof. Assume that a metric space (X,d) has SAP. To show that X has LFAP, fix an open cover $\mathcal{U}$ of X and find a non-expanding function $\varepsilon : X \to (0,1)$ such that the cover $\{\bar{B}_d(x;\varepsilon(x)) : x \in X\}$ refines $\mathcal{U}$. Here $\bar{B}_d(x;\varepsilon) = \{x' \in X : d(x',x) \le \varepsilon\}$ stands for the closed ε-ball centered at x.

For every $k \in \omega$ consider the closed subset $X_k = \{x \in X : \varepsilon(x) \ge 2^{-k}\}$ of X. Put $\varepsilon_k = 1/4^{k+2}$ for $k \in \{0,1\}$ and let $f_0 : X\times\omega \to X$, $f_0 : (x,n) \mapsto x$, be the projection. By induction we shall construct a sequence $(\varepsilon_k)_{k\in\omega}$ of positive real numbers and a sequence of maps $f_k : X \times \omega \to X$, $k \in \omega$, such that the following conditions are satisfied:

(1) $\varepsilon_{k+1} \le \frac{1}{4}\varepsilon_k \le 1/4^{k+3}$,
(2) f_k is ε_k-homotopic to f_{k-1},
(3) $f_k|X_{k-3} \times \omega = f_{k-1}|X_{k-3} \times \omega$,
(4) $f_k|(X \setminus X_{k+1}) \times \omega = f_0|(X \setminus X_{k+1}) \times \omega$,
(5) $\inf_{n\ne m} d(f_k(X_k \times \{n\}), f_k(X_k \times \{m\})) \ge 4\varepsilon_{k+1}$.

Assume that the maps $f_i : X \times \omega \to X$ and numbers ε_{i+1} satisfying the conditions (1)–(5) have been constructed for all $i < k$. By SAP, there is an ε_k-homotopy $(h_t)_{t\in\mathbb{I}} : X \times \omega \to X$ such that $h_0 = f_0$ and

$$\delta = \inf_{n\ne m} d(h_1(X \times \{n\}), h_1(X \times \{m\}) > 0.$$

Choose a continuous function $\lambda : X \to [0,1]$ such that $X_k \setminus X_{k-2} \subset \lambda^{-1}(1)$ and $X_{k-3} \cup (X \setminus X_{k+1}) \subset \lambda^{-1}(0)$.

Take any positive number $\varepsilon_{k+1} \le \frac{1}{4}\min\{\delta,\varepsilon_k\}$ and define a function $f_k : X \times \omega \to X$ by

$$f_k(x,n) = h_{\lambda(x)}(f_{k-1}(x,n),n).$$

It is clear that the conditions (1)–(4) are satisfied. The condition (5) will follow as soon as we check that $d(f_k(x,n), f_k(y,m)) \ge 4\varepsilon_{k+1}$ for any $x,y \in X_k$ and $n \ne m$.

There are unique $i,j \le k$ such that $x \in X_i \setminus X_{i-1}$ and $y \in X_j \setminus X_{j-1}$. If $i,j < k$, then

$$d(f_k(x,n), f_k(y,m)) \ge d(f_{k-1}(x,n), f_{k-1}(y,m)) - 2\varepsilon_k \ge 4\varepsilon_k - 2\varepsilon_k \ge 4\varepsilon_{k+1}.$$

It remains to consider the case $\max\{i,j\} = k$. We lose no generality assuming that $i = k$. If $j \ge k-1$, then

$$d(f_k(x,n), f_k(y,m)) = d(h_1(f_{k-1}(x,n),n), h_1(f_{k-1}(y,m),m)) \ge \delta \ge 4\varepsilon_{k+1}.$$

Next, assume that $j \le k-2$. In this case $k \ge j+2 \ge 3$. Then

$$\varepsilon(x) < 2^{-i+1} = 2^{-k+1} < 2^{-k+2} \le 2^{-j} \le \varepsilon(y)$$

and the non-expanding property of ε implies that

$$d(x,y) \ge |\varepsilon(x) - \varepsilon(y)| \ge 2^{-j} - 2^{-k+1} \ge 2^{-j-1}.$$

It follows from (4) and (2) that

$$\begin{aligned} d(x, f_k(x,n)) &= d(f_{i-2}(x,n), f_k(x,n)) = d(f_{k-2}(x,n), f_k(x,n)) \\ &\le \varepsilon_{k-1} + \varepsilon_k \le 2\varepsilon_{k-1} \le \frac{2}{4^{k+1}} \end{aligned}$$

and

$$d(y, f_k(y,m)) = d(f_{j-2}(y,m), f_k(y,m)) \le \varepsilon_{j-1} + \cdots + \varepsilon_k \le 2\varepsilon_{j-1} \le \frac{2}{4^{j+1}}.$$

Then

$$\begin{aligned} &d(f_k(x,n), f_k(y,m)) \ge d(x,y) - d(x, f_k(x,n)) - d(y, f_k(y,m)) \\ &\quad \ge \frac{1}{2^{j+1}} - \frac{2}{4^{k+1}} - \frac{2}{4^{j+1}} \ge \frac{1}{2^{j+1}} - \frac{2}{4^{j+3}} - \frac{2}{4^{j+1}} \ge \frac{4}{4^{j+5}} \ge \frac{4}{4^{k+3}} \ge 4\varepsilon_{k+1}. \end{aligned}$$

This completes the inductive step.

After completing the inductive construction, let

$$f_\infty = \lim_{k\to\infty} f_k : X \times \omega \to X.$$

The conditions (1)–(3) guarantee that f_∞ is well-defined and continuous. Let us show that it is ε-near to f_0. Given any $(x,n) \in X \times \omega$, there is a unique $i \in \omega$ such that $x \in X_i \setminus X_{i-1}$. By (3) and (4), $f_\infty(x,n) = f_{i+2}(x,n)$ and $f_0(x,n) = f_{i-2}(x,n)$. Then

$$\begin{aligned} d(f_\infty(x,n), x) &= d(f_\infty(x,n), f_0(x,n)) = d(f_{i+2}(x,n), f_{i-2}(x,n)) \\ &\le \varepsilon_{i+2} + \cdots + \varepsilon_{i-1} \le 2\varepsilon_{i-1} \le \frac{2}{4^{i+1}} < \frac{1}{2^i} \le \varepsilon(x). \end{aligned}$$

The choice of ε guarantees that f_∞ is $\mathcal{U}$-near to $f_0 : X \times \omega \to X$.

It remains to prove that the family $(f_\infty(X \times \{n\})_{n\in\omega}$ is discrete in X. Given any $x \in X$, there is a unique $i \in \omega$ such that $x \in X_i \setminus X_{i-1}$. Consider the open ball $B(x; 1/2^{i+2}) = \{x' \in X : d(x,x') < 1/2^{i+2}\}$.

Claim 1. $B(x; 1/2^{i+2}) \cap f_\infty(X \times \omega) \subset f_\infty(X_{i+1} \times \omega)$.

Proof. Assume, contrary to our claim, that $d(x, f_\infty(y,m)) < 1/2^{i+2}$ for some $m \in \omega$ and $y \in X \setminus X_{i+1}$. There is a unique $j \in \omega$ such that $y \in X_j \setminus X_{j-1}$. It follows from $y \notin X_{i+1}$ that $j \ge i+2$. Since

$$d(f_\infty(y,m), y) = d(f_{j+2}(y,m), f_{j-2}(y,m)) \le 2\varepsilon_{j-1} \le \frac{2}{4^{j+1}} \le \frac{1}{2^{i+2}},$$

and $\varepsilon(y) < 1/2^{j-1} < 1/2^i \le \varepsilon(x)$, by the non-expanding property of ε, we get a contradiction:

$$\begin{aligned} \frac{1}{2^{i+1}} &\le \frac{1}{2^i} - \frac{1}{2^{j-1}} \le |\varepsilon(x) - \varepsilon(y)| \le d(x,y) \\ &\le d(x, f_\infty(y,m)) + d(f_\infty(y,m), y) < \frac{1}{2^{i+2}} + \frac{1}{2^{i+2}} = \frac{1}{2^{i+1}}. \ \blacksquare \end{aligned}$$

Now the condition (5) and the inequality $\varepsilon_{i+2} \le 1/4^{i+4} \le 1/2^{i+2}$ imply that the ball $B(x;\varepsilon_{i+2})$ meets at most one set $f_\infty(X_{i+1} \times \{n\})$ and hence at most one set $f_\infty(X \times \{n\})$, which means that the family $(f_\infty(X \times \{n\}))_{n\in\omega}$ is discrete in X and hence X has LFAP. ∎

2. SAP in non-separable convex cones. In this section we shall prove that non-separable convex cones in Fréchet spaces have SAP.

A subset C of a linear metric space (L, d) is called a *convex cone* if it is convex and $\mathbb{R}_+ \cdot C = C$ where $\mathbb{R}_+ = [0,\infty)$. The principal result of this section is

LEMMA 2. *Each non-separable convex cone C in a Fréchet space L has SAP.*

In the proof of this lemma we shall use an operator version of the Josefson–Nissenzweig Theorem proved in [1]:

LEMMA 3. *For a dense continuous non-compact linear operator $S : X \to Y$ between normed spaces there is a continuous linear operator $T : Y \to c_0$ such that the operator $TS : X \to c_0$ is not compact.*

Recall that an operator $T : X \to Y$ between linear topological spaces is

- *dense* if TX is dense in Y;
- *compact* if the image $T(U)$ of some open neighborhood $U \subset X$ of zero is totally bounded in Y.

A subset B of a linear topological space Y is *totally bounded* if for each open neighborhood $V \subset Y$ of zero there is a finite subset $F \subset Y$ such that $B \subset V + F$.

Proof of Lemma 2. Assume that C is a non-separable convex cone in a Fréchet space L. By [3, I.6.4], the topology of the Fréchet space L is generated by an invariant metric d_L such that for every $\varepsilon > 0$ the ε-ball $B_L(\varepsilon) = \{x \in L : d_L(x, 0) < \varepsilon\}$ centered at the origin is convex. We lose no generality assuming that the linear subspace $C - C$ is dense in L.

Given any $\varepsilon > 0$, we need to construct maps $f_k : C \to C$, $k \in \omega$, such that each f_k is ε-homotopic to id_C and $\inf_{k\ne n} d(f_k(C), f_n(C)) > 0$. Since the metric d has convex balls, any two ε-near maps into C are ε-homotopic.

CLAIM 2. *There is a linear continuous operator $R : L \to Y$ onto a normed space Y such that $R(C)$ is not separable and $R^{-1}(\bar{B}_Y) \subset B_L(\varepsilon/2)$ where $\bar{B}_Y = \{y \in Y : \|y\| \le 1\}$.*

Proof. By [3, I.6.4], the Fréchet space L can be identified with a closed linear subspace of a countable product $\prod_{i\in\omega} X_i$ of Banach spaces. For every $n \in \omega$ let $Y_n = \prod_{i<n} X_i$ and let $\mathrm{pr}_n : L \to Y_n$ be the natural projection. Since C is non-separable, there is $n \in \omega$ such that for every $m \ge n$ the

image $\mathrm{pr}_m(C) \subset Y_m$ is not separable. We can take $m \geq n$ so large that $B_L(\varepsilon/2) \supset \mathrm{pr}_m^{-1}(U)$ for some open neighborhood $U \subset Y_m$ of the origin. The neighborhood U contains the closed r-ball $\bar{B}_{Y_m}(r)$ for some $r > 0$. Finally, consider the linear space $Y = \mathrm{pr}_m(L) \subset Y_m$ endowed with the norm $\|y\| = r^{-1}\|y\|_m$ where $\|\cdot\|_m$ is the norm Y_m. Then the operator $R = \mathrm{pr}_m : L \to Y$ has the desired properties. ∎

In the convex cone C consider the convex subset $B_C = C \cap R^{-1}(\bar{B}_Y)$ and observe that $C = \mathbb{R}_+ \cdot B_C$ and hence $R(C) = \mathbb{R}_+ \cdot R(B_C)$. Since $R(C)$ is non-separable, so is $R(B_C)$. Consider the convex bounded symmetric subset $D = R(B_C) - R(B_C) \subset Y$ and observe that $\mathbb{R}\cdot D = R(C) - R(C) = R(C-C)$. Then the Minkowski functional

$$\|x\|_Z = \inf\{\lambda > 0 : x \in \lambda D\}$$

is a well-defined norm on the linear space $Z = \mathbb{R} \cdot D = R(C - C)$ and the identity inclusion $I : Z \to Y$ is a bounded linear operator from the normed space $(Z, \|\cdot\|_Z)$ to the normed space Y. Since $I(Z) = Z$ is non-separable, the operator I is not compact. By Lemma 3, there is a linear operator $T : Y \to c_0$ with $\|T\| = 1$ such that the composition $TI : Z \to c_0$ is not compact. The latter means that the image $T(D) = TR(B_C) - TR(B_C)$ is not totally bounded in c_0 and hence the bounded set $TR(B_C)$ is not totally bounded in c_0.

Consequently, there is $\delta \in (0, 1]$ such that for every $n \in \omega$,

$$TR(B_C) \not\subset \{(x_i)_{i\in\omega} \in c_0 : \max_{i\geq n} |x_i| < \delta\}. \tag{2.1}$$

For every $n \in \omega$ let $e_n^* \in c_0^*$, $e_n^* : (x_i)_{i\in\omega} \mapsto x_n$, be the nth coordinate functional of c_0 and let $z_n^* = (TR)^*(e_n^*) \in L^*$.

Claim 3. *There are an increasing number sequence $(m_k)_{k\in\omega}$ and a sequence $(z_k)_{k\in\omega} \subset B_C$ such that for every $k \in \omega$:*

(1) $|z_{m_k}^*(z_k)| \geq \delta$;
(2) $|z_{m_i}^*(z_k)| < \delta^3/100$ *for all* $i > k$.

Proof. The sequences (m_k) and (z_k) will be constructed by induction. By (2.1) there are $z_0 \in B_C$ and $m_0 \in \omega$ such that $|e_{m_0}^*(z_0)| \geq \delta$. Now assume that for some $k \in \omega$ the points $z_0, \ldots, z_k$ and numbers $m_0 < m_1 < \cdots < m_k$ have been constructed. Since the points $TR(z_i)$, $i \leq k$, belong to the Banach space c_0, there is an $m > m_k$ so large that $|e_n^*(TR(z_i))| < \delta^3/100$ for all $n \geq m$ and $i \leq k$. By (2.1), there are $z_{k+1} \in B_C$ and $m_{k+1} \geq m$ such that $|z_{m_{k+1}}^*(z_{k+1})| = |e_{m_{k+1}}^*(TR(z_{k+1}))| \geq \delta$. This completes the inductive step. ∎

Decompose ω into a countable union $\omega = \bigcup_{k\in\omega} N_k$ of pairwise disjoint infinite subsets and by induction define a function $\xi : \omega \times \omega \to \omega$ such that

$\xi(i,k) \in N_k$ and $\xi(i+1,k) > \xi(i,k) > i$ for all $i,k \in \omega$. For any $i,k \in \omega$ let

$$z_{i,k} := z_{\xi(i,k)} \quad \text{and} \quad z^*_{i,k} := z^*_{m_{\xi(i,k)}} = (TR)^*(e^*_{m_{\xi(i,k)}}),$$

where $(z_i)_{i\in\omega}$ and $(m_k)_{k\in\omega}$ are given by Claim 3. It follows that the double sequences $(z_{i,k})_{i,k\in\omega}$ and $(z^*_{i,k})_{i,k\in\omega}$ have the following properties (to be used in the proof of Claim 8 below):

Claim 4. *If* $(i,k),(j,n) \in \omega\times\omega$, *then*

(1) $|z^*_{i,k}(z_{i,k})| \ge \delta$;
(2) $|z^*_{j,k}(z_{i,n})| < \delta^3/100$ *provided* $\xi(j,k) > \xi(i,n)$;
(3) $|z^*_{i,k}(z)| \le 1$ *for any* $z \in B_C$.

Claim 5. *There is a map* $f : C \to C$ *such that* $d(f, \mathrm{id}) < \varepsilon/2$ *and each* $x \in C$ *has a neighborhood* $O(x)$ *whose image* $f(O(x))$ *lies in the convex hull* $\mathrm{conv}(F_x)$ *of some finite subset* $F_x \subset C$.

Proof. Using the paracompactness of the metrizable space C, find a locally finite open cover $\mathcal{U}$ of X that refines the cover of C by open $\varepsilon/4$-balls. In each $U \in \mathcal{U}$ pick a point $c_U \in U$. Let $\{\lambda_U : C \to [0,1]\}_{U\in\mathcal{U}}$ be a partition of unity subordinated to the cover $\mathcal{U}$ in the sense that $\lambda_U^{-1}((0,1]) \subset U$ for all $U \in \mathcal{U}$. Finally, define a map $f : C \to C$ by the formula

$$f(x) = \sum_{U\in\mathcal{U}} \lambda_U(x) c_U.$$

It is standard to check that f has the desired property. ∎

For every $k \in \mathbb{Z}$ denote by C_k the set of points $x \in C$ that have a neighborhood $O(x) \subset C$ such that $|z^*_m f(x')| < \delta^3/100$ for each $x' \in O(x)$ and every $m \ge k$. It is clear that each C_k is open in C and lies in C_{k+1}.

Claim 6. $C = \bigcup_{k\in\omega} C_k$.

Proof. By Claim 5, each $x \in C$ has a neighborhood $O(x) \subset C$ such that $f(O(x)) \subset \mathrm{conv}(F)$ for some finite $F \subset C$. Taking into account that $TR(F)$ is a finite subset of the Banach space c_0, we can find $m \in \omega$ such that $|e^*_n TR(z)| < \delta^3/100$ for all $n \ge m$ and $z \in F$. Then also $|e^*_n TR(z)| < \delta^3/100$ for all $z \in \mathrm{conv}(F)$, in particular, $|e^*_n TRf(x')| < \delta^3/100$ for any $x' \in O(x)$. This means that $x \in C_m$ by the definition of C_m. ∎

Claim 7. *There is an open cover* $(U_k)_{k\in\omega}$ *of the space* C *such that* $U_k \subset \bar{U}_k \subset C_{k-1} \cap U_{k+1}$ *for all* $k \in \omega$.

Proof. If $C_{k_0} = C$ for some $k_0 \in \omega$, then put $U_k = \emptyset$ for $k < k_0$ and $U_k = C$ for $k \ge k_0$. If $C_k \ne C$ for all $k \in \omega$, then put $U_k = \{x \in C : d_L(x, C \setminus C_{k-1}) > 2^{-k}\}$ for all $k \in \omega$. ∎

By Theorem 5.1.9 of [6] there is a partition of unity $\{\lambda_k : C \to [0,1]\}_{k\in\omega}$ subordinated to the cover $\{U_{k+1}\setminus\bar U_{k-1}\}_{k\in\omega}$ of C in the sense that $\lambda_k^{-1}(0,1] \subset U_{k+1}\setminus \bar U_{k-1}$ for all $k\in\omega$ (here we assume that $U_k=\emptyset$ for $k<0$).

Now, for every $k\in\omega$ define a map $f_k : C\to C$ by the formula

$$f_k(x) = f(x) + \sum_{i\in\omega} \lambda_i(x) z_{i,k} = f(x) + \lambda_i(x) z_{i,k} + (1-\lambda_i(x)) z_{i+1,k},$$

where i is the unique number such that $x \in U_{i+1}\setminus U_i$. Since $f_k(x) - f(x) \in B_C \subset B_d(\varepsilon/2)$, we conclude that $d(f(x), f_k(x)) < \varepsilon/2$ and hence

$$d(x, f_k(x)) \le d(x, f(x)) + d(f(x), f_k(x)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for all $x\in C$. So, each f_k is ε-near and ε-homotopic to id_C.

Claim 8. *The family $(f_k(C))_{k\in\omega}$ is separated.*

Proof. By the continuity of the operator $TR : L \to c_0$, there is $\eta>0$ such that $TR(B_L(\eta)) \subset B_{c_0}(\delta^3/20)$. We claim that $\inf_{n\ne k} d(f_n(C), f_k(C)) \ge \eta$.

Fix any distinct $n,k\in\omega$ and $x,y\in C$. By the choice of η, the inequality $d(f_k(x), f_n(y)) \ge \eta$ will follow as soon as we check that $\|TR(f_k(x) - f_n(y))\| > \delta^3/20$. The latter will follow as soon as we find $m\in\omega$ such that $|e_m^* TR(f_k(x) - f_n(y))| > \delta^3/20$. Since $e_m^* TR(z) = z_m^*(z)$ for all $z \in L$, it suffices to show that $|z_m^*(f_k(x) - f_n(y))| > \delta^3/20$ for some $m\in\omega$.

Since $C = \bigcup_{i\in\omega}(U_{i+1}\setminus U_i)$, there are unique $i,j\in\omega$ such that $x\in U_{i+1}\setminus U_i$ and $y \in U_{j+1}\setminus U_j$. Then

$$\begin{aligned} f_k(x) &= f(x) + \lambda_i(x) z_{i,k} + \lambda_{i+1}(x) z_{i+1,k},\\ f_n(y) &= f(y) + \lambda_j(y) z_{j,n} + \lambda_{j+1}(y) z_{j+1,n}. \end{aligned}$$

Without loss of generality, $\xi(i+1,k) < \xi(j+1,n)$.

Since $x,y\in U_{\max\{i,j\}+1} \subset C_{\max\{i,j\}}$, we conclude that

$$\max\{|z_m^*(f(x))|, |z_m^*(f(y))|\} < \delta^3/100 \quad \text{for all } m \ge \max\{i,j\} \tag{2.2}$$

according to the definition of $C_{\max\{i,j\}}$.

We shall consider five cases.

1) $\lambda_{j+1}(y) > \delta^2/10$. In this case, put $m = m_{\xi(j+1,n)}$ and observe that $|z_m^*(z_{j+1,n})| = |z_{j+1,n}^*(z_{j+1,n})| \ge \delta$. Since $\max\{\xi(j,n), \xi(i+1,k), \xi(i,k)\} < \xi(j+1,n)$, we conclude that

$$\max\{|z_m^*(z_{j,n})|, |z_m^*(z_{i+1,k})|, |z_m^*(z_{i,k})|\} < \delta^3/100$$

by Claim 4(2). It follows from (2.2) and $\max\{i,j\} \le \max\{\xi(i,k), \xi(j,n)\}$ that

$$\max\{|z_m^*(f(x))|, |z_m^*(f(y))|\} < \delta^3/100.$$

Now we see that

$$\begin{aligned}
&|z^*_m(f_n(y) - f_k(x))| \\
&= |z^*_m(\lambda_{j+1}(y)z_{j+1,n} + \lambda_j(y)z_{j,n} + f(y) - f(x) - \lambda_{i+1}(x)z_{i+1,k} - \lambda_i(x)z_{i,k})| \\
&\ge \lambda_{j+1}(y)|z^*_m(z_{j+1,n})| \\
&\quad - |z^*_m(\lambda_j(y)z_{j,n} + f(y) - f(x) - \lambda_i(x)z_{i,k} + \lambda_{i+1}(x))z_{i+1,k})| \\
&> \frac{\delta^2}{10}\delta - 5\frac{\delta^3}{100} \ge \frac{\delta^3}{20}.
\end{aligned}$$

2) $\lambda_{j+1}(y) \le \delta^2/10$ and $\xi(j,n) > \xi(i+1,k)$. In this case put $m = \xi(j,n)$. Arguing as in the preceding case, we can show that

$$\max\{|z^*_m(f(x))|, |z^*_m(f(y))|\} < \frac{\delta^3}{100}, \quad \max\{z^*_m(z_{i+1,k})|, |z^*_m(z_{i,k})|\} < \frac{\delta^3}{100}.$$

Then

$$\begin{aligned}
&|e^*_m(f_n(y) - f_k(x))| \\
&= |e^*_m(\lambda_j(y)z_{j,n} + \lambda_{j+1}(y)z_{j+1,n} + f(y) - f(x) - \lambda_{i+1}(x)z_{i+1,k} - \lambda_i(x)z_{i,k})| \\
&\ge \lambda_j(y)|z^*_m(z_{j,n})| - \lambda_{j+1}(y)|z^*_m(z_{j+1,n})| \\
&\quad - |z^*_m(f(y) - f(x) - \lambda_i(x)z_{i,k} + \lambda_{i+1}(x)z_{i+1,k})| \\
&\ge (1 - \lambda_{j+1}(y))\delta - \frac{\delta^2}{10} - 4\frac{\delta^3}{100} \ge \left(1 - \frac{\delta^2}{10}\right)\delta - \frac{\delta^2}{10} - \frac{\delta^3}{25} > \frac{\delta^3}{20}.
\end{aligned}$$

3) $\lambda_{j+1}(y) \le \delta^2/10$, $\xi(j,n) < \xi(i+1,k)$, and $\lambda_{i+1}(x) > \delta/4$. In this case put $m = \xi(i+1,k)$ and observe that

$$\begin{aligned}
|z^*_m(f_k(x) - f_n(y))| &\ge \lambda_{i+1}(x)|z^*_m(z_{i+1,k})| - \lambda_{j+1}(y)|z^*_m(z_{j+1,n})| \\
&\quad - |z^*_m(f(x) + \lambda_i(x)z_{i,k} - f(y) - \lambda_j(y)z_{j,n})| \\
&> \frac{\delta}{4}\delta - \frac{\delta^2}{10} - 4\frac{\delta^3}{100} > \frac{\delta^3}{20}.
\end{aligned}$$

4) $\lambda_{j+1}(y) \le \delta^2/10$, $\xi(j,n) < \xi(i+1,k)$, $\lambda_{i+1}(x) \le \delta/4$, and $\xi(i,k) < \xi(j,n)$. In this case put $m = m_{\xi(j,n)}$ and observe that $\lambda_j(y) = (1-\lambda_{j+1}(y)) > 1 - \delta^2/10 \ge 9/10$ and thus

$$\begin{aligned}
|z^*_m(f_k(x) - f_n(y))| &\ge \lambda_j(y)|z^*_m(z_{j,n})| - \lambda_{j+1}(y)|z^*_m(z_{j+1,n})| \\
&\quad - \lambda_{i+1}(x)|z^*_m(z_{i+1,k})| - |z^*_m(f(x) - f(y) - \lambda_i(x)z_{i,k})| \\
&\ge \frac{9}{10}\delta - \frac{\delta^2}{10} - \frac{\delta}{4} - 3\frac{\delta^3}{100} > \frac{\delta^3}{20}.
\end{aligned}$$

5) $\lambda_{j+1}(y) \le \delta^2/10$, $\xi(j,n) < \xi(i+1,k)$, $\lambda_{i+1}(x) \le \delta/4$, and $\xi(i,k) > \xi(j,n)$. In this case put $m = m_{\xi(i,k)}$ and observe that $\lambda_i(x) = 1 - \lambda_{i+1}(x) \ge$

$1-\delta/4 \geq 3/4$. Then

$$\begin{aligned}|z_m^*(f_k(x)-f_n(y))| \geq \lambda_i(x)|z_m^*(z_{i,k})| - \lambda_{i+1}(x)|z_m^*(z_{i+1,k})| \\ - \lambda_{j+1}(y)|z_m^*(z_{j+1,n})| - |z_m^*(f(x)-f(y)-\lambda_j(y)z_{j,n})| \\ \geq \frac{3}{4}\delta - \frac{\delta}{4} - \frac{\delta^2}{10} - 3\frac{\delta^3}{100} > \frac{\delta^3}{20}. \ \blacksquare\end{aligned}$$

3. Proof of Theorem 5. Given a non-separable convex set X in a Fréchet space L, consider the convex cone

$$C = \{(tx,t) : x \in X,\ t \in [0,\infty)\} \subset L \times \mathbb{R}$$

in $L \times \mathbb{R}$ with base $X \times \{1\}$ which will be identified with X.

Let $\mathrm{pr} : C \to \mathbb{R}_+$, $\mathrm{pr} : (x,t) \mapsto t$, denote the projection onto the second coordinate. Observe that the map $r : C \setminus \{0\} \to X$, $r : (x,t) \mapsto x/t$, determines a retraction of $C \setminus \{0\}$ onto X. This retraction restricted to the set $C_{[1/3,3]} = \mathrm{pr}^{-1}([1/3,3])$ is a perfect map.

To prove that X has LFAP, fix an open cover $\mathcal{U}$ of X. For each open set $U \in \mathcal{U}$ consider the set $\tilde{U} = \{(tx,t) : x \in U,\ 1/3 < t < 3\}$. Then $\tilde{\mathcal{U}} = \{\mathrm{pr}^{-1}(\mathbb{R} \setminus [1/2,2]), \tilde{U} : U \in \mathcal{U}\}$ is an open cover of C.

By Lemma 2, the convex cone C has SAP, and hence LFAP by Lemma 1. Consequently, there is a map $f : C \times \omega \to C$ that is $\tilde{\mathcal{U}}$-near to the projection $f_0 : C \times \omega \to C$, $f_0 : (x,n) \mapsto x$, and the family $(f(C \times \{n\}))_{n\in\omega}$ is locally finite in C. Let $\tilde{f} = f|X \times \omega$ and $\tilde{f}_0 = f_0|X \times \omega$. It follows from the choice of the cover $\tilde{\mathcal{U}}$ that $\tilde{f}(X \times \omega) \subset C_{[1/3,3]}$ and the map $g = r \circ \tilde{f} : X \times \omega \to X$ is $\mathcal{U}$-near to the projection $\tilde{f}_0 : X \times \omega \to X$.

Since the family $(\tilde{f}(X \times \{n\})_{n\in\omega}$ is locally finite in $C_{[1/3,3]}$ and the map $r : C_{[1/3,3]} \to X$ is perfect, the family $(r \circ \tilde{f}(X \times \{n\}))_{n\in\omega}$ is locally finite in X, witnessing that X has LFAP.

4. Open problems. The proof of Theorem 5 heavily exploits the machinery of Banach space theory and does not work in the non-locally convex case. This leaves the following problem open:

PROBLEM 1. *Is each non-separable completely metrizable convex AR-subset of a linear metric space homeomorphic to a Hilbert space?*

Even a weaker problem seems to be open:

PROBLEM 2. *Is each complete linear metric AR-space homeomorphic to a Hilbert space?*

This is true in the separable case: see [4], [5].

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Taras Banakh
Wydział Matematyczno-Przyrodniczy
Uniwersytet Jana Kochanowskiego
Świętokrzyska 15
25-406 Kielce, Poland
and
Department of Mathematics
Ivan Franko National University of Lviv
Universytetska 1
79000 Lviv, Ukraine
E-mail: t.o.banakh@gmail.com

Robert Cauty
Faculté de Mathématiques
Université Paris VI
4, place Jussieu
75005 Paris, France
E-mail: cauty@math.jussieu.fr

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