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## The structure of Lindenstrauss–Pełczyński spaces

by

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Abstract. Lindenstrauss–Pełczyński (for short  $\mathcal{LP}$ ) spaces were introduced by these authors [Studia Math. 174 (2006)] as those Banach spaces X such that every operator from a subspace of  $c_0$  into X can be extended to the whole  $c_0$ . Here we obtain the following structure theorem: a separable Banach space X is an  $\mathcal{LP}$ -space if and only if every subspace of  $c_0$  is placed in X in a unique position, up to automorphisms of X. This, in combination with a result of Kalton [New York J. Math. 13 (2007)], provides a negative answer to a problem posed by Lindenstrauss and Pełczyński [J. Funct. Anal. 8 (1971)]. We show that the class of  $\mathcal{LP}$ -spaces does not have the 3-space property, which corrects a theorem in an earlier paper of the authors [Studia Math. 174 (2006)]. We then solve a problem in that paper showing that  $\mathcal{L}_{\infty}$  spaces not containing  $l_1$  are not necessarily  $\mathcal{LP}$ -spaces.

1.  $\mathcal{LP}$ -spaces have all subspaces of  $c_0$  in a unique position. In [6] we introduced the class of Lindenstrauss–Pełczyński spaces (for short  $\mathcal{LP}$ ) as those Banach spaces E such that all operators from subspaces of  $c_0$  into Ecan be extended to  $c_0$ . The spaces are so named because Lindenstrauss and Pełczyński first proved in [9] that C(K)-spaces have this property. In [6] it was shown that every  $\mathcal{LP}$ -space is an  $\mathcal{L}_{\infty}$ -space, that not all  $\mathcal{L}_{\infty}$ -spaces are  $\mathcal{LP}$ -spaces, and that complemented subspaces of Lindenstrauss spaces (see also [9, 7]), separably injective spaces and  $\mathcal{L}_{\infty}$ -spaces not containing  $c_0$  are  $\mathcal{LP}$ -spaces.

We now prove a fundamental structure theorem for this class; namely, separable  $\mathcal{LP}$ -spaces are characterized as those  $\mathcal{L}_{\infty}$  Banach spaces having all subspaces of  $c_0$  placed in a unique position. Precisely, let Y, X be Banach spaces. Following [5] we say that X is Y-automorphic if any isomorphism between two subspaces of X isomorphic to Y can be extended to an automorphism of X. We agree that if X contains no copies of Y then it is Y-automorphic. Lindenstrauss and Pełczyński prove in [9] that C[0, 1] is H-automorphic for all subspaces H of  $c_0$  and pose the question of whether

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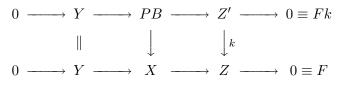
this property characterizes the subspaces of  $c_0$ . Kalton shows in [8] that the answer is no since C[0, 1] is also  $l_1$ -automorphic. In the opposite direction, there is the question of whether the property of being *H*-automorphic for all subspaces of  $c_0$  characterizes separable C(K)-spaces. The answer is no. In fact, amongst separable Banach spaces which contain an isomorphic copy of  $c_0$ , it characterizes being  $\mathcal{LP}$ .

Theorem 1.

- (i) A Banach space that contains c<sub>0</sub> and is H-automorphic for all subspaces H of c<sub>0</sub> is an LP-space.
- (ii) Every separable  $\mathcal{LP}$ -space is H-automorphic for all subspaces H of  $c_0$ .

Before entering into the proof, recall (see [4, 6]) the identification of exact sequences  $0 \to Y \to X \to Z \to 0$  of Banach spaces with z-linear maps  $F: Z \to Y$ , i.e. homogeneous maps such that for some constant K > 0 and every finite set  $x_1, \ldots, x_n$  one has  $||F(\sum x_j) - \sum Fx_j|| \le K \sum ||x_j||$ . The identification will be written as  $0 \to Y \to X \to Z \to 0 \equiv F$ . Two exact sequences  $0 \to Y \to X \to Z \to 0$  and  $0 \to Y \to X' \to Z \to 0$  of Banach spaces are said to be *equivalent* if there is a continuous linear operator  $T: X \to X'$  providing a commutative diagram

Two z-linear maps F, G are said to be equivalent, and written  $F \equiv G$ , when the associated exact sequences are equivalent. Under these identifications, given an exact sequence  $0 \to Y \to X \to Z \to 0 \equiv F$  and an operator  $k: Z' \to Z$ , the upper sequence in the associated pull-back diagram



corresponds to the z-linear map Fk (standard composition of maps). We will need the following lemma of independent interest.

LEMMA 1. Let  $F : Z \to Y$  be a z-linear map and let  $k : Z \to Z$  be a compact operator. Then  $Fk \equiv F$  implies  $F \equiv 0$ .

*Proof.* If  $Fk \equiv F$  then  $F - Fk \equiv F(1-k) \equiv 0$ . If 1 is not an eigenvalue of k then 1-k is an automorphism of Z, so  $F(1-k) \equiv 0$  implies  $F \equiv 0$ . If 1 is an eigenvalue of k then let  $z_1^1, \ldots, z_{n_1}^1$  be a basis for the associated space of eigenvectors. Let  $Z_1 = [z_1^1, \ldots, z_{n_1}^1]$  and consider the exact sequence

$$0 \to Z_1 \to Z \xrightarrow{q_1} Z/Z_1 \to 0.$$

Let  $s_1: Z/Z_1 \to Z$  be a continuous linear section for  $q_1$ . The operator  $q_1ks_1: Z/Z_1 \to Z/Z_1$  is compact. If 1 is not an eigenvalue of  $q_1ks_1$  then  $1_{Z/Z_1} - q_1ks_1 = q_1(1_Z - k)s_1$  is an automorphism of  $Z/Z_1$ . Since  $Z_1$  is finite-dimensional,  $F|_{Z_1} \equiv 0$  and there exists a z-linear map  $F_1: Z/Z_1 \to Y$  such that  $F_1q \equiv F$ . Now,  $F(1-k) \equiv 0$  implies  $F_1q_1(1_Z - k)s_1 \equiv 0$ , and therefore  $F_1 \equiv 0$ . Hence  $F \equiv 0$ . It remains to treat the case where 1 is an eigenvalue of  $q_1ks_1$ . Take then a basis  $q_1(z_1^2), \ldots, q_1(z_{n_2}^2)$  for the associated space of eigenvectors and form the closed linear span

$$Z_2 = [z_1^1, \dots, z_{n_1}^1, s_1q_1z_1^2, \dots, s_1q_1z_{n_2}^2].$$

The exact sequence

$$0 \to Z_2 \to Z \xrightarrow{q_2} Z/Z_2 \to 0$$

admits a continuous linear section  $s_2$ . If 1 is not an eigenvalue of the operator  $q_2ks_2$  the argument as before yields  $F \equiv 0$ . It remains to treat the case where 1 is an eigenvalue of  $q_2ks_2$ . We then proceed as follows. Assume that after n steps, 1 is an eigenvalue of  $q_nks_n$ . Take a basis  $q_n(z_1^{n+1}), \ldots, q_n(z_{n+1}^{n+1})$  for the associated space of eigenvectors and form the closed linear span

$$Z_{n+1} = [z_1^1, \dots, z_{n_1}^1, s_1q_1z_1^2, \dots, s_1q_1z_{n_2}^2, \dots, s_nq_nz_1^{n+1}, \dots, s_nq_nz_{n_{n+1}}^{n+1}].$$

The exact sequence

$$0 \to Z_{n+1} \to Z \xrightarrow{q_{n+1}} Z/Z_{n+1} \to 0$$

admits a continuous linear section  $s_{n+1}$ . If 1 is not an eigenvalue of  $q_{n+1}ks_{n+1}$  the same argument as before yields  $F \equiv 0$ . It remains to treat the case where 1 is an eigenvalue of  $q_{n+1}ks_{n+1}$ .

The process must stop because  $Z_n \subset \ker (1-k)^n$  and k has finite ascent, i.e. there is a natural N such that  $\ker (1-k)^N = \ker (1-k)^{N+1}$ .

Proof of Theorem 1. To prove (i) we adapt the arguments of [10, Thm. 3.2]. Let X be H-automorphic for all subspaces H of  $c_0$ , and assume that there is an embedding  $j: c_0 \to X$ . Assume there is a subspace  $i: H \subset c_0$  and a norm one operator  $T: H \to X$  that cannot be extended to  $c_0$  through i. Then for small  $\varepsilon > 0$  the operator  $ji + \varepsilon T : H \to X$  is an into isomorphism that cannot be extended to an operator  $R: X \to X$  through ji, as otherwise  $Rji = ji + \varepsilon T$  and  $\varepsilon^{-1}(Rj - j)$  would be an extension of T through i.

We show (ii). Let X be a separable  $\mathcal{LP}$ -space. If X does not contain  $c_0$ , then the result is (vacuously) true. So let  $i: H \to X$  be an embedding where  $j: H \to c_0$  is a subspace of  $c_0$ . The extension  $J: c_0 \to X$ , which exists because X is  $\mathcal{LP}$ , yields the commutative diagram

We now show that the operator qJ is not weakly compact. Otherwise it would be compact, hence J'p = qJ would be compact and thus J' would also be compact. Since X is separable, the embedding *i* can be extended to  $c_0$ , which yields a commutative diagram

$$(2) \qquad \begin{array}{cccc} 0 & \longrightarrow & H & \xrightarrow{i} & X & \xrightarrow{q} & X/H & \longrightarrow & 0 \\ & & \parallel & & \downarrow I & & \downarrow I' \\ 0 & \longrightarrow & H & \xrightarrow{j} & c_0 & \xrightarrow{p} & c_0/H & \longrightarrow & 0 \end{array}$$

Putting the two diagrams together one gets a commutative pull-back diagram

in which k = I'J' is compact. Lemma 1 shows that is impossible.

A Banach space C is said to have *Pełczyński's property* (V) if every operator on C is either weakly compact or an isomorphism on a copy of  $c_0$ . Since C(K)-spaces have property (V) [11] and we have shown that the operator qJis not weakly compact, it must be an isomorphism on a subspace isomorphic to  $c_0$ . Therefore q is also an isomorphism on a subspace isomorphic to  $c_0$ , which will necessarily be complemented in both X/H and X. This means the existence of a commutative diagram

in which both  $\beta$  and  $\gamma$  are isomorphisms. An application of the diagonal principles developed in [5] to the diagrams (1) and (2) yields a commutative diagram

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in which both  $\sigma, \mu$  are isomorphisms. Now, starting with a different embedding  $i': H \to X$  would lead to a similar diagram with j replaced by some embedding  $j': H \to c_0$ . But since  $c_0$  is H-automorphic by the classical Lindenstrauss–Pełczyński theorem, we are done.

The conclusion of Theorem 1(ii) clearly fails for nonseparable spaces since  $c_0 \oplus l_{\infty}$  contains a complemented and an uncomplemented copy of  $c_0$ . From the proof one is tempted to believe that  $\mathcal{LP}$ -spaces containing  $c_0$  have Pełczyński's property (V), which is not the case: let X be a Schur  $\mathcal{LP}$ -space (see [6]) and select a quotient map  $q : X \to c_0$  to construct the quotient  $Q : X \oplus c_0 \to c_0$  given by Q(x, y) = q(x). An immediate consequence of Theorem 1 and the fact that  $\mathcal{L}_{\infty}$ -spaces not containing  $c_0$  are  $\mathcal{LP}$ -spaces is

COROLLARY 1. Every  $\mathcal{L}_{\infty}$ -space which is H-automorphic for every subspace H of  $c_0$  is an  $\mathcal{LP}$ -space.

A Banach space X was defined in [10] to be *extensible* if every operator  $Y \to X$  from a subspace Y of X can be extended to X. It is clear that an extensible space that contains  $c_0$  must be an  $\mathcal{LP}$ -space; hence

COROLLARY 2. An extensible  $\mathcal{L}_{\infty}$ -space is an  $\mathcal{LP}$ -space.

Thus, even the product of separable automorphic spaces such as  $l_2 \oplus c_0$  may fail to be extensible.

**2.** Counterexamples. Our first counterexample is to show that, unlike C[0, 1], separable  $\mathcal{LP}$ -spaces may fail to be  $l_1$ -automorphic:

A separable  $\mathcal{LP}$ -space that is not  $l_1$ -automorphic. Consider an embedding  $i: l_1 \to C[0, 1]$  and another embedding  $j: l_1 \to X$  of  $l_1$  into its corresponding Bourgain–Pisier space  $\mathcal{L}_{\infty}$ -space X (see [2]). It was proved in [6] that X is a Schur  $\mathcal{LP}$ -space. This means that j cannot be extended through i to the whole C[0, 1] since Pełczyński's property (V) of C[0, 1] would make such an extension a weakly compact operator. The  $\mathcal{LP}$ -space  $X \oplus C[0, 1]$  is not  $l_1$ -automorphic because the embeddings  $(0, i): l_1 \to X \oplus C[0, 1]$  and  $(j, 0): l_1 \to X \oplus C[0, 1]$  cannot be transformed to each other by an automorphism  $\sigma: X \oplus C[0, 1] \to X \oplus C[0, 1]$ . Otherwise, if  $\sigma(0, i) = (j, 0)$ , and  $\pi: X \oplus C[0, 1] \to X$  is the projection, the operator  $\pi\sigma_{|C[0,1]}: C[0, 1] \to X$  would be an extension of j through i.

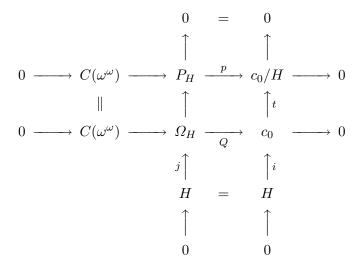
Our second counterexample shows that the statement of [6, Thm. 2] that "the class of  $\mathcal{LP}$ -spaces has the 3-space property" is wrong.

PROPOSITION 2.1. For every subspace H of  $c_0$  different from  $c_0$ , there is a twisted sum

$$0 \to C(\omega^{\omega}) \to \Omega_H \to c_0 \to 0,$$

and an operator  $H \to \Omega_H$  that cannot be extended to  $c_0$ . Hence the space  $\Omega_H$  is not an  $\mathcal{LP}$ -space.

Proof. Consider the exact sequence  $0 \to C(\omega^{\omega}) \to \Omega \to c_0 \to 0 \equiv M$ constructed in [3] which has the additional property of having the quotient map  $q: \Omega \to c_0$  strictly singular. Since every quotient of  $c_0$  is isomorphic to a subspace of  $c_0$ , we can assume that there is an embedding  $u_H: c_0/H \to c_0$ . The pull-back sequence  $0 \to C(\omega^{\omega}) \to P_H \xrightarrow{p} c_0/H \to 0 \equiv Gu_H$  also has strictly singular quotient map. We form the commutative diagram



To show that  $\Omega_H$  is not an  $\mathcal{LP}$ -space we show that j cannot be extended to  $c_0$  through i. Indeed, suppose J is such an extension, and denote by  $\nu$ the induced operator between the quotient spaces. There is a commutative diagram

The diagram means that  $Fp\nu \equiv F$ . But since p is strictly singular,  $p\nu$  is also strictly singular, hence compact. Lemma 1 can be used to conclude the argument.

The previous example also provides a negative answer to a question posed in [6, p. 227]: Is every  $\mathcal{L}_{\infty}$ -space not containing  $l_1$  an  $\mathcal{L}P$ -space? The space  $\Omega_H$  does not contain  $l_1$  since "not containing  $l_1$ " is a 3-space property (see [4, Thm. 3.2.d]).

Our last example provides a partial answer to the question of whether the original space  $\Omega$  constructed in [3] and which is the starting point in the proof of Proposition 2.1 is an  $\mathcal{LP}$ -space.

PROPOSITION 2.2. There exists an  $\mathcal{LP}$ -space X admitting two nontrivial exact sequences

$$0 \to X \to A_i \xrightarrow{q_i} c_0 \to 0$$

such that

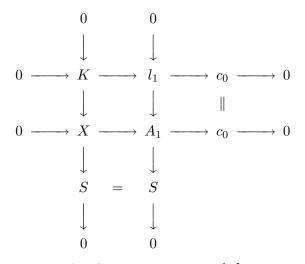
(1)  $A_1$  is an  $\mathcal{LP}$ -space and  $q_1$  is strictly singular.

(2)  $A_2$  is not an  $\mathcal{LP}$ -space.

*Proof.* Consider the projective presentation of  $c_0$ ,

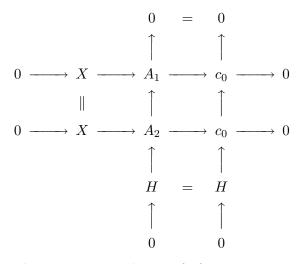
$$0 \to K \to \ell_1 \to c_0 \to 0,$$

and embed K into its corresponding Bourgain–Pisier space  $\mathcal{L}_{\infty}$ -space X (see [2]). It was proved in [6] that X is an  $\mathcal{LP}$ -space. To construct  $A_1$  we consider the push-out diagram



Since the Schur property is a 3-space property and  $\mathcal{L}_{\infty}$ -spaces with the Schur property are  $\mathcal{LP}$ -spaces,  $A_1$  is an  $\mathcal{LP}$ -space, and the quotient  $A_1 \to c_0$  must be strictly singular.

To obtain  $A_2$ , let  $0 \to X \to A_1 \to c_0 \to 0$  be the previously constructed sequence having strictly singular quotient, and let  $0 \to H \to c_0 \to c_0 \to 0$  be the nontrivial sequence constructed by Bourgain in [1]. We form the pull-back diagram



and then follow the argument in the proof of Proposition 2.1 to show that  $A_2$  is not an  $\mathcal{LP}$ -space.

**3.** Positive results. The first positive result exhibits two situations in which a twisted sum of two  $\mathcal{LP}$ -spaces is an  $\mathcal{LP}$ -space. The counterexamples in Section 2 show that these results are optimal.

PROPOSITION 3.1. Let  $0 \to Y \xrightarrow{i} X \xrightarrow{q} Z \to 0$  be an exact sequence in which both Y, Z are  $\mathcal{LP}$ -spaces. Then X is an  $\mathcal{LP}$ -space in the following cases:

- (1) Z does not contain  $c_0$ .
- (2) Y is separably injective.

Proof. Let  $j : H \to c_0$  be a subspace of  $c_0$  and let  $t : H \to X$  be an operator, and consider an extension  $(qt)^e$  of qt to  $c_0$ . To prove (1) observe that  $(qt)^e$  is weakly compact, hence compact. Since Y is an  $\mathcal{L}_{\infty}$ -space,  $(qt)^e$  can be lifted to an operator  $E : c_0 \to X$  through q, so  $qE = (qt)^e$ . The operator Ej - t thus takes values in Y, and can therefore be extended to an operator  $(Ej - t)^e : c_0 \to Y$ . The operator  $E - i(Ej - t)^e : c_0 \to X$  is the desired extension of  $t: (E - i(Ej - t)^e)j = Ej - i(Ej - t) = t$ .

The proof for (2) is analogous: in this case  $(qt)^e$  can be lifted to an operator  $E: c_0 \to X$  through q since Y is separably injective.

The second positive result is a correct statement and proof of the 3-space result presented in [6]. The argument there touches the poorly developed topic of relative homology with respect to an operator ideal. Precisely, classical Banach space homology works with the ideal  $\mathfrak{L}$  of continuous linear operators in the sense that, given an exact sequence  $0 \to Y \to X \to Z \to 0$ and a Banach space E, it produces the homology sequence

$$0 \to \mathfrak{L}(Z, E) \to \mathfrak{L}(X, E) \to \mathfrak{L}(Y, E)$$
$$\to \operatorname{Ext}(Z, E) \to \operatorname{Ext}(X, E) \to \operatorname{Ext}(Y, E) \to \cdots$$

formed by the derived functors of  $\mathfrak{L}$ . One could expect that a surjective and injective operator ideal  $\mathcal{A}$  would also produce a relative homology sequence

$$0 \to \mathcal{A}(Z, E) \to \mathcal{A}(X, E) \to \mathcal{A}(Y, E)$$
$$\to \mathcal{A}'(Z, E) \to \mathcal{A}'(X, E) \to \mathcal{A}'(Y, E) \to \cdots$$

formed by derived functors of  $\mathcal{A}$ . The problem, however, is that derivation is a process that can be done via injective or projective presentations, and the results of the two processes might not coincide. In the classical setting, the injective and projective derivations of  $\mathfrak{L}$  are equivalent; in the relative setting the equivalence depends on the following extra property of the ideal  $\mathcal{A}$ .

DEFINITION. An injective and surjective operator ideal  $\mathcal{A}$  will be called *balanced* if any commutative diagram

has the property that there is an operator  $\alpha' \in \mathcal{A}$  such that  $\alpha - \alpha'$  can be extended to  $l_1(\Gamma)$  if and only if there exists  $\gamma' \in \mathcal{A}$  such that  $\gamma - \gamma'$  can be lifted to  $l_{\infty}(\Lambda)$ .

The condition is clearly equivalent to the fact that projective and injective derivations coincide. Let us now define a Banach space X to be  $\mathcal{A}$ -injective (resp. separably  $\mathcal{A}$ -injective) if  $\mathcal{A}'(\cdot, X) = 0$  (resp.  $\mathcal{A}'(S, X) = 0$  for every separable space S). The following proposition contains the right statement of Theorem 2 in [6].

PROPOSITION 3.2. For any surjective and injective balanced operator ideal  $\mathcal{A}$ , being  $\mathcal{A}$ -injective (resp. separably  $\mathcal{A}$ -injective) is a 3-space property.

*Proof.* We include the proof (an abstract version of results in [6]) for the sake of completeness. Let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be an exact sequence. By assumption, both  $\mathcal{A}(\cdot, A)$  and  $\mathcal{A}(\cdot, C)$  are exact functors and we need to prove that also  $\mathcal{A}(\cdot, B)$  is exact. We construct the commutative diagram

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The rows are exact by the surjectivity of  $\mathcal{A}$ , while the first three columns are also exact by injectivity of  $\mathcal{A}$ . The fourth column is exact because  $\mathcal{A}$  is balanced. By hypothesis,

$$\mathcal{A}'(Z,A) = \mathcal{A}'(Z,C) = 0,$$

and the exactness of the fourth column implies that

$$\mathcal{A}'(Z,B) = 0,$$

hence  $\mathcal{A}(\cdot, B)$  is exact.

In [6] it is established that  $\mathcal{LP}$ -spaces are precisely the relatively separably injective objects associated with the ideal  $\Gamma_0$  of operators that factorize through a subspace of  $c_0$ . The mistake in the proof there is that the ideal  $\Gamma_0$  is not balanced.

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