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Embedding theorems for Lipschitz and Lorentz spaces on lower Ahlfors regular sets

by

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Abstract. We prove norm inequalities between Lorentz and Besov–Lipschitz spaces of fractional smoothness.

1. Main results. In what follows we let (F,ρ) be a metric space with a positive σ-finite Borel measure μ. By $B(x,r)$ we denote the open ball centred at x with radius r. We always assume that there exist $d>0$ and $C_1>0$ such that

$$\mu(B(x,r)) \ge C_1 r^d \quad \text{for all } 0<r\le 1 \text{ and } x\in F, \tag{1}$$

i.e., the lower Ahlfors d-regularity of F. In particular, F may be a d-set in $\mathbb{R}^n$ and μ the d-dimensional Hausdorff measure, or F may be an h-set with $h(r)\ge r^d$ for $0<r\le 1$ and μ an h-measure [10, 11, 5, 6, 18].

We denote $L^p = L^p(F,\mu)$. We obtain the following inequality of Sobolev type.

Theorem 1. *If $0<p<\infty$ and $0<\alpha<d/p$, then there exists a constant $c=c(d,C_1,p,\alpha)$ such that*

$$\|u\|_{L^{pd/(d-\alpha p)}} \le c\biggl(\|u\|_{L^p} + \biggl(\iint_{\rho(x,y)<1} \frac{|u(x)-u(y)|^p}{\rho(x,y)^{d+\alpha p}}\,\mu(dy)\,\mu(dx)\biggr)^{1/p}\biggr) \tag{2}$$

for all $u\in L^p$.

Under certain additional assumptions we can get rid of the L^p norm on the right hand side.

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Corollary 2. *Let $F \subset \mathbb{R}^n$ and $F = aF := \{ax : x \in F\}$ for some $a > 1$. Let μ be the d-dimensional Hausdorff measure and assume that it is σ-finite on F. Let $0 < p < \infty$ and $0 < \alpha < d/p$. There exists a constant $c = c(d, C_1, p, \alpha)$ such that*

$$\|u\|_{L^{pd/(d-\alpha p)}} \le c\biggl(\int_F\int_F \frac{|u(x)-u(y)|^p}{|x-y|^{d+\alpha p}}\,\mu(dy)\,\mu(dx)\biggr)^{1/p} \tag{3}$$

for all $u \in L^p$.

The result applies e.g. if F is a half-space in $\mathbb{R}^n$ (or more generally, an open cone) and $d = n$.

Inequality (2) for $p = 2$, $\alpha < 1$, and a d-set $F \subset \mathbb{R}^n$ was stated in [7, (2.3)] and applied in [7] to estimate the heat kernel of jump type processes (see also [4]). Such applications are our primary motivation to study such inequalities. They are also of interest in the study of function spaces on d-sets [17]. Furthermore, inequalities of this type have a close connection to Nash inequalities and heat kernel estimates (see [15, 8, 23, 1]).

Note that our proofs are different and more elementary than those in [7, 17]. Interestingly, in our inequalities we allow for all $p > 0$, rather than $p \ge 1$, $\alpha \in (0, d/p)$ may be larger than 1, and we only assume *lower* Ahlfors d-regularity. Moreover, our methods yield an extension to Besov–Lipschitz spaces, given below.

We recall the definition of *Lorentz spaces* $L_{p,q}$ [17, 2]. We define the decreasing rearrangement u^* of u in the usual way,

$$u^*(t) = \inf\{s : \mu(\{x : |u(x)| > s\}) \le t\}.$$

For $0 < p, q < \infty$ we define

$$\|u; L_{p,q}\| = \biggl(\int_0^\infty (t^{1/p}u^*(t))^q\,\frac{dt}{t}\biggr)^{1/q}, \qquad \|u; L_{p,\infty}\| = \sup_{t>0}(t^{1/p}u^*(t)).$$

We say that $u \in L_{p,q}$ if $\|u; L_{p,q}\| < \infty$.

For $0 < p < \infty$, $0 < q \le \infty$ and $\alpha > 0$ we define the Besov–Lipschitz type space $\mathrm{Lip}_0(\alpha, p, q, F) = \{u \in L^p : \|u; \mathrm{Lip}_0(\alpha, p, q, F)\| < \infty\}$, where

$$\|u; \mathrm{Lip}_0(\alpha, p, q, F)\| = \|u\|_{L^p} + \|(b_\nu)_{\nu=0}^\infty\|_{\ell^q}, \tag{4}$$

and the sequence $(b_\nu)_{\nu=0}^\infty$ is defined by

$$b_\nu = 2^{\nu\alpha}\biggl(2^{\nu d}\iint_{\rho(x,y)<2^{-\nu}} |u(x)-u(y)|^p\,\mu(dy)\,\mu(dx)\biggr)^{1/p}. \tag{5}$$

If $p, q \ge 1$, then (4) is a genuine norm.

The main result of this note is the following embedding theorem, which extends Proposition 6 in [17, p. 216].

THEOREM 3. *Let F, μ, ρ and d be as in Theorem* 1. *Let $0 < p < \infty$, $p \le q \le \infty$ and $0 < \alpha < d/p$. Then there exists a constant $c = c(d, C_1, p, q, \alpha)$ such that for all $u \in \mathrm{Lip}_0(\alpha, p, q, F)$,*

$$\|u; L_{p^*,q}\| \le c\|u; \mathrm{Lip}_0(\alpha, p, q, F)\|, \tag{6}$$

where $p^* = pd/(d - \alpha p)$.

We may regard Theorem 3 as a *subcritical* case of a *limiting embedding* (see [22, Remark 11.5] for definitions and a further discussion).

We mention that the Hardy inequality of [12, 9, 3] is similar to (3), except that it estimates the *weighted* L^p norm (and not L^{p^*}) by $\mathcal{E}$.

We note that the definition of $\mathrm{Lip}_0(\alpha, p, q, F)$ is very similar to the definition of the space $\Lambda^{d,\alpha}_{p,q}$ of Grigor'yan [13]. By the definition

$$\|u; \mathrm{Lip}_0(\alpha, p, q, F)\| \le c\|u; \Lambda^{d,\alpha}_{p,q}\|, \tag{7}$$

and these two norms are equivalent for bounded d-sets F. Correspondingly, (6) holds with the norm $\mathrm{Lip}_0(\alpha, p, q, F)$ replaced by the norm of $\Lambda^{d,\alpha}_{p,q}$ in Theorem 3. See [13, 14] for a further discussion.

We now recall the definition of $\mathrm{Lip}(\alpha, p, q, F)$ of Jonsson and Wallin [17]. Assume that $F \subset \mathbb{R}^n$ and ρ is the Euclidean distance. Let $\alpha > 0$ and $k \in \mathbb{Z}$ satisfy $k < \alpha \le k + 1$. Let $\{f^{(j)}\}_{|j|\le k}$ be a family of functions defined μ-a.e. on F, where $j = (j_1, \ldots, j_n)$ is a multiindex and $|j| = j_1 + \cdots + j_n$. We define P_j and R_j by requiring that

$$P_j(x, y) = \sum_{|j+l|\le k} \frac{f^{(j+l)}(y)}{l!}(x - y)^l, \qquad x, y \in F,$$

and that $f^{(j)}(x) = P_j(x, y) + R_j(x, y)$. The collection $\{f^{(j)}\}_{|j|\le k}$ belongs to *the Lipschitz space* $\mathrm{Lip}(\alpha, p, q, F)$ if and only if $f^{(j)} \in L^p$ for $|j| \le k$, and for $\nu = 0, 1, 2, \ldots$ and $|j| \le k$,

$$\Big(2^{\nu d} \iint_{|x-y|<2^{-\nu}} |R_j(x, y)|^p \,\mu(dx)\,\mu(dy)\Big)^{1/p} \le 2^{-\nu(\alpha-|j|)} a_\nu \tag{8}$$

for some sequence $(a_\nu) \in \ell^q$. The norm of $\{f^{(j)}\}_{|j|\le k}$ in $\mathrm{Lip}(\alpha, p, q, F)$ is

$$\sum_{|j|\le k} \|f^{(j)}\|_{L^p} + \inf \|(a_\nu)\|_{\ell^q}, \tag{9}$$

where the infimum is taken over all possible sequences (a_ν). We see that the definition of $\mathrm{Lip}(\alpha, p, q, F)$ uses (a substitute of) Taylor expansion of kth order, while $\mathrm{Lip}_0(\alpha, p, q, F)$ uses only increments of the function (0-order Taylor expansion). This motivates the notation Lip_0.

For a function f we put $\tilde{f}^{(0)} = f$ and $\tilde{f}^{(j)} = 0$ if $|j| > 0$. Clearly,

$$\|f^{(0)}; \mathrm{Lip}_0(\alpha, p, q, F)\| = \|\{\tilde{f}^{(j)}\}; \mathrm{Lip}(\alpha, p, q, F)\|.$$

In particular, we have $\mathrm{Lip}(\alpha, p, q, F) = \mathrm{Lip}_0(\alpha, p, q, F)$ for $\alpha \le 1$.

It seems that $\mathrm{Lip}_0(\alpha, p, q, F)$ is more appropriate to study jump processes on metric spaces (see [16, 20, 21]). For a d-set F the space $\mathrm{Lip}_0(\alpha d_w/4, 2, 2, F)$ is the domain of the Dirichlet form of a symmetric α-stable process on F [21], where $\alpha \in (0, 2)$ and d_w is the so-called walk dimension of F [20]. Also, $\mathrm{Lip}_0(d_w/2, 2, \infty, F)$ is the domain of the Dirichlet form of the Brownian motion e.g. on the Sierpiński gasket $F \subset \mathbb{R}^n$, (see [16]). Our results shed light on domains of non-local Dirichlet forms defined on more general sets.

Notation $c = c(a, b, \dots, z)$ means that the constant $0 < c < \infty$ depends only on $a, b, \dots, z$. All functions are assumed to be Borel measurable and complex-valued. In fact our results remain valid for Banach-space-valued functions u (see (13), (14)).

2. Proof of Theorem 3. In the following lemma we adopt the convention that $\frac{0}{0} = 0$.

Lemma 4. *For every $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} a_n \le a_0 + 3 \cdot 4^{\varepsilon} \sum_{n=1}^{\infty} \frac{a_n^{1+\varepsilon}}{(a_{n-1} + a_n + a_{n+1})^{\varepsilon}} \tag{10}$$

if $a_n \ge 0$, $n = 0, 1, \dots$ and $a_n = 0$ for large n.

Proof. Let

$$A = \left\{ n \in \{1, 2, \dots\} : a_n \ge \frac{1}{3}(a_{n-1} + a_{n+1}) \right\}, \qquad B = \{1, 2, \dots\} \setminus A,$$

and let N be such that $B \subset \{1, \dots, N\}$. For $n \in A$ we have $a_{n-1} + a_n + a_{n+1} \le 4a_n$, hence

$$\sum_{n \in A} a_n \le 4^{\varepsilon} \sum_{n \in A} \frac{a_n^{1+\varepsilon}}{(a_{n-1} + a_n + a_{n+1})^{\varepsilon}}. \tag{11}$$

On the other hand, we have

$$\sum_{n \in B} a_n \le \frac{1}{3} \sum_{n \in B} (a_{n-1} + a_{n+1}) \le \frac{1}{3} a_0 + \frac{2}{3} \sum_{n=1}^{N} a_n + \frac{1}{3} a_{N+1},$$

thus

$$\frac{1}{3} \sum_{n \in B} a_n < \frac{1}{3} a_0 + \frac{2}{3} \sum_{n \in A, n \le N} a_n + \frac{1}{3} a_{N+1}. \tag{12}$$

Since $N + 1 \in A$, we obtain from (12),

$$\sum_{n \in B} a_n < a_0 + 2 \sum_{n \in A, n \le N+1} a_n,$$

and this together with (11) completes the proof. ∎

REMARK 1. We note that (10) does not hold for *all* sequences $a_n \geq 0$. Indeed, for $a_n = \exp(b^n)$, the right hand side of (10) is finite if b is large enough, while the left hand side is infinite. One can prove that (10) holds, with some constant $c = c(\varepsilon)$ instead of $3 \cdot 4^\varepsilon$ in (10), for all sequences $a_n = o(q^n)$, where $q > 0$; however, the proof is more complicated and will be omitted.

Proof of Theorem 3. Let $u \in \mathrm{Lip}_0(\alpha, p, q, F)$. Our goal is to prove (6) with c independent of u. Note that

$$\| \, |u|; L_{p^*,q} \| = \|u; L_{p^*,q}\| \tag{13}$$

and

$$\| \, |u|; \mathrm{Lip}_0(\alpha, p, q, F)\| \leq \|u; \mathrm{Lip}_0(\alpha, p, q, F)\|, \tag{14}$$

hence it suffices to prove (6) for $u \geq 0$.

Furthermore, since for any $t > 0$ we have

$$\|u \wedge t; \mathrm{Lip}_0(\alpha, p, q, F)\| \leq \|u; \mathrm{Lip}_0(\alpha, p, q, F)\|,$$

by the bounded convergence theorem we may also assume that u is bounded. Finally, we may and will assume that $\|u\|_{L^p} = 1$.

Let

$$E_n = \{x \in F : u(x) \in [2^n, 2^{n+1})\},$$
$$\mu_n = \mu(E_n), \qquad n \in \mathbb{Z}.$$

The idea of the proof is to estimate the norms in (6) by means of μ_n only, and then use special inequalities for sequences, including (10) and the Hardy inequality. While estimates for the L^p and $L_{p^*,q}$ norms of u by means of μ_n are straightforward, this is not the case for the ℓ^q norm of (b_ν). This is the place where the somewhat unusual terms $\mu_n/(\mu_{n-1}+\mu_n+\mu_{n+1})$ arise, which result from considering x and y *not* in neighbouring sets E_n (see (5) and (16)). We estimate the terms by using Lemma 4. The assumption $\|u\|_{L^p} = 1$ implies that $\mu_{n-1} + \mu_n + \mu_{n+1} \leq 2^{-(n-1)p}$, thus $\mu_{n-1} + \mu_n + \mu_{n+1} \leq C_1/2$ for $n \geq n_0 = n_0(C_1, p)$.

We claim that for any $n \geq n_0$ there exists $\nu \in \{0, 1, 2, \ldots\}$ (depending on n, u, ...) such that

$$2^n \mu_n^{1/p} (\mu_{n-1} + \mu_n + \mu_{n+1})^{-\alpha/d} \leq c 2^{\nu\alpha} \Big(2^{\nu d} \int_{E_n} \int_{B(x, 2^{-\nu})} |u(x) - u(y)|^p \, \mu(dy) \, \mu(dx) \Big)^{1/p} \tag{15}$$

with constant $c = c(d, C_1, p, \alpha)$ independent of n. Here we adopt the convention that $0^a = 0$ for $a < 0$, hence the claim is obvious if $\mu_{n-1}+\mu_n+\mu_{n+1} = 0$.

We now prove the claim in the case when $\mu_{n-1}+\mu_n+\mu_{n+1}>0$. We have

$$
\begin{aligned}
(16)\qquad b_{n,\nu} &:= \int_{E_n}\int_{B(x,2^{-\nu})} |u(x)-u(y)|^p\,\mu(dy)\,\mu(dx)\\
&\ge \int_{E_n}\int_{B(x,2^{-\nu})\setminus(E_{n-1}\cup E_n\cup E_{n+1})} |u(x)-u(y)|^p\,\mu(dy)\,\mu(dx)\\
&\ge 2^{(n-1)p}\mu_n\cdot\mu(B(x,2^{-\nu})\setminus(E_{n-1}\cup E_n\cup E_{n+1}))\\
&\ge 2^{(n-1)p}\mu_n(C_1 2^{-\nu d}-(\mu_{n-1}+\mu_n+\mu_{n+1})).
\end{aligned}
$$

We take $\nu\in\{0,1,2,\ldots\}$ such that

$$2(\mu_{n-1}+\mu_n+\mu_{n+1})\le C_1 2^{-\nu d}<2^{d+1}(\mu_{n-1}+\mu_n+\mu_{n+1}).$$

Then

$$b_{n,\nu}\ge\frac{C_1}{2}\,2^{(n-1)p}\mu_n 2^{-\nu d},$$

hence

$$2^{\nu\alpha}(2^{\nu d}b_{n,\nu})^{1/p}\ge c(d,C_1,p,\alpha)(\mu_{n-1}+\mu_n+\mu_{n+1})^{-\alpha/d}2^n\mu_n^{1/p},$$

and the claim is proven.

We will first prove (6) in the case when $q<\infty$. Observe that $2^n\le u^*(t)<2^{n+1}$ if $\sum_{k>n}\mu_k<t<\sum_{k\ge n}\mu_k$. Hence

$$
\begin{aligned}
(17)\qquad \|u;L_{p^*,q}\|^q &= \int_0^\infty (t^{1/p^*}u^*(t))^q\frac{dt}{t}\\
&\le 2^q\sum_{n\in\mathbb{Z}}\int_{\sum_{k>n}\mu_k}^{\sum_{k\ge n}\mu_k} t^{q/p^*-1}2^{nq}\,dt\\
&= \frac{2^q p^*}{q}\sum_{n\in\mathbb{Z}}\Big(\Big(\sum_{k\ge n}\mu_k\Big)^{q/p^*}-\Big(\sum_{k>n}\mu_k\Big)^{q/p^*}\Big)2^{nq}\\
&\le \frac{2^q p^*}{q}\sum_{n\in\mathbb{Z}}\Big(\sum_{k\ge n}\mu_k\Big)^{q/p^*}2^{nq}.
\end{aligned}
$$

We use the following variant of the Hardy inequality ([17, Lemma 3, p. 121], [19]), valid for $s,q>0$:

$$\sum_{n=n_0}^\infty\Big(\sum_{k\ge n}\mu_k\Big)^s 2^{nq}\le c(n_0,s,q)\sum_{n=n_0}^\infty \mu_n^s 2^{nq},$$

and the estimate $\sum_{k\ge n}\mu_k\le 2^{-np}$, which follows from $\|u\|_{L^p}=1$. We deduce from (17) that

$$\|u; L_{p^*,q}\|^q \le c \sum_{n<n_0} 2^{-npq/p^*} 2^{nq} + c \sum_{n=n_0}^{\infty} \mu_n^{q/p^*} 2^{nq} \tag{18}$$
$$\le c\|u\|_{L_p}^q + c \sum_{n=n_0}^{\infty} \mu_n^{q/p^*} 2^{nq},$$

where $c = c(d, C_1, p, q, \alpha)$.

Since u is bounded, $\mu_n = 0$ for all large n. We are going to apply Lemma 4 to $a_n = \mu_n^\gamma 2^{nq}$, where $\gamma = q/p^*$, and $\varepsilon = \alpha q/(\gamma d) > 0$. Observe that $\gamma(1+\varepsilon) = q/p$. Note that

$$\frac{a_n^{1+\varepsilon}}{(a_{n-1} + a_n + a_{n+1})^\varepsilon} \le c \, \frac{\mu_n^{\gamma(1+\varepsilon)}}{(\mu_{n-1} + \mu_n + \mu_{n+1})^{\gamma\varepsilon}} \cdot 2^{nq}$$

with $c = c(d, p, q, \alpha)$. Thus by Lemma 4 and the inequality (15) raised to the qth power we obtain

$$\sum_{n=n_0}^{\infty} 2^{nq} \mu_n^\gamma \le c \sum_{n=n_0}^{\infty} \frac{\mu_n^{q/p}}{(\mu_{n-1} + \mu_n + \mu_{n+1})^{\alpha q/d}} \, 2^{nq} + 2^{(n_0-1)q} \mu_{n_0-1}^\gamma$$
$$\le c \sum_{n=n_0}^{\infty} 2^{\nu(n)q\alpha} (2^{\nu(n)d} b_{n,\nu(n)})^{q/p} + 2^{(n_0-1)q} \mu_{n_0-1}^\gamma \tag{19}$$
$$\le c \sum_{n=n_0}^{\infty} \sum_{\nu=0}^{\infty} 2^{\nu q\alpha} (2^{\nu d} b_{n,\nu})^{q/p} + c\|u\|_{L^p}^q \tag{20}$$

with $c = c(d, C_1, p, q, \alpha)$. We note that in (19) above $\nu(n)$ depends also on n and u, but the dependence vanishes in (20). The first term in (20) is now estimated as follows:

$$\sum_{n=n_0}^{\infty} \sum_{\nu=0}^{\infty} 2^{\nu q\alpha} (2^{\nu d} b_{n,\nu})^{q/p} \tag{21}$$
$$= \sum_{\nu=0}^{\infty} 2^{\nu q\alpha} \sum_{n=n_0}^{\infty} (2^{\nu d} b_{n,\nu})^{q/p} \le \sum_{\nu=0}^{\infty} 2^{\nu q\alpha} \Big(2^{\nu d} \sum_{n=n_0}^{\infty} b_{n,\nu}\Big)^{q/p}$$
$$\le \sum_{\nu=0}^{\infty} 2^{\nu q\alpha} \Big(2^{\nu d} \iint_{\rho(x,y)<2^{-\nu}} |u(x) - u(y)|^p \, \mu(dy) \, \mu(dx)\Big)^{q/p}$$
$$\le \|u; \mathrm{Lip}_0(\alpha, p, q, F)\|^q.$$

Putting (18), (20) and (21) together we obtain (6).

It remains to show (6) in the case when $q = \infty$. We have

$$\|u; L_{p^*,\infty}\| \le 2 \sup_n \Big(\sum_{k\ge n} \mu_k\Big)^{1/p^*} 2^n.$$

Observe that for $n \le n_0$,

$$\Big(\sum_{k\ge n} \mu_k\Big)^{1/p^*} 2^n \le (2^{-np})^{1/p^*} 2^n \le 2^{n_0(1-p/p^*)},$$

hence

$$\sup_{n\le n_0} \Big(\sum_{k\ge n} \mu_k\Big)^{1/p^*} 2^n \le c(d, C_1, p, \alpha)\, \|u\|_{L^p}. \tag{22}$$

Now let

$$S = \sup_{n\ge n_0} \Big(\sum_{k\ge n} \mu_k\Big)^{1/p^*} 2^n.$$

We have $S < \infty$, because u is bounded. Let $N \ge n_0$ be such that

$$\Big(\sum_{k\ge N} \mu_k\Big)^{1/p^*} 2^N \ge \frac{3}{4}\, S. \tag{23}$$

If $N = n_0$, then $S \le c(d, C_1, p, \alpha)\|u\|_{L^p}$ by (22). Henceforth we assume that $N > n_0$. By (15) we get

$$\sup_{n\ge n_0} 2^n \mu_n^{1/p} (\mu_{n-1} + \mu_n + \mu_{n+1})^{-\alpha/d} \le c\|u; \mathrm{Lip}_0(\alpha, p, \infty, F)\| \tag{24}$$

(see (5)). From (23) and the inequalities $(\sum_{k\ge n} \mu_k)^{1/p^*} 2^n \le S$ for $n = N-1$ and $n = N+1$, we obtain, respectively,

$$\mu_{N-1} + \mu_N + \mu_{N+1} \le \sum_{k\ge N-1} \mu_k \le \Big(\frac{8}{3}\Big)^{p^*} \sum_{k\ge N} \mu_k,$$

$$\mu_N \ge \Big(\Big(\frac{3}{2}\Big)^{p^*} - 1\Big) \sum_{k\ge N+1} \mu_k.$$

Thus $\sum_{k\ge N} \mu_k \le c(p^*)\mu_N$, hence by $1/p^* = -\alpha/d + 1/p$ and (24),

$$\begin{aligned}\frac{3}{4}\, S &\le \Big(\sum_{k\ge N} \mu_k\Big)^{1/p^*} 2^N \le c(\mu_{N-1} + \mu_N + \mu_{N+1})^{-\alpha/d} 2^N \mu_N^{1/p} \\ &\le c\|u; \mathrm{Lip}_0(\alpha, p, \infty, F)\|. \ \blacksquare\end{aligned}$$

Proof of Theorem 1. By Theorem 3 applied to $p = q < \infty$ we have

$$\begin{aligned}\|u\|_{L_{p^*,p}} &\le c\|u; \mathrm{Lip}_0(\alpha, p, p, F)\| \\ &\le c\Big(\|u\|_{L^p} + \Big(\iint_{\rho(x,y)<1} \frac{|u(x) - u(y)|^p}{\rho(x,y)^{d+\alpha p}}\, \mu(dy)\, \mu(dx)\Big)^{1/p}\Big),\end{aligned}$$

and the theorem follows from the embedding $L_{p^*,p} \subset L^{p^*}$ for $p < p^*$ [2, Proposition 4.2, p. 217]. ■

Proof of Corollary 2. Denote $u^{(a)}(x) = u(ax)$ and

$$\mathcal{E}(u) = \iint_{FF} \frac{|u(x)-u(y)|^p}{|x-y|^{d+\alpha p}}\, \mu(dy)\, \mu(dx).$$

It is easy to check that $\|u^{(a)}\|_{L^s} = a^{-d/s}\|u\|_{L^s}$ and $\mathcal{E}(u^{(a)}) = a^{-d+\alpha p}\mathcal{E}(u)$. Hence by (2) applied to $u^{(a^n)}$ we obtain

$$\|u\|_{L^{pd/(d-\alpha p)}} \le c(a^{-n\alpha}\|u\|_{L^p} + \mathcal{E}(u)^{1/p})$$

and the corollary follows by letting $n \to \infty$. ■

Note. One can simplify the proof of Corollary 2 to get a stronger result. Namely, assume instead of (1) that for some $C_1, d, r_0 > 0$,

$$\mu(B(x,r)) \ge C_1 r^d \quad \text{for all } 0 < r \le r_0 \text{ and } x \in F. \tag{25}$$

Then the new measure $\tilde{\mu}(A) := \mu(A) r_0^{-d}$ and the new metric $\tilde{\rho}(x,y) := \rho(x,y)/r_0$ satisfy (1), hence (2) holds. Coming back to μ and ρ we get the following corollary.

COROLLARY 5. *Assume that* (25) *holds. If* $0 < p < \infty$ *and* $0 < \alpha < d/p$, *then there exists a constant* $c = c(d, C_1, p, \alpha)$ *such that*

$$\|u\|_{L^{pd/(d-\alpha p)}} \le c\bigg(r_0^{-\alpha}\|u\|_{L^p} + \bigg(\iint_{\rho(x,y)<r_0} \frac{|u(x)-u(y)|^p}{\rho(x,y)^{d+\alpha p}}\, \mu(dy)\, \mu(dx)\bigg)^{1/p}\bigg) \tag{26}$$

for all $u \in L^p$. *In particular, if* (25) *holds for all* $r_0 > 0$, *then*

$$\|u\|_{L^{pd/(d-\alpha p)}} \le c\bigg(\iint_{FF} \frac{|u(x)-u(y)|^p}{\rho(x,y)^{d+\alpha p}}\, \mu(dy)\, \mu(dx)\bigg)^{1/p} \tag{27}$$

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