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Multi-dimensional Fejér summability and local Hardy spaces

by

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Abstract. It is proved that the multi-dimensional maximal Fejér operator defined in a cone is bounded from the amalgam Hardy space $W(h_p, \ell_\infty)$ to $W(L_p, \ell_\infty)$. This implies the almost everywhere convergence of the Fejér means in a cone for all $f \in W(L_1, \ell_\infty)$, which is larger than $L_1(\mathbb{R}^d)$.

1. Introduction. It is known that the Fejér means $\sigma_T f$ of a function $f \in L_1(\mathbb{R})$ converge a.e. to f as $T \to \infty$. Moreover, the maximal operator of the Fejér means $\sigma_* := \sup_{T>0} |\sigma_T|$ is of weak type $(1,1)$, i.e.

$$\sup_{\rho>0} \rho\lambda(\sigma_* f > \rho) \le C\|f\|_1 \quad (f \in L_1(\mathbb{R}))$$

(see Zygmund [17, Vol. I. p. 154]). Móricz [9, 10] also verified that σ_* is bounded from $H_p(\mathbb{R})$ to $L_p(\mathbb{R})$ for $p = 1$. The author extended those results to all $1/2 < p \le 1$ in [14].

For multi-dimensional trigonometric-Fourier series Marcinkiewicz and Zygmund [8] proved that the Fejér means $\sigma_n f$ of a function $f \in L_1(\mathbb{T}^d)$ converge a.e. to f as $n \to \infty$ provided that n is in a positive cone, i.e., $2^{-\tau} \le n_k/n_j \le 2^\tau$ for all $k, j = 1, \ldots, d$ and some $\tau \ge 0$ ($n = (n_1, \ldots, n_d) \in \mathbb{N}^d$). The analogous convergence also holds for the Fejér means of Fourier transforms, i.e. $\sigma_T f \to f$ a.e. as $T \to \infty$ and $2^{-\tau} \le T_k/T_j \le 2^\tau$ ($k, j = 1, \ldots, d$) for all $f \in L_1(\mathbb{R}^d)$ (see Weisz [15]). Moreover, the (restricted) maximal operator σ_* defined in a cone is bounded from $H_p(\mathbb{R}^d)$ to $L_p(\mathbb{R}^d)$ whenever $d/(d+1) < p < \infty$.

In this paper we will prove sharper inequalities and convergence results. Goldberg [4] has introduced and investigated the so-called local Hardy spaces $h_p(\mathbb{R}^d)$. We will show that σ_* is bounded from $h_p(\mathbb{R}^d)$ to $L_p(\mathbb{R}^d)$

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$(d/(d+1) < p < \infty)$. This extends the results just mentioned, because the norm of $h_p(\mathbb{R}^d)$ is smaller than the one of $H_p(\mathbb{R}^d)$.

Starting with the local Hardy spaces we introduce new Wiener amalgam spaces $W(h_p, \ell_\infty)(\mathbb{R}^d)$. Our main result is that σ_* is bounded from the amalgam space $W(h_p, \ell_\infty)(\mathbb{R}^d)$ to $W(L_p, \ell_\infty)(\mathbb{R}^d)$ $(d/(d+1) < p < \infty)$ and from $W(L_1, \ell_\infty)(\mathbb{R}^d)$ to the weak space $W(L_{1,\infty}, \ell_\infty)(\mathbb{R}^d)$. By a density argument we show that the Fejér means $\sigma_T f$ converge a.e. to f as $T \to \infty$ and $2^{-\tau} \le T_k/T_j \le 2^\tau$ $(k,j = 1,\dots,d)$, provided that $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$. This is a significant generalization of the result for integrable functions, because $W(L_1, \ell_\infty)(\mathbb{R}^d)$ is a much larger space than $L_1(\mathbb{R}^d)$.

2. Local Hardy spaces and Wiener amalgams. Let us fix $d \ge 1$, $d \in \mathbb{N}$. Let λ be the Lebesgue measure. We use the notation $|I|$ for the Lebesgue measure of the set I. For $x = (x_1,\dots,x_d), u = (u_1,\dots,u_d) \in \mathbb{R}^d$ we set $u \cdot x := \sum_{k=1}^d u_k x_k$. We write briefly $L_p(\mathbb{R}^d)$ for the real $L_p(\mathbb{R}^d, \lambda)$ space with the norm (or quasinorm)

$$\|f\|_p := \Big(\int_{\mathbb{R}^d} |f|^p \, d\lambda\Big)^{1/p} \qquad (0 < p \le \infty).$$

The *weak* L_p *space*, $L_{p,\infty}(\mathbb{R}^d)$ $(0 < p < \infty)$, consists of all measurable functions f for which

$$\|f\|_{L_{p,\infty}} := \sup_{\rho>0} \rho\lambda(|f| > \rho)^{1/p} < \infty.$$

Note that $L_{p,\infty}(\mathbb{R}^d)$ is a quasi-normed space (see Bergh and Löfström [1, p. 8]). It is easy to see that for each $0 < p \le \infty$,

$$L_p(\mathbb{R}) \subset L_{p,\infty}(\mathbb{R}) \quad \text{and} \quad \|\cdot\|_{L_{p,\infty}} \le \|\cdot\|_p.$$

The space of continuous functions with the supremum norm is denoted by $C(\mathbb{R})$, and $C_c(\mathbb{R})$ denotes the subspace of continuous functions having compact support.

For a measurable function ϕ on $\mathbb{R}^d$ let

$$\phi_t(x) := t^{-d}\phi(x/t) \qquad (x \in \mathbb{R}^d,\ t > 0).$$

Given a Schwartz function $\phi \in \mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \phi \, d\lambda \ne 0$ and $\operatorname{supp}\phi \subset [0,1/2]^d$, the *local Hardy space* $h_p(\mathbb{R}^d)$ $(0 < p \le \infty)$ consists of all tempered distributions f for which

$$\|f\|_{h_p(\mathbb{R}^d)} := \Big\| \sup_{0<t<1} |f * \phi_t| \Big\|_{L_p(\mathbb{R}^d)} < \infty.$$

Replacing the $L_p(\mathbb{R}^d)$ norm by the $L_{p,\infty}(\mathbb{R}^d)$ norm, we get the *weak local Hardy space* $h_{p,\infty}(\mathbb{R}^d)$ $(0 < p \le \infty)$. Taking the supremum over all $0 < t < \infty$ we obtain the definition of the *classical Hardy space* $H_p(\mathbb{R}^d)$. Other non-zero Schwartz functions ϕ define the same spaces and equivalent norms.

Usually the classical Hardy spaces are investigated. The local Hardy spaces were introduced in Goldberg [4]. Using these spaces we will get convergence of Fejér sums for functions from $W(L_1, \ell_\infty)(\mathbb{R}^d)$, which is a much larger space than $L_1(\mathbb{R}^d)$.

It is known that the Hardy spaces $h_p(\mathbb{R}^d)$ and $H_p(\mathbb{R}^d)$ are equivalent to the $L_p(\mathbb{R}^d)$ space when $1 < p \le \infty$, and $H_1(\mathbb{R}^d) \subset h_1(\mathbb{R}^d) \subset L_1(\mathbb{R}^d)$. Moreover,

$$\|f\|_{h_{1,\infty}} \le \|f\|_{H_{1,\infty}} \le C\|f\|_1 \quad (f \in L_1(\mathbb{R}^d)) \tag{1}$$

(see e.g. Stein [12, p. 91], Weisz [16, p. 68] and Goldberg [4]).

The atomic decomposition is a useful characterization of Hardy spaces. A bounded function a is an $h_p(\mathbb{R}^d)$-*atom* if there exists a cube $I \subset \mathbb{R}^d$ such that

(i) supp $a \subset I$,
(ii) $\|a\|_\infty \le |I|^{-1/p}$,
(iii) if $|I|<1$ then $\int_I a(x)x^k \, d\lambda(x) = 0$ for all multi-indices $k = (k_1, \dots, k_d)$ with $|k| \le M$, where $M \ge [d(1/p - 1)]$ and $[x]$ denotes the integer part of $x \in \mathbb{R}$.

We will say that a is a *type* 1 *atom* if $|I| < 1$ and a *type* 2 *atom* if $|I| \ge 1$. If we require the moment condition in (iii) for all $|I| \ge 1$ then we obtain the definition of $H_p(\mathbb{R}^d)$-*atom* (see Lu [7, Chapter 1], Stein [12, Chapter 3] and Goldberg [4]).

THEOREM 1. *A tempered distribution f is in $h_p(\mathbb{R}^d)$ $(0 < p \le 1)$ if and only if there exist a sequence $(a_k, k \in \mathbb{N})$ of $h_p(\mathbb{R}^d)$-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that*

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty \quad \text{and} \quad \sum_{k=0}^{\infty} \mu_k a_k = f \quad \text{in the sense of distributions.} \tag{2}$$

Moreover,

$$\|f\|_{h_p} \sim \inf \Big(\sum_{k=0}^{\infty} |\mu_k|^p \Big)^{1/p}$$

where the infimum is taken over all decompositions of f of the form (2).

Given a (quasi-)Banach space X on $\mathbb{R}^d$, a measurable function f belongs to the *Wiener amalgam space* $W(X, \ell_q)(\mathbb{R}^d)$ $(0 < q \le \infty)$ if

$$\|f\|_{W(X,\ell_q)} := \Big(\sum_{k \in \mathbb{Z}^d} \|f|_{[k,k+1)}\|_X^q \Big)^{1/q} < \infty$$

with the obvious modification for $q = \infty$, where $k+1 := (k_1+1, \dots, k_d+1)$. $W(X, c_0)(\mathbb{R}^d)$ is defined analogously, where c_0 denotes the set of sequences with limit 0. In this paper we will use the Wiener amalgam spaces for $X =$

$L_p(\mathbb{R}^d)$, $L_{p,\infty}(\mathbb{R}^d)$, $h_p(\mathbb{R}^d)$, $h_{p,\infty}(\mathbb{R}^d)$. The closed subspace of $W(L_\infty, \ell_q)(\mathbb{R}^d)$ containing continuous functions is denoted by $W(C, \ell_q)(\mathbb{R}^d)$ $(1 \le q \le \infty)$. The space $W(C, \ell_1)(\mathbb{R}^d)$ is called the *Wiener algebra*. It is used quite often in Gabor analysis, because it provides a convenient and general class of windows (see e.g. Gröchenig and Heil [5]). As can be seen in Feichtinger and Weisz [2, 3], it also plays an important role in summability theory. It is easy to see that

$$W(L_\infty, \ell_1)(\mathbb{R}^d) \subset L_p(\mathbb{R}^d) \subset W(L_1, \ell_\infty)(\mathbb{R}^d) \qquad (1 \le p \le \infty)$$

and $W(L_p, \ell_p)(\mathbb{R}^d) = L_p(\mathbb{R}^d)$.

3. Fejér means of Fourier transforms. The *Fourier transform* of $f \in L_1(\mathbb{R}^d)$ is

$$\hat{f}(x) := \int_{\mathbb{R}^d} f(t) e^{-2\pi\imath x \cdot t}\, dt \qquad (x \in \mathbb{R}^d),$$

where $\imath = \sqrt{-1}$. If $f \in L_p(\mathbb{R}^d)$ for some $1 \le p \le 2$ then the Fourier inversion formula

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(u) e^{2\pi\imath x \cdot u}\, du \qquad (x \in \mathbb{R}^d)$$

holds if $\hat{f} \in L_1(\mathbb{R}^d)$. This motivates the definition of the *Dirichlet integral* $s_t f$:

$$s_t f(x) := \int_{-t_1}^{t_1} \dots \int_{-t_d}^{t_d} \hat{f}(u) e^{2\pi\imath x \cdot u}\, du \qquad (t = (t_1, \dots, t_d) \in \mathbb{R}_+^d).$$

The *Fejér means* are defined by

$$\sigma_T f(x) := \frac{1}{\prod_{j=1}^d T_j} \int_0^{T_1} \dots \int_0^{T_d} s_t f(x)\, dt \qquad (T = (T_1, \dots, T_d) \in \mathbb{R}_+^d).$$

It is easy to see that

$$\sigma_T f(x) = \int_{\mathbb{R}^d} f(t) (K_{T_1}(x_1 - t_1) \times \dots \times K_{T_d}(x_d - t_d))\, dt \tag{3}$$

where

$$K_S(u) := S \operatorname{sinc}^2(Su) := S \left(\frac{\sin(\pi S u)}{\pi S u} \right)^2 \qquad (S > 0,\ u \in \mathbb{R},\ u \ne 0) \tag{4}$$

is the Fejér kernel. Remark that

$$\int_{\mathbb{R}} K_S(u)\, du = \int_{\mathbb{R}} \operatorname{sinc}^2(u)\, du = 1 \qquad (S > 0) \tag{5}$$

(see Zygmund [17, Vol. II, pp. 250–251]).

The definition of the Fejér means can be extended easily from $L_p(\mathbb{R}^d)$ $(1 \le p \le 2)$ functions to all $f \in W(L_1, \ell_\infty)(\mathbb{R})$ by (3), i.e.

$$\sigma_T f := f * (K_{T_1} \times \cdots \times K_{T_d}) \quad (T \in \mathbb{R}_+^d).$$

For a fixed $\tau \ge 0$ the (*restricted*) *maximal operator* is defined by

$$\sigma_* f := \sup_{\substack{2^{-\tau} \le T_k/T_j \le 2^\tau \\ k,j=1,\dots,d}} |\sigma_T f|.$$

Equality (3) implies

$$\|\sigma_* f\|_\infty \le \|\mathrm{sinc}^2\|_1^d \|f\|_\infty = \|f\|_\infty \quad (f \in L_\infty(\mathbb{R})). \tag{6}$$

In this paper the constants C and C_p may vary from line to line, and the constants C_p depend only on p.

4. Fejér summability and local Hardy spaces. In this section we generalize the results of Weisz [15] from $H_p(\mathbb{R}^d)$ spaces to $h_p(\mathbb{R}^d)$ and to the Wiener amalgam Hardy spaces.

THEOREM 2. *For $d/(d+1) < p < \infty$,*

$$\|\sigma_* f\|_p \le C_p \|f\|_{h_p} \quad (f \in h_p(\mathbb{R}^d))$$

and

$$\|\sigma_* f\|_{L_{1,\infty}} \le C\|f\|_1 \quad (f \in L_1(\mathbb{R}^d)).$$

Proof. For simplicity we prove the result for $d = 2$ only. For $d > 2$ the verification is very similar. Assume that $2/3 < p \le 1$ and a is an arbitrary $h_p(\mathbb{R})$-atom with support I, where I is a cube $I = I_1 \times I_2$ and $2^{-K-1} < |I_1| = |I_2| \le 2^{-K}$ for some $K \in \mathbb{Z}$. Let $4I_1$ denote the interval with the same center as I_1 and with length $4|I_1|$, and $4I := 4I_1 \times 4I_2$. We may suppose that the center of I is zero. If a is a type 1 atom, i.e. $K \ge 0$, then

$$\|\sigma_* a\|_p \le C_p, \tag{7}$$

as proved in [15].

Now suppose that a is a type 2 atom with $K < 0$. By (5),

$$\begin{aligned}\int_{4I} |\sigma_* a(x)|^p\,dx &\le \int_{4I} \sup_{T_1,T_2 \ge 1} \Big| \int_{\mathbb{R}^2} a(t) K_{T_1}(x_1 - t_1) K_{T_2}(x_2 - t_2)\,dt \Big|^p\,dx \\ &\le C_p 2^{2K} 2^{-2K}.\end{aligned}$$

Next we integrate over $(\mathbb{R} \setminus 4I_1) \times 4I_2$. We can see from (4) that

$$|K_S(u)| \le CS, \qquad |K_S(u)| \le \frac{C}{S|u|^2} \quad (u \in \mathbb{R} \setminus \{0\}, S \in \mathbb{R}_+). \tag{8}$$

Equality (3) implies that

$$|\sigma_T a(x)| \le C_p 2^{2K/p} \int_{I_1} \frac{C}{T_1(x_1-t_1)^2}\, dt_1.$$

We may suppose that $x_i > 0$ $(i=1,2)$. If $x_1 \in [i2^{-K},(i+1)2^{-K})$ $(i \ge 1)$ then

$$|\sigma_* a(x)| \le C_p 2^{2K/p} \int_{I_1} \frac{1}{(i2^{-K}-2^{-K-1})^2}\, dt_1 \le C_p 2^{K(2/p+1)} i^{-2}$$

and so

$$\int_{\mathbb{R}\setminus 4I_1} \int_{4I_2} |\sigma_* a(x)|^p\, dx \le \sum_{i=1}^{\infty} \int_{i2^{-K}}^{(i+1)2^{-K}} \int_{4I_2} |\sigma_* a(x)|^p\, dx \le C_p \sum_{i=1}^{\infty} 2^{Kp} i^{-2p},$$

which is bounded, because $K<0$. The integral over $4I_1 \times (\mathbb{R}\setminus 4I_2)$ can be estimated in the same way.

Similarly, if $x_1 \in [i_1 2^{-K},(i_1+1)2^{-K})$ and $x_2 \in [i_2 2^{-K},(i_2+1)2^{-K})$ $(i_1,i_2 \ge 1)$ then

$$|\sigma_T a(x)| \le C_p 2^{2K/p} \int_{I_1} \frac{C}{T_1(x_1-t_1)^2}\, dt_1 \int_{I_2} \frac{C}{T_2(x_2-t_2)^2}\, dt_2$$

and

$$|\sigma_* a(x)| \le C_p 2^{K(2/p+2)} i_1^{-2} i_2^{-2}.$$

Thus

$$\int_{\mathbb{R}\setminus 4I_1} \int_{\mathbb{R}\setminus 4I_2} |\sigma_* a(x)|^p\, dx \le \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \int_{i_1 2^{-K}}^{(i_1+1)2^{-K}} \int_{i_2 2^{-K}}^{(i_2+1)2^{-K}} |\sigma_* a(x)|^p\, dx$$

$$\le C_p \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} 2^{2Kp} i_1^{-2p} i_2^{-2p} \le C_p,$$

which shows (7) for type 2 atoms as well as the theorem. ∎

Recall that Theorem 2 was known for $H_p(\mathbb{R}^d)$ spaces (see Weisz [15]). The result for $h_p(\mathbb{R}^d)$ is stronger, because $\|\cdot\|_{h_p} \le \|\cdot\|_{H_p}$. Next we show the boundedness of σ_* on Wiener amalgam Hardy spaces.

THEOREM 3. *If $d/(d+1) < p < \infty$ then*

(9) $$\|\sigma_* f\|_{W(L_p,\ell_\infty)} \le C_p \|f\|_{W(h_p,\ell_\infty)} \qquad (f \in W(h_p,\ell_\infty)(\mathbb{R}^d))$$

and

(10) $$\|\sigma_* f\|_{W(L_{1,\infty},\ell_\infty)} \le C_p \|f\|_{W(L_1,\ell_\infty)} \qquad (f \in W(L_1,\ell_\infty)(\mathbb{R}^d)).$$

Proof. We are going to prove the result again for $d = 2$ only. The proof is similar for higher dimensions. Let $2/3 < p \le 1$ and $f|_{[j,j+1)} = \sum_{l=0}^{\infty} \mu_{j,l} a_{j,l}$ be an atomic decomposition of $f|_{[j,j+1)} \in h_p(\mathbb{R}^2)$ such that

$$\sum_{l=0}^{\infty} |\mu_{j,l}|^p \le C_p \|f|_{[j,j+1)}\|_{h_p}^p \le C_p \|f\|_{W(h_p,\ell_\infty)}^p.$$

Since $\operatorname{supp} \phi \subset [0,1/2]^2$, it follows that $\operatorname{supp} f|_{[j,j+1)} * \phi_t \subset [j, j+3/2)$, where $j \in \mathbb{Z}^2$, $0 < t < 1$. So we may suppose that $\operatorname{supp} a_{j,l} \subset [j, j+3/2)$. Thus

$$\begin{aligned} \int_{[k,k+1)} |\sigma_* f(x)|^p \, dx &\le \sum_{j\in\mathbb{Z}^2} \int_{[k,k+1)} |\sigma_*(f|_{[j,j+1)})(x)|^p \, dx \\ &\le \sum_{j\in\mathbb{Z}^2} \sum_{l=0}^{\infty} |\mu_{j,l}|^p \int_{[k,k+1)} |\sigma_* a_{j,l}(x)|^p \, dx \end{aligned} \tag{11}$$

for each fixed $k \in \mathbb{Z}^2$. Denote one of the atoms $a_{j,l}$ by a and suppose that it is supported in a cube $I = I_1 \times I_2$ with $2^{-K-1} < |I_1| = |I_2| \le 2^{-K}$ for some $K \in \mathbb{Z}$. Then $I \subset [j, j+3/2)$.

Throughout this proof we may assume that $j_i \ge k_i$ for $i = 1, 2$. In (11) we have to integrate over

$$\begin{aligned} [k, k+1) = ([k,k+1) \cap 4I) &\cup ([k,k+1) \cap (\mathbb{R} \setminus 4I_1) \times 4I_2) \\ &\cup ([k,k+1) \cap 4I_1 \times (\mathbb{R} \setminus 4I_2)) \\ &\cup ([k,k+1) \cap (\mathbb{R} \setminus 4I_1) \times (\mathbb{R} \setminus 4I_2)). \end{aligned}$$

We do this in four steps.

STEP 1: *Integrating over $[k, k+1) \cap 4I$.* It is easy to see that this set is empty if $j_1 \ge k_1 + 4$ or $j_2 \ge k_2 + 4$. For $k_i \le j_i \le k_i + 3$ $(i = 1, 2)$ we have, by (5),

$$\begin{aligned} &\int_{[k,k+1)\cap 4I} |\sigma_* a(x)|^p \, dx \\ &\quad \le \int_{4I} \sup_{T_1,T_2\ge 1} \Big| \int_{\mathbb{R}^2} a(t) K_{T_1}(x_1 - t_1) K_{T_2}(x_2 - t_2) \, dt \Big|^p \, dx \le C_p 2^{2K} 2^{-2K} \end{aligned}$$

and so

$$\begin{aligned} \sum_{j\in\mathbb{Z}^2} \sum_{l=0}^{\infty} |\mu_{j,l}|^p \int_{[k,k+1)\cap 4I} |\sigma_* a_{j,l}(x)|^p \, dx &\le C_p \sum_{j_1=k_1}^{k_1+3} \sum_{j_2=k_2}^{k_2+3} \sum_{l=0}^{\infty} |\mu_{j,l}|^p \\ &\le C_p \|f\|_{W(h_p,\ell_\infty)}^p. \end{aligned}$$

Step 2: *Integrating over* $[k,k+1)\cap(\mathbb{R}\setminus 4I_1)\times 4I_2$. This set is empty if $j_2 \geq k_2+4$. If a is a type 1 atom then

$$\int_{[k,k+1)\cap(\mathbb{R}\setminus 4I_1)\times 4I_2} |\sigma_* a(x)|^p\,dx \leq \int_{(\mathbb{R}\setminus 4I_1)\times 4I_2} |\sigma_* a(x)|^p\,dx \leq C_p$$

by (7). If a is a type 2 atom and $j_1 = k_1, k_1+1$, then the left hand side is 0, because $|I_1| \geq 1$ and so $[k,k+1)\cap(\mathbb{R}\setminus 4I_1)\times 4I_2$ is empty. Thus

$$\sum_{j_1=k_1}^{k_1+1}\sum_{j_2\in\mathbb{Z}}\sum_{l=0}^{\infty}|\mu_{j,l}|^p \int_{[k,k+1)\cap(\mathbb{R}\setminus 4I_1)\times 4I_2} |\sigma_* a_{j,l}(x)|^p\,dx$$
$$\leq C_p \sum_{j_1=k_1}^{k_1+1}\sum_{j_2=k_2}^{k_2+3}\sum_{l=0}^{\infty}|\mu_{j,l}|^p \leq C_p\|f\|^p_{W(h_p,\ell_\infty)}.$$

Assume that $j_1 \geq k_1+2$. If a is a type 1 atom then let

$$A_1(x_1,t_2) := \int_{-\infty}^{x_1} a(t_1,t_2)\,dt_1, \qquad A(x_1,x_2) := \int_{-\infty}^{x_2} A_1(x_1,t_2)\,dt_2$$

$(x\in\mathbb{R}^2)$. Observe that

$$\|A_1\|_\infty \leq C_p 2^{-K(1-2/p)}, \qquad \|A\|_\infty \leq C_p 2^{-K(2-2/p)}. \tag{12}$$

Integrating by parts in t_1 we can see that

$$\begin{aligned}\sigma_T a(x) &= \int_{I_1}\int_{I_2} a(t)K_{T_1}(x_1-t_1)K_{T_2}(x_2-t_2)\,dt \\ &= \int_{I_2} A_1(\nu_1,t_2)K_{T_1}(x_1-\nu_1)K_{T_2}(x_2-t_2)\,dt_2 \\ &\quad + \int_{I_1}\int_{I_2} A_1(t)K'_{T_1}(x_1-t_1)K_{T_2}(x_2-t_2)\,dt \\ &=: B_{1,T}(x)+B_{2,T}(x),\end{aligned} \tag{13}$$

where $I_i = [\mu_i,\nu_i]$ $(i=1,2)$. Using (8) we conclude that

$$|B_{1,T}(x)| \leq C_p 2^{-K(1-2/p)}\frac{C}{T_1(x_1-\nu_1)^2}T_2 2^{-K} \leq C_p 2^{-K(2-2/p)}(j_1-k_1)^{-2},$$

because $x_1\in(k_1,k_1+1)$ and $\nu_1\in I_1\subset[j_1,j_1+3/2)$. It is easy to see that

$$|K'_S(u)| \leq CS^2, \qquad |K'_S(u)| \leq \frac{C}{|u|^2} \qquad (u\in\mathbb{R}\setminus\{0\},\ S\in\mathbb{R}_+). \tag{14}$$

This, (5) and (12) imply that

$$|B_{2,T}(x)| \leq C_p 2^{-K(1-2/p)}\int_{I_1}\frac{C}{(x_1-t_1)^2}\,dt_1 \leq C_p 2^{-K(2-2/p)}(j_1-k_1)^{-2}.$$

Hence

$$\int\limits_{[k,k+1)\cap(\mathbb{R}\setminus 4I_1)\times 4I_2} |\sigma_* a(x)|^p \, dx \le \int\limits_{(k_1,k_1+1)\times 4I_2} |\sigma_* a(x)|^p \, dx$$
$$\le C_p 2^{-K(2p-2)}(j_1-k_1)^{-2p}2^{-K}$$

and

$$\sum_{j_1=k_1+2}^{\infty} \sum_{j_2\in\mathbb{Z}} \sum_{l=0}^{\infty} |\mu_{j,l}|^p \int\limits_{[k,k+1)\cap(\mathbb{R}\setminus 4I_1)\times 4I_2} |\sigma_* a_{j,l}(x)|^p \, dx$$
$$\le C_p \sum_{j_1=k_1+2}^{\infty} \sum_{j_2=k_2}^{k_2+3} \sum_{l=0}^{\infty} |\mu_{j,l}|^p 2^{-K(2p-1)}(j_1-k_1)^{-2p} \le C_p \|f\|^p_{W(h_p,\ell_\infty)}.$$

Similarly, if a is a type 2 atom then the first line of (13) yields

$$|\sigma_T a(x)| \le C_p 2^{2K/p} \int\limits_{I_1} \frac{C}{T_1(x_1-t_1)^2} \, dt_1 \le C_p 2^{K(2/p-1)}(j_1-k_1)^{-2}$$

and so

$$\sum_{j_1=k_1+2}^{\infty} \sum_{j_2\in\mathbb{Z}} \sum_{l=0}^{\infty} |\mu_{j,l}|^p \int\limits_{[k,k+1)\cap(\mathbb{R}\setminus 4I_1)\times 4I_2} |\sigma_* a_{j,l}(x)|^p \, dx$$
$$\le C_p \sum_{j_1=k_1+2}^{\infty} \sum_{j_2=k_2}^{k_2+3} \sum_{l=0}^{\infty} |\mu_{j,l}|^p 2^{K(1-p)}(j_1-k_1)^{-2p} \le C_p \|f\|^p_{W(h_p,\ell_\infty)},$$

because $K \le 0$.

STEP 3: *Integrating over* $[k,k+1)\cap 4I_1 \times (\mathbb{R}\setminus 4I_2)$. This case can be handled similarly to Step 2.

STEP 4: *Integrating over* $[k,k+1)\cap(\mathbb{R}\setminus 4I_1)\times(\mathbb{R}\setminus 4I_2)$. First assume that $j_i = k_i, k_i+1$ $(i=1,2)$. Similarly to Step 2, (7) implies

$$\int\limits_{[k,k+1)\cap(\mathbb{R}\setminus 4I_1)\times(\mathbb{R}\setminus 4I_2)} |\sigma_* a(x)|^p \, dx \le \int\limits_{(\mathbb{R}\setminus 4I_1)\times(\mathbb{R}\setminus 4I_2)} |\sigma_* a(x)|^p \, dx \le C_p$$

if a is a type 1 atom. For a type 2 atom the left hand side is again 0. Thus

$$\sum_{j_1=k_1}^{k_1+1} \sum_{j_2=k_2}^{k_2+1} \sum_{l=0}^{\infty} |\mu_{j,l}|^p \int\limits_{[k,k+1)\cap(\mathbb{R}\setminus 4I_1)\times(\mathbb{R}\setminus 4I_2)} |\sigma_* a_{j,l}(x)|^p \, dx$$
$$\le C_p \sum_{j_1=k_1}^{k_1+1} \sum_{j_2=k_2}^{k_2+1} \sum_{l=0}^{\infty} |\mu_{j,l}|^p \le C_p \|f\|^p_{W(h_p,\ell_\infty)}.$$

Next suppose that $j_1 \ge k_1+2$ and $j_2 = k_2, k_2+1$. The left hand side is 0 as above if a is a type 2 atom. For a type 1 atom we integrate by parts in (13) with respect to t_2:

$$\begin{aligned}\sigma_T a(x) &= -\int_{I_2} A(\nu_1,t_2)K_{T_1}(x_1-\nu_1)K'_{T_2}(x_2-t_2)\,dt_2\\ &\quad+\int_{I_1} A(t_1,\nu_2)K'_{T_1}(x_1-t_1)K_{T_2}(x_2-\nu_2)\,dt_1\\ &\quad-\int_{I_1}\int_{I_2} A(t)K'_{T_1}(x_1-t_1)K'_{T_2}(x_2-t_2)\,dt\\ &=: C_{1,T}(x)+C_{2,T}(x)+C_{3,T}(x).\end{aligned}$$

One can easily prove from the estimates in (8) that

$$|K_S(t)| \le \frac{CS^{2\eta-1}}{|t|^{2(1-\eta)}} \qquad (t\in\mathbb{R}\setminus\{0\},\ S\in\mathbb{R}_+) \tag{15}$$

where $0\le\eta\le 1$ is arbitrary. Similarly, for $0\le\eta\le1$, (14) implies

$$|K'_S(t)| \le \frac{CS^{2\eta}}{|t|^{2(1-\eta)}} \qquad (t\in\mathbb{R}\setminus\{0\},\ S\in\mathbb{R}_+). \tag{16}$$

If $x_1\in(k,k+1)$ and $x_2\in((\mu_2+\nu_2)/2-(i+1)2^{-K},(\mu_2+\nu_2)/2-i2^{-K})$ $(i\ge1)$ (the center of I_2 is $(\mu_2+\nu_2)/2$) then use (15) and (16) with $\eta=1/4$ to obtain

$$\begin{aligned}|C_{1,T}(x)| &\le C_p 2^{-K(2-2/p)}\frac{CT_1^{2\eta-1}}{(x_1-\nu_1)^{2(1-\eta)}}\int_{I_2}\frac{CT_2^{2\eta}}{(x_2-t_2)^{2(1-\eta)}}\,dt_2\\ &\le C_p2^{-K(2-2/p)}(j_1-k_1)^{-2(1-\eta)}2^{2K(1-\eta)}i^{-2(1-\eta)}2^{-K}\\ &= C_p2^{-K(3/2-2/p)}(j_1-k_1)^{-3/2}i^{-3/2}\end{aligned}$$

and

$$\begin{aligned}|C_{2,T}(x)| &\le C_p2^{-K(2-2/p)}\int_{I_1}\frac{CT_1^{2\eta}}{(x_1-t_1)^{2(1-\eta)}}\,dt_1\frac{CT_2^{2\eta-1}}{(x_2-\nu_2)^{2(1-\eta)}}\\ &= C_p2^{-K(3/2-2/p)}(j_1-k_1)^{-3/2}i^{-3/2}.\end{aligned}$$

Hence for $m=1,2$,

$$\begin{aligned}&\sum_{j_1=k_1+2}^{\infty}\sum_{j_2=k_2}^{k_2+1}\sum_{l=0}^{\infty}|\mu_{j,l}|^p\int_{[k,k+1)\cap(\mathbb{R}\setminus4I_1)\times(\mathbb{R}\setminus4I_2)}|C_{m,T}(x)|^p\,dx\\ &\qquad\le\sum_{j_1=k_1+2}^{\infty}\sum_{j_2=k_2}^{k_2+1}\sum_{l=0}^{\infty}|\mu_{j,l}|^p\int_{[k_1,k_1+1)\times(\mathbb{R}\setminus4I_2)}|C_{m,T}(x)|^p\,dx\end{aligned}$$

$$\leq \sum_{j_1=k_1+2}^{\infty} \sum_{j_2=k_2}^{k_2+1} \sum_{l=0}^{\infty} |\mu_{j,l}|^p \sum_{i=1}^{\infty} \int_{k_1}^{k_1+1} \int_{(\mu_2+\nu_2)/2-(i+1)2^{-K}}^{(\mu_2+\nu_2)/2-i2^{-K}} |C_{m,T}(x)|^p \, dx$$

$$\leq C_p \sum_{j_1=k_1+2}^{\infty} \sum_{j_2=k_2}^{k_2+1} \sum_{l=0}^{\infty} \sum_{i=1}^{\infty} |\mu_{j,l}|^p 2^{-K(3p/2-2)} (j_1-k_1)^{-3p/2} i^{-3p/2} 2^{-K}$$

$$\leq C_p \|f\|_{W(h_p,\ell_\infty)}^p.$$

For the third term we have

$$\begin{aligned} |C_{3,T}(x)| &\leq C_p 2^{-K(2-2/p)} \int_{I_1} \frac{C}{(x_1-t_1)^2} \, dt_1 \int_{I_2} \frac{C}{(x_2-t_2)^2} \, dt_2 \\ &\leq C_p 2^{-K(2-2/p)} 2^{-K} (j_1-k_1)^{-2} 2^{2K} i^{-2} 2^{-K} \\ &= C_p 2^{-K(2-2/p)} (j_1-k_1)^{-2} i^{-2} \end{aligned}$$

and

$$\sum_{j_1=k_1+2}^{\infty} \sum_{j_2=k_2}^{k_2+1} \sum_{l=0}^{\infty} |\mu_{j,l}|^p \int_{[k,k+1)\cap(\mathbb{R}\setminus 4I_1)\times(\mathbb{R}\setminus 4I_2)} |C_{3,T}(x)|^p \, dx$$

$$\leq \sum_{j_1=k_1+2}^{\infty} \sum_{j_2=k_2}^{k_2+1} \sum_{l=0}^{\infty} |\mu_{j,l}|^p \sum_{i=1}^{\infty} \int_{k_1}^{k_1+1} \int_{(\mu_2+\nu_2)/2-(i+1)2^{-K}}^{(\mu_2+\nu_2)/2-i2^{-K}} |C_{3,T}(x)|^p \, dx$$

$$\leq C_p \sum_{j_1=k_1+2}^{\infty} \sum_{j_2=k_2}^{k_2+1} \sum_{l=0}^{\infty} \sum_{i=1}^{\infty} |\mu_{j,l}|^p 2^{-K(2p-2)} (j_1-k_1)^{-2p} i^{-2p} 2^{-K}$$

$$\leq C_p \|f\|_{W(h_p,\ell_\infty)}^p.$$

The case $j_1 = k_1, k_1+1$ and $j_2 \geq k_2+2$ can be handled similarly.

Finally, we suppose that $j_i \geq k_i + 2$ for $i = 1, 2$. If a is a type 1 atom and $x \in (k, k+1)$ then

$$\begin{aligned} |C_{1,T}(x)| &\leq C_p 2^{-K(2-2/p)} \frac{C}{T_1(x_1-\nu_1)^2} \int_{I_2} \frac{C}{(x_2-t_2)^2} \, dt_2 \\ &\leq C_p 2^{-K(3-2/p)} (j_1-k_1)^{-2} (j_2-k_2)^{-2}, \\ |C_{2,T}(x)| &\leq C_p 2^{-K(2-2/p)} \int_{I_1} \frac{C}{(x_1-t_1)^2} \, dt_1 \frac{C}{(x_2-\nu_2)^2} \\ &\leq C_p 2^{-K(3-2/p)} (j_1-k_1)^{-2} (j_2-k_2)^{-2} \end{aligned}$$

and so

$$\sum_{j_1=k_1+2}^{\infty}\sum_{j_2=k_2+2}^{\infty}\sum_{l=0}^{\infty}|\mu_{j,l}|^p \int_{[k,k+1)\cap(\mathbb{R}\setminus 4I_1)\times(\mathbb{R}\setminus 4I_2)} |C_{m,T}(x)|^p\,dx$$
$$\leq C_p \sum_{j_1=k_1+2}^{\infty}\sum_{j_2=k_2+2}^{\infty}\sum_{l=0}^{\infty}|\mu_{j,l}|^p 2^{-K(3p-2)}(j_1-k_1)^{-2p}(j_2-k_2)^{-2p}$$
$$\leq C_p\|f\|^p_{W(h_p,\ell_\infty)}$$

for $m=1,2$. Similarly,

$$|C_{3,T}(x)|\leq C_p 2^{-K(2-2/p)}\int_{I_1}\frac{C}{(x_1-t_1)^2}\,dt_1\int_{I_2}\frac{C}{(x_2-t_2)^2}\,dt_2$$
$$\leq C_p 2^{-K(4-2/p)}(j_1-k_1)^{-2}(j_2-k_2)^{-2}$$

and

$$\sum_{j_1=k_1+2}^{\infty}\sum_{j_2=k_2+2}^{\infty}\sum_{l=0}^{\infty}|\mu_{j,l}|^p \int_{[k,k+1)\cap(\mathbb{R}\setminus 4I_1)\times(\mathbb{R}\setminus 4I_2)} |C_{3,T}(x)|^p\,dx$$
$$\leq C_p \sum_{j_1=k_1+2}^{\infty}\sum_{j_2=k_2+2}^{\infty}\sum_{l=0}^{\infty}|\mu_{j,l}|^p 2^{-K(4p-2)}(j_1-k_1)^{-2p}(j_2-k_2)^{-2p}$$
$$\leq C_p\|f\|^p_{W(h_p,\ell_\infty)}.$$

For a type 2 atom we can see from (13) that

$$|\sigma_T a(x)|\leq C_p 2^{2K/p}\int_{I_1}\frac{C}{(x_1-t_1)^2}\,dt_1\int_{I_2}\frac{C}{(x_2-t_2)^2}\,dt_2$$
$$\leq C_p 2^{-K(2-2/p)}(j_1-k_1)^{-2}(j_2-k_2)^{-2}$$

and

$$\sum_{j_1=k_1+2}^{\infty}\sum_{j_2=k_2+2}^{\infty}\sum_{l=0}^{\infty}|\mu_{j,l}|^p \int_{[k,k+1)\cap(\mathbb{R}\setminus 4I_1)\times(\mathbb{R}\setminus 4I_2)} |\sigma_T a(x)|^p\,dx$$
$$\leq C_p \sum_{j_1=k_1+2}^{\infty}\sum_{j_2=k_2+2}^{\infty}\sum_{l=0}^{\infty}|\mu_{j,l}|^p 2^{-K(2p-2)}(j_1-k_1)^{-2p}(j_2-k_2)^{-2p}$$
$$\leq C_p\|f\|^p_{W(h_p,\ell_\infty)}.$$

Taking into account (11) we have thus finished the proof of (9) for $2/3<p\leq 1$.

It is known that $h_\infty(\mathbb{R}^2)=L_\infty(\mathbb{R}^2)$ (see e.g. Stein [12, p. 91]) and $W(L_\infty,\ell_\infty)(\mathbb{R}^2)=L_\infty(\mathbb{R}^2)$. The boundedness of σ_* from $W(h_\infty,\ell_\infty)(\mathbb{R}^2)$ to $W(L_\infty,\ell_\infty)(\mathbb{R}^2)$ follows from (6). Applying this and (9) for $p=1$ we use complex interpolation to deduce that

$$\sigma_* : (W(h_1,\ell_\infty)(\mathbb{R}^2), W(h_\infty,\ell_\infty)(\mathbb{R}^2))_{[\eta]} \to (W(L_1,\ell_\infty)(\mathbb{R}^2), W(L_\infty,\ell_\infty)(\mathbb{R}^2))_{[\eta]}$$

is bounded, where $0 < \eta < 1$ is arbitrary. Using the method of Triebel [13, p. 121] we can prove that

$$(W(h_1,\ell_\infty)(\mathbb{R}^2), W(h_\infty,\ell_\infty)(\mathbb{R}^2))_{[\eta]} = W((h_1,h_\infty)_{[\eta]},\ell_\infty)(\mathbb{R}^2)$$

and that the same holds with L_1 instead of h_1. Choosing $\eta = 1-1/p$ $(1<p<\infty)$ we obtain the boundedness of

$$\sigma_* : W(h_p,\ell_\infty)(\mathbb{R}^2) \to W(L_p,\ell_\infty)(\mathbb{R}^2),$$

which is exactly (9).

By the real method of interpolation we have

$$(W(h_p,\ell_\infty)(\mathbb{R}^2), W(h_\infty,\ell_\infty)(\mathbb{R}^2))_{\eta,\infty} = W((h_p,h_\infty)_{\eta,\infty},\ell_\infty)(\mathbb{R}^2)$$

and the analogue for L_p (see Sagher [11], Kisliakov and Xu [6], Berg and Löfström [1, Chapter 5]). We conclude that

$$\sigma_* : W((h_p,h_\infty)_{\eta,\infty},\ell_\infty)(\mathbb{R}^2) \to W((L_p,L_\infty)_{\eta,\infty},\ell_\infty)(\mathbb{R}^2)$$

is bounded, where $0<\eta<1$ is arbitrary. If $p<1$ then the choice $\eta = 1-p$ implies the boundedness of

$$\sigma_* : W(h_{1,\infty},\ell_\infty)(\mathbb{R}^2) \to W(L_{1,\infty},\ell_\infty)(\mathbb{R}^2)$$

and inequality (1) proves (10). This completes the proof of Theorem 3. ∎

Remark 1. The exact value of C_p in (10) is $C_p = C_p \|\mathrm{sinc}^2\|_1^{d(1-p)}$ because of (6) and the basic theorems of interpolation theory, where $1/2 < p < 1$ is fixed.

Remark 1 will be used in the next corollary for $\mathrm{sinc}^2|_{(-k,k)^c}$ instead of sinc^2. Since $W(L_1,\ell_\infty)(\mathbb{R}^d) \supset L_1(\mathbb{R}^d)$, the next corollary is much more general than the results in [14].

Corollary 1. *For all* $f \in W(L_1,\ell_\infty)(\mathbb{R}^d)$,

$$\lim_{T\to\infty} \sigma_T f = f \qquad a.e. \tag{17}$$

as $\min(T_1,\dots,T_d)\to\infty$ *and* $2^{-\tau} \le T_k/T_j \le 2^\tau$ $(k,j=1,\dots,d)$.

Proof. For $f \in C_c(\mathbb{R}^2)$ we obtain the convergence from

$$|\sigma_T f(x) - f(x)| = \Big|\sigma_T f(x) - \Big(\int_{\mathbb{R}} \mathrm{sinc}^2\, d\lambda\Big)^2 f(x)\Big| \le \int_{\mathbb{R}^2} \Big| f\Big(x_1 - \frac{t_1}{T_1}, x_2 - \frac{t_2}{T_2}\Big) - f(x)\Big| \,\mathrm{sinc}^2(t_1)\,\mathrm{sinc}^2(t_2)\, dt$$

and from the Lebesgue dominated convergence theorem. Since $C_c(\mathbb{R}^2)$ is dense in $W(L_1, c_0)(\mathbb{R}^2)$, the corollary follows for all $f \in W(L_1, c_0)(\mathbb{R}^2)$ from (10) and the usual density argument due to Marcinkiewicz and Zygmund [8]. Observe that Theorem 3 and Remark 1 can be applied for the functions $\operatorname{sinc}^2|_{(-k,k)}$ $(k \in \mathbb{N})$ instead of sinc^2. Hence

$$\lim_{T\to\infty,\, 2^{-\tau}\le T_k/T_j\le 2^\tau} \sigma_{T,\operatorname{sinc}^2\times \operatorname{sinc}^2|_{(-k,k)^2}} f = \Big(\int_{\mathbb{R}} \operatorname{sinc}^2|_{(-k,k)}\, d\lambda\Big)^2 f \quad \text{a.e.} \tag{18}$$

for all $f \in W(L_1, c_0)(\mathbb{R}^2)$, where

$$\sigma_{T,h_1\times h_2} f(x) := \int_{\mathbb{R}^2} f\Big(x_1 - \frac{t_1}{T_1}, x_2 - \frac{t_2}{T_2}\Big) h_1(t_1) h_2(t_2)\, dt.$$

It is easy to see that if $x \in [-j,j]^2$ then for $T_i \ge 1$ $(i = 1,2)$,

$$\sigma_{T,\operatorname{sinc}^2\times \operatorname{sinc}^2|_{(-k,k)^2}} f(x) = \sigma_{T,\operatorname{sinc}^2\times \operatorname{sinc}^2|_{(-k,k)^2}} (f|_{[-j-k,j+k]^2})(x).$$

However, $f|_{[-j-k,j+k]^2} \in W(L_1, c_0)(\mathbb{R}^2)$ if $f \in W(L_1, \ell_\infty)(\mathbb{R}^2)$, and so (18) holds for all $f \in W(L_1, \ell_\infty)(\mathbb{R}^2)$. Fix $f \in W(L_1, \ell_\infty)(\mathbb{R}^2)$ and set

$$\xi := \limsup_{T\to\infty,\, 2^{-\tau}\le T_k/T_j\le 2^\tau} |\sigma_T f - f|.$$

We have (with the same meaning of lim sup)

$$\begin{aligned}
\xi &\le \limsup |\sigma_T f - \sigma_{T,\operatorname{sinc}^2\times \operatorname{sinc}^2|_{(-k,k)^2}} f| \\
&\quad + \limsup \Big|\sigma_{T,\operatorname{sinc}^2\times \operatorname{sinc}^2|_{(-k,k)^2}} f - \Big(\int_{\mathbb{R}^2} \operatorname{sinc}^2 \times \operatorname{sinc}^2|_{(-k,k)^2}\, d\lambda\Big) f\Big| \\
&\quad + \Big|\Big(\int_{\mathbb{R}^2} \operatorname{sinc}^2 \times \operatorname{sinc}^2|_{(-k,k)^2}\, d\lambda\Big)^2 f - f\Big| \\
&\le \limsup |\sigma_{T,\operatorname{sinc}^2\times \operatorname{sinc}^2|_{\mathbb{R}^2\setminus(-k,k)^2}} f| + \Big(\int_{\mathbb{R}^2\setminus(-k,k)^2} \operatorname{sinc}^2 \times \operatorname{sinc}^2\, d\lambda\Big)|f|
\end{aligned}$$

for all $k \in \mathbb{N}$. By Theorem 3 and Remark 1 we conclude that

$$\begin{aligned}
\|\xi\|_{W(L_{1,\infty},\ell_\infty)} &\le \Big\| \sup_{T\ge 1,\, 2^{-\tau}\le T_k/T_j\le 2^\tau} |\sigma_{T,\operatorname{sinc}^2\times \operatorname{sinc}^2|_{\mathbb{R}^2\setminus(-k,k)^2}} f| \Big\|_{W(L_{1,\infty},\ell_\infty)} \\
&\quad + \Big(\int_{\mathbb{R}^2\setminus(-k,k)^2} \operatorname{sinc}^2 \times \operatorname{sinc}^2\, d\lambda\Big) \|f\|_{W(L_{1,\infty},\ell_\infty)} \\
&\le C_p \|\operatorname{sinc}^2 \times \operatorname{sinc}^2|_{\mathbb{R}^2\setminus(-k,k)^2}\|_1^{1-p} \|f\|_{W(L_1,\ell_\infty)} \\
&\quad + \|\operatorname{sinc}^2 \times \operatorname{sinc}^2|_{\mathbb{R}^2\setminus(-k,k)^2}\|_1 \|f\|_{W(L_1,\ell_\infty)}
\end{aligned}$$

for all $k \in \mathbb{N}$, where $2/3 < p < 1$. Since $\|\operatorname{sinc}^2 \times \operatorname{sinc}^2|_{\mathbb{R}^2\setminus(-k,k)^2}\|_1 \to 0$ as $k \to \infty$, $\|\xi\|_{W(L_{1,\infty},\ell_\infty)} = 0$ and so $\xi = 0$ a.e., which finishes the proof. ∎

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