STUDIA MATHEMATICA 205 (1) (2011)

Asymptotic Fourier and Laplace transformations for hyperfunctions

by

Michael Langenbruch (Oldenburg)

Abstract. We develop an elementary theory of Fourier and Laplace transformations for exponentially decreasing hyperfunctions. Since any hyperfunction can be extended to an exponentially decreasing hyperfunction, this provides simple notions of asymptotic Fourier and Laplace transformations for hyperfunctions, improving the existing models. This is used to prove criteria for the uniqueness and solvability of the abstract Cauchy problem in Fréchet spaces.

1. Introduction. Fourier and Laplace transformations have a wide scope of applications in analysis and especially in the theory of partial differential operators and convolution equations. However, the use of these basic tools is somehow restricted by the fact that exponential bounds are needed to apply the transformations directly. To use these methods also in the case where no bounds are at hand, several models have been proposed to extend the transformations to asymptotic versions keeping the main structural properties needed in applications. We will mention here only a few models connected to the present paper. The reader is referred to the huge literature for further information. Based on earlier work of Vignaux [33], Lumer and Neubrander [20, 21] studied an asymptotic Laplace transformation in $L^1_{\mathrm{loc}}([0,\infty[)$ by considering the asymptotic behavior of the local Laplace transforms defined by

$$L_j(f)(z) := \int_0^j e^{-zt} f(t)\,dt \qquad \text{for } j \to \infty.$$

Komatsu proposed a different way in a series of papers [12–14] (see Section 6 for definitions needed below): he first extended the hyperfunction $[u] \in \mathcal{B}(\mathbb{R})$ to a Laplace hyperfunction $\overline{[u]} \in \mathcal{B}^{\exp}_{[0,\infty]}$, i.e. he chose a represent-

2010 *Mathematics Subject Classification*: Primary 44A10, 42A38; Secondary 46F15, 34A12, 47A10.

Key words and phrases: asymptotic Fourier transformation, asymptotic Laplace transformation, hyperfunction, abstract Cauchy problem, asymptotic resolvent.

DOI: 10.4064/sm205-1-4

ing function $h \in [u]$ with exponential growth outside each cone near $[0,\infty[$, and then introduced his Laplace transformation $\mathfrak{L}_{\mathrm{Kom}}$ on $\mathcal{B}^{\exp}_{[0,\infty]}$ obtaining the space $\mathfrak{L}_{\mathrm{Kom}}\mathcal{B}^{\exp}_{[0,\infty]}$ of germs of holomorphic functions of exponential type 0 near the half-circle $S_\infty := \{\infty e^{i\varphi} \mid |\varphi| < \pi/2\}$ at ∞. The extension of $[u]$ to $\overline{[u]} \in \mathcal{B}^{\exp}_{[0,\infty]}$ is unique only up to $\mathcal{B}^{\exp}_{\infty}$, i.e. up to Laplace hyperfunctions supported at ∞. Thus Komatsu's asymptotic Laplace transform $\mathfrak{L}_{\mathcal{B},\mathrm{Kom}}$ is a bijection

$$\mathfrak{L}_{\mathcal{B},\mathrm{Kom}} : \mathcal{B}([0,\infty[) \to \mathfrak{L}_{\mathrm{Kom}}\mathcal{B}^{\exp}_{[0,\infty[}/\mathfrak{L}_{\mathrm{Kom}}\mathcal{B}^{\exp}_{\infty}$$

where $\mathfrak{L}_{\mathrm{Kom}}\mathcal{B}^{\exp}_{\infty}$ is a space of exponentially decreasing holomorphic germs near S_∞ (see Section 6 for the precise definitions). Instead of Laplace hyperfunctions, Fourier hyperfunctions (see [11]) or modified hyperfunctions (see [28]) may be used in this procedure leading to two spaces of holomorphic functions of exponential type 0 on the right half-plane as Laplace images. Bounded hyperfunctions have been considered by Kunstmann [15] in connection with the Post–Widder inversion formula.

Accordingly, there has been some discussion on the appropriate way of defining an asymptotic Laplace transform (see e.g. [13, Section 4], [21, Section 2] and [15, end of Section 2]). The reader will agree that a satisfactory theory should meet the following conditions:

- (I) The model contains a wide class of generalized functions and is based on an elementary version of the Laplace transform defined on a space of generalized functions which has a simple topological structure.
- (II) For (generalized) functions with compact support, the Laplace transform should coincide with the Fourier–Laplace transform. Moreover, the Laplace transform should be compatible with convolution and multiplication by (a large class of) functions.
- (III) The Laplace transform should be asymptotic, i.e. in applications calculations should be needed near S_∞ only.

Apparently, the above theories satisfy these requirements only partially. In the present paper we will present a model for Fourier and Laplace transformations and their asymptotic versions which is satisfactory in the above sense. We will explain this here for the asymptotic Laplace transform which is based on the space $\mathcal{G}([0,\infty])$ of exponentially decreasing hyperfunctions of type $-\infty$ (supported in $[0,\infty]$), defined as follows. Let

$$\mathcal{H}_{-\infty}(\mathbb{C} \setminus [0,\infty[) := \{f \in H(\mathbb{C} \setminus [0,\infty[) \mid \forall k \in \mathbb{N} : \sup_{z \in W_k} |f(z)| e^{k|\mathrm{Re}\, z|} < \infty\}$$

where

$$W_k := \{z \in \mathbb{C} \mid |\mathrm{Im}\, z| \le k, \mathrm{dist}(z, [0,\infty[) > 1/k\}.$$

Then

$$\mathcal{G}([0,\infty]) := \mathcal{H}_{-\infty}(\mathbb{C}\setminus[0,\infty[)/\mathcal{H}_{-\infty}(\mathbb{C})$$

is the space of corresponding formal boundary values ($\mathcal{H}_{-\infty}(\mathbb{C})$ is defined at the beginning of Section 2). Since $g \in \mathcal{H}_{-\infty}(\mathbb{C}\setminus[0,\infty[)$ is exponentially decreasing of any order on W_k, the Laplace transform $\mathfrak{L}([g])$ of $[g] \in \mathcal{G}([0,\infty])$ can be defined by the natural absolutely convergent integral

$$\mathfrak{L}([g])(z) := \int_{\gamma_c} e^{-z\xi} g(\xi)\,d\xi, \qquad z \in \mathbb{C},$$

where $\gamma_c := \{t \pm ic \mid t \geq -c\} \cup \{-c+it \mid |t| \leq c\}$ (with clockwise orientation) and where $c > 0$ is arbitrary. The Laplace transform on $\mathcal{G}([0,\infty])$ is introduced by resorting to the elementary theory of Fourier transformation on Schwartz's space $\mathcal{S}(\mathbb{R})$, and it is easily seen that the Laplace transformation is a topological isomorphism from $\mathcal{G}([0,\infty])$ onto the weighted space

$$\mathfrak{L}\mathcal{G}_{[0,\infty]} := \{f \in H(\mathbb{C}) \mid \forall k \in \mathbb{N} : \sup_{\operatorname{Re} z \geq -k} |f(z)|e^{-|z|/k} < \infty\}$$

of entire functions (see Section 4). $\mathcal{G}([0,\infty])$ and $\mathfrak{L}\mathcal{G}_{[0,\infty]}$ are nuclear Fréchet spaces.

Convolution and multiplication by entire functions of exponential type can be defined on $\mathcal{G}([0,\infty])$ like for functions (see the respective definitions before Proposition 2.7 and in Proposition 2.9), and for $[f],[g] \in \mathcal{G}([0,\infty])$ and entire functions h of exponential type we have

$$\mathfrak{L}([f]*[g]) = \mathfrak{L}([f])\mathfrak{L}([g]) \quad \text{and} \quad \mathfrak{L}(h[f]) = h(-\partial)\mathfrak{L}([f]). \tag{1.1}$$

Since any hyperfunction can be extended to an exponentially decreasing hyperfunction by [18], our asymptotic Laplace transform $\mathfrak{L}_{\mathcal{B}}$ is a linear bijection

$$\mathfrak{L}_{\mathcal{B}} : \mathcal{B}([0,\infty[) \to \mathfrak{L}\mathcal{G}_{[0,\infty]}/\mathfrak{L}\mathcal{G}_{\infty}$$

where

$$\mathfrak{L}\mathcal{G}_{\infty} := \{f \in H(\mathbb{C}) \mid \forall k \in \mathbb{N} : \sup_{\operatorname{Re} z \geq -k} |f(z)|e^{-|z|/k+k|\operatorname{Re} z|} < \infty\}.$$

The formulas (1.1) also hold for the asymptotic Laplace transform $\mathfrak{L}_{\mathcal{B}}$ and general hyperfunctions $[f],[g] \in \mathcal{B}([0,\infty[)$.

The requirement (III) seems to be impossible in our model since the asymptotic Laplace image consists of the quotient space $\mathfrak{L}\mathcal{G}_{[0,\infty]}/\mathfrak{L}\mathcal{G}_{\infty}$ of entire functions. However we can prove that the canonical inclusions

$$\mathfrak{L}\mathcal{G}_{[0,\infty]} \subset \mathfrak{L}_{\mathrm{Kom}}\mathcal{B}^{\exp}_{[0,\infty[} \quad \text{and} \quad \mathfrak{L}\mathcal{G}_{\infty} \subset \mathfrak{L}_{\mathrm{Kom}}\mathcal{B}^{\exp}_{\infty}$$

define a bijective linear mapping

$$I : \mathfrak{L}\mathcal{G}_{[0,\infty]}/\mathfrak{L}\mathcal{G}_{\infty} \to \mathfrak{L}_{\mathrm{Kom}}\mathcal{B}^{\exp}_{[0,\infty[}/\mathfrak{L}_{\mathrm{Kom}}\mathcal{B}^{\exp}_{\infty}$$

such that $I \circ \mathfrak{L}_{\mathcal{B}} = \mathfrak{L}_{\mathcal{B},\mathrm{Kom}}$. Thus, Komatsu's and our Laplace range spaces are canonically linearly isomorphic respecting the Laplace transforms, which is somehow unexpected (compare also the remarks of Komatsu [13, Section 4]). Hence also (III) is satisfied.

The paper is organized as follows: The Fourier transformation and the asymptotic Fourier transformation are developed in Sections 2 and 3, respectively. These are used to study the Laplace transform and the asymptotic Laplace transform in Sections 4 and 5. The connection of our asymptotic Laplace transform to Komatsu's version is clarified in Section 6. Finally, we will give some application of our model to existence and uniqueness of solutions of the abstract Cauchy problem in Fréchet spaces. Here we profit from the fact that our model space $\mathcal{G}([0,\infty])$ is a nuclear Fréchet space. To illustrate these results we consider an infinite system of a first order differential equation with constant coefficients, i.e. we study the (ACP) defined by continuous linear operators in the space ω of all sequences (i.e. by matrices with finite rows; see [19], [31] and [7] for corresponding results on such systems and references to earlier work on the (ACP) in ω).

2. Fourier transformation. In this section we will study the Fourier transformation on the space

$$\mathcal{G}(\overline{\mathbb{R}}) := \mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R})/\mathcal{H}_{-\infty}(\mathbb{C})$$

of exponentially decreasing hyperfunctions of type $-\infty$. Here

$$\mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R}) := \{f \in H(\mathbb{C}\setminus\mathbb{R}) \mid \forall k \in \mathbb{N} : \|f\|_k := \sup_{z\in F_k} |f(z)|e^{k|\mathrm{Re}\, z|} < \infty\}$$

where $F_k := \{z \in \mathbb{C} \mid 1/k \le |\mathrm{Im}\, z| \le k\}$ and

$$\mathcal{H}_{-\infty}(\mathbb{C}) := \{f \in H(\mathbb{C}) \mid \forall k \in \mathbb{N} : \sup_{|\mathrm{Im}\, z|\le k} |f(z)|e^{k|\mathrm{Re}\, z|} < \infty\}.$$

In fact, the Fourier transformation on $\mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R})$ is introduced by resorting to the elementary theory of Fourier transformation on Schwartz's space $\mathcal{S}(\mathbb{R})$.

With obvious notation we have $\mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R}) = \mathcal{H}_{-\infty}(\mathbb{C}_+) \oplus \mathcal{H}_{-\infty}(\mathbb{C}_-)$ where $\mathbb{C}_\pm := \{z \in \mathbb{C} \mid \pm\mathrm{Im}\, z > 0\}$. It is thus convenient to study the Fourier transform first on $\mathcal{H}_{-\infty}(\mathbb{C}_+)$ and on $\mathcal{H}_{-\infty}(\mathbb{C}_-)$ separately. Since any function in $\mathcal{H}_{-\infty}(\mathbb{C}_\pm)$ is exponentially decreasing of any order on any strip in $\mathbb{C}_\pm$, the Fourier transform of $g \in \mathcal{H}_{-\infty}(\mathbb{C}_\pm)$ can be defined by the formula

$$\mathfrak{F}^{\pm}(g)(z) := \int_{\mathrm{Im}\,\xi=\pm c} e^{-iz\xi} g(\xi)\, d\xi, \qquad z \in \mathbb{C},$$

where $c > 0$ is arbitrary. The integral is absolutely convergent and independent of c for any $z \in \mathbb{C}$ by Cauchy's theorem. The Fourier image of

$\mathcal{H}_{-\infty}(\mathbb{C}_\pm)$ can be precisely described using the following weighted spaces of entire functions. Let $w_j^+(x) := x/j$ if $x \geq 0$ and $w_j^+(x) := jx$ if $x \leq 0$, and set $w_j^-(x) := w_j^+(-x)$. Define

$$\mathcal{H}_0^\pm := \{f \in H(\mathbb{C}) \mid \forall j \in \mathbb{N} : |f|_j^\pm := \sup_{|\operatorname{Im} z| \leq j} |f(z)| e^{-w_j^\pm(\operatorname{Re} z)} < \infty\}.$$

THEOREM 2.1. *The Fourier transformation*

$$\mathfrak{F}^\pm : \mathcal{H}_{-\infty}(\mathbb{C}_\pm) \to \mathcal{H}_0^\pm$$

is a topological isomorphism. The inverse Fourier transform is provided by

$$(\mathfrak{F}^\pm)^{-1}(h)(z) := S(h)(z) := \frac{1}{2\pi} \int_{\operatorname{Im} \xi = c} e^{iz\xi} h(\xi)\, d\xi \quad \text{for } z \in \mathbb{C}_\pm,\ h \in \mathcal{H}_0^\pm,$$

where $c \in \mathbb{R}$ is arbitrary.

Proof. We only give the proof for $\mathfrak{F}^+$ and $\mathcal{H}_{-\infty}(\mathbb{C}_+)$ since $\mathfrak{F}^-(g)(z) = \mathfrak{F}^+(\check{g})(-z)$ for $g \in \mathcal{H}_{-\infty}(\mathbb{C}_-)$.

(a) For $1/j \leq c \leq j$ and for $g \in \mathcal{H}_{-\infty}(\mathbb{C}_+)$ we have

$$\begin{aligned} |\mathfrak{F}^+(g)(x+iy)| &= \Big| \int_{\mathbb{R}} e^{(-ix+y)(t+ic)} g(t+ic)\, dt \Big| \\ &\leq 2e^{cx} \|g\|_{j+1} \quad \text{for } |y| \leq j. \end{aligned} \tag{2.1}$$

Hence $\mathfrak{F}^+(g) \in \mathcal{H}_0^+$ and $\mathfrak{F}^+ : \mathcal{H}_{-\infty}(\mathbb{C}_+) \to \mathcal{H}_0^+$ is continuous.

(b) The integral defining $S(h)$ is independent of c by Cauchy's theorem. For any $c \in \mathbb{R}$ and $h \in \mathcal{H}_0^+$ we thus get

$$\begin{aligned} |S(h)(x+iy)| &= \frac{1}{2\pi} \Big| \int_{\mathbb{R}} e^{(ix-y)(t+ic)} h(t+ic)\, dt \Big| \\ &\leq C_j e^{-cx} |h|_{j+1} \quad \text{if } 1/j \leq y \leq j \text{ and } |c| \leq j. \end{aligned}$$

Hence $S(h) \in \mathcal{H}_{-\infty}(\mathbb{C}_+)$ and $S : \mathcal{H}_0^+ \to \mathcal{H}_{-\infty}(\mathbb{C}_+)$ is continuous.

(c) $\mathfrak{F}^+$ is injective since (with $c = 1$)

$$\mathfrak{F}^+(g)(z) = e^z \widehat{g(\cdot + i)}(z) \quad \text{for } g \in \mathcal{H}_{-\infty}(\mathbb{C}_+) \tag{2.2}$$

and since the Fourier transform is injective on $\mathcal{S}$. For $h \in \mathcal{H}_0^+$ we have $e^{-\cdot}\, h \in \mathcal{S}$. By the Fourier inversion formula on $\mathcal{S}$ we thus get (with $c = 0$)

$$S(h)(x+i) = \mathfrak{F}^{-1}(e^{-\cdot}\, h)(x) \quad \text{for } x \in \mathbb{R}. \tag{2.3}$$

Combining (2.2) and (2.3) we get $\mathfrak{F}^+(S(h))(\tau) = h(\tau)$ for $\tau \in \mathbb{R}$ and $h \in \mathcal{H}_0^+$. Hence $\mathfrak{F}^+ \circ S = \mathrm{Id}$ on $\mathcal{H}_0^+$, $\mathfrak{F}^+$ is surjective and $S = (\mathfrak{F}^+)^{-1}$. ∎

For $[g] \in \mathcal{H}_{-\infty}(\mathbb{C} \setminus \mathbb{R})/\mathcal{H}_{-\infty}(\mathbb{C})$ the natural definition of the Fourier transform $\mathfrak{F}([g])$ is provided by the formula

$$\mathfrak{F}([g])(z) := \int_{|\operatorname{Im} \xi| = c} e^{-iz\xi} g(\xi)\, d\xi, \quad z \in \mathbb{C},$$

with clockwise orientation, where $c > 0$ is arbitrary. The integral is absolutely convergent and independent of c for any $z \in \mathbb{C}$. Moreover, it is well defined on $\mathcal{H}_{-\infty}(\mathbb{C} \setminus \mathbb{R})/\mathcal{H}_{-\infty}(\mathbb{C})$ since

$$\int\limits_{|\mathrm{Im}\,\xi|=c} e^{-iz\xi} h(\xi)\,d\xi = 0 \quad \text{for any } h \in \mathcal{H}_{-\infty}(\mathbb{C})$$

by Cauchy's theorem. By Theorem 2.1 the Fourier image of the space $\mathcal{H}_{-\infty}(\mathbb{C} \setminus \mathbb{R})/\mathcal{H}_{-\infty}(\mathbb{C})$ is contained in the weighted space of entire functions

$$\mathfrak{F}\mathcal{G}(\overline{\mathbb{R}}) := \{f \in H(\mathbb{C}) \mid \forall k \in \mathbb{N} : |f|_k := \sup_{|\mathrm{Im}\,z| \le k} |f(z)| e^{-|\mathrm{Re}\,z|/k} < \infty\}.$$

To see that $\mathfrak{F}(\mathcal{H}_{-\infty}(\mathbb{C} \setminus \mathbb{R})/\mathcal{H}_{-\infty}(\mathbb{C})) = \mathfrak{F}\mathcal{G}(\overline{\mathbb{R}})$ we notice that we have the following canonical decomposition of $\mathfrak{F}\mathcal{G}(\overline{\mathbb{R}})$:

Proposition 2.2. *Let*

$$\begin{aligned} S &: \mathcal{H}_{-\infty}(\mathbb{C}) \to \mathcal{H}_0^+ \times \mathcal{H}_0^-, & S(f) &:= (f|_{\mathbb{C}_+}, f|_{\mathbb{C}_-}), \\ T &: \mathcal{H}_0^+ \times \mathcal{H}_0^- \to \mathfrak{F}\mathcal{G}(\overline{\mathbb{R}}), & T(f,g) &:= f - g. \end{aligned}$$

Then the sequence

$$0 \to \mathcal{H}_{-\infty}(\mathbb{C}) \xrightarrow{S} \mathcal{H}_0^+ \times \mathcal{H}_0^- \xrightarrow{T} \mathfrak{F}\mathcal{G}(\overline{\mathbb{R}}) \to 0$$

is exact and splits.

Proof. S and T are continuous and $\ker(T) = S(\mathcal{H}_{-\infty}(\mathbb{C}))$. Let

$$\varphi(z) := \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{z} e^{-\xi^2}\,d\xi. \tag{2.4}$$

Then we have for $x \le -1$ by Cauchy's theorem

$$\begin{aligned} |\varphi(x+iy)| &= \frac{1}{\sqrt{\pi}} \Big| \int\limits_{-\infty}^{x} e^{-(t+iy)^2}\,dt \Big| \\ &\le \frac{1}{\sqrt{\pi}} e^{y^2} \int\limits_{-\infty}^{x} (-t) e^{-t^2}\,dt = \frac{1}{2\sqrt{\pi}} e^{-x^2+y^2}. \end{aligned} \tag{2.5}$$

Since $\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}$ we similarly get, for $x \ge 1$,

$$|1 - \varphi(x+iy)| = \frac{1}{\sqrt{\pi}} \Big| \int\limits_{x}^{\infty} e^{-(t+iy)^2}\,dt \Big| \le \frac{1}{2\sqrt{\pi}} e^{-x^2+y^2}. \tag{2.6}$$

Set $W(h) := (h\varphi, h(\varphi - 1))$ for $h \in \mathfrak{F}\mathcal{G}(\overline{\mathbb{R}})$. Then $W : \mathfrak{F}\mathcal{G}(\overline{\mathbb{R}}) \to \mathcal{H}_0^+ \times \mathcal{H}_0^-$ is defined and continuous by (2.5) and (2.6), and clearly $T \circ W$ is the identity mapping on $\mathfrak{F}\mathcal{G}(\overline{\mathbb{R}})$. ∎

Combining Theorem 2.1 and Proposition 2.2 we get

THEOREM 2.3. *The Fourier transformation*

$$\mathfrak{F} : \mathcal{G}(\overline{\mathbb{R}}) \to \mathfrak{F}\mathcal{G}(\overline{\mathbb{R}})$$

is a topological isomorphism.

Proof. The mapping $M : (f,g) \to (\mathfrak{F}^+(f), \mathfrak{F}^-(g))$ defines a topological isomorphism $\mathcal{H}_{-\infty}(\mathbb{C} \setminus \mathbb{R}) \to \mathcal{H}_0^+ \times \mathcal{H}_0^-$ by Theorem 2.1. Since $\mathfrak{F}^+$ and $\mathfrak{F}^-$ coincide on $\mathcal{H}_{-\infty}(\mathbb{C})$ with the Fourier transform which is a topological isomorphism on $\mathcal{H}_{-\infty}(\mathbb{C})$ by [17], the claim now follows from Proposition 2.2 since $\mathfrak{F} = T \circ M$. ∎

The inverse Fourier transform on $\mathfrak{F}\mathcal{G}(\overline{\mathbb{R}})$ is given by (the equivalence class of)

$$\mathfrak{F}^{-1}(h)(z) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{iz\xi} h(\xi)\varphi(\xi)\, d\xi \qquad \text{for } z \in \mathbb{C}_+, \tag{2.7}$$

$$\mathfrak{F}^{-1}(h)(z) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{iz\xi} h(\xi)(\varphi(\xi) - 1)\, d\xi \qquad \text{for } z \in \mathbb{C}_-, \tag{2.8}$$

with φ from (2.4).

The space $\mathcal{B}(K) := A(K)'$ of hyperfunctions with support in a compact $K \subset \mathbb{R}$ is canonically embedded in $\mathcal{G}(\overline{\mathbb{R}})$ by means of the following canonical representing functions for $\nu \in A(K)'$:

$$u_\nu(z) := \left\langle {}_\xi\nu, \frac{ie^{-(z-\xi)^2}}{2\pi(z-\xi)} \right\rangle \qquad \text{for } z \in \mathbb{C} \setminus K. \tag{2.9}$$

EXAMPLE 2.4. Let $K \subset \mathbb{R}$ be compact and let $\nu \in A(K)'$. Then

$$\mathfrak{F}([u_\nu])(z) = \langle {}_\xi\nu, e^{-iz\xi} \rangle =: \widehat{\nu}(z) \qquad \text{for } z \in \mathbb{C}$$

is the Fourier–Laplace transform of ν.

Proof. Since the Riemann sums converge uniformly for ξ near K, we have

$$\mathfrak{F}([u_\nu])(z) = \left\langle {}_\xi\nu, \int_{|\mathrm{Im}\, w| = c} \frac{ie^{-(w-\xi)^2}}{2\pi(w-\xi)} e^{-izw}\, dw \right\rangle = \langle {}_\xi\nu, e^{-iz\xi} \rangle$$

by Cauchy's integral formula since $g(w) := e^{-(w-\xi)^2 - izw}/(w-\xi)$ decreases exponentially at ∞. ∎

We now discuss some standard operations and their connection with the Fourier transformation, beginning with two easy examples:

EXAMPLE 2.5. Let $\tau_h(f) := f(\cdot - h)$, $h \in \mathbb{R}$, be the shift operator on $\mathcal{H}_{-\infty}(\mathbb{C} \setminus \mathbb{R})$. Then

$$\mathfrak{F}(\tau_h(f))(z) = \mathfrak{F}(f)(z) e^{-ihz}.$$

Example 2.6. Let $P(-i\partial) := \sum_{k=0}^{\infty} \frac{c_k}{k!}(-i\partial)^k$ where P is of exponential type 0, i.e.

$$\forall\varepsilon > 0\ \exists C_\varepsilon > 0\ \forall k \in \mathbb{N}_0 : \quad |c_k| = |P^{(k)}(0)| \le C_\varepsilon \varepsilon^k.$$

Then $P(-i\partial) \in A(\{0\})'$ is a continuous operator on $\mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R})$ and on $\mathcal{H}_{-\infty}(\mathbb{C})$, hence also on $\mathcal{G}(\overline{\mathbb{R}})$, and for $[f] \in \mathcal{G}(\overline{\mathbb{R}})$ we have

$$\mathfrak{F}(P(-i\partial)[f])(z) = P(z)\mathfrak{F}([f])(z) \quad \text{for } z \in \mathbb{C}.$$

Proof. For differential operators of finite order this is clear by partial integration. Since the sum defining $P(-i\partial)$ converges on $\mathcal{G}(\overline{\mathbb{R}})$ and since $\mathfrak{F}$ is continuous, this proves the claim. ∎

More generally, we define convolution on $\mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R})$ by

$$(f * g)(z) := \int\limits_{|\mathrm{Im}(z-w)|=c} f(w)g(z-w)\,dw \quad \text{for } 0 < c < |\mathrm{Im}\, z|,\ z \in \mathbb{C}\setminus\mathbb{R}$$

with counterclockwise orientation.

Proposition 2.7.

(a) *$f*g \in \mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R})$ if $f, g \in \mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R})$. The convolution is bilinear, continuous, commutative and associative.*

(b) *For $f, g \in \mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R})$ we have*

$$\mathfrak{F}(f * g) = \mathfrak{F}(f)\mathfrak{F}(g). \tag{2.10}$$

(c) *$f * g \in \mathcal{H}_{-\infty}(\mathbb{C})$ if $f \in \mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R})$ and $g \in \mathcal{H}_{-\infty}(\mathbb{C})$.*

(d) *The convolution $[f] * [g] := [f * g]$ is a well defined bilinear, continuous, commutative and associative operation on $\mathcal{G}(\overline{\mathbb{R}})$ and*

$$\mathfrak{F}([f] * [g]) = \mathfrak{F}([f])\mathfrak{F}([g]) \quad \textit{for } [f], [g] \in \mathcal{G}(\overline{\mathbb{R}}). \tag{2.11}$$

Proof. (a) The first claim follows by an easy estimate directly from the definitions. The next claims follow from the corresponding results in $\mathcal{S}(\mathbb{R})$.

(b) This is proved by changing the order of integration.

(c) Since $g \in \mathcal{H}_{-\infty}(\mathbb{C})$ we may shift the path of integration:

$$\begin{aligned}(f * g)(z) &= (g * f)(z) \\ &= \int\limits_{|\mathrm{Im}\, w| = |\mathrm{Im}\, z| + c} g(w)f(z-w)\,dw \quad \text{for } 0 < c,\ z \in \mathbb{C}\setminus\mathbb{R}.\end{aligned}$$

The resulting integral clearly is in $\mathcal{H}_{-\infty}(\mathbb{C})$.

(d) The convolution is well defined on $\mathcal{G}(\overline{\mathbb{R}})$ by (c); the rest follows from (a) and (b). ∎

We next discuss multiplication by functions, again starting with a simple example:

Example 2.8. For $w \in \mathbb{C}$ and $f \in \mathcal{G}(\overline{\mathbb{R}})$ we have

$$\mathfrak{F}([e^{iw\cdot} f])(z) = \mathfrak{F}([f])(z-w) = e^{iw(i\partial_z)}\mathfrak{F}([f])(z) \quad \text{for } z \in \mathbb{C}.$$

Proposition 2.9. *Let h be an entire function of exponential type, i.e.*

$$\exists C, C_1 > 0\ \forall k \in \mathbb{N}_0: \quad |h^{(k)}(0)| \le C_1 C^k.$$

The multiplication operator $M_h(f) := hf$ is continuous on $\mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R})$ and on $\mathcal{H}_{-\infty}(\mathbb{C})$, hence also on $\mathcal{G}(\overline{\mathbb{R}})$, and for $[f] \in \mathcal{G}(\overline{\mathbb{R}})$ we have

$$\mathfrak{F}(h[f])(z) = h(i\partial)\mathfrak{F}([f])(z) \quad \textit{for } z \in \mathbb{C}. \tag{2.12}$$

Proof. The first claims are obvious. (2.12) holds for polynomials h by differentiation with respect to the parameter z. Since the Taylor series h_n of h converges with respect to $\sup_{z\in\mathbb{C}} |h(z)|e^{-j|z|}$ for some $j \in \mathbb{N}$, we have $h_n[f] \to h[f]$ in $\mathcal{G}(\overline{\mathbb{R}})$. Since $\mathfrak{F}$ is continuous on $\mathcal{G}(\overline{\mathbb{R}})$ and since $h_n(i\partial)g \to h(i\partial)g$ in $\mathcal{H}(\mathbb{C})$, this proves the claim. ∎

3. Asymptotic Fourier transformation. We will use the results of the preceding section to define an asymptotic Fourier transform on the space

$$\mathcal{B}(\mathbb{R}) := H(\mathbb{C}\setminus\mathbb{R})/H(\mathbb{C})$$

of hyperfunctions on $\mathbb{R}$ in a way similar to Komatsu's procedure sketched in the introduction.

The embeddings $\mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R}) \hookrightarrow H(\mathbb{C}\setminus\mathbb{R})$ and $\mathcal{H}_{-\infty}(\mathbb{C}) \hookrightarrow H(\mathbb{C})$ define the canonical (restriction) mapping

$$R: \mathcal{G}(\overline{\mathbb{R}}) := \mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R})/\mathcal{H}_{-\infty}(\mathbb{C}) \to H(\mathbb{C}\setminus\mathbb{R})/H(\mathbb{C}) =: \mathcal{B}(\mathbb{R}).$$

We have shown in [18, Cor. 2.4] that R is surjective. The kernel of R is the space of exponentially decreasing hyperfunctions supported at $\{\pm\infty\}$, i.e.

$$\ker R = \mathcal{E}\mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R})/\mathcal{H}_{-\infty}(\mathbb{C}) =: \mathcal{G}(\{\pm\infty\})$$

where $\mathcal{E}\mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R})$ is the space of entire functions in $\mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R})$, i.e.

$$\mathcal{E}\mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R}) := \mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R}) \cap H(\mathbb{C}).$$

Hence we have

Theorem 3.1. *The canonical* (*restriction*) *mapping defines a linear isomorphism*

$$R: \mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R})/\mathcal{E}\mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R}) \to \mathcal{B}(\mathbb{R}).$$

The linear bijection $E := R^{-1}$ is called the *extension* mapping.

We next show that the Fourier inversion formula (see (2.7) and (2.8)) can be simplified if bounds for the representing functions are ignored.

Proposition 3.2. *For $h \in \mathfrak{F}\mathcal{G}(\overline{\mathbb{R}})$ let $g \in H(\mathbb{C} \setminus \mathbb{R})$ be defined by*

$$g(z) := \frac{1}{2\pi} \int_0^{\infty} e^{iz\xi} h(\xi)\, d\xi \qquad \text{for } z \in \mathbb{C}_+, \tag{3.1}$$

$$g(z) := \frac{1}{2\pi} \int_0^{-\infty} e^{iz\xi} h(\xi)\, d\xi \qquad \text{for } z \in \mathbb{C}_-. \tag{3.2}$$

Then $\mathfrak{F}^{-1}(h) - g \in H(\mathbb{C})$, i.e. $[\mathfrak{F}^{-1}(h)] = [g]$ as hyperfunctions.

Proof. For $z \in \mathbb{C} \setminus \mathbb{R}$ we get, by (2.7) and (2.8),

$$\begin{aligned} 2\pi K(h)(z) &:= 2\pi(\mathfrak{F}^{-1}(h) - g)(z) \\ &= \int_0^{\infty} e^{iz\xi} h(\xi)(\varphi(\xi) - 1)\, d\xi + \int_{-\infty}^{0} e^{iz\xi} h(\xi)\varphi(\xi)\, d\xi. \end{aligned} \tag{3.3}$$

$K(h)$ is an entire function and

$$|K(h)(z)| \le C \int_0^{\infty} e^{|\operatorname{Im} z|t + t - t^2}\, dt \le C_j \quad \text{if } |\operatorname{Im} z| \le j \tag{3.4}$$

by (2.6) and (2.5). ∎

The Fourier image of $\mathcal{G}(\{\pm\infty\})$ is easily determined. Let

$$\mathcal{H}_1^{\pm} := \{f \in H(\mathbb{C}) \mid \forall j \in \mathbb{N} : |f|_j^{\pm} := \sup_{\pm \operatorname{Im} z \le j} |f(z)| e^{-|\operatorname{Re} z|/j + j|\operatorname{Im} z|} < \infty\}.$$

Proposition 3.3. *The Fourier transform is a topological isomorphism*

$$\mathfrak{F} : \mathcal{G}(\{\pm\infty\}) \to \mathcal{H}_1^- \oplus \mathcal{H}_1^+ =: \mathfrak{F}\mathcal{G}(\{\pm\infty\}).$$

Proof. (a) If $f \in \mathcal{H}_1^- \cap \mathcal{H}_1^+$ then

$$\forall j \in \mathbb{N}_0 : \quad |f(iz)| \le C_j e^{|\operatorname{Im} z|/j - j|\operatorname{Re} z|} \quad \text{on } \mathbb{C}.$$

The Paley–Wiener theorem implies that $f = 0$.

(b) For $f \in \mathcal{E}\mathcal{H}_{-\infty}(\mathbb{C} \setminus \mathbb{R})$ we have (by Cauchy's theorem)

$$\mathfrak{F}([f])(z) = \int_{\gamma_-} e^{-iz\xi} f(\xi)\, d\xi + \int_{\gamma_+} e^{-iz\xi} f(\xi)\, d\xi =: F_-(z) + F_+(z),\ z \in \mathbb{C},$$

where $\gamma_\pm := \partial\{z \in \mathbb{C} \mid \pm\operatorname{Re} z > j \text{ and } |\operatorname{Im} z| < 1/j\}$. As in (2.1) we see that $F_\pm \in H_1^{\pm}$.

(c) By Theorem 2.3 we know that $\mathfrak{F}^{-1}(h) \in \mathcal{G}(\overline{\mathbb{R}})$ if $h := h_+ + h_- \in \mathcal{H}_1^+ \oplus \mathcal{H}_1^- \subset \mathfrak{F}\mathcal{G}(\overline{\mathbb{R}})$. By Proposition 3.2 we have to show that (3.1) and (3.2) define one entire function. By Cauchy's theorem we have, for $h_+ \in H_1^+$,

$$\int_0^{\infty} e^{i(x+iy)\xi} h_+(\xi)\, d\xi = \int_0^{-i\infty} e^{i(x+iy)\xi} h_+(\xi)\, d\xi \qquad \text{for } x = 0 \text{ and } y > 0,$$

and the right hand side is an entire function. Similarly,

$$\int_0^{-\infty} e^{i(x+iy)\xi} h_+(\xi)\,d\xi = \int_0^{-i\infty} e^{i(x+iy)\xi} h_+(\xi)\,d\xi \quad \text{for } x=0 \text{ and } y<0.$$

The argument for $h_- \in H_1^-$ is similar. This shows the claim. ∎

Combining Theorems 2.3 and 3.1 and Propositions 3.2 and 3.3 we get

THEOREM 3.4. *The asymptotic Fourier transform*

$$\mathfrak{F}_{\mathcal{B}} := \mathfrak{F} \circ E : \mathcal{B}(\mathbb{R}) \to \mathfrak{F}\mathcal{G}(\overline{\mathbb{R}})/\mathfrak{F}\mathcal{G}(\{\pm\infty\})$$

is a linear isomorphism. For $[h] \in \mathfrak{F}\mathcal{G}(\overline{\mathbb{R}})/\mathfrak{F}\mathcal{G}(\{\pm\infty\})$ *the inverse image* $\mathfrak{F}_{\mathcal{B}}^{-1}([h])$ *is given by* (*the equivalence class of*)

$$f(z) := \frac{1}{2\pi}\int_0^{\infty} e^{iz\xi} h(\xi)\,d\xi \qquad \textit{for } z \in \mathbb{C}_+, \tag{3.5}$$

$$f(z) := \frac{1}{2\pi}\int_0^{-\infty} e^{iz\xi} h(\xi)\,d\xi \qquad \textit{for } z \in \mathbb{C}_-. \tag{3.6}$$

Recall that the space $\mathcal{B}(K)$ of hyperfunctions with support in the compact $K \subset \mathbb{R}$ coincides with the space $A(K)'$ of analytic functionals on K. $\mathfrak{F}_{\mathcal{B}}$ coincides on $A(K)'$ with the Fourier–Laplace transform:

EXAMPLE 3.5. Let $K \subset \mathbb{R}$ be compact and let $\nu \in A(K)'$. Then

$$\mathfrak{F}_{\mathcal{B}}(\nu) = [\widehat{\nu}],$$

i.e. $\mathfrak{F}_{\mathcal{B}}(\nu)$ is the equivalence class of $\widehat{\nu}$ in $\mathfrak{F}\mathcal{G}(\overline{\mathbb{R}})/\mathfrak{F}\mathcal{G}(\{\pm\infty\})$.

Proof. This follows from Example 2.4. ∎

To define a convolution on $\mathcal{B}(\mathbb{R})$ by means of the convolution on $\mathcal{G}(\overline{\mathbb{R}})$ we need the following

LEMMA 3.6. $f*g \in \mathcal{E}\mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R})$ *if* $f \in \mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R})$ *and* $g \in \mathcal{E}\mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R})$.

Proof. Since $g \in \mathcal{E}\mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R})$ we may change the original path of integration to a path $\gamma_c, c>0$, which in the strip $]-c,c[\,\times i\mathbb{R}$ consists of the two lines $\{z \in \mathbb{C} \mid |\mathrm{Im}\, z| = c\}$:

$$(f*g)(z) = (g*f)(z) = \int_{\gamma_c} g(w) f(z-w)\,dw \quad \text{for } 0<|\mathrm{Im}\, z|<c, z \in \mathbb{C}\setminus\mathbb{R}.$$

The resulting integral clearly defines an entire function, hence a function in $\mathcal{E}\mathcal{H}_{-\infty}(\mathbb{C}\setminus\mathbb{R})$ by Proposition 2.7(a). ∎

For $[u],[v] \in \mathcal{B}(\mathbb{R})$ we may now define a convolution by

$$[u] *_{\mathcal{B}} [v] := R(E([u]) * E([v])).$$

Theorem 3.7.

(a) *The convolution* $*_{\mathcal{B}}$ *is well defined, bilinear, commutative and associative on* $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$.

(b) *For* $[u], [v] \in \mathcal{B}(\mathbb{R})$ *we have*

$$\mathfrak{F}_{\mathcal{B}}([u] *_{\mathcal{B}} [v]) = \mathfrak{F}_{\mathcal{B}}([u])\mathfrak{F}_{\mathcal{B}}([v]). \tag{3.7}$$

(c) $[\nu] *_{\mathcal{B}} [u]$ *coincides for* $[u] \in \mathcal{B}(\mathbb{R})$ *and* $\nu \in A(K)', K \subset \mathbb{R}$ *compact, with the usual convolution* $[\nu] * [u]$ *of hyperfunctions.*

Proof. (a)&(b) The operation $*_{\mathcal{B}}$ is well defined by Lemma 3.6. The remaining statements in (a)–(b) follow from Proposition 2.7.

(c) By the definition of convolution of hyperfunctions, $[\nu] * [u] = [u] * [\nu]$ is defined by

$$\int_{z+\gamma} w(\xi)v(z-\xi)\,d\xi$$

where w (and v) are representing functions for $[u]$ (and $[\nu]$, respectively) and where γ is a path around K. Hence we may choose $w \in E([u])$ and $v \in E([\nu])$ and then change the path of integration to obtain $[u] *_{\mathcal{B}} [\nu] = [\nu] *_{\mathcal{B}} [u]$. ■

Example 3.8.

(a) Let $\tau_h([u]) := [u(\cdot - h)], h \in \mathbb{R}$, be the shift operator on $\mathcal{B}(\mathbb{R})$. Then

$$\mathfrak{F}_{\mathcal{B}}(\tau_h([u]))(z) = \mathfrak{F}_{\mathcal{B}}([u])(z)e^{-ihz} \quad \text{for } [u] \in \mathcal{B}(\mathbb{R}).$$

(b) Let $P(-i\partial) := \sum_{k=0}^{\infty} \frac{c_k}{k!}(-i\partial)^k$ where P is of exponential type 0. Then

$$\mathfrak{F}_{\mathcal{B}}(P(-i\partial)[u])(z) = P(z)\mathfrak{F}_{\mathcal{B}}([u])(z) \quad \text{for } [u] \in \mathcal{B}(\mathbb{R}).$$

Proof. (a) This follows from Example 2.5 since $R(\tau_h E([u])) = \tau_h[u]$ and therefore $E(\tau_h[u]) = \tau_h E([u])$.

(b) This follows from Example 2.6 since $R(P(-i\partial)E([u])) = P(-i\partial)[u]$ and therefore $E(P(-i\partial)[u]) = P(-i\partial)E([u])$. ■

Multiplication of hyperfunctions by entire functions is naturally defined by multiplication of the defining functions. This extends multiplication of (ultra)distributions by entire functions. We include a short proof of this fact because of the difficulties mentioned in [21, Section 5]:

Remark 3.9. For $T \in \mathcal{D}'(\mathbb{R})$ let $[u_T]$ be the canonical representation of T as a hyperfunction. Then

$$f[u_T] = [u_{(fT)}] \quad \text{for } f \in \mathcal{H}(\mathbb{C}).$$

Proof. It is well known (see [32] and also [16]) that $[u_T]$ is uniquely determined by the fact that u_T can be extended to a distribution $\overline{u}_T \in \mathcal{D}'(\mathbb{C})$

such that $\overline{\partial}(\overline{u}_T) = T \otimes \delta_y$ where δ_y is Dirac's delta distribution (in imaginary direction). Since

$$f[u_T] := [fu_T] \quad \text{and} \quad \overline{\partial}(fu_T) = f\overline{\partial}(u_T) = f(T \otimes \delta_y) = (fT) \otimes \delta_y$$

the claim is proved. ∎

PROPOSITION 3.10. *For any entire function h of exponential type and $[u] \in \mathcal{B}(\mathbb{R})$ we have*

$$\mathfrak{F}_{\mathcal{B}}(h[u])(z) = h(i\partial)\mathfrak{F}_{\mathcal{B}}([u])(z). \tag{3.8}$$

Proof. This follows from Proposition 2.9 since $R(hE([u])) = h[u]$ and therefore $E(h[u]) = hE([u])$. ∎

4. Laplace transformation. We will use the above results on the Fourier transformation to obtain corresponding results for the Laplace transformation. Recall that

$$\mathcal{H}_{-\infty}(\mathbb{C} \setminus [0,\infty[) := \{f \in H(\mathbb{C} \setminus [0,\infty[) \mid \forall k \in \mathbb{N} : |f|_k < \infty\}$$

where

$$|f|_k := \sup_{z \in W_k} |f(z)|e^{k|\operatorname{Re} z|}, \qquad W_k := \{z \in \mathbb{C} \mid |\operatorname{Im} z| \le k,\ \operatorname{dist}(z, [0,\infty[) \ge 1/k\}.$$

For an exponentially decreasing hyperfunction $[g]$ supported in $[0,\infty]$, i.e. for

$$[g] \in \mathcal{G}([0,\infty]) := \mathcal{H}_{-\infty}(\mathbb{C} \setminus [0,\infty[)/\mathcal{H}_{-\infty}(\mathbb{C})$$

the path defining $\mathfrak{F}([g])$ can be changed by Cauchy's theorem so that the Laplace transform $\mathfrak{L}([g])$ is defined by

$$\mathfrak{L}([g])(z) := \mathfrak{F}([g])(-iz) = \int_{\Gamma_c} e^{-z\xi} g(\xi)\, d\xi, \qquad z \in \mathbb{C}, \tag{4.1}$$

with clockwise orientation where $c > 0$ is arbitrary and $\Gamma_c := \{z \in \mathbb{C} \mid \operatorname{dist}(z, [0,\infty[) = c\}$. The Laplace image of $\mathcal{G}([0,\infty])$ is now precisely the space

$$\mathfrak{L}\mathcal{G}_{[0,\infty]} := \{f \in H(\mathbb{C}) \mid \forall k \in \mathbb{N} : |f|_k := \sup_{\operatorname{Re} z \ge -k} |f(z)|e^{-|z|/k} < \infty\}.$$

THEOREM 4.1. *The Laplace transformation*

$$\mathfrak{L} : \mathcal{G}([0,\infty]) \to \mathfrak{L}\mathcal{G}_{[0,\infty]}$$

is a topological isomorphism.

Proof. By (4.1) we have, for any $j \in \mathbb{N}$,

$$|\mathfrak{L}([g])(z)| \le C_1 |g|_j e^{|z|/j} \quad \text{if } \operatorname{Re} z \ge -j+1,$$

hence $\mathfrak{L}([g]) \in \mathfrak{L}\mathcal{G}_{[0,\infty]}$ and $\mathfrak{L}$ is continuous. $\mathfrak{L}$ is injective on $\mathcal{G}([0,\infty])$ by Theorem 2.3. The inverse Laplace transform $\mathfrak{L}^{-1}(h) = \mathfrak{F}^{-1}(h(i\,\cdot))$,

$h \in \mathfrak{L}\mathcal{G}_{[0,\infty]}$, is given by

$$\mathfrak{L}^{-1}(h)(z) = K(h(i\,\cdot))(z) + g(z)$$

where $K(h(i\,\cdot))$ is the entire function given by (3.3) and where g is given by (3.1) and (3.2), respectively (for $h(i\,\cdot)$ instead of h). Notice that $h(i\,\cdot) \in \mathfrak{F}\mathcal{G}(\overline{\mathbb{R}})$ if $h \in \mathfrak{L}\mathcal{G}_{[0,\infty]}$. Hence by (3.4) we have

$$|K(h(i\,\cdot))(z)| \le C_j \quad \text{if } |\operatorname{Im} z| \le j. \tag{4.2}$$

Also (by Cauchy's theorem for $\operatorname{Re} z < -1$ and $\operatorname{Im} z = 1$ and using (3.1)),

$$2\pi g(z) = \int_0^\infty e^{iz\xi} h(i\xi)\, d\xi = \int_0^{-i\infty} e^{iz\xi} h(i\xi)\, d\xi = -i \int_0^\infty e^{zt} h(t)\, dt. \tag{4.3}$$

This also holds for $\operatorname{Re} z < -1$ and $\operatorname{Im} z = -1$ (using (3.2)). Thus $g(z)$ for $\operatorname{Re} z < 0$ is the holomorphic function defined by the right hand side of (4.3) and we have

$$|g(z)| \le C_j \quad \text{if } \operatorname{Re} z \le -1/j. \tag{4.4}$$

Using also (4.2) we see that $\mathfrak{L}^{-1}(h)(z)$ is holomorphic for $\operatorname{Re} z < 0$ and

$$|\mathfrak{L}^{-1}(h)(z)| \le C_j \quad \text{if } \operatorname{Re} z \le -1/j \text{ and } |\operatorname{Im} z| \le j. \tag{4.5}$$

Since $\mathfrak{L}(e^{-j\,\cdot} f)(z) = \mathfrak{L}(f)(z + j)$ for $j \in \mathbb{N}$ by the definition of $\mathfrak{L}$ we get $\mathfrak{L}^{-1}(h)(z) = e^{jz}\mathfrak{L}^{-1}(h(\,\cdot + j))(z)$ for $h \in \mathfrak{L}\mathcal{G}_{[0,\infty]}$, hence (4.5) and Theorem 2.3 show that $\mathfrak{L}^{-1}(h) \in \mathcal{G}([0,\infty])$ for $h \in \mathfrak{L}\mathcal{G}_{[0,\infty]}$. ∎

Formulas for the Laplace transform of convolutions and multiplication with entire functions of exponential type on $\mathcal{G}([0,\infty])$ can be easily derived from (4.1) and Propositions 2.7 and 2.9:

$$\mathfrak{L}([f] * [g]) = \mathfrak{L}([f])\mathfrak{L}([g]) \quad \text{for } [f], [g] \in \mathcal{G}([0,\infty]), \tag{4.6}$$

$$\mathfrak{L}(h[f]) = h(-\partial)\mathfrak{L}([f]) \quad \text{for } [f] \in \mathcal{G}([0,\infty]) \tag{4.7}$$

if h is an entire function of exponential type.

5. Asymptotic Laplace transformation. The asymptotic Laplace transform on

$$\mathcal{B}([0,\infty[) := H(\mathbb{C} \setminus [0,\infty[)/H(\mathbb{C})$$

is defined similarly to the asymptotic Fourier transform in Section 3: Let

$$\mathcal{E}\mathcal{H}_{-\infty}(\mathbb{C} \setminus [0,\infty[) := H(\mathbb{C}) \cap \mathcal{H}_{-\infty}(\mathbb{C} \setminus [0,\infty[).$$

Then we get, as in Section 3 using [18, Cor. 4.2],

Theorem 5.1. *The canonical* (*restriction*) *mapping defines a linear isomorphism*

$$R_+ : \mathcal{H}_{-\infty}(\mathbb{C} \setminus [0,\infty[)/\mathcal{E}\mathcal{H}_{-\infty}(\mathbb{C} \setminus [0,\infty[) \to \mathcal{B}([0,\infty[).$$

The inverse of R_+ is denoted by E_+ to emphasize the fact that we are considering supports in $[0,\infty]$ in this section. The Laplace image of

$$\mathcal{G}(\{\infty\}) := \mathcal{E}\mathcal{H}_{-\infty}(\mathbb{C}\setminus[0,\infty[)/\mathcal{H}_{-\infty}(\mathbb{C})$$

is precisely

$$\mathfrak{L}\mathcal{G}_\infty := \{f\in H(\mathbb{C}) \mid \forall k\in\mathbb{N}: \sup_{\operatorname{Re} z\ge -k} |f(z)|e^{k|\operatorname{Re} z|-|z|/k} < \infty\}.$$

PROPOSITION 5.2. *The Laplace transform is a topological isomorphism*

$$\mathfrak{L}:\mathcal{G}(\{\infty\})\to\mathfrak{L}\mathcal{G}_\infty.$$

Proof. For $[f]\in\mathcal{G}(\{\infty\})$ we have, by Cauchy's theorem,

$$\mathfrak{L}([f])(z) = \int_{\gamma_j} e^{-z\xi} f(\xi)\,d\xi,\qquad z\in\mathbb{C},$$

where $\gamma_j := \{z\in\mathbb{C}\mid \operatorname{dist}(z,[j,\infty[) = 1/j\}$. This shows that $\mathfrak{L}([f])\in\mathfrak{L}\mathcal{G}_\infty$ and that $\mathfrak{L}:\mathcal{G}(\{\infty\})\to\mathfrak{L}\mathcal{G}_\infty$ is continuous. It is injective by Theorem 4.1. For $h\in\mathfrak{L}\mathcal{G}_\infty$ we have $\mathfrak{L}^{-1}(h)\in\mathcal{G}([0,\infty])$ by Theorem 4.1 and $\mathfrak{L}^{-1}(h) = \mathfrak{F}^{-1}(h(i\,\cdot))\in\mathcal{H}(\mathbb{C})$ by Proposition 3.3 since $h(\,\cdot)\in\mathcal{H}_1^-$. ∎

The asymptotic Laplace transform $\mathfrak{L}_\mathcal{B}$ is now naturally defined by

$$\mathfrak{L}_\mathcal{B} := \mathfrak{L}\circ E_+ : \mathcal{B}([0,\infty[)\to\mathfrak{L}\mathcal{G}_{[0,\infty]}/\mathfrak{L}\mathcal{G}_\infty.$$

When transferring the results on Fourier and Laplace transformations to the asymptotic Laplace transformation we repeatedly use the formula

$$\mathfrak{L}_\mathcal{B}([u])(z) = [\mathfrak{F}(E_+([u]))(-iz)]\quad\text{for } [u]\in\mathcal{B}([0,\infty[). \tag{5.1}$$

THEOREM 5.3. *The asymptotic Laplace transform is a linear isomorphism*

$$\mathfrak{L}_\mathcal{B}:\mathcal{B}([0,\infty[)\to\mathfrak{L}\mathcal{G}_{[0,\infty]}/\mathfrak{L}\mathcal{G}_\infty.$$

The inverse $\mathfrak{L}_\mathcal{B}^{-1}([h])$ *is given by* (*the equivalence class of*)

$$f(z) := \frac{1}{2\pi i}\int_0^{i\infty} e^{z\xi}h(\xi)\,d\xi\qquad \textit{for } z\in\mathbb{C}_+, \tag{5.2}$$

$$f(z) := \frac{1}{2\pi i}\int_0^{-i\infty} e^{z\xi}h(\xi)\,d\xi\qquad \textit{for } z\in\mathbb{C}_-. \tag{5.3}$$

Proof. This follows from (5.1) and Proposition 3.2. ∎

The reader should also recall formula (4.3) for $\mathfrak{L}_\mathcal{B}^{-1}([h])(z)$ for $\operatorname{Re} z<0$.

Since the sheaf of hyperfunctions is flabby, for any $[u]\in\mathcal{B}([0,\infty[)$ and any $j\in\mathbb{N}$ we can find $\nu_j\in A([0,j])'$ such that

$$[u]|_{]-\infty,j[} = [u_j]|_{]-\infty,j[},\quad\text{i.e.}\quad [u]-[u_j]\in\mathcal{B}([j,\infty[), \tag{5.4}$$

where $u_j := u_{\nu_j}$ is the canonical representation of ν_j in $\mathcal{G}([0,\infty])$ defined by (2.9). As the notation indicates, the asymptotic Laplace transform $\mathfrak{L}_{\mathcal{B}}([u])$ can be viewed as a representation of the asymptotic behavior of the Fourier–Laplace transforms

$$\widetilde{\nu}_j(z) := \langle {}_\xi\nu_j, e^{-z\xi}\rangle$$

of the local parts ν_j of $[u]$. In the proof we need the trivial fact that by the definition of $\mathfrak{L}$,

$$\mathfrak{L}_{\mathcal{B}}(\tau_j([u])) = e^{-j\,\cdot}\mathfrak{L}_{\mathcal{B}}([u]) \quad \text{for } [u] \in \mathcal{B}([0,\infty[) \tag{5.5}$$

where $\tau_j([u]) := [u(\cdot - j)]$, $j \geq 0$, is the shift operator from $\mathcal{B}([0,\infty[)$ onto $\mathcal{B}([j,\infty[)$, i.e.

$$\mathfrak{L}_{\mathcal{B}}(\mathcal{B}([j,\infty[)) = (e^{-j\,\cdot}\mathfrak{L}\mathcal{G}_{[0,\infty]})/\mathfrak{L}\mathcal{G}_\infty. \tag{5.6}$$

Lemma 5.4.

(a) *For $[u] \in \mathcal{B}([0,\infty[)$ and $\nu_j \in A([0,j])'$ as above let $h \in \mathfrak{L}_{\mathcal{B}}([u])$. Then for any $l \in \mathbb{N}$,*

$$|h(z) - \widetilde{\nu}_j(z)| \leq C_j e^{|z|/l - j|\operatorname{Re} z|} \quad \textit{if } \operatorname{Re} z \geq -l. \tag{5.7}$$

(b) *Conversely, if h is an entire function satisfying* (5.7) *for any $j \in \mathbb{N}$ then $h \in \mathfrak{L}_{\mathcal{B}}([u])$.*

Proof. (a) By (5.6) and Example 2.4 we have

$$e^{j\,\cdot}[h - \widetilde{\nu}_j] = e^{j\,\cdot}(\mathfrak{L}_{\mathcal{B}}([u]) - \mathfrak{L}_{\mathcal{B}}([u_j])) = e^{j\,\cdot}\mathfrak{L}_{\mathcal{B}}([u - u_j]) \in \mathfrak{L}\mathcal{G}_{[0,\infty]}/\mathfrak{L}\mathcal{G}_\infty.$$

This shows (5.7).

(b) To prove the converse statement, it is clear that $h \in \mathfrak{L}\mathcal{G}_{[0,\infty]}$ and $[h - \widetilde{\nu}_j] \in \mathfrak{L}_{\mathcal{B}}(\mathcal{B}([j,\infty[))$ by (5.7) and (5.6) since $\widetilde{\nu}_j \in \mathfrak{L}\mathcal{G}_{[0,\infty]}$. We thus get, for any $j \in \mathbb{N}$,

$$\mathfrak{L}_{\mathcal{B}}^{-1}([h]) - [u] = [(\mathfrak{L}_{\mathcal{B}}^{-1}([h]) - [u_j]) - ([u] - [u_j])] \in \mathcal{B}([j,\infty[)$$

using Example 2.4 again. Hence $\mathfrak{L}_{\mathcal{B}}^{-1}([h]) = [u]$. ∎

As in Lemma 5.4, we will now show that convolution on $\mathcal{B}([0,\infty[)$ perfectly fits the asymptotic Laplace transformation. Recall that the convolution $[u] * [v]$ for $[u],[v] \in \mathcal{B}([0,\infty[)$ is defined by

$$([u] * [v])|_{]-\infty,j[} := ([u_j] * [v])|_{]-\infty,j[} \tag{5.8}$$

where $[u_j] \in A([0,j])'$ is chosen by (5.4). Notice that

$$[u - u_j] * [v] \in \mathcal{B}([j,\infty[) \tag{5.9}$$

by (5.8). The convolution is a bilinear commutative and associative operation on $\mathcal{B}([0,\infty[)$.

Theorem 5.5. *For $[u],[v] \in \mathcal{B}([0,\infty[)$ we have*

$$\mathfrak{L}_{\mathcal{B}}([u] * [v]) = \mathfrak{L}_{\mathcal{B}}([u])\mathfrak{L}_{\mathcal{B}}([v]). \tag{5.10}$$

Proof. The right hand side of (5.10) is well defined since $\mathfrak{L}\mathcal{G}_\infty$ is a two-sided ideal in $\mathfrak{L}\mathcal{G}_{[0,\infty]}$. By Theorem 3.7 we get, for any $j \in \mathbb{N}$,

$$\begin{aligned}\mathfrak{L}_{\mathcal{B}}([u] * [v]) - \mathfrak{L}_{\mathcal{B}}([u])\mathfrak{L}_{\mathcal{B}}([v])\qquad&\\ = \mathfrak{L}_{\mathcal{B}}([u - u_j] * [v]) + (\mathfrak{L}_{\mathcal{B}}([u_j]) - \mathfrak{L}_{\mathcal{B}}([u]))\mathfrak{L}_{\mathcal{B}}([v])&\\ = \mathfrak{L}_{\mathcal{B}}([u - u_j] * [v]) + \mathfrak{L}_{\mathcal{B}}([u_j - u])\mathfrak{L}_{\mathcal{B}}([v]).&\end{aligned}$$

The right hand side is in $\mathfrak{L}_{\mathcal{B}}(\mathcal{B}([j,\infty[)) = (e^{-j\,\cdot}\mathfrak{L}\mathcal{G}_{[0,\infty]})/\mathfrak{L}\mathcal{G}_\infty$ by (5.6) and (5.9) and since $\mathfrak{L}_{\mathcal{B}}(\mathcal{B}([j,\infty[))$ is an ideal in $\mathfrak{L}\mathcal{G}_{[0,\infty]}/\mathfrak{L}\mathcal{G}_\infty$. Thus, an estimate like (5.7) holds for any j and therefore $\mathfrak{L}_{\mathcal{B}}([u] * [v]) = \mathfrak{L}_{\mathcal{B}}([u])\mathfrak{L}_{\mathcal{B}}([v])$. ∎

Example 5.6. Let $P(\partial) := \sum_{k=0}^{\infty} \frac{c_k}{k!}\partial^k$ where P is of exponential type 0. Then

$$\mathfrak{L}_{\mathcal{B}}(P(\partial)[u])(z) = P(z)\mathfrak{L}_{\mathcal{B}}([u])(z), \quad z \in \mathbb{C}, \quad \text{for } [u] \in \mathcal{B}([0,\infty[).$$

From (4.7) we directly get, for $[u] \in \mathcal{B}([0,\infty[)$,

$$\mathfrak{L}_{\mathcal{B}}(h[u]) = h(-\partial)\mathfrak{L}_{\mathcal{B}}([u]) \tag{5.11}$$

if h is an entire function of exponential type.

6. Komatsu's asymptotic Laplace transform. We will now discuss the connection of our asymptotic Laplace transform $\mathfrak{L}_{\mathcal{B}}$ with the asymptotic Laplace transform $\mathfrak{L}_{\mathrm{Kom}}$ defined by Komatsu in [14, 13], mentioned in the introduction. We briefly recall the relevant notation. For $0 < \varphi < \pi/2$ and $r \geq 0$ let

$$\Gamma_{r,\varphi} := \{\rho e^{i\psi} \mid \rho \geq r,\ |\psi| \leq \varphi\}.$$

Let $U \subset \mathbb{C}$ be a *postsectorial* open set (see [20, 21]), i.e.

$$\forall 0 < \varphi < \pi/2\ \exists r > 0 : \quad \Gamma_{r,\varphi} \subset U.$$

Define

$$\mathfrak{L}\mathcal{G}_{[0,\infty]}(U) := \{f \in H(U) \mid \forall j \in \mathbb{N} : \sup_{z\in\Gamma_{r,\varphi}} |f(z)|e^{-|z|/j} < \infty \text{ if } \Gamma_{r,\varphi} \subset U\},$$

$$\mathfrak{L}\mathcal{G}_{\infty}(U) := \{f \in H(U) \mid \forall j \in \mathbb{N} : \sup_{z\in\Gamma_{r,\varphi}} |f(z)|e^{-|z|/j+j|\mathrm{Re}\, z|} < \infty \text{ if } \Gamma_{r,\varphi} \subset U\}.$$

Set

$$\mathfrak{L}_{\mathrm{Kom}}\mathcal{B}^{\exp}_{[0,\infty[} := \operatorname*{ind}_{U} \mathfrak{L}\mathcal{G}_{[0,\infty]}(U) \quad \text{and} \quad \mathfrak{L}_{\mathrm{Kom}}\mathcal{B}^{\exp}_{\infty} := \operatorname*{ind}_{U} \mathfrak{L}\mathcal{G}_{\infty}(U)$$

where the inductive limit runs over all postsectorial open sets. Then (cf. also [21, Lemma 1.2])

$$\mathfrak{L}_{\mathrm{Kom}} : \mathcal{B}([0,\infty[) \to \mathfrak{L}_{\mathrm{Kom}}\mathcal{B}^{\exp}_{[0,\infty[}/\mathfrak{L}_{\mathrm{Kom}}\mathcal{B}^{\exp}_{\infty} \text{ is a linear bijection,} \tag{6.1}$$

in other words, Komatsu's Laplace transform consists of equivalence classes of holomorphic functions defined near $S_\infty := \{\infty e^{i\varphi} \mid |\varphi| < \pi/2\}$ and satisfying the growth conditions from $\mathfrak{L}\mathcal{G}_{[0,\infty]}$ (and $\mathfrak{L}\mathcal{G}_\infty$, respectively) only there.

Therefore,

$$\mathfrak{L}\mathcal{G}_{[0,\infty]} \subset \mathfrak{L}_{\mathrm{Kom}}\mathcal{B}^{\exp}_{[0,\infty[} \quad \text{and} \quad \mathfrak{L}\mathcal{G}_{\infty} \subset \mathfrak{L}_{\mathrm{Kom}}\mathcal{B}^{\exp}_{\infty} \tag{6.2}$$

canonically. Surprisingly enough we can prove much more (see Theorem 6.3 below). The proof is based on the solution of the $\overline{\partial}$-problem in the following L^2-variant of $\mathfrak{L}\mathcal{G}_\infty$. Let $V_n := \{z \in \mathbb{C} \mid \operatorname{Re} z > -n\}$ and

$$L^2(\mathfrak{L}\mathcal{G}_\infty) := \Big\{ f \in L^2_{\mathrm{loc}}(\mathbb{C}) \;\Big|\; \forall n \in \mathbb{N} : |f|_n^2 := \int_{V_n} |f(z)|^2 e^{-2|z|/n + 2n|\operatorname{Re} z|}\, dz < \infty \Big\}.$$

Lemma 6.1. *For any $f \in L^2(\mathfrak{L}\mathcal{G}_\infty)$ with* $\operatorname{supp} f \subset V_0$ *there is $g \in L^2(\mathfrak{L}\mathcal{G}_\infty)$ such that $\overline{\partial} g = f$.*

Proof. (a) Fix $j \in \mathbb{N}$. By [9, Theorem 4.4.2] there is $g_n \in L^2_{\mathrm{loc}}(\mathbb{C})$ such that $\overline{\partial} g_n(z) = f(z)e^{jz}$ and

$$\begin{aligned}\int_{\mathbb{C}} |g_n(z)|^2 e^{-2|z|/n}(1+|z|^2)^{-2}\, dz &\le \int_{\mathbb{C}} |f(z)|^2 e^{-2|z|/n + 2j\operatorname{Re} z}\, dz \\ &= \int_{V_0} |f(z)|^2 e^{-2|z|/n + 2j|\operatorname{Re} z|}\, dz < \infty\end{aligned}$$

since $\operatorname{supp} f \subset V_0$ and $f \in L^2(\mathfrak{L}\mathcal{G}_\infty)$. Since the (Taylor) polynomials are contained in $\{h \in \mathcal{H}(\mathbb{C}) \mid \forall n \in \mathbb{N} : \sup_{z\in\mathbb{C}} |h(z)|e^{-|z|/n} < \infty\}$ and dense in $\{f \in \mathcal{H}(\mathbb{C}) \mid \sup_{z\in\mathbb{C}} |h(z)|e^{-|z|/n} < \infty\}$ for any n and since passing to sup-norms is allowed by the mean value property, the Mittag-Leffler procedure provides $g \in L^2_{\mathrm{loc}}(\mathbb{C})$ such that

$$\overline{\partial} g(z) = f(z)e^{jz} \quad \text{and} \quad \int_{\mathbb{C}} |g(z)|^2 e^{-2|z|/n}\, dz < \infty \qquad \text{for any } n.$$

Hence, $G_j := ge^{-jz}$ satisfies $\overline{\partial} G_j = f$ on $\mathbb{C}$ and $G_j - G_{j-1} \in H_{j-1}$ where

$$H_j := \Big\{ h \in \mathcal{H}(\mathbb{C}) \;\Big|\; \forall n \in \mathbb{N} : \int_{V_n} |h(z)|^2 e^{-2|z|/n + 2j|\operatorname{Re} z|}\, dz < \infty \Big\}.$$

(b) $\mathfrak{L}\mathcal{G}_\infty$ is dense in H_j with respect to $|\ \ |_j$. To prove this we may again pass from L^2-norms to sup-norms. Choose $g \in \mathfrak{L}\mathcal{G}_\infty$ such that $g(0) = 1$. The existence of g follows e.g. from [24, Example 3] where a Fourier hyperfunction $G \ne 0$ with support at ∞ is constructed. Then $0 \ne \widetilde{G}(z) := G(z)e^{-z^2} \in \mathcal{G}(\{\infty\})$ and hence $0 \ne \mathfrak{L}(\widetilde{G}) \in \mathfrak{L}\mathcal{G}_\infty$. We may thus set $g := \mathfrak{L}(\widetilde{G})$ (modulo a real shift). Let $f \in H_j$ and set $f_k(z) := f(z)g(z/k)$. Then $f_k \in \mathfrak{L}\mathcal{G}_\infty$ by the definition of H_j and $\mathfrak{L}\mathcal{G}_\infty$, and $f_k \to f$ uniformly on compact sets. Since for $k \ge 4j$ and $\operatorname{Re} z > -j$,

$$\begin{aligned}|f_k(z) - f(z)| &= |f(z)|\,|1 - g(z/k)| \\ &\le C_1 e^{|z|/(4j) - j|\operatorname{Re} z| + |z|/k} \le C_2 e^{|z|/(2j) - j|\operatorname{Re} z|}\end{aligned}$$

we have $|f(z)-f_k(z)|e^{-|z|/j+j|\mathrm{Re}\,z|} < \varepsilon$ outside a compact set for these k. Thus $f_k \to f$ with respect to $|\ |_j$.

(c) Notice that $\mathfrak{L}\mathcal{G}_\infty = L^2(\mathfrak{L}\mathcal{G}_\infty) \cap \ker \overline{\partial}$ by passing from sup-norms to L^2-norms in the definition of $\mathfrak{L}\mathcal{G}_\infty$. By (b) and the Mittag-Leffler argument applied to the solutions G_j and the seminorms $|\ |_j$, the claim is proved. ∎

Corollary 6.2. *Let U be a postsectorial open set. Then the mapping*

$$T : \mathfrak{L}\mathcal{G}_{[0,\infty]} \times \mathfrak{L}\mathcal{G}_\infty(U) \to \mathfrak{L}\mathcal{G}_{[0,\infty]}(U), \qquad (f,g) \to f+g,$$

is surjective.

Proof. (a) Let U be a postsectorial open set and $f \in \mathfrak{L}\mathcal{G}_{[0,\infty]}(U)$. Then there are $r_j, C_j > 0$ such that

$$|f(z)| \le C_j e^{|z|/(j+1)} \le e^{|z|/j} \quad \text{ if } z \in \Gamma_j := \Gamma_{r_j,\pi/2-1/j}.$$

Let $\Gamma := \bigcup_{j\in\mathbb{N}} \Gamma_j$ and choose $\varphi \in C^\infty(\mathbb{C})$ such that $\operatorname{supp}\varphi \subset \operatorname{int}(\Gamma)$, $\varphi(z) = 1$ if $z \in \Gamma$ and $\operatorname{dist}(z,\partial\Gamma) \ge 1/2$ and such that $\|\varphi\|_\infty + \|\nabla\varphi\|_\infty < \infty$. Notice that $\operatorname{supp}\overline{\partial}(f\varphi) \subset \Gamma \subset V_0$ and $\overline{\partial}(f\varphi) \in L^2(\mathfrak{L}\mathcal{G}_\infty)$ since for any n there is C_n such that

$$n\,\mathrm{Re}\,z \le C_n + |z|/n \quad \text{ if } \operatorname{dist}(z,\partial\Gamma) \le 1. \tag{6.3}$$

By Lemma 6.1 there is $g \in L^2(\mathfrak{L}\mathcal{G}_\infty)$ such that $\overline{\partial}g = \overline{\partial}(f\varphi)$. Hence, $G := f\varphi - F \in \mathfrak{L}\mathcal{G}_{[0,\infty]}$ and $f - g = f(1-\varphi) + g \in \mathfrak{L}\mathcal{G}_\infty(U)$ since (6.3) also holds on $\Gamma_{r,\varphi} \setminus \Gamma$ for any $r \ge 0$ and any $0 < \varphi < \pi/2$. ∎

Theorem 6.3. *The inclusions* (6.2) *define a bijective linear mapping*

$$I : \mathfrak{L}\mathcal{G}_{[0,\infty]}/\mathfrak{L}\mathcal{G}_\infty \to \mathfrak{L}_{\mathrm{Kom}}\mathcal{B}^{\exp}_{[0,\infty[}/\mathfrak{L}_{\mathrm{Kom}}\mathcal{B}^{\exp}_\infty$$

such that $I \circ \mathfrak{L}_{\mathcal{B}} = \mathfrak{L}_{\mathrm{Kom}}$.

Proof. (a) I is well defined by (6.2). Let $f \in \mathfrak{L}\mathcal{G}_{[0,\infty]}$ be such that $f|_U \in \mathfrak{L}\mathcal{G}_\infty(U)$ for some postsectorial open set U. For $j \in \mathbb{N}$ choose $0 < \varphi_0 < \pi/2$ such that $\cos(\varphi_0) \le 1/(2j^2)$. For $z \notin \Gamma_{r_0,\varphi_0}$ and $\mathrm{Re}\,z > 0$ we then have

$$j\,\mathrm{Re}\,z \le C_0 + j\cos(\varphi_0)|z| \le C_0 + |z|/(2j)$$

and therefore, since $f \in \mathfrak{L}\mathcal{G}_{[0,\infty]}$,

$$|f(z)| \le C_j e^{|z|/(2j)} \le \widetilde{C}_j e^{|z|/j - j\,\mathrm{Re}\,z} \quad \text{ if } \mathrm{Re}\,z > 0 \text{ and } z \notin \Gamma_{r_0,\varphi_0}.$$

Since U is postsectorial and hence $\Gamma_{r_0,\varphi_0} \subset U$ for some r_0 this shows that $f \in \mathfrak{L}\mathcal{G}_\infty$. Thus I is injective. It is surjective by Theorem 6.2.

(b) We now prove that $I \circ \mathfrak{L}_{\mathcal{B}} = \mathfrak{L}_{\mathrm{Kom}}$. Let $[u] \in \mathcal{B}([0,\infty[)$ and let $\mathfrak{L}_{\mathrm{Kom}}([u])$ be defined on the postsectorial open set U. Since Komatsu's Laplace transform $\mathfrak{L}_{\mathrm{Kom}}$ also satisfies (5.7) on $\Gamma_{r_0,\varphi_0} \subset U$, by Lemma 5.4(a) we get

$$\begin{aligned}|I\circ\mathfrak{L}_{\mathcal{B}}([u])(z)-\mathfrak{L}_{\mathrm{Kom}}([u])(z)|\\ &=|(I\circ\mathfrak{L}_{\mathcal{B}}([u])(z)-\widetilde{\nu}_j(z))-(\mathfrak{L}_{\mathrm{Kom}}([u])(z)-\widetilde{\nu}_j(z))|\\ &\le C_je^{|z|/j-j|\mathrm{Re}\,z|}\quad\text{ on } \Gamma_{r_0,\varphi_0}\subset U,\end{aligned}$$

that is, $I\circ\mathfrak{L}_{\mathcal{B}}(u)-\mathfrak{L}_{\mathrm{Kom}}(u)\in\mathfrak{L}_{\mathrm{Kom}}\mathcal{B}_\infty^{\exp}$. ∎

The surjectivity of I also follows from the results of Komatsu (i.e. from (6.1)) and the equality $I\circ\mathfrak{L}_{\mathcal{B}}=\mathfrak{L}_{\mathrm{Kom}}$.

We have in fact proved in Theorem 6.3 that the mappings

$$\mathfrak{L}\mathcal{G}_{[0,\infty]}/\mathfrak{L}\mathcal{G}_\infty\to\mathfrak{L}\mathcal{G}_{[0,\infty]}(U)/\mathfrak{L}\mathcal{G}_\infty(U)\to\mathfrak{L}\mathcal{G}_{[0,\infty]}(V)/\mathfrak{L}\mathcal{G}_\infty(V)$$

are both linear bijections if $V\subset U$ are postsectorial open sets.

As mentioned in the introduction, Lumer and Neubrander (see [20]) introduced an asymptotic Laplace transform on $L^1_{\mathrm{loc}}([0,\infty[)$ and a modified version $\mathfrak{L}_{\mathrm{LN}}$ (see [21]) and clarified the connection with Komatsu's Laplace transform $\mathfrak{L}_{\mathrm{Kom}}$, namely

$$\mathfrak{L}_{\mathrm{Kom}}(f)\subset\mathfrak{L}_{\mathrm{LN}}(f)\quad\text{ if } f\in L^1_{\mathrm{loc}}([0,\infty[).$$

We therefore get, from Theorem 6.3,

Corollary 6.4. *For* $f\in L^1_{\mathrm{loc}}([0,\infty[)$ *we have* $\mathfrak{L}_{\mathcal{B}}(f)\subset\mathfrak{L}_{\mathrm{LN}}(f)$.

7. Abstract Cauchy problem in Fréchet spaces. In this section our elementary theory of asymptotic Laplace transform on $\mathcal{B}([0,\infty[)$ will be applied to the abstract Cauchy problem (ACP) for hyperfunctions with values in Fréchet spaces. This application is based on tensor product methods and the fact that the spaces used in our model for the (asymptotic) Laplace transformation are nuclear Fréchet spaces.

In the following, F will always denote a Fréchet space. The space of F-valued holomorphic functions on an open set $U\subset\mathbb{C}$ is denoted by $H(U,F)$. The space of F-valued hyperfunctions on $[0,\infty[$ is by definition

$$\mathcal{B}([0,\infty[,F):=H(\mathbb{C}\setminus[0,\infty[,F)/H(\mathbb{C},F).$$

Let

$$x'(t)=Cx(t),\qquad x(0)=x_0,\ x_0\in E, \tag{7.1}$$

be an abstract Cauchy problem (ACP), where E is a Fréchet space and

$$C:F:=D(C)\subseteq E\to E$$

is a closed operator with domain $F:=D(C)$. Then F is a Fréchet space when equipped with the graph topology and $C:F\to E$ is continuous.

An F-valued hyperfunction $[u] \in \mathcal{B}([0,\infty[, F)$ is called a *solution of* (7.1) (in the sense of hyperfunctions) if

$$\frac{d}{dt}[u] - C[u] = x_0 \otimes \delta_0 \tag{7.2}$$

where δ_0 is Dirac's δ-distribution considered as a hyperfunction.

The (ACP) (7.1) is said to have the *uniqueness property* (in the sense of hyperfunctions) if $[u] \equiv 0$ is the only solution of (7.2) for $x_0 = 0$. To characterize the uniqueness property using the asymptotic Laplace transform we need vector valued versions of our model spaces. These consist of F-valued holomorphic functions f such that $\|f\|_n$ satisfies the bounds corresponding to the scalar spaces for any $n \in \mathbb{N}$ where $(\|\ \|_k)_{k\in\mathbb{N}}$ is a system of seminorms defining the topology of F. In this way,

$$\mathcal{G}([0,\infty], F) := \mathcal{H}_{-\infty}(\mathbb{C} \setminus [0,\infty[, F)/\mathcal{H}_{-\infty}(\mathbb{C}, F).$$

Using the π-tensor product and the fact that $\mathcal{G}([0,\infty])$ and $\mathcal{G}(\{\infty\})$ are nuclear Fréchet spaces we have shown in [18, Cor. 5.2] that for any Fréchet space F,

$$R_+ : \mathcal{G}([0,\infty], F)/\mathcal{G}(\{\infty\}, F) \to \mathcal{B}([0,\infty[, F) \text{ is a bijection}$$

with inverse E_+. Also, $\mathcal{G}([0,\infty], F) = \mathcal{G}([0,\infty]) \mathbin{\widehat{\otimes}_\pi} F$ and $\mathcal{G}(\{\infty\}, F) = \mathcal{G}(\{\infty\}) \mathbin{\widehat{\otimes}_\pi} F$ and therefore

$$\mathfrak{L}_{\mathcal{B}} : \mathcal{G}([0,\infty], F) \to \mathfrak{L}\mathcal{G}_{[0,\infty]}(F) \quad \text{and} \quad \mathfrak{L}_{\mathcal{B}} : \mathcal{G}(\{\infty\}, F) \to \mathfrak{L}\mathcal{G}_{\infty}(F)$$

are topological isomorphisms by Theorem 4.1 and Proposition 5.2. For any Fréchet space F we thus get a linear isomorphism

$$\mathfrak{L}_{\mathcal{B}} := \mathfrak{L} \circ E_+ : \mathcal{B}([0,\infty[, F) \to \mathfrak{L}\mathcal{G}_{[0,\infty]}(F)/\mathfrak{L}\mathcal{G}_{\infty}(F). \tag{7.3}$$

THEOREM 7.1. *For E, F and C as above the following are equivalent:*

(a) *The* (*ACP*) (7.1) *has the uniqueness property* (*in the sense of hyperfunctions*).
(b) *If $h \in \mathfrak{L}\mathcal{G}_{[0,\infty]}(F)$ and $(z - C)h \in \mathfrak{L}\mathcal{G}_{\infty}(E)$ then $h \in \mathfrak{L}\mathcal{G}_{\infty}(F)$.*
(c) *If $h \in \mathfrak{L}\mathcal{G}_{[0,\infty]}(F)$ and $(z - C)h \in \mathfrak{L}\mathcal{G}_{\infty}(E)$ then $\{h(t)e^{nt} \mid t \geq 0\}$ is weakly bounded in F for any $n \in \mathbb{N}$.*

Proof. (a)$\Rightarrow$(b). By (7.3) there is $[u] \in \mathcal{B}([0,\infty[, F)$ such that $[h] = \mathfrak{L}_{\mathcal{B}}([u]) \in \mathfrak{L}\mathcal{G}_{[0,\infty]}(F)/\mathfrak{L}\mathcal{G}_{\infty}(F)$. By assumption and Example 5.6 we have

$$\mathfrak{L}_{\mathcal{B}}\left(\frac{d}{dt}[u] - C[u]\right) = (z - C)\mathfrak{L}_{\mathcal{B}}([u]) = (z - C)[h] = 0,$$

hence $[u] = 0$ by (a) and (7.3), and therefore $[h] = 0$, i.e. $h \in \mathfrak{L}\mathcal{G}_{\infty}(F)$.

(b)$\Rightarrow$(c). By the definition of $\mathfrak{L}\mathcal{G}_{\infty}(F)$.

(c)⇒(a). Let $[u] \in \mathcal{B}([0,\infty[, F)$ and $\frac{d}{dt}[u] - C[u] = 0$. Then $[h] := \mathfrak{L}_{\mathcal{B}}([u])$ satisfies

$$0 = \mathfrak{L}_{\mathcal{B}}\left(\frac{d}{dt}[u] - C[u]\right) = (z - C)[h],$$

hence $(z - C)h \in \mathfrak{L}\mathcal{G}_\infty(E)$. By (c), $y \circ h$ is exponentially decreasing of any order on $[0,\infty[$ for any $y \in F'$. Since $y \circ h \in \mathfrak{L}\mathcal{G}_{[0,\infty]}$ this implies that $y \circ h \in \mathfrak{L}\mathcal{G}_\infty$ for any $y \in F'$ by the Phragmén–Lindelöf theorem, and therefore $h \in \mathfrak{L}\mathcal{G}_\infty(F)$ since $\mathfrak{L}\mathcal{G}_\infty$ is nuclear. Thus $[u] = 0$ by (7.3) again. ■

From Theorem 7.1 we can deduce an extension of Lyubich's uniqueness theorem (see [22]) to Fréchet spaces (see Theorem 7.2 below). For this we need an appropriate notion of an asymptotic left resolvent for operators in Fréchet spaces (cf. [1] and [4] for general notions of resolvents in locally convex spaces). Let $F_n := (F/\ker\|\ \|_n)^\sim$ denote the canonical local Banach space for $\|\ \|_n$ and let $\kappa_n^F : F \to F_n$ be the corresponding spectral mapping. A sequence of operators $(R_n(t,C))_{n\in\mathbb{N}} \in (L(E,F_n))_{n\in\mathbb{N}}$ is an *asymptotic left resolvent* on a decreasing sequence of sets $\Sigma_n \subset \mathbb{C}$ if

$$R_n(t,C)(t - C) = \kappa_n^F + S_n(t) \quad \text{for } t \in \Sigma_n \tag{7.4}$$

where

$$\forall n \in \mathbb{N}\ \exists m \in \mathbb{N}: \quad \|S_n(t)\|_{L(E_m,F_n)} \le C_n e^{|t|/n - n|\mathrm{Re}\,t|} \quad \text{on } \Sigma_n. \tag{7.5}$$

Theorem 7.2. *Let E, F and C be as above. The* (*ACP*) (7.1) *has the uniqueness property* (*in the sense of hyperfunctions*) *if there is an asymptotic left resolvent $(R_n(t,C))_{n\in\mathbb{N}}$ for $\Sigma_n := [t_n,\infty[$ such that*

$$\forall n \in \mathbb{N}\ \exists k \in \mathbb{N}: \quad \|R_n(t,C)\|_{L(E_k,F_n)} \le C_n e^{kt} \quad \textit{if } t \ge t_n. \tag{7.6}$$

Proof. We use the criterion from Theorem 7.1(c). Let $h \in \mathfrak{L}\mathcal{G}_{[0,\infty]}(F)$ and $(z - C)h \in \mathfrak{L}\mathcal{G}_\infty(E)$. We then have, for any n and $t \ge t_n$,

$$\|\kappa_n^F(h(t))\|_n \le \|R_n(t,C)(t - C)h(t)\|_n + \|S_n(t)(h(t))\|_n \le \widetilde{C}_n e^{-(n-1)t}$$

by (7.4)—(7.6). Hence $\{h(t)e^{nt} \mid t \ge 0\}$ is bounded in F for any $n \in \mathbb{N}$. ■

We can also formulate a sufficient criterion for the uniqueness property by means of an asymptotic existence assumption for the dual operator:

Theorem 7.3. *Let E, F and C be as above. The* (*ACP*) (7.1) *has the uniqueness property* (*in the sense of hyperfunctions*) *if for any $y \in F'$ and any $n \in \mathbb{N}$ there is $t_{y,n} \ge 0$ such that for any $t \ge t_{y,n}$ there are $y_n(t), s_{y,n}(t) \in F'$, $k \in \mathbb{N}$ and $C_1 > 0$ such that for $t \ge t_{y,n}$,*

$$(t - {}^tC)y_n(t) = y + s_{y,n}(t), \quad |\langle y_n(t), x\rangle| \le C_1\|x\|_k e^{kt},$$
$$|\langle s_{y,n}(t), x\rangle| \le C_1\|x\|_k e^{-nt}.$$

Proof. Let $h \in \mathfrak{LG}_{[0,\infty]}(F)$ and $v := (z - C)h \in \mathfrak{LG}_\infty(E)$. By assumption we have, for any $y \in F'$,

$$\langle y, h(t)\rangle = \langle (t - {}^tC)y_n(t), h(t)\rangle - \langle s_{y,n}(t), h(t)\rangle = \langle y_n(t), v(t)\rangle - \langle s_{y,n}(t), h(t)\rangle$$

for large t. Using the known estimates for $y_n(t), s_{y,n}(t), h(t)$ and $v(t)$ for large t we conclude that $\{h(t)e^{nt} \mid t \geq 0\}$ is weakly bounded in F for any $n \in \mathbb{N}$ and use Theorem 7.1. ∎

As an illustration of the above results we will briefly discuss the uniqueness of the solutions of the (ACP) for continuous linear operators C in the space $F := E := \omega := \mathbb{C}^{\mathbb{N}}$ of all sequences endowed with the canonical product topology (see [19], [31] and [7] and the references there for earlier work on the (ACP) in this space). Any $C \in L(\omega)$ is given by an infinite, matrix $A := (a_{ij})_{i,j\in\mathbb{N}}$ with finite rows, i.e. for any $j \in \mathbb{N}$,

$$l_j := l((a_{js})_{s\in\mathbb{N}}) := \max\{s \in \mathbb{N} \mid a_{js} \neq 0\} < \infty \tag{7.7}$$

(here $l_j(v) := 0$ if $v = 0$).

In the following we will not distinguish between the operator $C \in L(\omega)$ and the corresponding matrix A. We will also consider the (ACP) in the classical sense, that is, the problem

$$x'(t) = Ax(t), \qquad t > 0, \qquad x(0) = x_0,\ x_0 \in \omega, \tag{7.8}$$

where $x \in C^1([0, \infty[, \omega)$.

THEOREM 7.4. *Let $C \in L(\omega)$ be given by the infinite matrix A. The following are equivalent:*

(a) *The* (*ACP*) (7.1) *has the uniqueness property* (*in the sense of hyperfunctions*).
(b) *The* (*ACP*) (7.8) *has the uniqueness property* (*in the classical sense*).
(c) *For any $n \in \mathbb{N}$,*

$$\sup_{k\in\mathbb{N}} l_n(A^k) < \infty. \tag{7.9}$$

Proof. (c)⇒(a). We use the criterion of Theorem 7.3(c). Since $F' = E' = \omega'$ is the space φ of finite sequences, the condition must be shown only for the canonical unit vectors $e_j =: y$. By (7.9) we know that $G := \operatorname{span}\{{}^tA^k e_j \mid k \in \mathbb{N}\}$ is finite-dimensional. Since G is ${}^tA$-invariant, any operator norm $\|{}^tA\|$ is finite, hence the Neumann series

$$Y(t) := \sum_{l=0}^{\infty} {}^tA^l e_j t^{-l-1}$$

exists in $G \subset \varphi$ for $t \geq t_j$. Clearly,

$$|\langle Y(t), x\rangle| \leq C_1\|x\|_{l_0} \quad \text{and} \quad (t - {}^tA)Y(t) = e_j \quad \text{for } t \geq t_j$$

where $l_0 := \max_{v\in G} l(v)$. Hence the criterion in 7.3(c) is satisfied for $y_n := Y$ and $s_{e_j,n} := 0$.

(a)⇒(b). Let $x \in C^1([0,\infty[,\omega)$ be a solution of (7.8) for $x_0 := 0$. Then x defines a hyperfunction $[u] \in \mathcal{B}([0,\infty[)$ (e.g. by [2]) which is a solution of (7.2) for $x_0 := 0$, hence $[u] = 0$ by assumption and thus $x \equiv 0$ on $[0,\infty[$.

(b)⇒(c). Let $x \in C^1(\mathbb{R},\omega)$ be a solution of

$$x'(t) = Ax(t), \quad t \in \mathbb{R}, \quad x(0) = x_0,\ x_0 \in \omega, \tag{7.10}$$

for $x_0 := 0$. Then $x|_{[0,\infty[}$ solves (7.8) for $x_0 = 0$, hence $x(t) = 0$ for $t \geq 0$. (7.10) is solvable for any $x_0 \in \omega$ by [31, Theorem 2.3] (see also Theorem 7.8) and we have just shown that the solution is unique on $[0, 1]$. The proof of [19, Satz 3 (b)⇒(d)] now implies that [19, Satz 3(d)] holds, hence the spectrum of A (in the sense used in [19]) is at most countable by [19, Satz 3]. This implies (7.9) by the remarks before [19, Satz 3]. ∎

We now discuss the existence of solutions of the (ACP) (7.1). For this we will have to solve the equation $(\lambda - C)S(\lambda) = x_0$ only approximately near the half-circle S_∞ at ∞, and moreover the approximate solution is needed only in the local Banach spaces F_n of F. To present the precise formulation and its proof in Theorem 7.6 below, we will use the Laplace transform of F-valued Laplace hyperfunctions developed in [3]. We briefly recall the corresponding notation and results from [3]. Let

$$H := \operatorname*{ind}_K(\operatorname*{proj}_k H_{K,k}) := \operatorname*{ind}_K H_K$$

where

$$H_{K,k} := \{f \in H(\Omega_K) : \|f\|_{K,k} := \sup_{z\in\Omega_K} |f(z)| \exp(k \operatorname{Re} z) < \infty\}$$

and

$$\Omega_K := \left\{ z \in \mathbb{C} : |\operatorname{Im} z| < \frac{\operatorname{Re} z}{K} + \frac{1}{K^2} \right\}.$$

Then an F-valued Laplace hyperfunction is a continuous linear operator $T : H \to F$. The Laplace transform $\mathcal{L}(T)$ is not a single holomorphic function, but a spectral-valued holomorphic function introduced in [3] as follows: let $F := \operatorname{proj}_n F_n$ with spectral mappings $\kappa_n^F : F \to F_n$ as above and let $\kappa_n^m : F_m \to F_n$ for $m \geq n$ be the corresponding linking maps.

Let $\mathcal{G} := (G_m)_{m\in\mathbb{N}}$ be a decreasing family of non-void domains in $\mathbb{C}$ and let $\mathcal{F} := (F_n)_{n\in\mathbb{N}}$. A family $\mathcal{S} = (S_m)_{m\in\mathbb{N}}$ is called a *spectral-valued* (or *$\mathcal{F}$-valued*) *holomorphic function* (denoted by $\mathcal{S} : \mathcal{G} \to \mathcal{F}$) if

(i) $S_m : G_m \to F_m$ is holomorphic;
(ii) (compatibility) $\forall m \geq n : \kappa_n^m \circ S_m = S_n|_{G_m}$.

Finally, we need the space $H_{\exp}(\mathcal{F})$ which is the set of all holomorphic $\mathcal{F}$-valued maps $\mathcal{S} : \mathcal{G} \to \mathcal{F}$ where $\mathcal{G}$ consists of postsectorial sets and

$$\forall m, K \in \mathbb{N}\ \forall |\varphi| < \pi/2 : \sup_{\lambda \in \Gamma_{r,\varphi}} \|S_m(T)(\lambda)\|_m e^{-\operatorname{Re}\lambda/K} < \infty \quad \text{if } \Gamma_{r,\varphi} \subset G_m.$$

Notice that $H_{\exp}(\mathcal{F})$ is considered rather as a set of germs near S_∞ and thus is a vector space canonically.

In the case of Fréchet spaces the main result of [3] is the following (see [3, Theorem 2.4 and Corollary 3.5]):

THEOREM 7.5. *The Laplace transform $\mathcal{L} : L(H, F) \to H_{\exp}(\mathcal{F})$ is a linear bijection such that $\mathcal{L}\big(\frac{d}{dt}T\big) = \lambda\mathcal{L}(T)$.*

By continuity of $C : F \to E$, for any $m \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that C defines a continuous linear mapping $C_m^k : F_k \to E_m$.

A general criterion for the solvability of the (ACP) (7.1) in Fréchet spaces is the following

THEOREM 7.6. *Let E, F and C be as above. For $x_0 \in E$ the following are equivalent:*

(a) *The* (*ACP*) (7.1) *has a solution* (*in the sense of hyperfunctions*).
(b) *There is a spectral-valued holomorphic function $\mathcal{S} := (S_m)_{m\in\mathbb{N}} \in H_{\exp}(\mathcal{F})$ such that for any $m \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that*

$$(\lambda - C_m^k)S_k(\lambda) = \kappa_m^E(x_0) + s_m(\lambda) \quad \text{on } G_k \tag{7.11}$$

where for any $j \in \mathbb{N}$ and any $|\varphi| < \pi/2$ with $\Gamma_{r,\varphi} \subset G_k$ there is $C_1 > 0$ such that

$$\|s_m(\lambda)\|_m \le C_1 e^{-j\operatorname{Re}\lambda + |\operatorname{Im}\lambda|/j} \quad \text{on } \Gamma_{r,\varphi}. \tag{7.12}$$

Proof. (a)$\Rightarrow$(b). Let $[u] \in \mathcal{B}([0,\infty[, F)$ be a solution of (7.2) and $[h] := \mathfrak{L}_{\mathcal{B}}([u]) \in \mathfrak{L}\mathcal{G}_{[0,\infty]}(F)/\mathfrak{L}\mathcal{G}_\infty(F)$. Then $(\lambda - C)[h] = \mathfrak{L}_{\mathcal{B}}(x_0 \otimes \delta_0)$ coincides with the constant function $[x_0]$ in the sense of $\mathfrak{L}\mathcal{G}_{[0,\infty]}(F)/\mathfrak{L}\mathcal{G}_\infty(F)$. Hence (b) is satisfied for $G_m := \mathbb{C}$ and $S_m := \kappa_m(h)$.

(b)$\Rightarrow$(a). By Theorem 7.5 there is $T \in L(H, F)$ such that $\mathcal{S} = \mathcal{L}(T)$. By (7.12) and Theorem 7.5 again, there is $\widetilde{T} \in L(H, F)$ such that $s := (s_m)_{m\in\mathbb{N}} = \mathcal{L}(\widetilde{T})$ and

$$\left(\frac{d}{dt} - C\right)T = x_0 \otimes \delta_0 + \widetilde{T}. \tag{7.13}$$

To translate this equation from Laplace hyperfunctions to hyperfunctions, i.e. to general boundary values ignoring growth conditions, we use the functions $f_\lambda(t) := \frac{-1}{2\pi i} e^{(t-\lambda)^2}/(t-\lambda)$ for $\lambda \notin [0,\infty[$. Since $f_\lambda \in H$ the function

$$u_T : \mathbb{C} \setminus [0,\infty[\to F, \quad u_T(\lambda) := \langle T, f_\lambda \rangle,$$

is defined. Since the difference quotients with respect to λ converge in H, u_T is holomorphic and

$$\frac{d}{d\lambda}u_T(\lambda) = \Big\langle T, \frac{d}{d\lambda}f_\lambda\Big\rangle = \Big\langle T, -\frac{d}{dt}f_\lambda\Big\rangle = \Big\langle \frac{d}{dt}T, f_\lambda\Big\rangle.$$

By (7.13) and Theorem 7.5 this implies that

$$(\lambda - C)u_T(\lambda) = -x_0 \otimes f_0(\lambda) + u_{\widetilde{T}}(\lambda).$$

Since $-f_0$ represents Dirac's δ-distribution we only need to show that $u_{\widetilde{T}}$ is an entire function. This follows by an argument similar to that in (5.5) and (5.6): For $j \in \mathbb{N}$ and $T \in L(H,F)$ let $\langle \tau_{-j}T, f\rangle := \langle T, f(\,\cdot + j)\rangle$. Then

$$\mathcal{L}(\tau_{-j}T) = e^{-j\,\cdot}\mathcal{L}(T) \tag{7.14}$$

by the definition of $\mathcal{L}(T)$ in [3]. By (7.12) we know that $e^{j\,\cdot}s \in H_{\exp}(\mathcal{F})$, hence there is $T_j \in L(H,F)$ such that $\mathcal{L}(T) = e^{j\,\cdot}s$ and therefore $\mathcal{L}(\tau_{-j}T_j) = s$ by (7.14) and finally $\tau_{-j}T_j = \widetilde{T}$ by Theorem 7.5 since $\mathcal{L}(\widetilde{T}) = s$. Hence, for any $j \in \mathbb{N}$,

$$u_{\widetilde{T}}(\lambda) = \langle \tau_{-j}T_j, f_\lambda\rangle = \langle T_j, f_\lambda(\,\cdot\, + j)\rangle = \langle T_j, f_{\lambda-j}\rangle$$

is holomorphic for $\lambda \notin [j,\infty[$ since $T_j \in L(H,F)$. The theorem is proved. ∎

As before we can formulate a criterion for the general solvability of the (ACP) (7.1) using a suitable notion of asymptotic right resolvent. Here a spectral-valued holomorphic operator function $\mathcal{R} := (R_m)_{m\in\mathbb{N}} : \mathcal{G} := (G_m)_{m\in\mathbb{N}} \to \mathcal{L}(E,F) := (L(E,F_m)_{m\in\mathbb{N}}$ is called an *asymptotic right resolvent* if $\mathcal{R} \in H_{\exp}(\mathcal{L}(E,F))$ and if there is a spectral, valued holomorphic function $\mathcal{T} := (T_m)_{m\in\mathbb{N}} : \widetilde{\mathcal{G}} := (\widetilde{G}_m)_{m\in\mathbb{N}} \to \mathcal{L}(E) := (L(E,E_m))_{m\in\mathbb{N}}$ such that for any $m \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that

$$(\lambda - C_m^k)R_k(\lambda) = \kappa_m^E + T_m(\lambda) \quad \text{on } G_k \tag{7.15}$$

where for any $j \in \mathbb{N}$ and any $|\varphi| < \pi/2$ with $\Gamma_{r,\varphi} \subset G_k$,

$$\|T_m(\lambda)\|_{L(E_k,E_m)} \le C_1 e^{-j\operatorname{Re}\lambda + |\operatorname{Im}\lambda|/j} \quad \text{on } \Gamma_{r,\varphi}. \tag{7.16}$$

Theorem 7.7. *Let E, F and C be as above. Then the* (*ACP*) (7.1) *has a solution* (*in the sense of hyperfunctions*) *for any $x_0 \in E$ if C admits an asymptotic right resolvent.*

Proof. For $x_0 \in E$ set $S_k(\lambda) := R_k(\lambda)x_0$ and apply Theorem 7.6. ∎

We again discuss the (ACP) for continuous linear operators C in the space ω as an example, using Theorem 7.6 for a new short proof (see [31] for the (ACP) of the inhomogeneous equation $x'(t) = Cx(t) + f(t)$).

Theorem 7.8. *For any $\alpha \in \omega$ the* (*ACP*)

$$x'(t) = Ax(t), \quad t > 0, \quad x(0) = \alpha, \tag{7.17}$$

has a classical solution $x \in C^1(\mathbb{R},\omega)$.

Proof. For $\alpha \in \omega$ let $\varphi_\alpha \in C_0^\infty(\mathbb{R})$ be a solution of the Borel problem $\varphi_\alpha^{(j)}(0) = \alpha_j$ for $j \in \mathbb{N}_0$. Set $\widetilde{\varphi}_\alpha := (\varphi_\alpha, \varphi'_\alpha, \ldots)$. Then the Laplace transform $f_\alpha(\lambda) := \int_0^\infty e^{-\lambda t}\widetilde{\varphi}_\alpha(t)\,dt$ is an entire ω-valued function satisfying

(7.18) $\quad (\lambda - L)(f_\alpha(\lambda)) = \alpha,$

(7.19) $\quad \forall E \in \mathrm{FS}(\varphi)\ \exists C_E > 0\ \forall y \in E:$
$$|\langle y, f_\alpha(\lambda)\rangle| \le C_E\|y\|_2/|\lambda| \quad \text{if } \operatorname{Re}\lambda > 0$$

where L is the left shift and $\mathrm{FS}(\varphi)$ is the set of finite-dimensional subspaces of φ . We will construct an ω-valued holomorphic function g, i.e. $g = (g_k)_{k\in\mathbb{N}}$ such that for any k, $g_k(\lambda)$ is holomorphic for $\operatorname{Re}\lambda > B_k$ and

(7.20) $\quad \forall y \in \varphi: \quad \langle y, (\lambda - A)g(\lambda)\rangle = \langle y, \alpha\rangle \quad \text{if } \operatorname{Re}\lambda > B_y,$

(7.21) $\quad \forall E \in FS(\varphi)\ \exists C_E, B_E > 0\ \forall y \in E:$
$$|\langle y, g(\lambda)\rangle| \le C_E\|y\|_2/|\lambda| \quad \text{if } \operatorname{Re}\lambda > B_E.$$

Then the inverse Laplace transform $x = (x_k)_k$ of g is a solution of the (ACP) (7.17) by Theorem 7.6 and x_k is a C^1-function for each k.

g is constructed inductively. Let $E_1 := \operatorname{span}\{w_k := ({}^tA)^{k-1}e_1 \mid k \in \mathbb{N}\}$; it is an ${}^tA$-invariant subspace of φ. If $\dim E_1 < \infty$, we set
$$\langle w, g(\lambda)\rangle := \Big\langle \sum_{k=0}^{\infty} \lambda^{-k-1}({}^tA)^k w, \alpha\Big\rangle \quad \text{for } w \in E_1.$$
This defines $g(\lambda)$ on E_1 for large $|\lambda|$, and (7.20) and (7.21) hold on E_1. If $\dim E_1 = \infty$ then $\{w_k \mid k \in \mathbb{N}\}$ is linearly independent and ${}^tAw_k = w_{k+1}$ like a right shift, hence we set
$$\langle w, g(\lambda)\rangle := \langle w, {}^tTf_\beta(\lambda)\rangle \quad \text{where} \quad \beta_j = \langle w_j, \alpha\rangle \text{ for any } j \in \mathbb{N}$$
and $T : E_1 \to \varphi$ is the isomorphism mapping w_k to the canonical unit vector e_k. Again, (7.20) and (7.21) are easily shown on E_1. Now assume that E_j has been constructed and that $E_j \ne \varphi$. Then there is l_0 minimal such that $e_{l_0} \notin E_j$. If $v_{n_0+1} := ({}^tA)^{n_0}e_{l_0} \in E_j$ for some (minimal) n_0 we set, for $v_d := ({}^tA)^{d-1}e_{l_0}, 1 \le d \le n_0$,
$$\langle v_d, g(\lambda)\rangle := \sum_{l=1}^{n_0+1-d} \lambda^{-l}\langle v_{d+l-1}, \alpha\rangle + \lambda^{d-n_0-1}\langle v_{n_0+1}, g(\lambda)\rangle.$$
(7.20) and (7.21) are easily shown on $\widetilde{E}_j := \operatorname{span}\{v_k \mid k \le n_0\}$, hence they hold on $E_{j+1} := \operatorname{span}(E_j, \widetilde{E}_j)$. If $({}^tA)^n e_{l_0} \notin E_j$ for any n we proceed as in the first step to define $g(\lambda)$ on $\widetilde{E}_j := \operatorname{span}\{v_k \mid k \in \mathbb{N}\}$. In this way we define $g(\lambda)$ inductively on φ. The theorem is proved. ∎

Acknowledgements. The author wants to thank P. Domański (Poznań) for interesting discussions on the subject of this paper and for his great hospitality during a visit in Poznań in March 2010, where part of this paper was written.

References

[1] H. Arikan, L. Runov and V. Zahariuta, *Holomorphic functional calculus for operators on a locally convex space*, Results Math. 43 (2003), 23–36.

[2] P. Domański and M. Langenbruch, *Vector valued hyperfunctions and boundary values of vector valued harmonic and holomorphic functions*, Publ. RIMS Kyoto Univ. 44 (2008), 1097–1142.

[3] —, —, *On the Laplace transform for vector valued hyperfunctions*, Funct. Approx. Comment. Math. 43 (2010), 129–159.

[4] —, —, *On the abstract Cauchy problem for operators in locally convex spaces*, preprint.

[5] —, —, *Asymptotic Laplace transform for vector valued hyperfunctions and the abstract Cauchy problem*, manuscript.

[6] I. M. Gelfand and G. E. Shilov, *Generalized Functions*, Vol. 2, Academic Press, New York, 1968.

[7] G. Herzog, *On ordinary linear differential equations in* $\mathbb{C}^J$, Demonstratio Math. 28 (1995), 383–398.

[8] L. Hörmander, *The Analysis of Linear Partial Differential Operators I, II*, Springer, Berlin, 1983.

[9] —, *An Introduction to Complex Analysis in Several Variables*, North-Holland, Amsterdam, 1990.

[10] A. Kaneko, *Introduction to Hyperfunctions*, Kluwer, Dordrecht, 1988.

[11] T. Kawai, *On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients*, J. Fac. Sci. Univ. Tokyo Sect. IA 17 (1970), 467–517.

[12] H. Komatsu, *Laplace transforms of hyperfunctions—A new foundation of the Heaviside operational calculus*, J. Fac. Sci. Univ. Tokyo 12 (1987), 357–372.

[13] —, *Laplace transforms of hyperfunctions—Another foundation of the Heaviside operational calculus*, in: Generalized Functions, Convergence Structures, and Their Applications, Plenum Press, New York, 1988, 57–70.

[14] —, *Operational calculus, hyperfunctions and ultradistributions*, in: Algebraic Analysis, Vol. 1, Academic Press, Elsevier, 1989, 357–372.

[15] P. C. Kunstmann, *Post–Widder inversion for Laplace transforms of hyperfunctions*, in: Functional Analysis and Evolution Equations, H. Amann et al. (eds.), Birkhäuser, Basel, 2007, 423–431.

[16] M. Langenbruch, *Randverteilungen von Nullösungen hypoelliptischer Differentialgleichungen*, Manuscripta Math. 26 (1978), 17–35.

[17] —, *Hermite functions and weighted spaces of generalized functions*, ibid. 119 (2006), 269–285.

[18] —, *Extension of Sato's hyperfunctions*, Funct. Approx. Comment. Math. 44 (2011), 33–44.

[19] R. Lemmert und Ä. Weckbach, *Charakterisierungen zeilenendlicher Matrizen mit abzählbarem Spektrum*, Math. Z. 188 (1984), 119–124.

[20] G. Lumer and F. Neubrander, *Asymptotic Laplace transforms and evolution equations*, in: M. Demuth et al. (ed.), Evolution Equations, Feshbach Resonances, Singular Hodge Theory, Math. Top. 16, Wiley-VCH, 1999, 37–57.

[21] G. Lumer and F. Neubrander, *The asymptotic Laplace transform: New results and relation to Komatsu's Laplace transform of hyperfunctions*, in: F. Ali Mehmeti et al. (eds.), Partial Differential Equations on Multistructures (Luminy, 1999), Lecture Notes Pure Appl. Math. 219, Dekker, 2001, 147–162.

[22] Yu. I. Lyubich, *The classical and local Laplace transformation in an abstract Cauchy problem*, Uspekhi Mat. Nauk 21 (1966), no. 3, 3–51 (in Russian).

[23] R. Meise and D. Vogt, *Introduction to Functional Analysis*, Clarendon Press, Oxford, 1997.

[24] M. Morimoto and K. Yoshino, *Some examples of analytic functionals with carrier at infinity*, Proc. Japan Acad. Ser. A 56 (1980), 357–361.

[25] S. Ōuchi, *Hyperfunction solutions of the abstract Cauchy problem*, Proc. Japan Acad. 47 (1971), 541–544.

[26] —, *On abstract Cauchy problems in the sense of hyperfunctions*, in: Lecture Notes in Math. 287, Springer, 1972, 135–152.

[27] —, *Semi-groups of operators in locally convex spaces*, J. Math. Soc. Japan 25 (1973), 265–276.

[28] Y. Saburi, *Fundamental properties of modified Fourier hyperfunctions*, Tokyo J. Math. 8 (1985), 231–273.

[29] M. Sato, *Theory of hyperfunctions*, Sugaku 10 (1958), 1–27 (in Japanese).

[30] —, *Theory of hyperfunctions I*, J. Fac. Sci. Univ. Tokyo Sect. IA 8 (1959), 139–193.

[31] S. A. Shkarin, *Some results on solvability of ordinary linear differential equations in locally convex spaces*, Math. USSR Sb. 71 (1992), no. 1, 29–40.

[32] H. G. Tillmann, *Darstellung der Schwartz'schen Distributionen durch analytische Funktionen*, Math. Z. 77 (1961), 106–124.

[33] J. C. Vignaux, *Sugli integrali di Laplace asintotici*, Atti Acad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. (6) 29 (1939), 396–402.

Michael Langenbruch
Department of Mathematics
University of Oldenburg
D-26111 Oldenburg, Germany
E-mail: Michael.Langenbruch@uni-oldenburg.de

Received January 28, 2011 (7094)