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Operators whose adjoints are quasi p-nuclear

by

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Abstract. For $p \geq 1$, a set K in a Banach space X is said to be relatively p-compact if there exists a p-summable sequence (x_n) in X with $K \subseteq \{\sum_n \alpha_n x_n : (\alpha_n) \in B_{\ell_{p'}}\}$. We prove that an operator $T\colon X \to Y$ is p-compact (i.e., T maps bounded sets to relatively p-compact sets) iff T^* is quasi p-nuclear. Further, we characterize p-summing operators as those operators whose adjoints map relatively compact sets to relatively p-compact sets.

1. Introduction. In [4], Grothendieck characterized the compact subsets of a Banach space as those sets lying in the closed convex hull of a null sequence. This result aroused interest in the study of sets sitting inside the convex hull of certain classes of null sequences.

In [13], Sinha and Karn introduced the notion of p-compact set ($p \geq 1$). A set K of a Banach space X is *relatively p-compact* if it is contained in the p-convex hull of a p-summable sequence (x_n) in X, i.e. $K \subset \{\sum_n \alpha_n x_n \colon (\alpha_n) \in B_{\ell_{p'}}\}$. This notion opens a new approach to the p-approximation property. The authors of [13] investigate when the identity map on X can be approximated by finite rank operators on p-compact subsets of X and they connect their results with the p-approximation properties defined by Saphar [12] and Reinov [10] (which were conceived via the tensor product route). To this end, there is a previous analysis of the ideal $\mathcal{K}_p$ of p-compact operators (the operators mapping bounded sets to relatively p-compact sets) and it is proved that the adjoint of a p-compact operator admits a factorization through a subspace of ℓ_p. Using this factorization, a complete norm κ_p is defined on the ideal $\mathcal{K}_p$. It is shown that $\mathcal{K}_p$ is contained in the ideal Π_p^d of operators with p-summing adjoint [13, Proposition 5.3] and that $\mathcal{K}_p(X,Y)$ contains the space $\mathcal{N}_p^d(X,Y)$ of operators with p-nuclear adjoint whenever Y is reflexive (see the remark after [13, Proposition 5.3]).

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The aim of this paper is to deepen the study of $\mathcal{K}_p$ and its possible applications. In Section 3, we show the close relationship between p-compact operators and quasi p-nuclear operators. Quasi p-nuclear operators, introduced by Persson and Pietsch in [6], are an important tool to obtain results and counterexamples related to the approximation property of order p (see [8]). We prove that an operator is quasi p-nuclear iff its adjoint is p-compact (Proposition 3.8); in fact, the dual result is also true, which improves Proposition 5.3 in [13]. Another important result of that section is the characterization of p-summing operators as those operators whose adjoints map relatively compact sets to relatively p-compact sets. In the last section, we deal with the Banach ideal $\mathcal{V}_p$ of p-completely continuous operators (operators mapping relatively weakly p-compact sets to relatively p-compact sets) and we show that, though $\Pi_p \subset \mathcal{V}_p$ [13, Proposition 5.4], the inclusion is strict in general for every $p \geq 1$.

2. Preliminaries and notations. Throughout this paper, X and Y will be Banach spaces. As usual, we denote the closed unit ball of X by B_X, the dual of X by X^*, and the space of all bounded (linear) operators from X into Y by $\mathcal{L}(X,Y)$. The subspace of $\mathcal{L}(X,Y)$ consisting of all compact (respectively, weakly compact) operators from X into Y is denoted by $\mathcal{K}(X,Y)$ (respectively, $\mathcal{W}(X,Y)$).

Given a real number $p \in [1,\infty)$ and an arbitrary set I, $\ell_p(I)$ (respectively, $\ell_\infty(I)$) stands for the Banach space of all scalar functions ξ defined on I satisfying $\sum_{i\in I}|\xi_i|^p < \infty$ (respectively, $\sup_{i\in I}|\xi_i| < \infty$) endowed with its natural norm. As usual, we write ℓ_p instead of $\ell_p(\mathbb{N})$.

Let $\ell_p^w(X)$ be the space of all weakly p-summable sequences (x_n) in X. It is a Banach space with the norm

$$\|(x_n)\|_p^w = \sup_{x^*\in B_{X^*}} \Big(\sum_n |\langle x_n, x^*\rangle|^p\Big)^{1/p} = \sup_{(\alpha_n)\in B_{\ell_{p'}}} \Big\|\sum_n \alpha_n x_n\Big\|.$$

The subspace of $\ell_p^w(X)$ consisting of the (strongly) p-summable sequences is denoted by $\ell_p(X)$, which is also a Banach space endowed with the norm

$$\|(x_n)\|_p = \Big(\sum_n \|x_n\|^p\Big)^{1/p}.$$

We write $\ell_\infty(X)$ for the Banach space of all bounded sequences (x_n) in X with the norm

$$\|(x_n)\|_\infty = \sup_n \|x_n\|.$$

We denote by $c_0(X)$ the space of all norm null sequences in X, which is a closed subspace of $\ell_\infty(X)$ with the above norm.

In addition to the classical Banach ideals $[\mathcal{L}, \|\cdot\|]$, $[\mathcal{K}, \|\cdot\|]$ and $[\mathcal{W}, \|\cdot\|]$, we deal with the ideals $[\Pi_p, \pi_p]$ of all p-summing operators and $[\mathcal{N}_p, \nu_p]$ of all p-nuclear operators. We also consider the injective hull of $[\mathcal{N}_p, \nu_p]$, which has been treated in the literature under the name of the Banach ideal of quasi p-nuclear operators [6]. We denote this Banach ideal by $\mathcal{QN}_p$. So, an operator $T\colon X \to Y$ is *quasi p-nuclear* iff $j_Y \circ T \in \mathcal{N}_p(X, \ell_\infty(B_{Y^*}))$, where j_Y is the natural isometric embedding from Y into $\ell_\infty(B_{Y^*})$. It is well known that $T \in \mathcal{QN}_p(X,Y)$ iff there exists a sequence $(x_n^*) \in \ell_p(X^*)$ such that

$$(1) \qquad \|Tx\| \le \Big(\sum_n |\langle x, x_n^*\rangle|^p\Big)^{1/p}$$

for all $x \in X$. The quasi p-nuclear norm is

$$\nu_p^Q(T) = \inf\{\|(x_n^*)\|_p \colon (1) \text{ holds for all } x \in X\}$$

for all $T \in \mathcal{QN}_p(X,Y)$. If $\mathcal{A}$ is a Banach ideal, then $\mathcal{A}^d$ denotes its dual ideal, that is, $\mathcal{A}^d(X,Y) = \{T \in \mathcal{L}(X,Y)\colon T^* \in \mathcal{A}(Y^*, X^*)\}$.

If $p > 1$ and $p' = p(p-1)^{-1}$, the map $\Phi_p\colon (x_n) \in \ell_p^w(X) \mapsto \Phi_p(x_n) \in \mathcal{L}(\ell_{p'}, X)$, where $\Phi_p(x_n)(\alpha_n) = \sum_n \alpha_n x_n$, is an isometric isomorphism which allows us to identify the spaces $\ell_p^w(X)$ and $\mathcal{L}(\ell_{p'}, X)$. For $p = 1$, $\ell_1^w(X)$ is isometrically isomorphic to $\mathcal{L}(c_0, X)$ under the corresponding map Φ_1.

The following notions were introduced by Sinha and Karn in [13] trying to extend the characterization of compact sets in Banach spaces as those sets lying inside of the closed convex hull of a norm null sequence [4]. If $p \in [1,\infty)$, the *p-convex hull* of a sequence $(x_n) \in \ell_p^w(X)$ is

$$p\text{-co}\,(x_n) = \Phi_p(x_n)(B_{\ell_{p'}}) = \Big\{\sum_n \alpha_n x_n \colon (\alpha_n) \in B_{\ell_{p'}}\Big\}$$

(c_0 instead of $\ell_{p'}$ if $p = 1$). It is clear that the p-convex hull of a sequence is an absolutely convex set; if $p > 1$, it is also weakly compact so, in particular, norm closed.

A set $K \subset X$ is *relatively p-compact* if there exists a sequence $(x_n) \in \ell_p(X)$ such that $K \subset p\text{-co}\,(x_n)$. Since $p\text{-co}\,(x_n)$ is a relatively compact set when $(x_n) \in \ell_p(X)$, relatively p-compact sets in X are relatively compact. If compact sets are viewed as ∞-compact sets, then it is easy to show that p-compact sets are q-compact for $1 \le p < q \le \infty$. Notice that the convex hull of a relatively p-compact set is relatively p-compact too.

A set $K \subset X$ is *relatively weakly p-compact* if there exists a sequence $(x_n) \in \ell_p^w(X)$ such that $K \subseteq p\text{-co}\,(x_n)$. If $p > 1$, relatively weakly p-compact sets in X are relatively weakly compact. However, $p = 1$ is a pathological case: B_{c_0} is weakly 1-compact since $B_{c_0} = p\text{-co}\,(e_n)$, where $(e_n) \in \ell_1^w(c_0)$ is the unit vector basis in c_0. Again, it is a standard argument to prove that weakly p-compact sets are weakly q-compact for $1 < p < q < \infty$.

Finally, we recall that an operator $T \in \mathcal{L}(X,Y)$ is said to be *p-compact* (respectively, *weakly p-compact*) if $T(B_X)$ is relatively p-compact (respectively, weakly p-compact) in Y. The set of p-compact (respectively, weakly p-compact) operators from X into Y is denoted by $\mathcal{K}_p(X,Y)$ (respectively, $\mathcal{W}_p(X,Y)$).

3. Main results. The next propositions are the keys to connect p-compactness and quasi p-nuclearity.

PROPOSITION 3.1. *Let $p \in [1,\infty)$, $T \in \mathcal{L}(X,Y)$ and $(y_n) \in \ell_p^w(Y)$. The following statements are equivalent:*

(a) $\|T^*y^*\| \le (\sum_n |\langle y_n, y^*\rangle|^p)^{1/p}$ *for all* $y^* \in Y^*$.

(b) $T(B_X) \subseteq \overline{p\text{-co}\,(y_n)}$.

Proof. (a)⇒(b). By contradiction, assume that there exists $x_0 \in B_X$ so that $Tx_0 \notin \overline{p\text{-co}\,(y_n)}$. As $\overline{p\text{-co}\,(y_n)}$ is absolutely convex, we can separate Tx_0 and $\overline{p\text{-co}\,(y_n)}$ strictly by a closed hyperplane; that is to say, there exist $\alpha > 0$ and $y^* \in Y^*$ such that $|\langle Tx_0, y^*\rangle| > \alpha$ and $|\langle y, y^*\rangle| < \alpha$ for all $y \in \overline{p\text{-co}\,(y_n)}$. Then

$$\begin{aligned}\alpha &< |\langle Tx_0, y^*\rangle| \le \|T^*y^*\| \\ &\le \Big(\sum_n |\langle y_n, y^*\rangle|^p\Big)^{1/p} = \sup_{(\alpha_n)\in B_{\ell_{p'}}} \Big|\Big\langle \sum_n \alpha_n y_n, y^*\Big\rangle\Big| \le \alpha,\end{aligned}$$

a contradiction.

(b)⇒(a). Given $\varepsilon > 0$ and $y^* \in B_{Y^*}$, choose $x \in B_X$ such that $\|T^*y^*\| < |\langle x, T^*y^*\rangle| + \varepsilon/2$. Now, take $(\alpha_n) \in B_{\ell_{p'}}$ so that $\|Tx - \sum_n \alpha_n y_n\| < \varepsilon/2$. Then

$$\begin{aligned}\|T^*y^*\| &< |\langle x, T^*y^*\rangle| + \varepsilon/2 \\ &\le \Big|\Big\langle Tx - \sum_n \alpha_n y_n, y^*\Big\rangle\Big| + \Big|\Big\langle \sum_n \alpha_n y_n, y^*\Big\rangle\Big| + \varepsilon/2 \\ &< \sum_n |\alpha_n|\,|\langle y_n, y^*\rangle| + \varepsilon \le \|(\alpha_n)\|_{p'} \cdot \Big(\sum_n |\langle y_n, y^*\rangle|^p\Big)^{1/p} + \varepsilon \\ &\le \Big(\sum_n |\langle y_n, y^*\rangle|^p\Big)^{1/p} + \varepsilon\end{aligned}$$

and letting $\varepsilon \to 0$ we obtain the conclusion. ∎

Arguing in a similar way, we obtain the dual version of the above result:

PROPOSITION 3.2. *Let $p \in [1,\infty)$, $T \in \mathcal{L}(X,Y)$ and $(x_n^*) \in \ell_p^w(X^*)$. The following statements are equivalent:*

(a) $\|Tx\| \le (\sum_n |\langle x, x_n^*\rangle|^p)^{1/p}$ *for all* $x \in X$.

(b) $T^*(B_{Y^*}) \subseteq \overline{p\text{-co}\,(x_n^*)}$.

REMARK 3.3. In Proposition 3.1 we can use p-co (y_n) instead of $\overline{p\text{-co}\,(y_n)}$ in case $p > 1$. On the other hand, if $p = 1$ and $(y_n) \in \ell_1(Y)$, we have $\overline{\{\sum_n \alpha_n y_n : (\alpha_n) \in B_{c_0}\}} = \{\sum_n \alpha_n y_n : (\alpha_n) \in B_{\ell_\infty}\}$ and this set is 1-compact too. In fact, for $\delta_n \to \infty$ such that $\sum_n |\delta_n|\,\|y_n\| < \infty$, we have the obvious inclusion

$$\Big\{\sum_n \alpha_n y_n : (\alpha_n) \in B_{\ell_\infty}\Big\} \subset \Big\{\sum_n \alpha_n(\delta_n y_n) : (\alpha_n) \in B_{c_0}\Big\}.$$

COROLLARY 3.4. *Let* $T \in \mathcal{L}(X,Y)$. *Then the following properties hold:*

(I) *If* $T \in \mathcal{K}_p(X,Y)$, *then* $T^* \in \mathcal{QN}_p(Y^*, X^*)$.

(II) $T \in \mathcal{QN}_p(X,Y)$ *iff* $T^* \in \mathcal{K}_p(Y^*, X^*)$.

In other words, $\mathcal{K}_p \subseteq \mathcal{QN}_p^d$ *and* $\mathcal{QN}_p = \mathcal{K}_p^d$.

The converse of Corollary 3.4(I) cannot be deduced directly from Proposition 3.1. Indeed, if $T^* \in \mathcal{QN}_p(Y^*, X^*)$, then there exists a sequence $(y_n^{**}) \in \ell_p(Y^{**})$ such that $\|T^* y^*\| \le (\sum_n |\langle y_n^{**}, y^*\rangle|^p)^{1/p}$ for all $y^* \in Y^*$, and consequently $T(B_X) \subseteq p\text{-co}\,(y_n^{**})$. In other words, $T \in \mathcal{K}_p(X, Y^{**})$ (although $T(X) \subset Y$). In addition, we will need to deal with the ideal of so-called $\mathcal{N}^p$-operators. We recall that $T \in \mathcal{N}^p(X,Y)$ if there exist sequences $(x_n^*) \in \ell_{p'}^w(X^*)$ and $(y_n) \in \ell_p(Y)$ such that T admits the representation $T = \sum_n x_n^* \otimes y_n$ (note that $\mathcal{N}^p(X,Y) \subseteq \mathcal{K}_p(X,Y)$). The norm in this ideal will be denoted by ν^p and is defined by

$$\nu^p(T) = \inf \|(y_n)\|_p \cdot \|(x_n^*)\|_{p'}^w$$

where the infimum is taken over all representations of T as above (see [10]). We will make use of the following theorem:

THEOREM ([10, Theorem 1]). *Let* $p \in [1,\infty]$, $T \in \mathcal{L}(X,Y)$ *and suppose that either* X^* *or* Y^{***} *has the approximation property. If* $T \in \mathcal{N}^p(X, Y^{**})$, *then* $T \in \mathcal{N}^p(X,Y)$. *In other words, under these conditions, the p-nuclearity of* T^* *implies that* $T \in \mathcal{N}^p(X,Y)$.

Let K be a bounded subset of X. We define the following bounded operators:

$$u_K : \ell_1(K) \to X, \qquad (\xi_x)_{x\in K} \mapsto \sum_{x\in K} \xi_x x,$$

$$j_K : X^* \to \ell_\infty(K), \qquad x^* \mapsto (\langle x, x^*\rangle)_{x\in K}.$$

Notice that $u_K^* = j_K$. We write u_X and j_X instead of u_{B_X} and j_{B_X}, respectively.

PROPOSITION 3.5. *Let* K *be a bounded subset of* X. *The following statements are equivalent:*

(a) *K is relatively p-compact.*
(b) *u_K is p-compact.*
(c) *j_K is p-nuclear.*

Proof. (a)⇔(b). This follows from the inclusions $K \subseteq u_K(B_{\ell_1(K)}) \subseteq \overline{\mathrm{co}}(K)$.

(b)⇔(c). Let u_K be p-compact. By Corollary 3.4, j_K is quasi p-nuclear, and since $\ell_\infty(K)$ is an injective space, j_K is p-nuclear [6, Theorem 38]. For the converse, suppose j_K is p-nuclear. According to [10, Theorem 1], the operator u_K belongs to $\mathcal{N}^p(\ell_1(K), X)$ and, a fortiori, it is p-compact. ∎

Corollary 3.6. *Le K be a subset of X. If K is relatively p-compact in X^{**}, then K is p-compact in X. In particular, an operator $T \in \mathcal{L}(X,Y)$ is p-compact iff T^{**} is p-compact.*

Proof. By Proposition 3.5, $J_K : x^{***} \in X^{***} \mapsto (\langle x, x^{***}\rangle)_{x\in K} \in \ell_\infty(K)$ is p-nuclear, hence so is $j_K = J_K|_{X^*} : x^* \in X^* \mapsto (\langle x, x^*\rangle)_{x\in K} \in \ell_\infty(K)$. Again a call to Proposition 3.5 tells us that K is p-compact in X. ∎

Remark 3.7. Let A be a bounded subset of X^*. As in the proof of Proposition 3.5, A is relatively p-compact iff the operator $\hat{j}_A : x \in X \mapsto (\langle x, x^*\rangle)_{x^*\in A} \in \ell_\infty(A)$ is p-nuclear.

In Corollary 3.4, it is shown that $\mathcal{K}_p \subseteq \mathcal{QN}_p^d$. Now if $T \in \mathcal{L}(X,Y)$ is such that $T^* \in \mathcal{QN}_p(Y^*, X^*)$ then $T^{**} \in \mathcal{K}_p(X^{**}, Y^{**})$ (Corollary 3.4). From the above result, it follows that $T \in \mathcal{K}_p(X,Y)$. This leads to the following proposition which improves Proposition 5.3 in [13].

Proposition 3.8. $\mathcal{K}_p = \mathcal{QN}_p^d$.

In a recent paper [14], Sinha and Karn have dealt with the Banach operator ideals $\mathcal{K}_p^d$ and $\mathcal{K}_p^{dd}$. The above results simplify the understanding of that paper, since $\mathcal{K}_p^d = \mathcal{QN}_p$ and $\mathcal{K}_p^{dd} = \mathcal{K}_p$.

Corollary 3.9. *An operator $T \in \mathcal{L}(X,Y)$ is such that $T^* \in \mathcal{QN}_p(Y^*, X^*)$ if and only if there exists $(y_n) \in \ell_p(Y)$ such that $\|T^* y^*\| \le (\sum_n |\langle y_n, y^*\rangle|^p)^{1/p}$ for all $y^* \in Y^*$.*

As we have mentioned in the introduction, p-compact operators have been characterized as those operators whose adjoints factor through a subspace of ℓ_p [13, Theorem 3.1]. This factorization yields a complete norm defined on $\mathcal{K}_p(X,Y)$. Having in mind the preceding results, we have obtained the same factorization for the adjoints of p-compact operators in a much simpler way. In fact, Theorem 3.1 in [13] can be stated in the following manner:

Proposition 3.10. *Let X and Y be Banach spaces and $p \in [1,\infty)$. The following statements are equivalent:*

(a) $T \in \mathcal{K}_p(X, Y)$.
(b) *There exists a closed subspace* H *of* ℓ_p *and operators* $R \in \mathcal{QN}_p(Y^*, H)$ *and* $S \in \mathcal{L}(H, X^*)$ *such that* $T^* = S \circ R$.

Proof. (a)$\Leftrightarrow$(b). If $T \in \mathcal{K}_p(X, Y)$, there exists a sequence $(y_n) \in \ell_p(Y)$ such that $\|T^* y^*\| \leq (\sum_n |\langle y_n, y^* \rangle|^p)^{1/p}$ for all $y^* \in Y^*$ (Proposition 3.1). Put

$$H = \overline{\{(\langle y_n, y^* \rangle) : y^* \in Y^*\}}$$

and define the operators $R\colon y^* \in Y^* \mapsto (\langle y_n, y^* \rangle) \in H$ and $S\colon (\langle y_n, y^* \rangle) \in H \mapsto T^* y^* \in Y^*$. It is easy to check that H, R and S satisfy the required conditions. The converse is trivial via Proposition 3.8. ∎

If $T \in \mathcal{K}_p(X, Y)$, we define

$$k_p(T) = \inf \|(y_n)\|_p$$

where the infimum is taken over all sequences $(y_n) \in \ell_p(Y)$ satisfying

$$T(B_X) \subseteq \Big\{ \sum_n \alpha_n y_n : (\alpha_n) \in B_{\ell_{p'}} \Big\}.$$

The inequality $k_p(T) \geq \nu_p^Q(T^*)$ (respectively, the equality $k_p(T^*) = \nu_p^Q(T)$) is a direct consequence of Proposition 3.1 (respectively, Proposition 3.2). Now, $[\mathcal{K}_p, k_p]$ becomes a Banach ideal and the proof is similar to that in [6, p. 31] showing that $[\mathcal{QN}_p, \nu_p^Q]$ is a Banach ideal (both proofs can be connected via Proposition 3.1). According to [7, Theorem 6.1.8], the norm k_p is equivalent to the norm κ_p defined by Sinha and Karn in [13]. Moreover, at the end of this section we prove that these norms coincide (Proposition 3.15).

Proposition 3.11. $[\mathcal{K}_p, k_p]$ *is the surjective hull of* $[\mathcal{N}^p, \nu^p]$ *for all* $p \in [1, \infty)$.

Proof. If $T \in \mathcal{L}(X, Y)$ and $T \circ u_X(B_X)$ $[\ell_1(B_X) \xrightarrow{u_X} X \xrightarrow{T} Y]$ is relatively p-compact, then so is $T(B_X)$. In other words, $\mathcal{K}_p$ is surjective, and since $\mathcal{N}^p \subseteq \mathcal{K}_p$, we have $(\mathcal{N}^p)^s \subseteq \mathcal{K}_p$.

On the other hand, if $T \in \mathcal{K}_p(X, Y)$, then $T^* \in \mathcal{QN}_p(Y^*, X^*)$ (Corollary 3.4). Thus, $j_X \circ T^* \in \mathcal{QN}_p(Y^*, \ell_\infty(B_X)) = \mathcal{N}_p(Y^*, \ell_\infty(B_X))$, and since $j_X \circ T^* = (T \circ u_X)^*$ and $\ell_\infty(B_X)$ has the approximation property, it follows that $T \circ u_X \in \mathcal{N}^p(\ell_1(B_X), Y)$ ([10, Theorem 1]). So, we have obtained the equality $(\mathcal{N}^p)^s = \mathcal{K}_p$.

Now, a standard argument shows that

$$(\mathcal{N}^p(\ell_1(I), Y), \nu^p) = (\mathcal{K}_p(\ell_1(I), Y), k_p) \quad \text{(isometrically)}$$

for all nonempty sets I. In particular, this proves that $k_p = (\nu^p)^s$. ∎

Now, we can state our main result. We will need the following theorem:

Theorem ([11, Proposition 6.14]). *Let $1 \le p < \infty$ and let X and Y be Banach spaces. An operator $T\colon X \to Y$ is p-summing if and only if there exists a positive constant C such that for every finite-dimensional subspace E of X and every finite-codimensional subspace F of Y, the finite-dimensional operator*

$$q_F \circ T \circ i_E \colon E \to X \to Y \to Y/F$$

satisfies $\pi_p(q_F \circ T \circ i_E) \le C$. Furthermore, we have $\pi_p(T) = \inf C$, where the infimum is taken over all such pairs E, F.

Theorem 3.12. *Let $T \in \mathcal{L}(X,Y)$ and $p \in [1,\infty)$. The following statements are equivalent:*

(a) *T is p-summing.*
(b) *T^* maps relatively compact subsets of Y^* to relatively p-compact subsets of X^*.*

Proof. (a)⇒(b). Let (y_n^*) be a null sequence in Y^* and define $S\colon y \in Y \mapsto (\langle y, y_n^*\rangle) \in c_0$. Obviously, S is ∞-nuclear; therefore, $S \circ T$ is p-nuclear and

$$\nu_p(S \circ T) \le \nu_\infty(S)\pi_p(T) \le \pi_p(T) \sup_n \|y_n^*\|$$

[16, Theorem 9.13]. Then $(S \circ T)^*\colon e_n \in \ell_1 \mapsto T^* y_n^* \in X^*$ belongs to $\mathcal{N}^p(\ell_1, X^*)$ and $\nu^p((S \circ T)^*) \le \nu_p(S \circ T)$. As mentioned before, $\mathcal{K}_p(\ell_1, X^*)$ and $\mathcal{N}^p(\ell_1, X^*)$ are isometric, so

$$k_p((S \circ T)^*) \le \nu_p(S \circ T) \le \pi_p(T) \sup_n \|y_n^*\|.$$

This proves that the linear map

$$U\colon c_0(Y^*) \to \mathcal{K}_p(\ell_1, X^*), \qquad (y_n^*) \mapsto \sum_n e_n^* \otimes T^* y_n^*,$$

is well defined and $\|U\| \le \pi_p(T)$ (this inequality will be used in the next proposition). Notice that, in particular, we have proved that the set $\{T^* y_n^* : n \in \mathbb{N}\}$ is relatively p-compact.

(b)⇒(a). To prove (a) we will use [11, Proposition 6.14]. Let E be a finite-dimensional subspace of X and F a subspace of Y whose codimension is finite. Given the sequence

$$E \xrightarrow{i_E} X \xrightarrow{T} Y \xrightarrow{q_F} Y/F,$$

we obtain

$$F^\perp \xrightarrow{q_F^*} Y^* \xrightarrow{T^*} X^* \xrightarrow{i_E^*} X^*/E^\perp.$$

For simplicity, we identify the operator $Q\colon e_n \in \ell_1 \mapsto y_n^* \in Y^*$ with the sequence (y_n^*). Now, consider the map

$$\phi\colon \mathcal{K}(\ell_1, Y^*) \to \mathcal{K}_p(\ell_1, X^*), \quad (y_n^*) \mapsto (T^* y_n^*).$$

The map ϕ is linear and has closed graph, so it is continuous. Thus, there exists a positive constant C such that $k_p(T^* y_n^*) < C$ for every relatively compact sequence (y_n^*) in B_{Y^*}.

Choose (y_n^*) dense in $B_{F^\perp}$. Since $k_p(T^* y_n^*) < C$, there exists a sequence (x_n^*) in $\ell_p(X^*)$ such that $\|(x_n^*)\|_p < C$ and $\{T^* y_n^*\} \subset \overline{p\text{-co}\,(x_n^*)}$. By density, we also have $T^*(B_{F^\perp}) \subset \overline{p\text{-co}\,(x_n^*)}$. This yields $k_p(T^* \circ q_F^*) \le \|(x_n^*)\|_p < C$ and therefore $k_p(i_E^* \circ T^* \circ q_F^*) < C$. Now, we can conclude that $k_p(i_E^* \circ T^* \circ q_F^*) = \nu_p^Q(q_F \circ T \circ i_E) < C$ (see the comment after the definition of k_p on page 297). Finally, recall that $\pi_p \le \nu_p^Q$. ∎

PROPOSITION 3.13. *Let X, Y and Z be Banach spaces and $p \ge 1$. If the operator $T\colon X \to Y$ is p-summing and $S\colon Z \to Y^*$ is compact, then $T^* \circ S$ is p-compact and $k_p(T^* \circ S) \le \pi_p(T)\|S\|$.*

Proof. Given $S \in \mathcal{K}(Z, Y^*)$ and $\varepsilon > 0$ there exists a null sequence (y_n^*) such that $S(B_Z) \subset \overline{\text{co}}\,(y_n^*)$ and

$$\sup_n \|y_n^*\| < \sup_{\|z\| \le 1} \|Sz\| + \varepsilon = \|S\| + \varepsilon.$$

Now, we define the operator $A\colon (\alpha_n) \in \ell_1 \mapsto \sum_n \alpha_n T^* y_n^* \in X^*$. In the above theorem we have proved that

$$k_p(A) \le \pi_p(T) \sup_n \|y_n^*\|.$$

Thus, given $\delta > 0$, there exists (x_n^*) in $\ell_p(X^*)$ such that $\overline{\text{co}}\,(T^* y_n^*) \subseteq p\text{-co}\,(x_n^*)$ and $\|(x_n^*)\|_p < \pi_p(T)\|(y_n^*)\|_\infty + \delta$. Consequently, $T^*(S(B_Z)) \subseteq \overline{\text{co}}\,(T^* y_n^*) \subseteq p\text{-co}\,(x_n^*)$ and these inclusions yield

$$k_p(T^* \circ S) \le \|(x_n^*)\|_p < \pi_p(T)\|(y_n^*)\|_\infty + \delta.$$

Letting $\delta \to 0$, we obtain $k_p(T^* \circ S) \le \pi_p(T)\|(y_n^*)\|_\infty$. Finally, since $\|(y_n^*)\|_\infty \le \|S\| + \varepsilon$ we deduce

$$k_p(T^* \circ S) \le \pi_p(T)(\|S\| + \varepsilon).$$

The proof concludes by letting $\varepsilon \to 0$. ∎

The dual version of the main theorem is also valid.

THEOREM 3.14. *Let $T \in \mathcal{L}(X, Y)$ and $p \in [1, \infty)$. The following statements are equivalent:*

(a) *T^* is p-summing.*
(b) *T maps relatively compact subsets of X to relatively p-compact subsets of Y.*

Proof. (a)⇒(b). This is an easy consequence of Theorem 3.12 and Corollary 3.6.

(b)⇒(a). By Proposition 3.5, we can consider the linear map

$$V\colon c_0(X) \to \mathcal{N}_p(Y^*, \ell_\infty), \qquad (x_n) \mapsto \sum_n Tx_n \otimes e_n$$

((e_n) is the canonical basis of c_0). The operator V is continuous because its graph is closed. Let J be the restriction of V^* to $\Pi_{p'}(\ell_\infty, Y^*)$. A straightforward argument shows that $J\colon \Pi_{p'}(\ell_\infty, Y^*) \to \ell_1(X^*)$ is the continuous linear map defined by $J(A) = (T^* \circ A(e_n))$. As $\pi_{p'}(A) \le \nu_{p'}(A)$ for all $A \in \mathcal{N}_{p'}(\ell_\infty, Y^*)$, it follows that the map

$$J_0\colon \mathcal{N}_{p'}(\ell_\infty, Y^*) \to \ell_1(X^*), \qquad A \mapsto (T^* \circ A(e_n)),$$

is continuous. Now we consider $J_0^*\colon \ell_\infty(X^{**}) \to \Pi_p(Y^*, \ell_\infty^{**})$ and $\phi = J_0^*|_{c_0(X^{**})}$. If $(x_n^{**}) \in c_0(X^{**})$, $y^* \in Y^*$ and $\mu \in \ell_\infty^*$, then

$$\begin{aligned}\langle J_0^*(x_n^{**})(y^*), \mu\rangle &= J_0^*(x_n^{**})(\mu \otimes y^*) = \langle (x_n^{**}), J_0(\mu \otimes y^*)\rangle \\ &= \langle (x_n^{**}), (T^*[\mu \otimes y^*(e_n)])\rangle = \sum_n \langle x_n^{**}, T^*(\langle \mu, e_n\rangle y^*)\rangle \\ &= \sum_n \langle T^{**}x_n^{**}, y^*\rangle \langle \mu, e_n\rangle = \Big\langle \sum_n \langle T^{**}x_n^{**}, y^*\rangle e_n, \mu \Big\rangle.\end{aligned}$$

This proves that ϕ maps $c_0(X^{**})$ into $\Pi_p(Y^*, \ell_\infty)$ and $\phi(x_n^{**}) = \sum_n T^{**}x_n^{**} \otimes e_n$. Finally, we will show that $\phi(c_0(X^{**})) \subseteq \mathcal{N}_p(Y^*, \ell_\infty)$. First, for each $n \in \mathbb{N}$, we define

$$(2) \qquad \phi_n\colon \ell_\infty^n(X^{**}) \to \Pi_p(Y^*, \ell_\infty^n), \qquad (x_k^{**})_{k=1}^n \mapsto \sum_{k=1}^n T^{**}x_k^{**} \otimes e_k.$$

By the ideal properties, we have $\|\phi_n\| \le \|\phi\|$ for all $n \in \mathbb{N}$. In view of [16, Corollary 9.5], $\pi_p(u) = \nu_p(u)$ for all $u \in \mathcal{L}(Y^*, \ell_\infty^n)$. Thus, we can write (2) in the form

$$(3) \qquad \phi_n\colon \ell_\infty^n(X^{**}) \to \mathcal{N}_p(Y^*, \ell_\infty^n), \qquad (x_k^{**})_{k=1}^n \mapsto \sum_{k=1}^n T^{**}x_k^{**} \otimes e_k.$$

Let us prove that $(\phi(x_1^{**}, \ldots, x_n^{**}, 0, \ldots))_n$ is a Cauchy sequence in $\mathcal{N}_p(Y^*, \ell_\infty)$ for all $(x_k^{**}) \in c_0(X^{**})$. According to (3) and the ideal properties of $\mathcal{N}_p$ we have

$$\begin{aligned}&\nu_p(\phi(x_1^{**}, \ldots, x_n^{**}, 0, \ldots) - \phi(x_1^{**}, \ldots, x_m^{**}, 0, \ldots)) \\ &\qquad = \nu_p(\phi(\ldots, 0, x_{m+1}^{**}, \ldots, x_n^{**}, 0, \ldots)) \le \|\phi\| \cdot \sup_{m<k\le n} \|x_k^{**}\|\end{aligned}$$

for $n > m$. Thus, $(\phi(x_1^{**}, \ldots, x_n^{**}, 0, \ldots))_n$ converges to an operator $S \in \mathcal{N}_p(Y^*, \ell_\infty)$ and this operator is necessarily equal to $\phi(x_n^{**}) = \sum_n T^{**}x_n^{**} \otimes e_n$. In particular, this implies that T^{**} maps relatively compact sets in X^{**} to relatively p-compact sets in Y^{**}. Now, a call to Theorem 3.12 tells us that T^* is p-summing. ■

We finish this section by showing that our definition of k_p coincides with that in [13]. An operator $T \in \mathcal{L}(X,Y)$ belongs to $\mathcal{K}_p(X,Y)$ if and only if there exists $\hat{y} = (y_n) \in \ell_p(Y)$ such that $T^* = S_{\hat{y}} \circ \phi_{\hat{y}}^*$, where $\phi_{\hat{y}}^* : y^* \in Y^* \mapsto (\langle y_n, y^* \rangle) \in H := \overline{\{(\langle y_n, y^*\rangle) : y^* \in Y^*\}}$ and $S_{\hat{y}} : (\langle y_n, y^*\rangle) \in H \mapsto T^*y^* \in X^*$ [13, Theorem 3.2]. Using this decomposition, we can endow $\mathcal{K}_p(X,Y)$ with the norm κ_p defined by

$$\kappa_p(T) = \inf\{\|S_{\hat{y}}\| \cdot \|\hat{y}\|_p : \hat{y} = (y_n) \in \ell_p(Y),\, T^* = S_{\hat{y}} \circ \phi_{\hat{y}}^*\}.$$

Proposition 3.15. *Let X and Y be Banach spaces and $p \geq 1$. Then $k_p(T) = \kappa_p(T)$ for all $T \in \mathcal{K}_p(X,Y)$.*

Proof. Given $T \in \mathcal{K}_p(X,Y)$ and $\hat{y} = (y_n) \in \ell_p(Y)$, we know that $\|T^*y^*\| \leq \|(\langle y_n, y^*\rangle)\|_p$ for all $y^* \in Y^*$ if and only if $T(B_X) \subset p\text{-co}\,(y_n)$ (Proposition 3.1). Since $\|S_{\hat{y}}(\langle y_n, y^*\rangle)\| = \|T^*y^*\|$, it follows that $\|S_{\hat{y}}\| \leq 1$ and $\kappa_p(T) \leq k_p(T)$.

Now, given $0 < \varepsilon < 1$, consider $\hat{y} = (y_n) \in \ell_p(Y)$ such that

$$\kappa_p(T) + \varepsilon > \|S_{\hat{y}}\| \cdot \|\hat{y}\|_p.$$

Moreover, $\hat{y}$ can be chosen so that $\|S_{\hat{y}}\| > 1 - \varepsilon$. Otherwise, $\|T^*y^*\| = \|S_{\hat{y}}(\langle y_n, y^*\rangle)\| \leq \|(\langle (1-\varepsilon)y_n, y^*\rangle)\|_p$ for all $y^* \in Y^*$ and this means that $T(B_X) \subset p\text{-co}\,((1-\varepsilon)y_n)$ (Proposition 3.1). But then

$$\begin{aligned}\|S_{(1-\varepsilon)\hat{y}}\| &= \sup\{\|T^*y^*\| : \|(\langle (1-\varepsilon)y_n, y^*\rangle)\|_p \leq 1\} \\ &= \sup\left\{\left\|T^*\left(\frac{1}{1-\varepsilon}\, z^*\right)\right\| : \left\|\left(\left\langle (1-\varepsilon)y_n, \frac{1}{1-\varepsilon}\, z^*\right\rangle\right)\right\|_p \leq 1\right\} \\ &= \frac{\|S_{\hat{y}}\|}{1-\varepsilon},\end{aligned}$$

which implies $\kappa_p(T) + \varepsilon > \|S_{(1-\varepsilon)\hat{y}}\| \cdot \|(1-\varepsilon)\hat{y}\|_p$. By induction, we have $T(B_X) \subset p\text{-co}\,((1-\varepsilon)^m y_n)$ for all $m \in \mathbb{N}$, which is impossible if $T \neq 0$. So

$$\kappa_p(T) + \varepsilon > (1-\varepsilon)\|\hat{y}\|_p > (1-\varepsilon)k_p(T),$$

and since ε can be chosen arbitrarily, $\kappa_p(T) \geq k_p(T)$. ■

4. The operator ideal $\mathcal{V}_p$. We will denote by $\mathcal{V}_p(X,Y)$ the vector space of all operators from X into Y that map relatively weakly p-compact subsets of X to relatively p-compacts subsets of Y. In [13], the authors proved that $\Pi_p(X,Y) \subset \mathcal{V}_p(X,Y)$. First of all, we give sufficient conditions for which the converse inclusion holds for $p = 1, 2$. We will denote by $\ell_p^u(X)$ the subspace of $\ell_p^w(X)$ consisting of all unconditionally p-summable sequences in X, that is, those sequences (x_n) satisfying

$$\lim_{n\to\infty}\Big(\sup_{\|x^*\|\le 1}\sum_{m\ge n}|\langle x_m,x^*\rangle|^p\Big)<\infty.$$

PROPOSITION 4.1. *If Y is an $\mathcal{L}_1$-space, then $\Pi_1(X,Y)=\mathcal{V}_1(X,Y)$ for every Banach space X.*

Proof. If $(x_n)\in\ell_1^u(X)$ and $T\in\mathcal{V}_1(X,Y)$, then the set

$$\Big\{\sum_n \alpha_n T(x_n) : (\alpha_n)\in B_{c_0}\Big\}$$

is relatively 1-compact in Y. So, the operator $A: e_n\in c_0\mapsto T(x_n)\in Y$ is 1-compact. By Corollary 3.4, its adjoint $A^*:Y^*\to\ell_1$ is quasi 1-nuclear, and therefore it is 1-summing. As Y^* is an $\mathcal{L}_\infty$-space, A^* is integral. Actually, A^* is nuclear because ℓ_1 is a dual space and has the Radon–Nikodym property. According to [3, Theorem VIII.7], A is nuclear. This yields $\sum_n\|T(x_n)\| < \infty$. ∎

PROPOSITION 4.2. *If Y is a Banach space isomorphic to a Hilbert space, then $\Pi_2(X,Y)=\mathcal{V}_2(X,Y)$ for every Banach space X.*

Proof. Let $T\in\mathcal{V}_2(X,Y)$ and $(x_n)\in\ell_2^w(X)$. By hypothesis, the operator $S:\ell_2\to Y$ defined by $S(e_n)=T(x_n)$ is 2-compact, and therefore its adjoint $S^*:y^*\in Y^*\mapsto(\langle T(x_n),y^*\rangle)\in\ell_2$ is quasi 2-nuclear (Corollary 3.4). According to [2, Theorem 4.19], S^* has a 2-summing adjoint because Y^* is isomorphic to a Hilbert space. In particular, S is 2-summing and this implies that $\sum_n\|T(x_n)\|^2<\infty$. So, we have proved that T is 2-summing. ∎

However, in general, $\Pi_p(X,Y)$ is strictly contained in $\mathcal{V}_p(X,Y)$ for all $p\in[1,\infty)$. The following relationships are obvious for all $p\ge 1$:

$$\Pi_p(\ell_{p'},X)\subset\Phi_p(\ell_p(X))\subset\mathcal{K}_p(\ell_{p'},X)=\mathcal{V}_p(\ell_{p'},X). \tag{4}$$

If $p>1$, the first inclusion is strict whenever X is not a subspace of a quotient of an L_p-space [15, Theorem 3.1]. So, only the case $p=1$ needs to be studied.

Let $1\le p<2$. Let $\mathcal{C}_p$ be the ideal of all operators mapping weakly p-summable sequences to unconditionally p-summable sequences. First of all, we will prove that $\Pi_p^d\circ\mathcal{C}_p\subset\mathcal{V}_p$ for every $p\ge 1$. So, let $T=T_2\circ T_1$, where T_1 belongs to $\mathcal{C}_p(X,Y)$ and $T_2\in\Pi_p^d(Y,Z)$. If (x_n) is a weakly p-summable sequence in X and $A=\{\sum_n\alpha_n x_n:(\alpha_n)\in B_{\ell_{p'}}\}$, notice that $T_1(A)$ is relatively compact in Y. Then $T_2(T_1(A))$ is relatively p-compact (Theorem 3.12).

Now we are going to show that the inclusion $\Pi_p\subset\mathcal{V}_p$ is, in general, strict for every $1\le p<2$. Denote by $I_{2,0}$ the identity map from ℓ_2 into c_0. According to [1, Lemma 6] the identity map from ℓ_2 onto ℓ_2 belongs to $\mathcal{C}_p$ for every $p < 2$. On the other hand, $(I_{2,0})^*$ is p-summing, so $I_{2,0} \in \Pi_p^d \circ \mathcal{C}_p \subset \mathcal{V}_p$ for all $p < 2$. Nevertheless, $I_{2,0}$ is not p-summing.

Finally, we have obtained the following result about the biadjoint of an operator $T \in \mathcal{V}_2$. Here, $\mathcal{I}_p$ denotes the Banach ideal of p-integral operators.

PROPOSITION 4.3. *Let X be a Banach space such that $I_{X^{**}} \in \mathcal{C}_2$. If $T \in \mathcal{V}_2(X,Y)$, then $T^{**} \in \mathcal{V}_2(X^{**},Y^{**})$.*

Proof. Given $T \in \mathcal{V}_2(X,Y)$, consider the linear map

$$U : (x_n) \in \ell_2^u(X) \mapsto \sum_n Tx_n \otimes e_n \in \mathcal{QN}_2(Y^*, \ell_2).$$

It is easy to prove that U has closed graph, and therefore it is continuous. Its adjoint maps $\mathcal{I}_2(\ell_2, Y^{***})$ into $\mathcal{I}_1(\ell_2, X^*)$. Put $V = U^*|_{\mathcal{N}_2(\ell_2,Y^*)}$. Since $\mathcal{N}_1(\ell_2, X^*)$ is isometric to a subspace of $\mathcal{I}_1(\ell_2, X^*)$ it follows easily that V maps $\mathcal{N}_2(\ell_2, Y^*)$ into $\mathcal{N}_1(\ell_2, X^*)$. We also denote by V the operator

$$\sum_n e_n^* \otimes y_n^* \in \mathcal{N}_2(\ell_2, Y^*) \mapsto \sum_n e_n^* \otimes T^* y_n^* \in \mathcal{N}_1(\ell_2, X^*).$$

Taking adjoints again we obtain the operator

$$(x_n^{**}) \in \mathcal{L}(X^*, \ell_2) \overset{V^*}{\mapsto} \sum_n T^{**} x_n^{**} \otimes e_n \in \Pi_2(Y^*, \ell_2). \tag{5}$$

As every 2-summing operator is 2-integral and the 2-summing norm coincides with the 2-integral norm, (5) can be written in the form

$$(x_n^{**}) \in \mathcal{L}(X^*, \ell_2) \overset{V^*}{\mapsto} \sum_n T^{**} x_n^{**} \otimes e_n \in \mathcal{I}_2(Y^*, \ell_2).$$

Now, as in the proof of (b)⇒(a) in Theoren 3.14, we can prove that V^* maps $\ell_2^u(X^{**})$ into $\mathcal{N}_2(Y^*, \ell_2)$. This shows that the operator $A : y^* \in Y^* \mapsto (\langle T^{**} x_n^{**}, y^* \rangle) \in \ell_2$ is 2-nuclear whenever (x_n^{**}) is unconditionally 2-summable in X^{**}, and therefore its adjoint $A^* : e_n \in \ell_2 \mapsto T^{**} x_n^{**} \in Y^{**}$ belongs to $\mathcal{N}^2$. So, A^* is 2-compact and this concludes the proof. ∎

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