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# On nested sequences of convex sets in Banach spaces

by

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**Abstract.** We study different aspects of the representation of weak*-compact convex sets of the bidual $X^{**}$ of a separable Banach space $X$ via a nested sequence of closed convex bounded sets of $X$.

**1. Introduction.** In this paper we solve several problems about nested intersections of convex closed bounded sets in Banach spaces.

We begin with a study of different aspects of the representation of weak*-compact convex sets of the bidual $X^{**}$ of a separable Banach space $X$ via a nested sequence of closed convex bounded sets of $X$. More precisely, let us say that a convex closed bounded subset $C \subset X^{**}$ is *representable* if it can be written as the intersection

$$C = \bigcap_{n\in\mathbb{N}} \overline{C_n}^{w^*}$$

for a nested sequence $(C_n)$ of bounded convex closed subsets of $X$. This topic was considered in [6, 7], where the problem of which weak*-closed convex sets of the bidual are representable was posed. In [5], Bernardes shows that when $X^*$ is separable, every weak*-compact convex subset of $X^{**}$ is representable. Here we will show that compact convex sets of $X^{**}$ are representable if and only if $X$ does not contain $\ell_1$, and also that there are spaces without copies of $\ell_1$ containing weak*-compact convex metrizable subsets of the bidual that are not representable.

In Section 3, we solve problem (2) in [7] by showing that when the sets are viewed as the distance types (in the sense of [6]) they define, i.e., as elements of $\mathbb{R}^X$, then every weak*-compact convex set $C \subset X^{**}$ is represented by a nested sequence $(C_n)$ of closed convex sets of $X$; which means that for all $x \in X$,

$$\operatorname{dist}(x, C) = \lim \operatorname{dist}(x, C_n).$$

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In Section 4 we present two examples: the first one solves Marino's question [13] about the possibility of enlarging nested sequences of convex sets to get better intersections; the second one solves Behrends' question about the validity for $\varepsilon = 0$ of the Helly–Bárány theorem [3].

## 2. Representation of convex sets in biduals

DEFINITION 2.1. A Banach space is said to enjoy the *Convex Representation Property*, for short CRP (resp. *Compact Convex Representation Property*, for short CCRP), if every weak*-compact (resp. compact) convex subset $C$ of $X^{**}$ can be represented as the intersection

$$C = \bigcap_{n\in\mathbb{N}} \overline{C_n}^{w^*}$$

for a nested sequence $(C_n)$ of bounded convex closed subsets of $X$.

PROPOSITION 2.2. *prop:2.2 A separable Banach space has CCRP if and only if it does not contain $\ell_1$.*

*Proof.* The necessity follows from the Odell–Rosenthal characterization [15] of separable Banach spaces containing $\ell_1$. Indeed, if $X$ contains $\ell_1$ then there is an element $\mu \in X^{**}$ which is not the weak*-limit of any sequence of elements of $X$. Hence, $\{\mu\} = \bigcap_{n\in\mathbb{N}} \overline{C_n}^{w^*}$ is impossible: taking elements $c_n \in C_n$ one would get $\emptyset \neq \bigcap_n \overline{\{c_k : k \geq n\}}^{w^*} \subset \bigcap_n \overline{C_n}^{w^*} = \{\mu\}$, which means that $\mu$ is the only weak*-cluster point of the sequence $(c_n)$ and thus $\mu = w^*\text{-}\lim c_n$.

As for the sufficiency, let $K$ be a compact convex subset of $X^{**}$. For every $n \in \mathbb{N}$, let $F_n = \{z_n^k : k \in I_n\}$ be a finite subset of $K$ for which $K \subset F_n + n^{-1}B_{X^{**}}$. There is no loss of generality in assuming that $F_n \subset F_{n+1}$. For each $z_n^k \in F_n$, let $(x_n^k(m))_m \subset X$ be a sequence in $X$ weak*-convergent to $z_n^k$. Set

$$C_n = \overline{\mathrm{conv}}\{x_n^k(m) : k \in I_n,\, m \geq n\} + n^{-1}B_X.$$

It is clear that $C_n$ is a nested sequence of closed convex subsets of $X$. Moreover,

$$K \subset F_n + n^{-1}B_{X^{**}} \subset \overline{\overline{\mathrm{conv}}\{x_n^k(m) : k \in I_n,\, m \geq n\} + n^{-1}B_X}^{w^*} = \overline{C_n}^{w^*},$$

and thus $K \subset \bigcap_n \overline{C_n}^{w^*}$.

Fix now $p \in \bigcap_n \overline{C_n}^{w^*}$. Since $p \in \overline{C_n}^{w^*}$, there is a finite convex combination $\sum_{i\in I_n} \theta_i z_n^i$ for which $\|p - \sum_{i\in I_n} \theta_i z_n^i\| \leq n^{-1}$. This implies that $p \in \overline{K} = K$ and thus $\bigcap_n \overline{C_n}^{w^*} \subset K$. ∎

This shows that Problem 1 in [7] has a negative answer. On the other hand, Bernardes obtains in [5] an affirmative answer when $X^*$ is separable, which is somehow the best that can be expected. Let us briefly review and extend Bernardes' result. Recall that a partially ordered set $\Gamma$ is called *filtering* when for any points $i, j \in \Gamma$ there is $k \in \Gamma$ such that $i \leq k$ and $j \leq k$. An indexed family of subsets $(C_\alpha)_{\alpha\in\Gamma}$ will be called filtering when it is filtering with respect to the natural (reverse) order $C_\beta \subset C_\alpha$ whenever $\alpha \leq \beta$. One has:

PROPOSITION 2.3. *If $C$ is a convex weak*-compact set in the bidual $X^{**}$ of a Banach space $X$ then there is a filtering family $(C_\alpha)_{\alpha\in\Gamma}$ of convex bounded and closed subsets of $X$ such that*

$$C = \bigcap_{\alpha\in\Gamma} \overline{C_\alpha}^{w^*}.$$

*Proof.* There is no loss of generality in assuming that $C \subset B_{X^{**}}$. Let $\Gamma$ be the partially ordered set of finite subsets of $B_{X^*}$. For each $\alpha \in \Gamma$ we denote by $|\alpha|$ the cardinality of the set $\alpha$. Set now

$$C_\alpha = \{x \in X : \exists z \in C\ \forall y \in \alpha,\ |(z-x)(y)| \leq |\alpha|^{-1}\}.$$

This family $(C_\alpha)_{\alpha\in\Gamma}$ is filtering, as also is $(\overline{C_\alpha}^{w^*})_{\alpha\in\Gamma}$, which ensures that $\bigcap_{\alpha\in\Gamma} \overline{C_\alpha}^{w^*}$ is nonempty. Let us show the equality

$$C = \bigcap_{\alpha\in\Gamma} \overline{C_\alpha}^{w^*}.$$

• $C \subset \bigcap_{\alpha\in\Gamma} \overline{C_\alpha}^{w^*}$: Let $z \in C \subset B_{X^{**}}$; given $\alpha \in \Gamma$, by the Banach–Alaoglu theorem, there is $x \in B_X$ such that $|(z-x)(y)| < |\alpha|^{-1}$ for all $y \in \alpha$. Hence $x \in C_\alpha$, and thus $z \in \overline{C_\alpha}^{w^*}$.

• $\bigcap_{\alpha\in\Gamma} \overline{C_\alpha}^{w^*} \subset C$: Let $z \in \bigcap_{\alpha\in\Gamma} \overline{C_\alpha}^{w^*}$ and let $V_{\alpha,\varepsilon}$ be the weak*-neighborhood of 0 determined by $\alpha \in \Gamma$ and $\varepsilon > 0$, i.e., $V_{\alpha,\varepsilon} = \{p \in X^{**} : \forall y \in \alpha, |p(y)| \leq \varepsilon\}$. Pick $\beta \in \Gamma$ with $\alpha \leq \beta$ and $|\beta|^{-1} \leq \varepsilon$. Since $z \in \overline{C_\beta}^{w^*}$, there is $x \in C_\beta$ such that $|(z-x)(y)| \leq \varepsilon$ for all $y \in \alpha$; this moreover means that there is some $z' \in C$ such that $|(z'-x)(y)| \leq |\beta|^{-1} \leq \varepsilon$ for all $y \in \beta$. Putting all together one gets, for all $y \in \alpha$,

$$|(z-z')(y)| = |(z-x)(y) + (x-z')(y)| \leq 2\varepsilon,$$

and thus $z - z' \in V_{\alpha,2\varepsilon}$. Hence $z \in \overline{C}^{w^*} = C$. ∎

The size of $\Gamma$ can be reduced just by taking first a dense subset $Y \subset B_{X^*}$ and then fixing as $\Gamma$ a fundamental family of finite sets of $Y$, in the sense that every finite subset of $Y$ is contained in some element of $\Gamma$. This reduction modifies the proof as follows: starting from the first finite set $\alpha$—no longer in $\Gamma$—determining $V_{\alpha,\varepsilon}$ one must take a set $\beta \in \Gamma$ such that for each $y \in \alpha$ there is $y' \in \beta$ such that $\|y-y'\| \leq |\beta|^{-1} \leq \varepsilon$. Get $x$ and $z'$ as above. Finally, for $y \in \alpha$, one gets

$$|(z-z')(y)| = |(z-z')(y-y') + (z-z')(y')| \le \varepsilon + 2\varepsilon = 3\varepsilon.$$

A consequence of this simplification is that when $X^*$ is separable then $\Gamma$ reduces to $\mathbb{N}$ and thus one gets the main result in [5]:

COROLLARY 2.4 (Bernardes). *Every Banach space with separable dual has CRP.*

One therefore has:

$$X^* \text{ separable } \Rightarrow \text{ CRP } \Rightarrow \text{ CCRP } \Leftrightarrow \ell_1 \not\subseteq X.$$

This suggests two questions: 1) whether CCRP implies CRP and 2) whether CRP implies having separable dual. One has

PROPOSITION 2.5. *CCRP does not imply CRP.*

To prove this we are going to show that the James-Tree space—perhaps the simplest space not containing $\ell_1$ but having nonseparable dual—fails CRP. For information about $JT$, we refer to [10, Chapter VIII]. We begin with a preparatory lemma that can be considered as a complement to Kalton's [12, Lemma 5.1].

LEMMA 2.6. *Let $(C_n)_n$ be a nested sequence of bounded closed convex subsets of a Banach space $X$. If $\bigcap_n \overline{C_n}^{w^*}$ is weak*-metrizable then:*

(1) *Every $g \in \bigcap_n \overline{C_n}^{w^*}$ is the weak*-limit of a sequence $(c_n)$ with $c_n \in C_n$.*
(2) *Every sequence $(c_n)$ with $c_n \in C_n$ admits a weak*-convergent subsequence.*

*Proof.* (1) is clear: Let $(V_n)_n$ be a sequence of weak*-neighborhoods of $g$ such that $\{g\} = \bigcap_n V_n \cap \bigcap_n \overline{C_n}^{w^*}$. Picking $c_n \in C_n \cap V_n$ one gets $\{g\} = \overline{\{c_n\}}^{w^*}$.

To prove (2), let us consider the following equivalence relation on the set $\mathcal{P}_\infty(\mathbb{N})$ of infinite subsets of $\mathbb{N}$: $A \sim B$ if and only if $A$ and $B$ coincide except for a finite set. Moreover, $K$ will denote the set of all compact subsets of $\bigcap_n \overline{C_n}^{w^*}$. Given a sequence $(c_n)$ with $c_n \in C_n$ we define a map $w : \mathcal{P}_\infty(\mathbb{N})/\!\sim \to K$ by

$$w([A]) = \bigcap_k \overline{\{c_n : n \in A,\, n > k\}}^{w^*}.$$

The set $\mathcal{P}_\infty(\mathbb{N})/\!\sim$ admits a natural order: $[A] \le [B]$ if $A$ is eventually contained in $B$. This order has the property that for every decreasing sequence $([A_n])_n$ there is an element $[B]$ with $[B] \le [A_n]$ for all $n$. Since $\bigcap_n \overline{C_n}^{w^*}$ is metrizable, it follows [2, Sect. 2] that there is $M \in \mathcal{P}_\infty(\mathbb{N})$ on which $w$ is stationary, i.e., $w([C]) = w([M])$ for all infinite subsets $C \subset M$. This immediately implies that $w(\{c_n\}_{n\in M})$ has only one point, and thus $\{c_n\}_{n\in M}$ is weak*-convergent. ∎

Let us denote by $G$ the set of all branches of the dyadic tree $T$. For each $r \in G$, let $e_r$ denote the corresponding element of the basis of $\ell_2(G)$ considered as a subspace of $JT^{**}$. Let $\{e_{k,l} : k \in \mathbb{N}_0,\ 1 \le l \le 2^k\}$ denote the unit vector basis of $JT$. The action of $e_r$ on $x^* \in JT^*$ is given by

$$\langle x^*, e_r\rangle = \lim_{\text{along } r} \langle e_{k,l}, x^*\rangle.$$

For each $m \in \mathbb{N}$ we denote by $P_m$ the norm-one projection in $JT$ defined by $P_m e_{k,l} = e_{k,l}$ if $k \ge m$, and $P_m e_{k,l} = 0$ otherwise. For each $r \in G$ we consider $f_r \in JT^*$ given by setting $\langle e_{k,l}, f_r\rangle$ equal to 1 if $(k,l) \in r$, and to 0 otherwise. Observe that $\langle f_r, e_s\rangle = \delta_{r,s}$. Let $S = \{s_n : n \in \mathbb{N}\}$ denote a countable subset of $G$ such that the branches in $S$ include all the nodes of the tree $T$.

*Proof of Proposition 2.5: The James-Tree space fails CRP.* Let us show that the closed unit ball $B$ of $\ell_2(S)$ cannot be represented. Assume that we can write $B = \bigcap_{n\in\mathbb{N}} \overline{C_n}^{w^*}$. The set $B$ is weak*-metrizable, because it is the unit ball of a separable reflexive subspace. By Lemma 2.6, each vector in $B$ is the weak*-limit of a sequence $(x_n)$ with $x_n \in C_n$. For each $s \in S$ we select $x_n^s \in C_n$ so that $w^*\text{-}\lim x_n^s = e_s$. Note that $\lim_n \|(I - P_k)x_n^s\| = 0$ for every $k$ and $s$.

We take $t_1 \in S$, $t_1 \ne s_1$. Also we take $x_1 = x_{n_1}^{t_1}$ with $|\langle x_1, f_{t_1}\rangle - 1| < 2^{-1}$, and select $(k_1, l_1) \in t_1 \setminus s_1$ such that $\|P_{k_1} x_1\| < 2^{-1}$. Next we take $t_2 \in S$ with $(k_1, l_1) \in t_2$ and $t_2 \ne s_2$. Also we take $x_2 = x_{n_2}^{t_2}$ with $\|(I - P_{k_1})x_2\| < 2^{-2}$ and $|\langle x_2, f_{t_2}\rangle - 1| < 2^{-2}$, and select $(k_2, l_2) \in t_2 \setminus s_2$ with $k_2 > k_1$ such that $\|P_{k_2} x_2\| < 2^{-2}$. Proceeding in this way we obtain a sequence $(x_i)$ that is eventually contained in each $C_n$, and an ordered sequence of different nodes $(k_i, l_i)$ that determine a branch $r \in G \setminus S$. Since $JT$ is separable and contains no copies of $\ell_1$, the sequence $(x_i)$ has a subsequence that is weak*-convergent to some $x^{**} \in JT^{**}$ [9, First Theorem, p. 215]. Thus, $x^{**} \in \bigcap_{n\in\mathbb{N}} \overline{C_n}^{w^*}$, but $x^{**} \notin B$ since $\langle f_r, x^{**}\rangle = 1$. ∎

Proposition 2.2 thus characterizes the CCRP, while Proposition 2.5 shows that even when compact convex sets are representable, arbitrary weak*-metrizable convex bounded closed sets do not have to be. The question of which convex sets are representable thus arises. Bigger than compact spaces are the so called small sets [4, 8, 1], but it was shown in [4] that a closed bounded convex small set is compact.

**3. Representation of convex sets in the hyperspace.** The theory of types in Banach spaces represents the elements of a Banach space $g \in X$ as functions $\tau_g(x) = \|x - g\|$. These are the *elementary types*, and the *types* are the closure of the set of elementary types in $\mathbb{R}^X$. It can be shown that *bidual types*, i.e., functions having the form $\tau_g(x) = \|x - g\|$ for $g \in X^{**}$, are also types [11]. In close parallelism, the theory of distance types was developed in [6]: in it, the elements to be represented are the closed bounded convex subsets $C$ of $X$ via the function $d_C(x) = \mathrm{dist}(x, C)$. These are the *elementary distance types*. The $\emptyset$-*distance types* are the functions of the form $d(x) = \lim d_{C_n}(x)$ where $(C_n)$ is a nested sequence of closed bounded convex subsets of $X$ with empty intersection. In [6, Thm. 4.1] it was shown that in every nonreflexive separable Banach space there exist $\emptyset$-distance types that are not types. It was also shown [6, Thm. 5.1] that bidual types on separable Banach spaces coincide with $\emptyset$-distance types defined by "flat" (in the sense of Milman and Milman [14]) nested sequences of bounded convex closed sets $(C_n)$. In [7, Thm. 1] it is shown that given a nested sequence $(C_n)$ of bounded convex closed sets in a separable space $X$, one always has

$$\mathrm{dist}\Big(x, \bigcap \overline{C_n}^{w^*}\Big) = \lim \mathrm{dist}(x, C_n).$$

Bernardes shows in [5, Thm. 1] that this happens in all Banach spaces.

All this suggests the problem [7, Problem 2] whether the analogue of Farmaki's result (bidual types are types) also holds for distance types; i.e., if given a weak*-compact convex subset $C$ of $X^{**}$, the *bidual distance type* it defines, $d_C(x) = \mathrm{dist}(x, C)$ on $X$, is a $\emptyset$-distance type. Let us give an affirmative answer.

Proposition 3.1. *Let $C$ be a weak*-compact convex subset of the bidual $X^{**}$ of a separable space $X$ such that $C \cap X = \emptyset$. There is a nested sequence $(C_n)$ of closed convex sets in $X$ such that $C \subset \bigcap_n \overline{C_n}^{w^*}$ and for all $x \in X$,*

$$\mathrm{dist}(x, C) = \lim \mathrm{dist}(x, C_n).$$

*Proof.* Let $(x_n)$ be a dense sequence in $X$. Since $C$ is bounded, it is contained in the ball $\gamma B_{X^{**}}$ for some $\gamma > 0$. We proceed inductively: Pick $x_1$, let $\alpha_1 = \mathrm{dist}(x_1, C)$, then choose an increasing sequence $(\alpha_n^1)$ convergent to $\alpha_1$. Pick functionals $\varphi_n^1 \in B_{X^*}$ that strictly separate $C$ and $x_1 + (\alpha_n^1) B_{X^{**}}$, say

$$\inf_{z \in C} z(\varphi_n^1) > \|x_1\| + \alpha_n^1 + 2\varepsilon_n^1.$$

Set $C_{n,1} = \{x \in X : \exists z \in C,\ |(z - x)(\varphi_n^1)| \le \varepsilon_n^1\} \cap \gamma B_{X^{**}}$. The sequence of convex sets $C_{n,1}$ is nested and every point $z \in C$ belongs to the weak*-closure of some set $\{x \in X : |(z - x)(\varphi_n^1)| \le n^{-1}\}$, which is in turn contained in $C_{n,1}$. Thus, $C \subset \bigcap_n \overline{C_{n,1}}^{w^*}$. Moreover, $x_1 + (\alpha_1) B_{X^{**}} \cap \bigcap_n \overline{C_{n,1}}^{w^*} = \emptyset$ because otherwise there whould be $c_n \in C_{n,1}$ with $(x_1 + \alpha_1 b - c_n)(\varphi_n^1) < \varepsilon_n^1$; since there must be $z_n \in C$ with $|(z_n - c_n)(\varphi_n^1)| \le \varepsilon_n^1$, pick a weak*-accumulation point $z \in C$ of $(z_n)$ to conclude that $(x_1 + \alpha_1 b - z)(\varphi_n^1) = (x_1 + \alpha_1 b - c_n + c_n - z)(\varphi_n^1) \le 2\varepsilon_n^1$, which immediately yields

$$z(\varphi_n^1) = (x_1 + \alpha_1 b)(\varphi_n^1) - (x_1 + \alpha_1 b - z)(\varphi_n^1) \le \|x_1\| + \alpha_n^1 + 2\varepsilon_n^1$$

in contradiction with the separation above.

Thus, by [7, Thm. 1] we get

$$\operatorname{dist}(x_1, C) = \operatorname{dist}\Big(x_1, \bigcap \overline{C_{n,1}}^{w^*}\Big) = \lim \operatorname{dist}(x_1, C_{n,1}).$$

We pass to $x_2$. Everything goes as before except that all the action is inside $\bigcap \overline{C_{n,1}}^{w^*}$. Precisely, once $\alpha_2, \alpha_n^2, \varphi_n^2, \varepsilon_n^2$ have been fixed by the same procedure as above, set

$$C_{n,2} = \Big\{x \in X : \exists z \in C,\ \max_{i=1,2} |(z - x)(\varphi_n^i)| \le \varepsilon_n^i\Big\} \cap \gamma B_{X^{**}}$$

to conclude that $C \subset \bigcap_n \overline{C_{n,2}}^{w^*} \subset \bigcap_n \overline{C_{n,1}}^{w^*}$ and

$$\operatorname{dist}(x_i, C) = \operatorname{dist}\Big(x_i, \bigcap \overline{C_{n,2}}^{w^*}\Big) = \lim \operatorname{dist}(x_i, C_{n,2})$$

for $i = 1, 2$. Proceed inductively. Since $C_{n,k+1} \subset C_{n,k}$, we can diagonalize the final sequence of sequences to get a sequence $(C_{k,k})$ which satisfies $C \subset \bigcap \overline{C_{k,k}}^{w^*}$, and moreover, for all $n$,

$$\operatorname{dist}(x_n, C) = \operatorname{dist}\Big(x_i, \bigcap \overline{C_{k,k}}^{w^*}\Big) = \lim \operatorname{dist}(x_n, C_{k,k}).$$

By continuity, the equality remains valid for all $x \in X$. ■

In the clasical case, as Farmaki remarks in [11], it is not obvious that *fourth-dual* types, i.e., maps of the form $\tau_g(x) = \|x + g\|$ for $g \in X^4$ on separable spaces $X$, are necessarily types. One may thus ask: Let $X$ be a separable Banach space and let $C \subset X^{2k}$ be a bounded weak*-closed convex. Must there be a sequence $(C_n)$ of bounded convex closed subsets of $X$ such that for every $x \in X$ one has $\operatorname{dist}(x, C) = \lim \operatorname{dist}(x, C_n)$?

## 4. Further properties of nested sequences

**4.1. Enlarging sets for better intersection: Marino's problem.** Let $A$ be a closed set. For $\varepsilon > 0$ we set

$$A^\varepsilon = \{x \in X : \operatorname{dist}(x, A) \le \varepsilon\}.$$

An extremely nice result of Marino [13] establishes that given any family $(G_\gamma)$ of convex sets with nonempty intersection, $\bigcap_\gamma G_\gamma^\varepsilon$ is either bounded for every $\varepsilon > 0$, or unbounded for every $\varepsilon > 0$. A question left open in [7, p. 583] is whether it is possible to have $\bigcap A_n = \emptyset$, some intersections $\bigcap A_n^\varepsilon$

nonempty and bounded, and others unbounded. The next example shows that it can be so:

EXAMPLE 2. Consider in $\ell_1$ the sequence

$$A_{2k} = \left\{x \in \ell_1 : x_{k+1} \le -\frac{2^{k+1}-1}{2^{k+1}}\right\} \quad \text{and} \quad A_{2k-1} = \{x \in \ell_1 : x_k \ge 1\}.$$

Then $\bigcap A_n = \emptyset = \bigcap A_n^\varepsilon = \emptyset$ for all $\varepsilon < 1$, while

$$\bigcap A_n^1 = \{x \in \ell_1 : \forall k,\ 0 \le x_k \le 1/2^k\},$$

and $\bigcap A_n^{1+\varepsilon}$ is unbounded for all $\varepsilon > 0$ since all $x \in \ell_1$ with $-\varepsilon \le x_i \le 0$ for every $i$ belong to that set.

The choice of $\ell_1$ for the example is not accidental: during the proof of [7, Prop. 9] it is shown that in reflexive spaces, $\bigcap A_n = \emptyset$ implies $\bigcap A_n^\varepsilon = \emptyset$ for all $\varepsilon > 0$. Marino's theorem in combination with [7, Prop. 9] shows that in a nonreflexive space, if $\alpha = \inf\{\varepsilon > 0 : \bigcap A_n^\varepsilon \ne \emptyset\}$ then $\bigcap A_n^\varepsilon$ is either bounded for all $\varepsilon > \alpha$, or unbounded for all $\varepsilon > \alpha$.

Let us show now that Marino's theorem remains "almost" valid for nested sequences with empty intersection in a finite-dimensional space. In this case, the boundedness of some $A_n$ immediately implies, by compactness, that $\bigcap_{n\in\mathbb{N}} A_n \ne \emptyset$. Assume thus that one has a nested sequence of unbounded convex sets. Let $T_k = \{x \in X : k \le \|x\| \le k+1\}$. One has:

LEMMA 4.1. *Let $(A_n)$ be a sequence of unbounded connected sets in a finite-dimensional space $X$. Then either $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$, or for all but finitely many $k \in \mathbb{N}$ and every $\varepsilon > 0$ there is an infinite subset $N_k \subset \mathbb{N}$ such that $T_k \cap \bigcap_{n\in N_k} A_n^\varepsilon \ne \emptyset$.*

*Proof.* If for every $k \in \mathbb{N}$ the ball $kB$ of radius $k$ does not intersect $\bigcap_{n\in\mathbb{N}} A_n$ then $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$. Otherwise, let $x_{n,k} \in A_n \cap kB$. Since $A_n$ is unbounded, there is a point $y_{n,k+1}$ with $\|y_{n,k+1}\| > k+1$. Since $A_n$ is connected, there is some $x_{n,k+1}$ in $A_n$ with $k \le \|x_{n,k+1}\| \le k+1$, and thus in $A_n \cap T_k$. The sequence $(x_{n,k+1})_n$ lies in the compact set $T_k$ and thus for some infinite subset $N_k \subset \mathbb{N}$ the subsequence $(x_{n,k+1})_{n\in N_k}$ is convergent to some point $x_{k+1} \in T_k$. Thus, $x_{k+1} + \varepsilon B$ intersects the sets $\{A_n : n \in N_k\}$ and hence $\bigcap_{n\in N_k} A_n^\varepsilon \cap T_k \ne \emptyset$. ∎

Thus we get:

PROPOSITION 4.2. *Let $(A_n)$ be a nested sequence of unbounded connected sets in a finite-dimensional space $X$. Then either $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$, or $\bigcap_{n\in\mathbb{N}} A_n^\varepsilon$ is unbounded for every $\varepsilon > 0$.*

The assertion obviously fails for disconnected sets and also fails in infinite-dimensional spaces:

EXAMPLE 1. In $\ell_2$ take $A_n = \{x \in \ell_2 : \forall k > n,\ 0 \le x_k \le 1, \text{ and } \forall k \le n, x_k = 0\}$. This is a nested sequence of unbounded convex closed sets such that $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$, while for all $\varepsilon > 0$ the set $\bigcap_{n\in\mathbb{N}} A_n^\varepsilon$ is bounded: indeed, if $y \in A_n^\varepsilon$ for all $n$ then there is $x_n \in A_n$ for which $\|y - x_n\| \le \varepsilon$; thus $\sum_{i=1}^n |y_i|^2 \le \varepsilon^2$ for all $n$, so $\|y\| \le \varepsilon^2$.

**4.2. On the Helly–Bárány theorem.** In one of the main theorems in [3], Behrends establishes a Helly–Bárány theorem for separable Banach spaces [3, Thm. 5.5]: *Let $X$ be a separable Banach space and $\mathcal{C}_n$ a family of nonvoid, closed and convex subsets of the unit ball $B$ for every $n$. Suppose that there is a positive $\varepsilon_0 \le 1$ such that $\bigcap_{C\in\mathcal{C}_n} C + \varepsilon B = \emptyset$ for every $n$ and every $0 < \varepsilon < \varepsilon_0$. Then there are $C_n \in \mathcal{C}_n$ such that $\bigcap_n C_n + \varepsilon B = \emptyset$.* Behrends asks [3, Remark 2, p. 248] whether one can put $\varepsilon = \varepsilon_0$ in this theorem. The following example shows that the answer is no:

EXAMPLE. In $c_0$, the family $\mathcal{C}_n$ contains two convex sets:

$$a_n^+ = \left\{x \in c_0 : \forall i \in \mathbb{N},\ |x_i| \le \frac{1}{2}\left(1 + \frac{1}{i}\right) \text{ and } |x_n| = \frac{1}{2}\left(1 + \frac{1}{n}\right)\right\}$$

and

$$a_n^- = \left\{x \in c_0 : \forall i \in \mathbb{N},\ |x_i| \le \frac{1}{2}\left(1 + \frac{1}{i}\right) \text{ and } |x_n| = -\frac{1}{2}\left(1 + \frac{1}{n}\right)\right\}.$$

One has $a_n^+ \cap a_n^- = \emptyset$ for all $n \in \mathbb{N}$. But for every $z \in \{-,+\}^{\mathbb{N}}$ the choice $a_n^{z(n)} \in \mathcal{C}_n$ has $x \in \bigcap_n a_n^{z(n)} \ne \emptyset$ for $x_i = z(i)\frac{1}{2i}$.

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