Distances between Hilbertian operator spaces

by

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Abstract. We compute the completely bounded Banach–Mazur distance between different finite-dimensional homogeneous Hilbertian operator spaces.

Introduction. The operator space analogue of the Banach–Mazur distance, d_{cb}, has been developed and used effectively by Pisier [1, 4, 5] and Zhang [7] in the study of operator spaces. In this paper we make some further contributions to this topic. We consider the n-dimensional Hilbertian row operator space R_n and measure its distance to the interpolated space, $E_\theta := (E, \overline{E^*})_\theta$, where E is also an n-dimensional homogeneous Hilbertian operator space; we also measure the distance from $R_\theta := (R_n)_\theta := (R_n, C_n)_\theta$ to E.

Our methods are basic and do not require complicated general estimates. It may be the case that some of these results are known to experts or are part of the folklore of the subject but we have been unable to find them in the literature. In a further article the second author has developed an approach that applies to the interpolated spaces between any two homogeneous Hilbertian operator spaces [6]. We refer to [1, 4, 5] for background material on operator spaces and completely bounded operators.

Banach–Mazur distance. We measure the distance between the operator spaces E and F using the (completely bounded version of the) Banach–Mazur distance.

DEFINITION 1. If E and F are operator spaces then

$$d_{\mathrm{cb}}(E, F) = \inf\{\|u\|_{\mathrm{cb}}\|u^{-1}\|_{\mathrm{cb}} \mid u : E \to F \text{ a complete isomorphism}\}.$$

If E and F are operator spaces and $u : E \to F$ is a completely bounded

operator, we let E^* and F^* denote the operator duals of E and F, respectively, and let $u^* : F^* \to E^*$ denote the transpose mapping. Since $\|u\|_{\mathrm{cb}} = \|u^*\|_{\mathrm{cb}}$ (see [5, p. 40]) we see that

$$d_{\mathrm{cb}}(E, F) = d_{\mathrm{cb}}(E^*, F^*).$$

Restricted to all n-dimensional operator spaces the mapping $\log d_{\mathrm{cb}}(\cdot, \cdot)$ is a metric and so we have the triangle inequality

$$\log d_{\mathrm{cb}}(E, G) \leq \log d_{\mathrm{cb}}(E, F) + \log d_{\mathrm{cb}}(F, G)$$

for any triple of operator spaces E, F, G. We shall frequently use the following multiplicative form of this inequality:

$$d_{\mathrm{cb}}(E, G) \leq d_{\mathrm{cb}}(E, F) \cdot d_{\mathrm{cb}}(F, G).$$

DEFINITION 2.

(a) An operator space is *Hilbertian* if, as a Banach space, it is isometric to a Hilbert space.
(b) ([5, p. 172]) An operator space E is *homogenous* if for all linear mappings $u : E \to E$ we have $\|u\| = \|u\|_{\mathrm{cb}}$.

In this article we confine ourselves to finite-dimensional homogeneous Hilbertian complex operator spaces. Examples include R_n, the n-dimensional row Hilbertian operator space, and its operator dual C_n, the n-dimensional column Hilbertian operator space; $R_n \cap C_n$ and its operator dual $R_n + C_n$; and $\min(\ell_2^n)$ and its operator dual $\max(\ell_2^n)$. We refer to [4, 5] for details.

If E and F are two homogeneous Hilbertian operator spaces of the same finite dimension then all isometries from E to F have the same completely bounded norm [5, p. 219], and we let $\|E \to F\|_{\mathrm{cb}}$ denote this norm.

We now state the following important result of Zhang [7].

PROPOSITION 1. *If E and F are two homogeneous Hilbertian operator spaces of the same finite dimension then*

$$d_{\mathrm{cb}}(E, F) = \|E \to F\|_{\mathrm{cb}} \cdot \|F \to E\|_{\mathrm{cb}}.$$

The following result shows that the combined distance from R_n and C_n to any n-dimensional homogeneous Hilbertian operator space is fixed.

PROPOSITION 2. *If E is an n-dimensional homogeneous Hilbertian operator space then*

$$d_{\mathrm{cb}}(R_n, C_n) = d_{\mathrm{cb}}(R_n, E) \cdot d_{\mathrm{cb}}(E, C_n) = n.$$

Proof. By [5, Proposition 9.2.1] a unitary operator from a Hilbertian homogeneous operator space to itself is a complete contraction. It suffices to apply this result, the averaging process used in [5, Proposition 10.1], and [4, Corollary 9.8] to complete the proof.

Let $(e_{1i})_{i=1}^n$ and $(e_{i1})_{i=1}^n$ denote, respectively, the standard basis for R_n and C_n in $\mathcal{B}(\ell_2^n)$. We denote by $R_n \cap C_n$ the operator subspace of $\mathcal{B}(\ell_2^n) \oplus_\infty \mathcal{B}(\ell_2^n)$ spanned by $(e_{1i} \oplus e_{i1})_{i=1}^n$. This is an n-dimensional homogeneous Hilbertian operator space. Its operator dual, $R_n + C_n$, is also an n-dimensional homogeneous Hilbertian operator space [5, p. 184].

We let $\overline{E}$ denote the complex conjugate of the operator space E (see [5, p. 63]). The operations of taking complex conjugates and dual commute, that is, $\overline{E^*} = \overline{E}^*$. We have

$$R_n = \overline{R_n} \quad \text{and} \quad C_n = \overline{C_n}$$

where we identify operator spaces which are completely isometric (see [5, pp. 63 and 65]). Hence

$$\overline{(R_n \cap C_n)^*} = \overline{R_n + C_n} = \overline{R_n} + \overline{C_n} = R_n + C_n \quad \text{and} \quad \overline{(R_n + C_n)^*} = R_n \cap C_n.$$

We let $\min(\ell_2^n)$ and $\max(\ell_2^n)$ denote, respectively, the n-dimensional homogeneous Hilbertian operator space with its minimal, respectively maximal, operator space structure (see [5, p. 71]). We have

$$(\min(E))^* = \max(E^*)$$

for any operator space E.

By [4, p. 28],

$$\overline{\min(\ell_2^n)^*} = \max(\overline{(\ell_2^n)^*}) = \max(\ell_2^n).$$

Operator spaces E_0 and E_1 are called *compatible* if they can be continuously injected into the same topological vector space. This allows one to define a continuum of operator spaces, $(E_0, E_1)_\theta$, $\theta \in [0,1]$, that interpolate between E_0 and E_1. The following interpolation estimate will be most useful [5, p. 57].

PROPOSITION 3. *Let (E_0, E_1) and (F_0, F_1) be two compatible couples of operator spaces. If a linear operator $T : E_0 \cap E_1 \to F_0 \cap F_1$ extends to completely bounded operators $T_0 : E_0 \to F_0$ and $T_1 : E_1 \to F_1$, then it extends to a completely bounded operator $T_\theta : (E_0, E_1)_\theta \to (F_0, F_1)_\theta$ and*

$$(1) \qquad \|T_\theta\|_{\mathrm{cb}(E_\theta, F_\theta)} \leq \|T_0\|_{\mathrm{cb}(E_0, F_0)}^{1-\theta} \|T_1\|_{\mathrm{cb}(E_1, F_1)}^\theta.$$

We rephrase Proposition 3 in a more suitable form for our purposes (this is essentially reiteration as given in [4, Proposition 2.1, p. 22]).

LEMMA 1. *Let (E_0, E_1) and (F_0, F_1) be two compatible couples of operator spaces. If a linear operator $T : E_0 \cap E_1 \to F_0 \cap F_1$ extends to completely bounded operators $T_0 : E_0 \to F_0$ and $T_1 : E_1 \to F_1$ and $0 \leq \alpha < \beta \leq 1$, then for $\alpha \leq \theta \leq \beta$ we have*

$$(2) \qquad \|T_\theta\|_{cb(E_\theta, F_\theta)} \leq \{\|T_\alpha\|_{cb(E_0, F_0)}\}^{\frac{\beta - \theta}{\beta - \alpha}} \cdot \{\|T_\beta\|_{cb(E_1, F_1)}\}^{\frac{\theta - \alpha}{\beta - \alpha}}.$$

Proof. For $0 \leq \omega \leq 1$ we have, by (1),

$$\|T_{(1-\omega)\alpha+\omega\beta}\|_{\mathrm{cb}} \leq \|T_\alpha\|_{\mathrm{cb}}^{1-\omega} \cdot \|T_\beta\|_{\mathrm{cb}}^{\omega}.$$

If we let $\theta = (1-\omega)\alpha + \omega\beta$ then $\omega = \frac{\theta-\alpha}{\beta-\alpha}$ and $1-\omega = \frac{\beta-\theta}{\beta-\alpha}$. Making these substitutions completes the proof.

The spaces E and $\overline{E^*}$ are compatible for any operator space E and, if $E_\theta := (E, \overline{E^*})_\theta$, then, since $\overline{E^*} = \overline{E}^*$, we have

$$\overline{E_\theta^*} = E_{1-\theta}.$$

Any two Hilbertian operator are compatible and the following general result is due to Pisier [4, Corollary 2.4].

PROPOSITION 4. *If E is an n-dimensional Hilbertian operator space and* OH_n *is Pisier's n-dimensional self-dual Hilbertian operator space then*

$$\mathrm{OH}_n = (E, \overline{E^*})_{1/2}.$$

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PROPOSITION 5. *If $0 \leq \theta \leq 1$, E is an n-dimensional homogeneous Hilbertian operator space and $E_\theta = (E, \overline{E^*})_\theta$, then*

$$d_{\mathrm{cb}}(E_\theta, R_n) = d_{\mathrm{cb}}(E, R_n)^{1-\theta} \cdot d_{\mathrm{cb}}(E, C_n)^{\theta}.$$

Proof. We first interpolate between the identity inclusions $R_n \to E$ and $R_n \to \overline{E^*}$. We have

$$\|R_n \to E_\theta\|_{\mathrm{cb}} \leq \|R_n \to E\|_{\mathrm{cb}}^{1-\theta} \cdot \|R_n \to \overline{E^*}\|_{\mathrm{cb}}^{\theta}$$
$$\leq \|R_n \to E\|_{\mathrm{cb}}^{1-\theta} \cdot \|E \to C_n\|_{\mathrm{cb}}^{\theta}.$$

Next, interpolating between the mappings $E \to R_n$ and $\overline{E^*} \to R_n$ we obtain

$$\|E_\theta \to R_n\|_{\mathrm{cb}} \leq \|E \to R_n\|_{\mathrm{cb}}^{1-\theta} \cdot \|\overline{E^*} \to R_n\|_{\mathrm{cb}}^{\theta}$$
$$\leq \|E \to R_n\|_{\mathrm{cb}}^{1-\theta} \cdot \|C_n \to E\|_{\mathrm{cb}}^{\theta}.$$

Combining these two estimates leads to

$$d_{\mathrm{cb}}(E_\theta, R_n) = \|E_\theta \to R_n\|_{\mathrm{cb}} \cdot \|R_n \to E_\theta\|_{\mathrm{cb}}$$
$$\leq \|E \to R_n\|_{\mathrm{cb}}^{1-\theta} \cdot \|C_n \to E\|_{\mathrm{cb}}^{\theta} \cdot \|R_n \to E\|_{\mathrm{cb}}^{1-\theta} \cdot \|E \to C_n\|_{\mathrm{cb}}^{\theta}$$
$$= d_{\mathrm{cb}}(E, R_n)^{1-\theta} \cdot d_{\mathrm{cb}}(E, C_n)^{\theta}.$$

A similar argument with C_n replacing R_n implies

$$d_{\mathrm{cb}}(E_\theta, C_n) \leq d_{\mathrm{cb}}(E, C_n)^{1-\theta} \cdot d_{\mathrm{cb}}(E, R_n)^{\theta}.$$

Hence, by Proposition 2,

$$n = d_{\mathrm{cb}}(R_n, C_n) \leq d_{\mathrm{cb}}(R_n, E_\theta) \cdot d_{\mathrm{cb}}(E_\theta, C_n)$$
$$\leq d_{\mathrm{cb}}(E, R_n)^{1-\theta} \cdot d_{\mathrm{cb}}(E, C_n)^{\theta} \cdot d_{\mathrm{cb}}(E, C_n)^{1-\theta} \cdot d_{\mathrm{cb}}(E, R_n)^{\theta}$$
$$= d_{\mathrm{cb}}(E, R_n) \cdot d_{\mathrm{cb}}(E, C_n) = n.$$

This implies we have equality at all stages. Hence, for $0 \le \theta \le 1$,

$$d_{\mathrm{cb}}(R_n, E_\theta) = d_{\mathrm{cb}}(R_n, E)^{1-\theta} \cdot d_{\mathrm{cb}}(C_n, E)^{\theta}.$$

This completes the proof.

EXAMPLE 1. (a) Let $E = R_n$. Then $\overline{E^*} = \overline{R_n^*} = \overline{C_n} = C_n$ and

$$d_{\mathrm{cb}}(R_n, R_n) = 1 \quad \text{and} \quad d_{\mathrm{cb}}(R_n, C_n) = n.$$

Letting $R_\theta = (R_n, \overline{R_n^*}) = (R_n, \overline{C_n})_\theta = (R_n, C_n)_\theta$ we obtain

$$d_{\mathrm{cb}}(R_\theta, R_n) = d_{\mathrm{cb}}(R_n, R_n)^{1-\theta} \cdot d_{\mathrm{cb}}(R_n, C_n)^{\theta} = n^{\theta}, \qquad d_{\mathrm{cb}}(R_\theta, C_n) = n^{1-\theta}.$$

This result is due to Zhang [7] (see also Mathes [2, 3]).

(b) Let $E = R_n \cap C_n$. Then, as previously noted, $\overline{E^*} = R_n + C_n$ and $E_\theta = (R_n \cap C_n, R_n + C_n)_\theta$. By [5, Theorem 10.5] we have

$$d_{\mathrm{cb}}(R_n \cap C_n, R_n) = d_{\mathrm{cb}}(R_n \cap C_n, C_n) = \sqrt{n}.$$

Proposition 5 implies

$$d_{\mathrm{cb}}((R_n \cap C_n)_\theta, R_n) = d_{\mathrm{cb}}(R_n \cap C_n, R_n)^{1-\theta} \cdot d_{\mathrm{cb}}(R_n \cap C_n, C_n)^{\theta}$$
$$= n^{(1-\theta)/2} \cdot n^{\theta/2} = \sqrt{n}.$$

In proving this result we have also shown

$$d_{\mathrm{cb}}((R_n + C_n)_\theta, R_n) = \sqrt{n}.$$

(c) Let $M = \min(\ell_2^n)$. By [5, Theorem 10.5] we have the same estimates as in part (b) and hence letting $M_\theta := (\min(\ell_2^n), \max(\ell_2^n))_\theta$ we obtain

$$d_{\mathrm{cb}}(M_\theta, R_n) = d_{\mathrm{cb}}(M_\theta, C_n) = \sqrt{n}$$

for all $0 \le \theta \le 1$.

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NOTATION. For $0 \le \alpha, \beta \le 1$ we define $\|\alpha, \beta\|$ by letting (^{1})

$$n^{\|\alpha,\beta\|} = \|R_\alpha \to R_\beta\|_{\mathrm{cb}}.$$

Clearly $\|\alpha, \beta\| \ge 0$ for all α and β, $\|\alpha, \alpha\| = 0$ for all α and, for $0 \le \alpha, \beta \le 1$,

$$d_{\mathrm{cb}}(R_\alpha, R_\beta) = n^{\|\alpha,\beta\| + \|\beta,\alpha\|}.$$

If $0 < \alpha < 1$ then, by Example 1(a), $d_{\mathrm{cb}}(R_\alpha, C_n) = n^{1-\alpha}$ and hence

(3) $$\|\alpha, 1\| + \|1, \alpha\| = 1 - \alpha.$$

Using Lemma 1 we obtain the following estimates.

(^{1}) The notation $\|\alpha, \beta\|_{R_n}$ would be more accurate, but since we only use this notation in connection with row and column operator spaces, this, hopefully, will not cause any confusion.

LEMMA 2. *For $0 \le \alpha \le \theta \le \beta \le 1$:*

(a)
$$\|\alpha, \theta\| \le \frac{\|\alpha, \beta\|}{\beta - \alpha} (\theta - \alpha),$$

(b)
$$\|\theta, \alpha\| \le \frac{\|\beta, \alpha\|}{\beta - \alpha} (\theta - \alpha),$$

(c)
$$\|\beta, \theta\| \le \frac{\|\beta, \alpha\|}{\beta - \alpha} (\beta - \theta),$$

(d)
$$\|\theta, \beta\| \le \frac{\|\alpha, \beta\|}{\beta - \alpha} (\beta - \theta).$$

Proof. We prove the first two parts; the others are proved in the same way. Interpolating between $I : R_\alpha \to R_\alpha$ and $I : R_\alpha \to R_\beta$ and using Lemma 1 we obtain

$$\|\alpha, \theta\| \le \|\alpha, \alpha\| \cdot \frac{\beta - \theta}{\beta - \alpha} + \|\alpha, \beta\| \cdot \frac{\theta - \alpha}{\beta - \alpha}.$$

Since $\|\alpha, \alpha\| = 0$ this proves (a).

For (b) we again use Lemma 1 but now interpolate between $I : R_\alpha \to R_\alpha$ and $I : R_\beta \to R_\alpha$ to obtain

$$\|\theta, \alpha\| \le \|\alpha, \alpha\| \cdot \frac{\beta - \theta}{\beta - \alpha} + \|\beta, \alpha\| \cdot \frac{\theta - \alpha}{\beta - \alpha} = \frac{\|\beta, \alpha\|}{\beta - \alpha} \cdot (\theta - \alpha).$$

We shall also need the following lemma. It is a simple consequence of Proposition 3.

LEMMA 3. *For $0 \le \alpha \le 1$ we have*

$$\lim_{\beta \to \alpha} \|\alpha, \beta\| = \lim_{\beta \to \alpha} \|\beta, \alpha\| = 0.$$

Proof. Suppose $\beta \to \alpha^+$. All other cases are proved the same way. Fix $\alpha_0 > \alpha$ and suppose $\alpha < \beta < \alpha_0$ for all β. We interpolate, using Proposition 2, between the isometric identity mappings $R_\alpha \to R_\alpha$ and $R_\alpha \to R_{\alpha_0}$ to obtain

$$1 \le \|R_\alpha \to R_\beta\|_{\mathrm{cb}} \le \|R_\alpha \to R_{\alpha_0}\|_{\mathrm{cb}}^{(\beta - \alpha)/(\alpha_0 - \alpha)},$$

and this gives the required result.

PROPOSITION 6. *If $0 \le \alpha, \beta \le 1$ then $\|\alpha, \beta\| + \|\beta, \alpha\| = |\beta - \alpha|$ and*

$$n^{|\beta - \alpha|} = d_{\mathrm{cb}}(R_\alpha, R_\beta) = n^{\|\alpha, \beta\| + \|\beta, \alpha\|}.$$

Proof. We may suppose $\alpha < \beta$. By Example 1(a) and the triangle inequality,

$$n = d_{\mathrm{cb}}(R_n, C_n) = d_{\mathrm{cb}}(R_0, R_1) \le d_{\mathrm{cb}}(R_0, R_\alpha) \cdot d_{\mathrm{cb}}(R_\alpha, R_\beta) \cdot d_{\mathrm{cb}}(R_\beta, R_1)$$
$$= n^\alpha \cdot d_{\mathrm{cb}}(R_\alpha, R_\beta) \cdot n^{1 - \beta}$$

and

(4) $$n^{\beta-\alpha} \leq d_{bc}(R_\alpha, R_\beta) = n^{\|\alpha,\beta\|+\|\beta,\alpha\|}.$$

From Lemma 2(a)&(b) and (3),

$$\|\alpha,\beta\| + \|\beta,\alpha\| \leq \frac{\|\alpha,1\|}{1-\alpha}(\beta-\alpha) + \frac{\|1,\alpha\|}{1-\alpha}(\beta-\alpha)$$
$$= \frac{(\|\alpha,1\| + \|1,\alpha\|)}{1-\alpha}(\beta-\alpha) = \beta-\alpha.$$

Combining this with (4) completes the proof.

Our next result is a sharpening of Proposition 6. We use the following notation. For n a non-negative integer we let $\mathbb{D}_n = \{j2^{-n} \mid 0 \leq j \leq 2^n\}$ and let $\mathbb{D} = \bigcup_{n=0}^{\infty} \mathbb{D}_n$, the set of dyadic numbers in $[0,1]$. Note that $\mathbb{D}_n \subset \mathbb{D}_{n+1}$ for all n and $\mathbb{D}_0 = \{0,1\}$.

PROPOSITION 7. *For $0 \leq \alpha \leq \beta \leq 1$ we have*

(5) $$\|\alpha,\beta\| = \|\beta,\alpha\| = (\beta-\alpha)/2.$$

Proof. We may suppose $\alpha < \beta$. By Proposition 6, $\|\alpha,\beta\|+\|\beta,\alpha\| = \beta-\alpha$ for all α,β and it suffices to prove $\|\alpha,\beta\| = (\beta-\alpha)/2$. We first prove this by induction for all α,β in $\mathbb{D}$. The result for $\mathbb{D}_0$ is well known since

$$n^{\|0,1\|} = \|R_n \to C_n\|_{\mathrm{cb}} = \|C_n \to R_n\|_{\mathrm{cb}} = \sqrt{n}.$$

Suppose the result holds for $\mathbb{D}_n$. Let $\alpha = p2^{-n-1}$ and $\beta = q2^{-n-1}$, where p and q are positive integers, lie in $\mathbb{D}_{n+1}$. We consider several cases.

CASE 1: *both p and q are even.* Then $\alpha,\beta \in \mathbb{D}_n$, and our induction hypothesis implies that $\|\alpha,\beta\| = (\beta-\alpha)/2$.

CASE 2: *p is odd and q is even.* Let $\alpha' = \alpha-2^{-n-1} = (p-1)2^{-n-1}$. Then $p-1$ is even and α' and β both belong to $\mathbb{D}_n$. By our induction hypothesis

$$\|\alpha',\beta\| = (\beta-\alpha')/2 = (\beta-\alpha+2^{-n-1})/2.$$

Since $\alpha' < \alpha < \beta$, Lemma 2 implies

$$\|\alpha,\beta\| \leq \frac{\|\alpha',\beta\|}{\beta-\alpha'} \cdot (\beta-\alpha) = \frac{\|\alpha',\beta\|}{\beta-\alpha+2^{-n-1}} \cdot (\beta-\alpha) = \frac{\beta-\alpha}{2}$$

and, using Proposition 6, a similar estimate for $\|\beta,\alpha\|$ proves (5) in this case.

CASE 3: *p is even and q is odd.* We let $\beta' = \beta + 2^{-n-1}$. Then $\beta' = (q+1)2^{-n-1}$ and both α and β' lie in $\mathbb{D}_n$. By our induction hypothesis

$$\|\alpha,\beta'\| = (\beta'-\alpha)/2 = (\beta+2^{-n-1}-\alpha)/2.$$

Since $\alpha < \beta < \beta'$, Lemma 2 implies

$$\|\alpha,\beta\| \leq \frac{\|\alpha,\beta'\|}{\beta'-\alpha} \cdot (\beta-\alpha) = \frac{\|\alpha,\beta'\|}{\beta+2^{-n-1}-\alpha} \cdot (\beta-\alpha) = \frac{\beta-\alpha}{2},$$

and, as above, (5) follows also in this case.

CASE 4: *p and q are both odd.* We use α' as in Case 2. By Case 3 we have $\|\alpha', \beta\| = (\beta - \alpha')/2$. Now $\alpha' < \alpha < \beta$ and hence Lemma 2 implies

$$\|\alpha, \beta\| \leq \frac{\|\alpha', \beta\|}{\beta - \alpha'} \cdot (\beta - \alpha) = \frac{\beta - \alpha}{2},$$

and, as above, (5) follows also in this case.

We have established (5) whenever α and β are dyadic rationals in $[0, 1]$. Now suppose α and β are arbitrary elements in $[0, 1]$, $\alpha < \beta$. Since $\mathbb{D}$ is dense in $[0, 1]$ we can choose a decreasing sequence $(\alpha_m)_{m=1}^{\infty}$ in $\mathbb{D}$ and an increasing sequence $(\beta_m)_{m=1}^{\infty}$ in $\mathbb{D}$ such that $\lim_{n \to \infty} \alpha_m = \alpha$ and $\lim_{n \to \infty} \beta_m = \beta$. We may also suppose that $\alpha_1 < \beta_1$. Since

$$[\alpha, \beta] \subset [\alpha, \alpha_n] \cup [\alpha_n, \beta_n] \cup [\beta_n, \beta]$$

for all integers n, we have, by Lemma 3 and the triangle inequality,

$$\begin{aligned}
n^{\|\alpha, \beta\|} = \|R_\alpha \to R_\beta\|_{\mathrm{cb}} \\
\leq \|R_\alpha \to R_{\alpha_n}\|_{\mathrm{cb}} \cdot \|R_{\alpha_n} \to R_{\beta_n}\|_{\mathrm{cb}} \cdot \|R_{\beta_n} \to R_\beta\|_{\mathrm{cb}} \\
\leq \liminf_{n \to \infty} \|R_{\alpha_n} \to R_{\beta_n}\|_{\mathrm{cb}} = \lim_{n \to \infty} n^{(\beta_n - \alpha_n)/2} = n^{(\beta - \alpha)/2}.
\end{aligned}$$

Hence $\|\alpha, \beta\| \leq (\beta - \alpha)/2$. Similarly one can show $\|\beta, \alpha\| \leq (\beta - \alpha)/2$; but, since $\|\alpha, \beta\| + \|\beta, \alpha\| = \beta - \alpha$, we obtain

$$\|\alpha, \beta\| = \|\beta, \alpha\| = (\beta - \alpha)/2$$

as required.

Distances to R_θ

PROPOSITION 8. *If $0 \leq \theta \leq 1$ then*

$$(6) \qquad d_{\mathrm{cb}}(R_\theta, R_n \cap C_n) = n^{(1 + |2\theta - 1|)/4}.$$

Proof. By [5, p. 221], $\|R_n \cap C_n \to R_n\|_{\mathrm{cb}} = 1$ and $\|R_n \cap C_n \to C_n\|_{\mathrm{cb}} = 1$. Interpolating we have, for all θ,

$$(7) \qquad \|R_n \cap C_n \to R_\theta\|_{\mathrm{cb}} = 1.$$

By [5, p. 222],

$$(8) \qquad \|R_n \to R_n \cap C_n\|_{\mathrm{cb}} = n^{1/2},$$

and by [5, p. 220],

$$\|\mathrm{OH}_n \to R_n \cap C_n\|_{\mathrm{cb}} = n^{1/4}.$$

By Lemma 1,

$$(9) \qquad \|(R_n, \mathrm{OH}_n)_\theta \to R_n \cap C_n\|_{\mathrm{cb}} \leq n^{(1-\theta)/2} \cdot n^{\theta/4} = n^{1/2 - \theta/4}.$$

Since $R_{1/2} = \mathrm{OH}_n$ reiteration implies $(R_n, \mathrm{OH}_n)_\theta = R_{\theta/2}$. Hence (9) can be rewritten as

$$\|R_{\theta/2} \to R_n \cap C_n\|_{\mathrm{cb}} \leq n^{1/2 - \theta/4}$$

for $0 \leq \theta \leq 1$, or as

(10) $$\|R_\theta \to R_n \cap C_n\|_{\mathrm{cb}} \leq n^{(1-\theta)/2}$$

for $0 \leq \theta \leq 1/2$. By Proposition 6, $\|R_n \to R_\theta\|_{\mathrm{cb}} = n^{\theta/2}$. By the triangle inequality

$$n^{1/2} = \|R_n \to R_n \cap C_n\|_{\mathrm{cb}} \leq \|R_n \to R_\theta\|_{\mathrm{cb}} \cdot \|R_\theta \to R_n \cap C_n\|_{\mathrm{cb}}$$
$$= n^{\theta/2} \cdot \|R_\theta \to R_n \cap C_n\|_{\mathrm{cb}},$$

and hence

(11) $$n^{(1-\theta)/2} \leq \|R_\theta \to R_n \cap C_n\|_{\mathrm{cb}}.$$

Combining (10) and (11) we obtain, for $0 \leq \theta \leq 1/2$,

$$n^{(1-\theta)/2} = \|R_\theta \to R_n \cap C_n\|_{\mathrm{cb}},$$

and hence, using (7),

(12) $$d_{\mathrm{cb}}(R_\theta, R_n \cap C_n) = n^{(1-\theta)/2}.$$

Interchanging R_n and C_n in (12) we see that, for $0 \leq \lambda \leq 1/2$, we have

$$d_{\mathrm{cb}}(C_\lambda, R_n \cap C_n) = n^{(1-\lambda)/2}.$$

Since $R_{1-\lambda} = C_\lambda$ we obtain, on letting $1 - \lambda = \theta$,

(13) $$d_{\mathrm{cb}}(R_\theta, R_n \cap C_n) = d_{\mathrm{cb}}(C_\lambda, R_n \cap C_n) = n^{(1-\lambda)/2} = n^{\theta/2}$$

for $1/2 \leq \theta \leq 1$. Combining (12) and (13) yields (6).

COROLLARY 1. *If* $0 \leq \theta \leq 1$ *then*

(14) $$d_{\mathrm{cb}}(R_\theta, R_n + C_n) = n^{(1+|2\theta-1|)/4}.$$

Proof. By Proposition 8 we have

$$d_{\mathrm{cb}}(R_\theta, R_n + C_n) = d_{\mathrm{cb}}(\overline{R_\theta^*}, \overline{(R_n + C_n)^*}) = d_{\mathrm{cb}}(R_{1-\theta}, R_n \cap C_n)$$
$$= n^{(1+|2(1-\theta)-1|)/4} = n^{(1+|2\theta-1|)/4}.$$

PROPOSITION 9. *There exists a positive constant A such that for all n and θ we have*

(15) $$A\sqrt{n} \leq d_{\mathrm{cb}}(\min(\ell_2^n), R_\theta) \leq \sqrt{n},$$
(16) $$A\sqrt{n} \leq d_{\mathrm{cb}}(\max(\ell_2^n), R_\theta) \leq \sqrt{n}.$$

Proof. We prove the first result; the second one can be proved in the same way. By [5, 10.23],

$$\|\min(\ell_2^n) \to R_n\|_{\mathrm{cb}} = \|\min(\ell_2^n) \to C_n\|_{\mathrm{cb}} = \sqrt{n}.$$

By interpolation, for $0 \leq \theta \leq 1$,

$$\|\min(\ell_2^n) \to R_\theta\|_{\mathrm{cb}} \leq \sqrt{n}.$$

By the definition of minimal operator spaces any isometry into $\min(\ell_2^n)$ has cb-norm 1. Hence, the above implies

(17) $$d_{\mathrm{cb}}(\min(\ell_2^n), R_\theta) \leq \sqrt{n}.$$

Since $(\min(\ell_2^n))^* = \max(\ell_2^n)$ and $R_\theta^* = R_{1-\theta}$ we have

$$\|R_\theta \to \max(\ell_2^n)\|_{\mathrm{cb}} = \|(\max(\ell_2^n))^* \to R_\theta^*\|_{\mathrm{cb}} = \|\min(\ell_2^n) \to R_{1-\theta}\|_{\mathrm{cb}} \leq \sqrt{n},$$

and since $\|\max(\ell_2^n) \to R_\theta\|_{\mathrm{cb}} \leq 1$ we have

(18) $$d_{\mathrm{cb}}(\max(\ell_2^n), R_\theta) \leq \sqrt{n}.$$

By [5, Theorem 3.8] there exists a positive constant c such that, for every positive integer n,

(19) $$cn \leq d_{\mathrm{cb}}(\max(\ell_2^n), \min(\ell_2^n)) \leq n.$$

For $0 \leq \theta \leq 1$ let

$$\alpha_\theta = \min\left\{ \frac{d_{\mathrm{cb}}(R_\theta, \min(\ell_2^n))}{\sqrt{n}} \;\middle|\; n \in \mathbb{N} \right\}$$

and let

$$\alpha := \inf\{\alpha_\theta \mid 0 \leq \theta \leq 1\}.$$

By (17), $0 \leq \alpha \leq 1$. Suppose $\alpha = 0$. Choose θ' such that $\alpha_{\theta'} < c$ and then choose n_0 so that

(20) $$\frac{d_{\mathrm{cb}}(R_{\theta'}, \min(\ell_2^{n_0}))}{\sqrt{n_0}} < c.$$

For θ' and n_0 we have, by (18)–(20),

$$cn_0 \leq d_{\mathrm{cb}}(\max(\ell_2^{n_0}), \min(\ell_2^{n_0})) \leq d_{\mathrm{cb}}(\max(\ell_2^{n_0}), R_{\theta'}) \cdot d_{\mathrm{cb}}(R_{\theta'}, \min(\ell_2^{n_0}))$$
$$< \sqrt{n_0} \cdot (c\sqrt{n_0}) = cn_0.$$

This is impossible, so $\alpha > 0$. Hence for all $\theta \in [0, 1]$ and all positive n,

(21) $$d_{\mathrm{cb}}(R_\theta, \min(\ell_2^n)) \geq \alpha\sqrt{n}.$$

Combining (17) and (21) we obtain (15).

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