STUDIA MATHEMATICA 222 (1) (2014)

Unconditionality of orthogonal spline systems in L^p

by

Markus Passenbrunner (Linz)

Abstract. We prove that given any natural number k and any dense point sequence (t_n), the corresponding orthonormal spline system is an unconditional basis in reflexive L^p.

1. Introduction. In this work, we are concerned with orthonormal spline systems of arbitrary order k with arbitrary partitions. We let $(t_n)_{n=2}^{\infty}$ be a dense sequence of points in the open unit interval $(0,1)$ such that each point occurs at most k times. Moreover, define $t_0 := 0$ and $t_1 := 1$. Such point sequences are called *admissible*.

For $n \geq 2$, we define $\mathcal{S}_n^{(k)}$ to be the space of polynomial splines of order k with grid points $(t_j)_{j=0}^{n}$, where the points 0 and 1 both have multiplicity k. For each $n \geq 2$, the space $\mathcal{S}_{n-1}^{(k)}$ has codimension 1 in $\mathcal{S}_n^{(k)}$, and therefore there exists $f_n^{(k)} \in \mathcal{S}_n^{(k)}$ that is orthonormal to $\mathcal{S}_{n-1}^{(k)}$. Observe that $f_n^{(k)}$ is unique up to sign. In addition, let $(f_n^{(k)})_{n=-k+2}^{1}$ be the collection of orthonormal polynomials in $L^2[0,1]$ such that the degree of $f_n^{(k)}$ is $k+n-2$. The system of functions $(f_n^{(k)})_{n=-k+2}^{\infty}$ is called the *orthonormal spline system of order k corresponding to* $(t_n)_{n=0}^{\infty}$. We will frequently omit the parameter k and write f_n instead of $f_n^{(k)}$.

The purpose of this article is to prove the following

Theorem 1.1. *Let $k \in \mathbb{N}$ and $(t_n)_{n\geq 0}$ be an admissible sequence of knots in $[0,1]$. Then the corresponding general orthonormal spline system of order k is an unconditional basis in $L^p[0,1]$ for every $1 < p < \infty$.*

A celebrated result of A. Shadrin [12] states that the orthogonal projection operator onto $\mathcal{S}_n^{(k)}$ is bounded on $L^\infty[0,1]$ by a constant that depends only on k. As a consequence, $(f_n)_{n\geq -k+2}$ is a basis in $L^p[0,1]$, $1 \leq p < \infty$. There are various results on the unconditionality of spline systems restrict-

---

2010 *Mathematics Subject Classification*: 42C10, 46E30.
Key words and phrases: orthonormal spline system, unconditional basis, L^p.

DOI: 10.4064/sm222-1-5

ing either the spline order k or the partition $(t_n)_{n\geq 0}$. The first result in this direction, in [1], states that the classical Franklin system—the orthonormal spline system of order 2 corresponding to dyadic knots—is an unconditional basis in $L^p[0,1]$, $1<p<\infty$. This was extended in [3] to unconditionality of orthonormal spline systems of arbitrary order, but still with dyadic knots. Considerable effort has been made to weaken the restriction to dyadic knot sequences. In the series of papers [7, 9, 8] this restriction was removed step-by-step for general Franklin systems, with the final result that for each admissible point sequence $(t_n)_{n\geq 0}$ with parameter $k=2$, the associated general Franklin system forms an unconditional basis in $L^p[0,1]$, $1<p<\infty$. We combine the methods used in [9, 8] with some new inequalities from [11] to prove that orthonormal spline systems are unconditional in $L^p[0,1]$, $1<p<\infty$, for any spline order k and any admissible point sequence $(t_n)_{n\geq 0}$.

The organization of the present article is as follows. In Section 2, we give some preliminary results concerning polynomials and splines. Section 3 develops some estimates for the orthonormal spline functions f_n using the crucial notion of associating to each function f_n a characteristic interval J_n in a delicate way. Section 4 treats a central combinatorial result concerning the number of indices n such that a given grid interval J can be a characteristic interval of f_n. In Section 5 we prove a few technical lemmata used in the proof of Theorem 1.1, and Section 6 finally proves Theorem 1.1. We remark that the results and proofs in Sections 5 and 6 closely follow [8].

2. Preliminaries. Let k be a positive integer. The parameter k will always be used for the order of the underlying polynomials or splines. We use the notation $A(t)\sim B(t)$ to indicate the existence of two constants $c_1,c_2>0$ that depend only on k, such that $c_1B(t)\leq A(t)\leq c_2B(t)$ for all t, where t denotes all implicit and explicit dependences that the expressions A and B might have. If the constants c_1,c_2 depend on an additional parameter p, we write $A(t)\sim_p B(t)$. Correspondingly, we use the symbols $\lesssim$, $\gtrsim$, $\lesssim_p$, $\gtrsim_p$. For a subset E of the real line, we denote by $|E|$ its Lebesgue measure and by $\mathbb{1}_E$ its characteristic function.

First, we recall a few elementary properties of polynomials.

Proposition 2.1. *Let $0<\rho<1$. Let I be an interval and A be a subset of I with $|A|\geq\rho|I|$. Then, for every polynomial Q of order k on I,*

$$\max_{t\in I}|Q(t)|\lesssim_\rho\sup_{t\in A}|Q(t)|\quad\textit{and}\quad\int_I|Q(t)|\,dt\lesssim_\rho\int_A|Q(t)|\,dt.$$

Lemma 2.2. *Let V be an open interval and f be a function satisfying $\int_V|f(t)|\,dt\leq\lambda|V|$ for some $\lambda>0$. Then, denoting by T_Vf the orthogonal* *projection of* $f \cdot \mathbb{1}_V$ *onto the space of polynomials of order* k *on* V,

$$\|T_V f\|_{L^2(V)}^2 \lesssim \lambda^2 |V|. \tag{2.1}$$

Moreover,

$$\|T_V f\|_{L^p(V)} \lesssim \|f\|_{L^p(V)}, \qquad 1 \le p \le \infty. \tag{2.2}$$

Proof. Let l_j, $0 \le j \le k-1$, be the jth Legendre polynomial on $[-1,1]$ with the normalization $l_j(1) = 1$. In view of the integral identity

$$l_j(x) = \frac{1}{\pi} \int_0^\pi (x + \sqrt{x^2-1}\cos\varphi)^j \, d\varphi, \qquad x \in \mathbb{C} \setminus \{-1, 1\},$$

l_j is uniformly bounded by 1 on $[-1,1]$. We have the orthogonality relation

$$\int_{-1}^{1} l_i(x) l_j(x)\, dx = \frac{2}{2j+1} \delta(i,j), \qquad 0 \le i, j \le k-1, \tag{2.3}$$

where $\delta(\cdot,\cdot)$ denotes the Kronecker delta. Now let $\alpha := \inf V$ and $\beta := \sup V$. For

$$l_j^V(x) := 2^{1/2} |V|^{-1/2} l_j\Big(\frac{2x-\alpha-\beta}{\beta-\alpha}\Big), \qquad x \in [\alpha, \beta],$$

relation (2.3) still holds for the sequence $(l_j^V)_{j=0}^{k-1}$, that is,

$$\int_\alpha^\beta l_i^V(x) l_j^V(x)\, dx = \frac{2}{2j+1} \delta(i,j), \qquad 0 \le i, j \le k-1.$$

So, $T_V f$ can be represented in the form

$$T_V f = \sum_{j=0}^{k-1} \frac{2j+1}{2} \langle f, l_j^V \rangle l_j^V.$$

Thus we obtain

$$\begin{aligned} \|T_V f\|_{L^2(V)} &\le \sum_{j=0}^{k-1} \frac{2j+1}{2} |\langle f, l_j^V\rangle| \, \|l_j^V\|_{L^2(V)} = \sum_{j=0}^{k-1} \sqrt{\frac{2j+1}{2}} \, |\langle f, l_j^V \rangle| \\ &\le \|f\|_{L^1(V)} \sum_{j=0}^{k-1} \sqrt{\frac{2j+1}{2}} \, \|l_j^V\|_{L^\infty(V)} \lesssim \|f\|_{L^1(V)} |V|^{-1/2}. \end{aligned}$$

Now, (2.1) is a consequence of the assumption $\int_V |f(t)|\, dt \le \lambda |V|$. If we set $p' = p/(p-1)$, the second inequality (2.2) follows from

$$\|T_V f\|_{L^p(V)} \le \sum_{j=0}^{k-1} \frac{2j+1}{2} \|f\|_{L^p(V)} \|l_j^V\|_{L^{p'}(V)} \|l_j^V\|_{L^p(V)} \lesssim \|f\|_{L^p(V)},$$

since $\|l_j^V\|_{L^p(V)} \lesssim |V|^{1/p - 1/2}$ for $0 \le j \le k-1$ and $1 \le p \le \infty$. ∎

We now let

$$\mathcal{T} = (0 = \tau_1 = \cdots = \tau_k < \tau_{k+1} \le \cdots \le \tau_M < \tau_{M+1} = \cdots = \tau_{M+k} = 1) \tag{2.4}$$

be a partition of $[0,1]$ consisting of knots of multiplicity at most k, that is, $\tau_i < \tau_{i+k}$ for all $1 \le i \le M$. Let $\mathcal{S}_{\mathcal{T}}^{(k)}$ be the space of polynomial splines of order k with knots $\mathcal{T}$. The basis of L^∞-normalized B-spline functions in $\mathcal{S}_{\mathcal{T}}^{(k)}$ is denoted by $(N_{i,k})_{i=1}^M$ or for short $(N_i)_{i=1}^M$. Corresponding to this basis, there exists a biorthogonal basis of $\mathcal{S}_{\mathcal{T}}^{(k)}$, denoted by $(N_{i,k}^*)_{i=1}^M$ or $(N_i^*)_{i=1}^M$. Moreover, we write $\nu_i = \tau_{i+k} - \tau_i$.

We now recall a few important results on the B-splines N_i and their dual functions N_i^*.

Proposition 2.3. *Let $1 \le p \le \infty$ and $g = \sum_{j=1}^M a_j N_j$. Then*

$$|a_j| \lesssim |J_j|^{-1/p} \|g\|_{L^p(J_j)}, \qquad 1 \le j \le M, \tag{2.5}$$

where J_j is the subinterval $[\tau_i, \tau_{i+1}]$ of $[\tau_j, \tau_{j+k}]$ of maximal length. Additionally,

$$\|g\|_p \sim \Big(\sum_{j=1}^M |a_j|^p \nu_j\Big)^{1/p} = \|(a_j \nu_j^{1/p})_{j=1}^M\|_{\ell^p}. \tag{2.6}$$

Moreover, if $h = \sum_{j=1}^M b_j N_j^*$, then

$$\|h\|_p \lesssim \Big(\sum_{j=1}^M |a_j|^p \nu_j^{1-p}\Big)^{1/p} = \|(a_j \nu_j^{1/p-1})_{j=1}^M\|_{\ell^p}. \tag{2.7}$$

The two inequalites (2.5) and (2.6) are Lemmata 4.1 and 4.2 in [6, Chapter 5], respectively. Inequality (2.7) is a consequence of the celebrated result of Shadrin [12] that the orthogonal projection operator onto $\mathcal{S}_{\mathcal{T}}^{(k)}$ is bounded on L^∞ independently of $\mathcal{T}$. For a deduction of (2.7) from this result, see [4, Property P.7].

The next task is to estimate the inverse of the Gram matrix $(\langle N_{i,k}, N_{j,k}\rangle)_{i,j=1}^M$. Before we do that, we recall the concept of totally positive matrices: Let $Q_{m,n}$ be the set of strictly increasing sequences of m integers from the set $\{1,\dots,n\}$, and A be an $n \times n$-matrix. For $\alpha, \beta \in Q_{m,n}$, we denote by $A[\alpha;\beta]$ the submatrix of A consisting of the rows indexed by α and the columns indexed by β. Furthermore, we let α' (the complement of α) be the uniquely determined element of $Q_{n-m,n}$ that consists of all integers in $\{1,\dots,n\}$ not occurring in α. In addition, we use the notation $A(\alpha;\beta) := A[\alpha';\beta']$.

Definition 2.4. Let A be an $n \times n$-matrix. Then A is called *totally positive* if

$$\det A[\alpha;\beta] \ge 0 \qquad \text{for } \alpha, \beta \in Q_{m,n},\ 1 \le m \le n.$$

The cofactor formula $b_{ij} = (-1)^{i+j} \det A(j;i)/\det A$ for the inverse $B = (b_{ij})_{i,j=1}^M$ of the matrix A leads to

PROPOSITION 2.5. *The inverse $B = (b_{ij})$ of a totally positive matrix $A = (a_{ij})$ has the checkerboard property:*

$$(-1)^{i+j} b_{ij} \geq 0 \quad \textit{for all } i, j.$$

THEOREM 2.6 ([5]). *Let $k \in \mathbb{N}$ and $\mathcal{T}$ be an arbitrary partition of $[0,1]$ as in (2.4). Then the Gram matrix $A = (\langle N_{i,k}, N_{j,k}\rangle)_{i,j=1}^M$ of the B-spline functions is totally positive.*

This theorem is a consequence of the so called basic composition formula [10, Chapter 1, equation (2.5)] and the fact that the kernel $N_{i,k}(x)$, depending on the variables i and x, is totally positive [10, Chapter 10, Theorem 4.1]. As a consequence, the inverse $B = (b_{ij})_{i,j=1}^M$ of A has the checkerboard property by Proposition 2.5.

THEOREM 2.7 ([11]). *Let $k \in \mathbb{N}$, let $\mathcal{T}$ be the partition defined as in (2.4) and $(b_{ij})_{i,j=1}^M$ be the inverse of the Gram matrix $(\langle N_{i,k}, N_{j,k}\rangle)_{i,j=1}^M$ of the B-spline functions $N_{i,k}$ of order k corresponding to $\mathcal{T}$. Then*

$$|b_{ij}| \leq C \frac{\gamma^{|i-j|}}{\tau_{\max(i,j)+k} - \tau_{\min(i,j)}}, \qquad 1 \leq i, j \leq M,$$

where the constants $C > 0$ and $0 < \gamma < 1$ depend only on k.

Let $f \in L^p[0,1]$ for some $1 \leq p < \infty$. Since the orthonormal spline system $(f_n)_{n \geq -k+2}$ is a basis in $L^p[0,1]$, we can write $f = \sum_{n=-k+2}^{\infty} a_n f_n$. Based on this expansion, we define the *square function* $Sf := \big(\sum_{n=-k+2}^{\infty} |a_n f_n|^2\big)^{1/2}$ and the *maximal function* $Mf := \sup_m \big|\sum_{n \leq m} a_n f_n\big|$. Moreover, given a measurable function g, we denote by $\mathcal{M}g$ the *Hardy–Littlewood maximal function* of g, defined as

$$\mathcal{M}g(x) := \sup_{I \ni x} |I|^{-1} \int_I |g(t)|\, dt,$$

where the supremum is taken over all intervals I containing x.

A corollary of Theorem 2.7 is the following relation between M and $\mathcal{M}$:

THEOREM 2.8 ([11]). *If $f \in L^1[0,1]$, we have*

$$Mf(t) \lesssim \mathcal{M}f(t), \qquad t \in [0,1].$$

3. Properties of orthogonal spline functions. This section deals with the calculation and estimation of one explicit orthonormal spline function $f_n^{(k)}$ for fixed $k \in \mathbb{N}$ and $n \geq 2$ induced by the admissible sequence $(t_n)_{n=0}^{\infty}$. Let i_0 be an index with $k+1 \leq i_0 \leq M$. The partition $\mathcal{T}$ is defined as follows:

$$\mathcal{T} = (0 = \tau_1 = \cdots = \tau_k < \tau_{k+1} \le \cdots \le \tau_{i_0} \\ \le \cdots \le \tau_M < \tau_{M+1} = \cdots = \tau_{M+k} = 1),$$

and $\widetilde{\mathcal{T}}$ is defined to be $\mathcal{T}$ with τ_{i_0} removed. In the same way we denote by $(N_i : 1 \le i \le M)$ the B-spline functions corresponding to $\mathcal{T}$, and by $(\widetilde{N}_i : 1 \le i \le M-1)$ those corresponding to $\widetilde{\mathcal{T}}$. Böhm's formula [2] gives the following relationship between N_i and $\widetilde{N}_i$:

$$\begin{cases} \widetilde{N}_i(t) = N_i(t) & \text{if } 1 \le i \le i_0-k-1, \\ \widetilde{N}_i(t) = \dfrac{\tau_{i_0}-\tau_i}{\tau_{i+k}-\tau_i} N_i(t) + \dfrac{\tau_{i+k+1}-\tau_{i_0}}{\tau_{i+k+1}-\tau_{i+1}} N_{i+1}(t) & \text{if } i_0-k \le i \le i_0-1, \\ \widetilde{N}_i(t) = N_{i+1}(t) & \text{if } i_0 \le i \le M-1. \end{cases} \tag{3.1}$$

To calculate the orthonormal spline functions corresponding to $\widetilde{\mathcal{T}}$ and $\mathcal{T}$, we first determine a function $g \in \operatorname{span}\{N_i : 1 \le i \le M\}$ such that $g \perp \widetilde{N}_j$ for all $1 \le j \le M-1$. That is, we assume that g is of the form

$$g = \sum_{j=1}^{M} \alpha_j N_j^*,$$

where $(N_j^* : 1 \le j \le M)$ is the system biorthogonal to $(N_i : 1 \le i \le M)$. In order for g to be orthogonal to $\widetilde{N}_j$, $1 \le j \le M-1$, it has to satisfy the identities

$$0 = \langle g, \widetilde{N}_i \rangle = \sum_{j=1}^{M} \alpha_j \langle N_j^*, \widetilde{N}_i \rangle, \qquad 1 \le i \le M-1.$$

Using (3.1), this implies $\alpha_j = 0$ if $1 \le i \le i_0 - k - 1$ or $i_0 + 1 \le i \le M$. For $i_0 - k \le i \le i_0 - 1$, we have the recursion formula

$$\alpha_{i+1} \frac{\tau_{i+k+1} - \tau_{i_0}}{\tau_{i+k+1} - \tau_{i+1}} + \alpha_i \frac{\tau_{i_0} - \tau_i}{\tau_{i+k} - \tau_i} = 0, \tag{3.2}$$

which determines the sequence (α_j) up to a multiplicative constant. We choose

$$\alpha_{i_0-k} = \prod_{\ell=i_0-k+1}^{i_0-1} \frac{\tau_{\ell+k} - \tau_{i_0}}{\tau_{\ell+k} - \tau_\ell}$$

for symmetry reasons. This starting value and the recursion (3.2) yield the explicit formula

(3.3)

$$\alpha_j = (-1)^{j-i_0+k} \Bigg(\prod_{\ell=i_0-k+1}^{j-1} \frac{\tau_{i_0} - \tau_\ell}{\tau_{\ell+k} - \tau_\ell} \Bigg) \Bigg(\prod_{\ell=j+1}^{i_0-1} \frac{\tau_{\ell+k} - \tau_{i_0}}{\tau_{\ell+k} - \tau_\ell} \Bigg), \quad i_0-k \le j \le i_0.$$

So,

$$g = \sum_{j=i_0-k}^{i_0} \alpha_j N_j^* = \sum_{j=i_0-k}^{i_0} \sum_{\ell=1}^{M} \alpha_j b_{j\ell} N_\ell,$$

where $(b_{j\ell})_{j,\ell=1}^M$ is the inverse of the Gram matrix $(\langle N_j, N_\ell\rangle)_{j,\ell=1}^M$. We remark that the sequence (α_j) alternates in sign and since the matrix $(b_{j\ell})_{j,\ell=1}^M$ is checkerboard, we see that the B-spline coefficients of g, namely

$$w_\ell := \sum_{j=i_0-k}^{i_0} \alpha_j b_{j\ell}, \qquad 1 \le \ell \le M, \tag{3.4}$$

satisfy

$$\Big| \sum_{j=i_0-k}^{i_0} \alpha_j b_{j\ell} \Big| = \sum_{j=i_0-k}^{i_0} |\alpha_j b_{j\ell}|, \qquad 1 \le j \le M. \tag{3.5}$$

In Definition 3.1 below, we assign to each orthonormal spline function a characteristic interval that is a grid point interval $[\tau_i, \tau_{i+1}]$ and lies close to the newly inserted point τ_{i_0}. We will see later that the choice of this interval is crucial proving important properties that are needed to show that the system $(f_n^{(k)})_{n=-k+2}^\infty$ is an unconditional basis in L^p, $1 < p < \infty$, for all admissible knot sequences $(t_n)_{n\ge 0}$. This approach was already used by G. G. Gevorkyan and A. Kamont [8] in the proof that general Franklin systems are unconditional in L^p, $1 < p < \infty$, where the characteristic intervals were called J-intervals. Since we give a slightly different construction here, we name them characteristic intervals.

Definition 3.1. Let $\mathcal{T}, \widetilde{\mathcal{T}}$ be as above and τ_{i_0} the new point in $\mathcal{T}$ that is not present in $\widetilde{\mathcal{T}}$. We define the *characteristic interval J corresponding to* τ_{i_0} as follows.

(1) Let

$$\Lambda^{(0)} := \Big\{ i_0 - k \le j \le i_0 : |[\tau_j, \tau_{j+k}]| \le 2 \min_{i_0-k\le\ell\le i_0} |[\tau_\ell, \tau_{\ell+k}]| \Big\}$$

be the set of all indices j for which the support of the B-spline function N_j is approximately minimal. Observe that $\Lambda^{(0)}$ is nonempty.

(2) Define

$$\Lambda^{(1)} := \Big\{ j \in \Lambda^{(0)} : |\alpha_j| = \max_{\ell\in\Lambda^{(0)}} |\alpha_\ell| \Big\}.$$

For an arbitrary, but fixed index $j^{(0)} \in \Lambda^{(1)}$, set $J^{(0)} := [\tau_{j^{(0)}}, \tau_{j^{(0)}+k}]$.

(3) The interval $J^{(0)}$ can now be written as the union of k grid intervals

$$J^{(0)} = \bigcup_{\ell=0}^{k-1} [\tau_{j^{(0)}+\ell}, \tau_{j^{(0)}+\ell+1}] \quad \text{with } j^{(0)} \text{ as above.}$$

We define the *characteristic interval* $J = J(\tau_{i_0})$ to be one of the above k intervals that has maximal length.

We remark that in the definition of $\Lambda^{(0)}$, we may replace the factor 2 by any other constant $C > 1$. It is essential, though, that $C > 1$ in order to obtain the following theorem which is crucial for further investigations.

THEOREM 3.2. *With the above definition* (3.4) *of* w_ℓ *for* $1 \le \ell \le M$ *and the index* $j^{(0)}$ *given in Definition* 3.1,

$$|w_{j^{(0)}}| \gtrsim b_{j^{(0)},j^{(0)}}. \tag{3.6}$$

Before we start the proof of this theorem, we state a few remarks and lemmata. For the choice of $j^{(0)}$ in Definition 3.1, we have, by construction, the following inequalities: for all $i_0 - k \le \ell \le i_0$ with $\ell \ne j^{(0)}$,

$$|\alpha_\ell| \le |\alpha_{j^{(0)}}| \quad \text{or} \quad |[\tau_\ell, \tau_{\ell+k}]| > 2 \min_{i_0-k \le s \le i_0} |[\tau_s, \tau_{s+k}]|. \tag{3.7}$$

We recall the identity

$$|\alpha_j| = \Bigg(\prod_{\ell=i_0-k+1}^{j-1} \frac{\tau_{i_0} - \tau_\ell}{\tau_{\ell+k} - \tau_\ell}\Bigg)\Bigg(\prod_{\ell=j+1}^{i_0-1} \frac{\tau_{\ell+k} - \tau_{i_0}}{\tau_{\ell+k} - \tau_\ell}\Bigg), \qquad i_0 - k \le j \le i_0. \tag{3.8}$$

Since by (3.5),

$$|w_{j^{(0)}}| = \sum_{j=i_0-k}^{i_0} |\alpha_j b_{j,j^{(0)}}| \ge |\alpha_{j^{(0)}}|\,|b_{j^{(0)},j^{(0)}}|,$$

in order to show (3.6), we prove the inequality

$$|\alpha_{j^{(0)}}| \ge D_k > 0$$

with a constant D_k only depending on k. By (3.8), this inequality follows from the more elementary inequalities

$$\begin{aligned} \tau_{i_0} - \tau_\ell &\gtrsim \tau_{\ell+k} - \tau_{i_0}, & i_0 - k + 1 \le \ell \le j^{(0)} - 1,\\ \tau_{\ell+k} - \tau_{i_0} &\gtrsim \tau_{i_0} - \tau_\ell, & j^{(0)} + 1 \le \ell \le i_0 - 1. \end{aligned} \tag{3.9}$$

We will only prove the second line of (3.9) for all choices of $j^{(0)}$. The first line is proved by a similar argument. We observe that if $j^{(0)} \ge i_0 - 1$, then there is nothing to prove, so we assume

$$j^{(0)} \le i_0 - 2. \tag{3.10}$$

Moreover, we need only show the single inequality

$$\tau_{j^{(0)}+k+1} - \tau_{i_0} \gtrsim \tau_{i_0} - \tau_{j^{(0)}+1}, \tag{3.11}$$

since if we assume (3.11), then for any $j^{(0)} + 1 \le \ell \le i_0 - 1$,

$$\tau_{\ell+k} - \tau_{i_0} \ge \tau_{j^{(0)}+k+1} - \tau_{i_0} \gtrsim \tau_{i_0} - \tau_{j^{(0)}+1} \ge \tau_{i_0} - \tau_\ell.$$

We now choose j to be the minimal index in the range $i_0 \ge j > j^{(0)}$ such that

$$|\alpha_j| \le |\alpha_{j^{(0)}}|. \tag{3.12}$$

If there is no such j, we set $j = i_0 + 1$.

If $j \le i_0$, we employ (3.8) to deduce that (3.12) is equivalent to

$$\begin{aligned}(3.13)\quad (\tau_{j+k} - \tau_j)^{1-\delta(j,i_0)} &\prod_{\ell=j^{(0)}\vee(i_0-k+1)}^{j-1} (\tau_{i_0} - \tau_\ell) \\ &\le (\tau_{j^{(0)}+k} - \tau_{j^{(0)}})^{1-\delta(j^{(0)},i_0-k)} \prod_{\ell=j^{(0)}+1}^{j\wedge(i_0-1)} (\tau_{\ell+k} - \tau_{i_0}),\end{aligned}$$

where $\delta(\cdot,\cdot)$ is the Kronecker delta. Furthermore, let m in the range $i_0 - k \le m \le i_0$ be such that $\tau_{m+k} - \tau_m = \min_{i_0-k\le s\le i_0}(\tau_{s+k} - \tau_s)$.

Now, from the minimality of j and (3.7), we obtain

$$\tau_{\ell+k} - \tau_\ell > 2(\tau_{m+k} - \tau_m), \qquad j^{(0)} + 1 \le \ell \le j - 1. \tag{3.14}$$

Thus, by definition,

$$m \le j^{(0)} \quad \text{or} \quad m \ge j. \tag{3.15}$$

LEMMA 3.3. *In the above notation, if $m \le j^{(0)}$ and $j - j^{(0)} \ge 2$, then we have* (3.11)*, or more precisely,*

$$\tau_{j^{(0)}+k+1} - \tau_{i_0} \ge \tau_{i_0} - \tau_{j^{(0)}+1}. \tag{3.16}$$

Proof. We expand the left hand side of (3.16) as

$$\tau_{j^{(0)}+k+1} - \tau_{i_0} = \tau_{j^{(0)}+k+1} - \tau_{j^{(0)}+1} - (\tau_{i_0} - \tau_{j^{(0)}+1}).$$

By (3.14) (observe that $j - j^{(0)} \ge 2$), we conclude that

$$\tau_{j^{(0)}+k+1} - \tau_{i_0} \ge 2(\tau_{m+k} - \tau_m) - (\tau_{i_0} - \tau_{j^{(0)}+1}).$$

Since $m + k \ge i_0$ and $m \le j^{(0)}$, we finally obtain

$$\tau_{j^{(0)}+k+1} - \tau_{i_0} \ge \tau_{i_0} - \tau_{j^{(0)}+1}. \ \blacksquare$$

LEMMA 3.4. *Let $j^{(0)}$, j and m be as above. If $j^{(0)} + 1 \le \ell \le j - 1$ and $m \ge j$, we have*

$$\tau_{i_0} - \tau_\ell \ge \tau_{\ell+1+k} - \tau_{i_0}.$$

Proof. Let $j^{(0)} + 1 \le \ell \le j - 1$. Then from (3.14) we obtain

$$\tau_{i_0} - \tau_\ell = \tau_{\ell+1+k} - \tau_\ell - (\tau_{\ell+1+k} - \tau_{i_0}) \ge 2(\tau_{m+k} - \tau_m) - (\tau_{\ell+1+k} - \tau_{i_0}). \tag{3.17}$$

Since we have assumed $m \ge j \ge \ell + 1$, we get $m + k \ge \ell + 1 + k$, and additionally we have $m \le i_0$ by definition of m. Thus (3.17) yields

$$\tau_{i_0} - \tau_\ell \ge \tau_{\ell+1+k} - \tau_{i_0}.$$

Since the index ℓ was arbitrary in the range $j^{(0)}+1 \le \ell \le j-1$, the proof of the lemma is complete. ∎

Proof of Theorem 3.2. We employ the above definition of $j^{(0)}$, j, and m and split our analysis into a few cases, distinguishing various possibilities for $j^{(0)}$ and j. In each case we will show (3.11).

CASE 1: *There is no $j > j^{(0)}$ such that $|\alpha_j| \le |\alpha_{j^{(0)}}|$.* In this case, (3.15) implies $m \le j^{(0)}$. Since $j^{(0)} \le i_0 - 2$ by (3.10), we apply Lemma 3.3 to deduce (3.11).

CASE 2: $i_0 - k + 1 \le j^{(0)} < j \le i_0 - 1$. Using the restrictions on $j^{(0)}$ and j, we see that (3.13) becomes

$$(\tau_{j^{(0)}+k} - \tau_{j^{(0)}}) \prod_{\ell=j^{(0)}+1}^{j} (\tau_{\ell+k} - \tau_{i_0}) \ge (\tau_{j+k} - \tau_j) \prod_{\ell=j^{(0)}}^{j-1} (\tau_{i_0} - \tau_\ell).$$

This implies

$$\tau_{j^{(0)}+k+1} - \tau_{i_0} \ge \frac{(\tau_{j+k} - \tau_j)(\tau_{i_0} - \tau_{j^{(0)}})}{\tau_{j^{(0)}+k} - \tau_{j^{(0)}}} \prod_{\ell=j^{(0)}+1}^{j-1} \frac{\tau_{i_0} - \tau_\ell}{\tau_{\ell+1+k} - \tau_{i_0}}.$$

Since by definition of $j^{(0)}$, we have in particular $\tau_{j^{(0)}+k} - \tau_{j^{(0)}} \le 2(\tau_{j+k} - \tau_j)$, we conclude further that

$$\tau_{j^{(0)}+k+1} - \tau_{i_0} \ge \frac{\tau_{i_0} - \tau_{j^{(0)}+1}}{2} \prod_{\ell=j^{(0)}+1}^{j-1} \frac{\tau_{i_0} - \tau_\ell}{\tau_{\ell+1+k} - \tau_{i_0}}. \tag{3.18}$$

If $j = j^{(0)} + 1$, the assertion (3.11) follows from (3.18), since the product is then empty.

If $j \ge j^{(0)} + 2$ and $m \le j^{(0)}$, we use Lemma 3.3 to obtain (3.11).

If $j \ge j^{(0)} + 2$ and $m \ge j$, we apply Lemma 3.4 to the terms in the product appearing in (3.18) to deduce (3.11).

This finishes the proof of Case 2.

CASE 3: $i_0 - k + 1 \le j^{(0)} < j = i_0$. Recall that $j^{(0)} \le i_0 - 2 = j - 2$ by (3.10). If $m \le j^{(0)}$, Lemma 3.3 gives (3.11). So we assume $m \ge j$. Since $i_0 = j$ and $m \le i_0$, we have $m = j$. The restrictions on $j^{(0)}$, j imply that condition (3.13) is nothing else than

$$(\tau_{j^{(0)}+k} - \tau_{j^{(0)}}) \prod_{\ell=j^{(0)}+1}^{i_0-1} (\tau_{\ell+k} - \tau_{i_0}) \ge \prod_{\ell=j^{(0)}}^{i_0-1} (\tau_{i_0} - \tau_\ell).$$

Thus, in order to show (3.11), it is enough to prove that there exists a constant $D_k > 0$ only depending on k such that

$$\frac{\tau_{i_0} - \tau_{j^{(0)}}}{\tau_{j^{(0)}+k} - \tau_{j^{(0)}}} \prod_{\ell=j^{(0)}+2}^{i_0-1} \frac{\tau_{i_0} - \tau_\ell}{\tau_{\ell+k} - \tau_{i_0}} \geq D_k. \tag{3.19}$$

First observe that by Lemma 3.4,

$$\tau_{i_0} - \tau_{j^{(0)}} \geq \tau_{j^{(0)}+k+2} - \tau_{i_0} \geq \tau_{j^{(0)}+k} - \tau_{i_0}.$$

Inserting this inequality in the left hand side of (3.19) and applying Lemma 3.4 directly to the terms in the product, we obtain (3.19).

CASE 4: $i_0 - k = j^{(0)} < j = i_0$. We have $j^{(0)} \leq i_0 - 2$ by (3.10). If $m \leq j^{(0)}$, just apply Lemma 3.3 to obtain (3.11). Thus we assume $m \geq j$. Since $i_0 = j$ and $m \leq i_0$, we have $m = j$. The restrictions on $j^{(0)}$, j imply that (3.13) takes the form

$$\prod_{\ell=i_0-k+1}^{i_0-1} (\tau_{\ell+k} - \tau_{i_0}) \geq \prod_{\ell=i_0-k+1}^{i_0-1} (\tau_{i_0} - \tau_\ell).$$

Thus, to show (3.11), it is enough to prove that there exists a constant $D_k > 0$ only depending on k such that

$$\prod_{\ell=i_0-k+2}^{i_0-1} \frac{\tau_{i_0} - \tau_\ell}{\tau_{\ell+k} - \tau_{i_0}} \geq D_k.$$

But this is a consequence of Lemma 3.4, finishing the proof of Case 4.

CASE 5: $i_0 - k = j^{(0)} < j \leq i_0 - 1$. In this case, (3.11) becomes

$$\tau_{i_0+1} - \tau_{i_0} \gtrsim \tau_{i_0} - \tau_{i_0-k+1}, \tag{3.20}$$

and (3.13) is nothing else than

$$\prod_{\ell=i_0-k+1}^{j} (\tau_{\ell+k} - \tau_{i_0}) \geq (\tau_{j+k} - \tau_j) \prod_{\ell=i_0-k+1}^{j-1} (\tau_{i_0} - \tau_\ell). \tag{3.21}$$

For $j = i_0 - k + 1$, (3.20) follows easily from (3.21). If we assume $j - j^{(0)} \geq 2$ and $m \leq j^{(0)}$, we just apply Lemma 3.3 to obtain (3.11). If $j - j^{(0)} \geq 2$ and $m \geq j$, then (3.20) is equivalent to the existence of a constant $D_k > 0$ only depending on k such that

$$\frac{(\tau_{j+k} - \tau_j) \prod_{\ell=i_0-k+2}^{j-1} (\tau_{i_0} - \tau_\ell)}{\prod_{\ell=i_0-k+2}^{j} (\tau_{\ell+k} - \tau_{i_0})} \geq D_k.$$

This follows from the obvious inequality $\tau_{j+k} - \tau_j \geq \tau_{j+k} - \tau_{i_0}$ and from Lemma 3.4. Thus, the proof of Case 5 is complete, thereby concluding the proof of Theorem 3.2. ∎

We will use this result to prove lemmata connecting the L^p norm of the function g and the corresponding characteristic interval J. Before we start, we need another simple

LEMMA 3.5. *Let $C = (c_{ij})_{i,j=1}^n$ be a symmetric positive definite matrix. Then for $(d_{ij})_{i,j=1}^n = C^{-1}$ we have*

$$c_{ii}^{-1} \le d_{ii}, \qquad 1 \le i \le n.$$

Proof. Since C is symmetric, it is diagonalizable:

$$C = S\Lambda S^T,$$

for some orthogonal matrix $S = (s_{ij})_{i,j=1}^n$ and for the diagonal matrix Λ consisting of the eigenvalues $\lambda_1, \dots, \lambda_n$ of C. These eigenvalues are positive, since C is positive definite. Clearly,

$$C^{-1} = S\Lambda^{-1}S^T.$$

Let i be an arbitrary integer in the range $1 \le i \le n$. Then

$$c_{ii} = \sum_{\ell=1}^n s_{i\ell}^2 \lambda_\ell \quad \text{and} \quad d_{ii} = \sum_{\ell=1}^n s_{i\ell}^2 \lambda_\ell^{-1}.$$

Since $\sum_{\ell=1}^n s_{i\ell}^2 = 1$ and the function $x \mapsto x^{-1}$ is convex on $(0,\infty)$, we conclude by Jensen's inequality that

$$c_{ii}^{-1} = \Big(\sum_{\ell=1}^n s_{i\ell}^2 \lambda_\ell\Big)^{-1} \le \sum_{\ell=1}^n s_{i\ell}^2 \lambda_\ell^{-1} = d_{ii}. \ \blacksquare$$

LEMMA 3.6. *Let $\mathcal{T}$, $\widetilde{\mathcal{T}}$ be as above and $g = \sum_{j=1}^M w_j N_j$ be the function in* $\operatorname{span}\{N_i : 1 \le i \le M\}$ *that is orthogonal to every $\widetilde{N}_i$, $1 \le i \le M-1$, with $(w_j)_{j=1}^M$ given in* (3.4). *Moreover, let $\varphi = g/\|g\|_2$ be the L^2-normalized orthogonal spline function corresponding to the mesh point τ_{i_0}. Then*

$$\|\varphi\|_{L^p(J)} \sim \|\varphi\|_p \sim |J|^{1/p-1/2}, \qquad 1 \le p \le \infty,$$

where J is the characteristic interval associated to the point τ_{i_0}, given in Definition 3.1.

Proof. As a consequence of (2.5), we get

$$\|g\|_{L^p(J)} \gtrsim |J|^{1/p} |w_{j^{(0)}}|. \tag{3.22}$$

By Theorem 3.2, $|w_{j^{(0)}}| \gtrsim b_{j^{(0)},j^{(0)}}$, where we recall that $(b_{ij})_{i,j=1}^M$ is the inverse of the Gram matrix $(a_{ij})_{i,j=1}^M = (\langle N_i, N_j\rangle)_{i,j=1}^M$. Now we invoke Lemma 3.5 and (2.6) to infer from (3.22) that

$$\begin{aligned}\|g\|_{L^p(J)} &\gtrsim |J|^{1/p} b_{j^{(0)},j^{(0)}} \ge |J|^{1/p} a_{j^{(0)},j^{(0)}}^{-1} \\ &= |J|^{1/p} \|N_{j^{(0)}}\|_2^{-2} \gtrsim |J|^{1/p} \nu_{j^{(0)}}^{-1}.\end{aligned}$$

Since, by construction, J is the maximal subinterval of $J^{(0)}$ and there are exactly k subintervals of $J^{(0)}$, we finally get

$$\|g\|_{L^p(J)} \gtrsim |J|^{1/p-1}. \tag{3.23}$$

On the other hand, $g = \sum_{j=i_0-k}^{i_0} \alpha_j N_j^*$, so we use (2.7) to obtain

$$\|g\|_p \lesssim \Big(\sum_{j=i_0-k}^{i_0} |\alpha_j|^p \nu_j^{1-p}\Big)^{1/p}.$$

Since $|\alpha_j| \le 1$ for all j and $\nu_{j^{(0)}}$ is minimal (up to the factor 2) among the values ν_j, $i_0 - k \le j \le i_0$, we can estimate this further by

$$\|g\|_p \lesssim \nu_{j^{(0)}}^{1/p-1}.$$

We now use the inequality $|J| \le \nu_{j^{(0)}} = |J^{(0)}|$ from the construction of J to get

$$\|g\|_p \lesssim |J|^{1/p-1}. \tag{3.24}$$

The assertion of the lemma now follows from (3.23) and (3.24) after renormalization. ■

We denote by $d_{\mathcal{T}}(x)$ the number of points in $\mathcal{T}$ between x and J counting the endpoints of J. Correspondingly, for an interval $V \subset [0,1]$, we denote by $d_{\mathcal{T}}(V)$ the number of points in $\mathcal{T}$ between V and J counting the endpoints of both J and V.

LEMMA 3.7. *Let $\mathcal{T}, \widetilde{\mathcal{T}}$ be as above and $g = \sum_{j=1}^M w_j N_j$ be orthogonal to every $\widetilde{N}_i$, $1 \le i \le M-1$, with $(w_j)_{j=1}^M$ as in* (3.4). *Moreover, let $\varphi = g/\|g\|_2$ be the normalized orthogonal spline function corresponding to τ_{i_0}, and $\gamma < 1$ the constant from Theorem* 2.7 *depending only on the spline order k. Then*

$$|w_j| \lesssim \frac{\gamma^{d_{\mathcal{T}}(\tau_j)}}{|J| + \operatorname{dist}(\operatorname{supp} N_j, J) + \nu_j} \quad \textit{for all } 1 \le j \le M. \tag{3.25}$$

Moreover, if $x < \inf J$, then

$$\|\varphi\|_{L^p(0,x)} \lesssim \frac{\gamma^{d_{\mathcal{T}}(x)}|J|^{1/2}}{(|J| + \operatorname{dist}(x,J))^{1-1/p}}, \qquad 1 \le p \le \infty. \tag{3.26}$$

Similarly, for $x > \sup J$,

$$\|\varphi\|_{L^p(x,1)} \lesssim \frac{\gamma^{d_{\mathcal{T}}(x)}|J|^{1/2}}{(|J| + \operatorname{dist}(x,J))^{1-1/p}}, \qquad 1 \le p \le \infty. \tag{3.27}$$

Proof. We begin by showing (3.25). By definition of w_j and α_ℓ (see (3.4) and (3.3)), we have

$$|w_j| \lesssim \max_{i_0-k \le \ell \le i_0} |b_{j\ell}|.$$

Now we invoke Theorem 2.7 to deduce

$$
\begin{aligned}
(3.28)\qquad |w_j| &\lesssim \frac{\max_{i_0-k\le\ell\le i_0}\gamma^{|\ell-j|}}{\min_{i_0-k\le\ell\le i_0}\big(\tau_{\max(\ell,j)+k}-\tau_{\min(\ell,j)}\big)}\\
&\lesssim \frac{\gamma^{d_{\mathcal{T}}(\tau_j)}}{\min_{i_0-k\le\ell\le i_0}\big(\tau_{\max(\ell,j)+k}-\tau_{\min(\ell,j)}\big)},
\end{aligned}
$$

where the second inequality follows from the location of J in the interval $[\tau_{i_0-k},\tau_{i_0+k}]$. It remains to estimate the minimum in the denominator. Fix ℓ with $i_0-k\le\ell\le i_0$. First we observe that

$$(3.29)\qquad \tau_{\max(\ell,j)+k}-\tau_{\min(\ell,j)}\ge \tau_{j+k}-\tau_j=|\operatorname{supp}N_j|=\nu_j.$$

Moreover, by definition of J,

$$(3.30)\qquad \tau_{\max(\ell,j)+k}-\tau_{\min(\ell,j)}\ge \min_{i_0-k\le r\le i_0}(\tau_{r+k}-\tau_r)\ge |J^{(0)}|/2\ge |J|/2.$$

If now $j\ge\ell$, then

$$
\begin{aligned}
(3.31)\qquad \tau_{\max(\ell,j)+k}-\tau_{\min(\ell,j)} &= \tau_{j+k}-\tau_\ell\ge\tau_{j+k}-\tau_{i_0}\\
&\ge \max(\tau_j-\sup J^{(0)},0),
\end{aligned}
$$

since $\tau_{i_0}\le\sup J^{(0)}$. But $\max(\tau_j-\sup J^{(0)},0)=\operatorname{dist}([\tau_j,\tau_{j+k}],J^{(0)})$ due to the fact that $\inf J^{(0)}\le\tau_{i_0}\le\tau_{\ell+k}\le\tau_{j+k}$ for the current choice of j. Additionally, $\operatorname{dist}([\tau_j,\tau_{j+k}],J)\le|J^{(0)}|+d([\tau_j,\tau_{j+k}],J^{(0)})$. So, as a consequence of (3.31),

$$(3.32)\qquad \tau_{\max(\ell,j)+k}-\tau_{\min(\ell,j)}\ge\operatorname{dist}([\tau_j,\tau_{j+k}],J)-|J^{(0)}|.$$

An analogous calculation proves (3.32) also in the case $j\le\ell$. We now combine (3.28) with (3.29), (3.30) and (3.32) to obtain (3.25).

Next we consider the integral $\big(\int_0^x|g(t)|^p\,dt\big)^{1/p}$ for $x<\inf J$. The analogous estimate (3.27) follows from a similar argument. Let τ_s be the first grid point in $\mathcal{T}$ to the right of x and observe that $\operatorname{supp}N_r\cap[0,\tau_s)=\emptyset$ for $r\ge s$. Then

$$\|g\|_{L^p(0,x)}\le\|g\|_{L^p(0,\tau_s)}\le\Big\|\sum_{i=1}^{s-1}w_iN_i\Big\|_p.$$

By (2.6),

$$\|g\|_{L^p(0,x)}\le\big\|\big(w_i\nu_i^{1/p}\big)_{i=1}^{s-1}\big\|_{\ell^p}.$$

We now use (3.25) for w_i to get

$$\|g\|_{L^p(0,x)}\lesssim\bigg\|\bigg(\frac{\gamma^{d_{\mathcal{T}}(\tau_i)}\nu_i^{1/p}}{|J|+\operatorname{dist}(\operatorname{supp}N_i,J)+\nu_i}\bigg)_{i=1}^{s-1}\bigg\|_{\ell^p}.$$

Since $\nu_i \le |J| + \operatorname{dist}(\operatorname{supp} N_i, J) + \nu_i$ for all $1 \le i \le M$ and $\operatorname{dist}(\operatorname{supp} N_i, J) + \nu_i \ge \operatorname{dist}(x, J)$ for all $1 \le i \le s-1$, the last display yields

$$\|g\|_{L^p(0,x)} \lesssim (|J| + \operatorname{dist}(x, J))^{-1+1/p} \|(\gamma^{d_{\mathcal{T}}(\tau_i)})_{i=1}^{s-1}\|_{\ell^p}.$$

The last ℓ^p-norm is a geometric sum with largest term $\gamma^{d_{\mathcal{T}}(x)}$, so

$$\|g\|_{L^p(0,x)} \lesssim \frac{\gamma^{d_{\mathcal{T}}(x)}}{(|J| + \operatorname{dist}(x, J))^{1-1/p}}.$$

This concludes the proof, since we have seen in the proof of Lemma 3.6 that $\|g\|_2 \sim |J|^{-1/2}$. ■

REMARK 3.8. Analogously we obtain

$$\begin{aligned}\sup_{\tau_{j-1} \le t \le \tau_j} |\varphi(t)| &\lesssim \max_{j-k \le i \le j-1} \frac{\gamma^{d_{\mathcal{T}}(\tau_i)} |J|^{1/2}}{|J| + \operatorname{dist}(\operatorname{supp} N_i, J) + \nu_i} \\ &\lesssim \frac{\gamma^{d_{\mathcal{T}}(\tau_j)} |J|^{1/2}}{|J| + \operatorname{dist}(J, [\tau_{j-1}, \tau_j]) + |[\tau_{j-1}, \tau_j]|},\end{aligned}$$

since $[\tau_{j-1}, \tau_j] \subset \operatorname{supp} N_i$ whenever $j - k \le i \le j - 1$.

4. Combinatorics of characteristic intervals. Let $(t_n)_{n=0}^{\infty}$ be an admissible sequence of points and $(f_n)_{n=-k+2}^{\infty}$ the corresponding orthonormal spline functions of order k. For $n \ge 2$, the associated partitions $\mathcal{T}_n$ to f_n are defined to consist of the grid points $(t_j)_{j=0}^{n}$, the knots $t_0 = 0$ and $t_1 = 1$ having both multiplicity k in $\mathcal{T}_n$. If $n \ge 2$, we denote by $J_n^{(0)}$ and J_n the characteristic intervals $J^{(0)}$ and J from Definition 3.1 associated to the new grid point t_n. If $-k + 2 \le n \le 1$, we additionally set $J_n := [0, 1]$. For any $x \in [0, 1]$, we define $d_n(x)$ to be the number of grid points in $\mathcal{T}_n$ between x and J_n counting the endpoints of J_n. Moreover, for a subinterval V of $[0, 1]$, we denote by $d_n(V)$ the number of knots in $\mathcal{T}_n$ between V and J_n counting the endpoints of both V and J_n. Finally, if

$$\begin{aligned}\mathcal{T}_n = (0 = \tau_{n,1} = \cdots = \tau_{n,k} < \tau_{n,k+1} \\ \le \cdots \le \tau_{n,n+k-1} < \tau_{n,n+k} = \cdots = \tau_{n,n+2k-1} = 1),\end{aligned}$$

and if $t_n = \tau_{n,i_0}$, then we denote by $t_n^{+\ell}$ the point $\tau_{n,i_0+\ell}$.

For the proof of the central Lemma 4.2 of this section, we need a combinatorial lemma of Erdős and Szekeres:

LEMMA 4.1 (Erdős–Szekeres). *Let n be an integer. Every sequence $(x_1, \ldots, x_{(n-1)^2+1})$ of real numbers of length $(n-1)^2+1$ contains a monotone sequence of length n.*

We now use this result to prove a lemma about the combinatorics of the characteristic intervals J_n:

Lemma 4.2. *Let $x, y \in (t_n)_{n=0}^\infty$ be such that $x < y$ and $0 \le \beta \le 1/2$. Then there exists a constant F_k only depending on k such that*

$$N_0 := \operatorname{card}\{n : J_n \subseteq [x,y],\ |J_n| \ge (1-\beta)|[x,y]|\} \le F_k,$$

where $\operatorname{card} E$ *denotes the cardinality of the set E.*

Proof. If n is such that $J_n \subseteq [x,y]$ and $|J_n| \ge (1-\beta)|[x,y]|$, then, by definition of J_n, we have $t_n \in [0,(1-\beta)x+\beta y] \cup [\beta x + (1-\beta)y, 1]$. Thus, by the pigeon-hole principle, in one of the two sets $[0,(1-\beta)x+\beta y]$ and $[\beta x+(1-\beta)y,1]$, there are at least

$$N_1 := \left\lfloor \frac{N_0-1}{2} \right\rfloor + 1$$

indices n with $J_n \subset [x,y]$ and $|J_n| \ge (1-\beta)|[x,y]|$. Assume without loss of generality that this set is $[\beta x+(1-\beta)y,1]$. Now, let $(n_i)_{i=1}^{N_1}$ be an increasing sequence of indices such that $t_{n_i} \in [\beta x+(1-\beta)y,1]$ and $J_{n_i} \subset [x,y]$, $|J_{n_i}| \ge (1-\beta)|[x,y]|$ for every $1 \le i \le N_1$. Observe that for such i, J_{n_i} is to the left of t_{n_i}. By the Erdős–Szekeres Lemma 4.1, the sequence $(t_{n_i})_{i=1}^{N_1}$ contains a monotone subsequence $(t_{m_i})_{i=1}^{N_2}$ of length

$$N_2 := \lfloor \sqrt{N_1 - 1} \rfloor + 1.$$

If $(t_{m_i})_{i=1}^{N_2}$ is increasing, then $N_2 \le k$. Indeed, if $N_2 \ge k+1$, there are at least k points (namely $t_{m_1}, \dots, t_{m_k}$) in the sequence $\mathcal{T}_{m_{k+1}}$ between $\inf J_{m_{k+1}}$ and $t_{m_{k+1}}$. This is in conflict with the location of $J_{m_{k+1}}$.

If $(t_{m_i})_{i=1}^{N_2}$ is decreasing, we let

$$s_1 \le \dots \le s_L$$

be an enumeration of the elements in $\mathcal{T}_{m_1}$ such that $\inf J_{m_1} \le s \le t_{m_1}$. By definition of J_{m_1}, we obtain $L \le k+1$. Thus, there are at most k intervals $[s_\ell, s_{\ell+1}]$, $1 \le \ell \le L-1$, contained in $[\inf J_{m_1}, t_{m_1}]$. Again, by the pigeon-hole principle, there exists one index $1 \le \ell \le L-1$ such that the interval $[s_\ell, s_{\ell+1}]$ contains (at least)

$$N_3 := \left\lfloor \frac{N_2-1}{k} \right\rfloor + 1$$

points of the sequence $(t_{m_i})_{i=1}^{N_2}$. Let $(t_{r_i})_{i=1}^{N_3}$ be a subsequence of length N_3 of such points. Furthermore, define

$$N_4 := \lfloor N_3/k \rfloor.$$

Since $(t_{r_i})_{i=1}^{N_3}$ is decreasing, we have a collection of N_4 disjoint intervals

$$I_\mu := (t_{r_{\mu\cdot k}}, t^{+k}_{r_{\mu\cdot k}}) \subseteq [s_\ell, s_{\ell+1}], \qquad 1 \le \mu \le N_4.$$

Consequently, there exists (at least) one index μ such that

$$|I_\mu| \le |[s_\ell, s_{\ell+1}]|/N_4.$$

We next observe that the definition of J_{m_1} yields

$$|J_{m_1}| \geq |[s_\ell, s_{\ell+1}]|.$$

We thus get

$$\begin{aligned} (4.1) \qquad |J^{(0)}_{r_{\mu}\cdot k}| \geq |J_{r_{\mu}\cdot k}| &\geq (1-\beta)|[x,y]| \geq (1-\beta)|J_{m_1}| \\ &\geq (1-\beta)|[s_\ell, s_{\ell+1}]| \geq (1-\beta)N_4|I_\mu|. \end{aligned}$$

On the other hand, the construction of $J^{(0)}_{r_{\mu}\cdot k}$ implies in particular

$$(4.2) \qquad |J^{(0)}_{r_{\mu}\cdot k}| \leq 2(t^{+k}_{r_{\mu}\cdot k} - t_{r_{\mu}\cdot k}) = 2|I_\mu|.$$

The inequalities (4.1) and (4.2) imply $N_4 \leq 2/(1-\beta) \leq 4$. Since N_4 only depends on k, this proves the assertion of the lemma. ∎

5. Technical estimates

Lemma 5.1. *Let* $f = \sum_{n=-k+2}^{\infty} a_n f_n$ *and* V *be an open subinterval of* $[0,1]$. *Then*

$$(5.1) \qquad \int_{V^c} \sum_{j\in\Gamma} |a_j f_j(t)|\, dt \lesssim \int_V \Big(\sum_{j\in\Gamma} |a_j f_j(t)|^2\Big)^{1/2} dt,$$

where $\Gamma := \{j : J_j \subset V$ *and* $-k+2 \leq j < \infty\}$.

Proof. If $|V| = 1$, then (5.1) holds trivially, so we assume that $|V| < 1$. We define $x := \inf V$, $y := \sup V$ and fix $n \in \Gamma$. The definition of Γ implies $n \geq 2$, since $J_j = [0,1]$ for $-k+2 \leq j \leq 1$. We only estimate the integral in (5.1) over $[y,1]$; the integral over $[0,x]$ is estimated similarly. Lemma 3.7 implies

$$\int_y^1 |f_n(t)|\, dt \lesssim \gamma^{d_n(y)} |J_n|^{1/2}.$$

Applying Lemma 3.6 yields

$$(5.2) \qquad \int_y^1 |f_n(t)|\, dt \lesssim \gamma^{d_n(y)} \int_{J_n} |f_n(t)|\, dt.$$

Now choose $\beta = 1/4$ and let J_n^β be the unique closed interval that satisfies

$$|J_n^\beta| = \beta |J_n| \quad \text{and} \quad \inf J_n^\beta = \inf J_n.$$

Since f_n is a polynomial of order k on J_n, we apply Proposition 2.1 to (5.2) and estimate further

$$(5.3) \qquad \int_y^1 |a_n f_n(t)|\, dt \lesssim \gamma^{d_n(y)} \int_{J_n^\beta} |a_n f_n(t)|\, dt \leq \gamma^{d_n(y)} \int_{J_n^\beta} \Big(\sum_{j\in\Gamma} |a_j f_j(t)|^2\Big)^{1/2} dt.$$

Define $\Gamma_s := \{j \in \Gamma : d_j(y) = s\}$ for $s \geq 0$. For fixed $s \geq 0$ and $j_1, j_2 \in \Gamma_s$, we have either

$$J_{j_1} \cap J_{j_2} = \emptyset \quad \text{or} \quad \sup J_{j_1} = \sup J_{j_2}.$$

So, Lemma 4.2 implies that there exists a constant F_k, only depending on k, such that each $t \in V$ belongs to at most F_k intervals J_j^β, $j \in \Gamma_s$. Thus, summing over $j \in \Gamma_s$, from (5.3) we get

$$\begin{aligned}\sum_{j\in\Gamma_s} \int_y^1 |a_j f_j(t)|\,dt &\lesssim \sum_{j\in\Gamma_s} \gamma^s \int_{J_j^\beta} \Big(\sum_{\ell\in\Gamma} |a_\ell f_\ell(t)|^2\Big)^{1/2}\,dt \\ &\lesssim \gamma^s \int_V \Big(\sum_{\ell\in\Gamma} |a_\ell f_\ell(t)|^2\Big)^{1/2}\,dt.\end{aligned}$$

Finally, we sum over $s \geq 0$ to obtain (5.1). ∎

Let g be a real-valued function defined on $[0,1]$. We denote by $[g > \lambda]$ the set $\{x \in [0,1] : g(x) > \lambda\}$ for any $\lambda > 0$.

Lemma 5.2. *Let $f = \sum_{n=-k+2}^\infty a_n f_n$ with only finitely many nonzero coefficients a_n, $\lambda > 0$, $r < 1$ and*

$$E_\lambda = [Sf > \lambda], \quad B_{\lambda,r} = [\mathcal{M}\mathbb{1}_{E_\lambda} > r].$$

Then

$$E_\lambda \subset B_{\lambda,r}.$$

Proof. Fix $t \in E_\lambda$. Since $Sf = \big(\sum_{n=-k+2}^\infty |a_n f_n|^2\big)^{1/2}$ is continuous except possibly at finitely many grid points, where it is continuous from the right, there exists an interval $I \subset E_\lambda$ such that $t \in I$. This implies

$$\begin{aligned}(\mathcal{M}\mathbb{1}_{E_\lambda})(t) &= \sup_{t\ni U} |U|^{-1} \int_U \mathbb{1}_{E_\lambda}(x)\,dx \\ &= \sup_{t\ni U} \frac{|E_\lambda \cap U|}{|U|} \geq \frac{|E_\lambda \cap I|}{|I|} = \frac{|I|}{|I|} = 1 > r,\end{aligned}$$

so $t \in B_{\lambda,r}$, proving the lemma. ∎

Lemma 5.3. *Under the assumptions of Lemma 5.2, define*

$$\Lambda = \{n : J_n \not\subset B_{\lambda,r} \text{ and } -k+2 \leq n < \infty\} \quad \text{and} \quad g = \sum_{n\in\Lambda} a_n f_n.$$

Then

$$\int_{E_\lambda} Sg(t)^2\,dt \lesssim_r \int_{E_\lambda^c} Sg(t)^2\,dt. \tag{5.4}$$

Proof. If $B_{\lambda,r} = [0,1]$, the index set Λ is empty, and thus (5.4) holds trivially; so assume $B_{\lambda,r} \neq [0,1]$. Then we apply Lemma 3.6 (for $n \geq 2$) and the fact that $J_n = [0,1]$ for $n \le 1$ to obtain

$$\int_{E_\lambda} Sg(t)^2\,dt = \sum_{n\in\Lambda}\int_{E_\lambda} |a_n f_n(t)|^2\,dt \lesssim \sum_{n\in\Lambda}\int_{J_n} |a_n f_n(t)|^2\,dt.$$

We split the last expression into

$$I_1 := \sum_{n\in\Lambda}\int_{J_n\cap E_\lambda^c} |a_n f_n(t)|^2\,dt, \qquad I_2 := \sum_{n\in\Lambda}\int_{J_n\cap E_\lambda} |a_n f_n(t)|^2\,dt.$$

For I_1, we clearly have

$$I_1 \le \sum_{n\in\Lambda}\int_{E_\lambda^c} |a_n f_n(t)|^2\,dt = \int_{E_\lambda^c} Sg(t)^2\,dt. \tag{5.5}$$

It remains to estimate I_2. First we observe that by Lemma 5.2, $E_\lambda \subset B_{\lambda,r}$. Since the set $B_{\lambda,r} = [\mathcal{M}\mathbb{1}_{E_\lambda} > r]$ is open in $[0,1]$, we decompose it into a countable collection $(V_j)_{j=1}^\infty$ of disjoint open subintervals of $[0,1]$. Utilizing this decomposition, we estimate

$$I_2 \le \sum_{n\in\Lambda}\sum_{j:\,|J_n\cap V_j|>0}\int_{J_n\cap V_j} |a_n f_n(t)|^2\,dt. \tag{5.6}$$

If $n \in \Lambda$ and $|J_n \cap V_j| > 0$, then, by definition of Λ, J_n is an interval containing at least one endpoint $x \in \{\inf V_j, \sup V_j\}$ of V_j for which

$$\mathcal{M}\mathbb{1}_{E_\lambda}(x) \le r.$$

This implies

$$|E_\lambda \cap J_n \cap V_j| \le r|J_n \cap V_j| \quad \text{or equivalently} \quad |E_\lambda^c \cap J_n \cap V_j| \ge (1-r)|J_n \cap V_j|.$$

This inequality and the fact that $|f_n|^2$ is a polynomial of order $2k-1$ on J_n allow us to use Proposition 2.1 to deduce from (5.6) that

$$\begin{aligned} I_2 &\lesssim_r \sum_{n\in\Lambda}\sum_{j:\,|J_n\cap V_j|>0}\int_{E_\lambda^c\cap J_n\cap V_j} |a_n f_n(t)|^2\,dt \\ &\le \sum_{n\in\Lambda}\int_{E_\lambda^c\cap J_n\cap B_{\lambda,r}} |a_n f_n(t)|^2\,dt \\ &\le \sum_{n\in\Lambda}\int_{E_\lambda^c} |a_n f_n(t)|^2\,dt = \int_{E_\lambda^c} Sg(t)^2\,dt. \end{aligned}$$

Combined with (5.5), this completes the proof. ∎

LEMMA 5.4. *Let V be an open subinterval of $[0,1]$, $x := \inf V$, $y := \sup V$ and $f = \sum_{n=-k+2}^\infty a_n f_n \in L^p[0,1]$ for $1 < p < 2$ with $\operatorname{supp} f \subset V$. Let $R > 1$ satisfy $R\gamma < 1$ for the constant γ from Theorem 2.7. Then*

$$\sum_{n=\mathrm{n}(V)}^\infty R^{p d_n(V)} |a_n|^p \|f_n\|^p_{L^p(\widetilde{V}^c)} \lesssim_{p,R} \|f\|_p^p, \tag{5.7}$$

where $\mathrm{n}(V) = \min\{n : \mathcal{T}_n \cap V \neq \emptyset\}$ *and* $\widetilde{V} = (\widetilde{x}, \widetilde{y})$ *with* $\widetilde{x} = x - 2|V|$ *and* $\widetilde{y} = y + 2|V|$.

Proof. First observe that $\widetilde{V}^c = [0, \widetilde{x}] \cup [\widetilde{y}, 1]$. We estimate only the part corresponding to $[0, \widetilde{x}]$ and assume that $\widetilde{x} > 0$. The other part is treated analogously.

Let $m \geq 0$ and define

$$T_m := \big\{n \in \mathbb{N} : n \geq \mathrm{n}(V),\ \mathrm{card}\{i \leq n : \widetilde{x} \leq t_i \leq x\} = m\big\}. \tag{5.8}$$

We remark that T_m is finite, since the sequence $(t_n)_{n=0}^{\infty}$ is dense in $[0,1]$.

We now split T_m into the following six subsets:

$$\begin{aligned}
T_m^{(1)} &= \{n \in T_m : J_n \subset [\widetilde{x}, x]\},\\
T_m^{(2)} &= \{n \in T_m : \widetilde{x} \in J_n,\ |J_n \cap [\widetilde{x}, x]| \geq |V|,\ J_n \not\subset [\widetilde{x}, x]\},\\
T_m^{(3)} &= \big\{n \in T_m : J_n \subset [0, \widetilde{x}] \text{ or}\\
&\qquad\qquad \big(\widetilde{x} \in J_n \text{ with } |J_n \cap [\widetilde{x}, x]| \leq |V| \text{ and } J_n \not\subset [\widetilde{x}, x]\big)\big\},\\
T_m^{(4)} &= \{n \in T_m : x \in J_n,\ |J_n \cap [\widetilde{x}, x]| \geq |V|,\ J_n \not\subset [\widetilde{x}, x]\},\\
T_m^{(5)} &= \big\{n \in T_m : J_n \subset [x, \widetilde{y}] \text{ or}\\
&\qquad\qquad \big(x \in J_n \text{ with } |J_n \cap [\widetilde{x}, x]| \leq |V| \text{ and } J_n \not\subset [\widetilde{x}, x]\big)\big\},\\
T_m^{(6)} &= \big\{n \in T_m : J_n \subset [\widetilde{y}, 1] \text{ or } \big(\widetilde{y} \in J_n \text{ with } J_n \not\subset [x, \widetilde{y}]\big)\big\}.
\end{aligned}$$

We treat each of these separately. Before examining sums like the one in (5.7) with n restricted to one of the above sets, we note that for all n we have, by definition of $a_n = \langle f, f_n \rangle$ and the support assumption on f,

$$|a_n|^p \leq \int_V |f(t)|^p\, dt \cdot \Big(\int_V |f_n(t)|^{p'}\, dt\Big)^{p-1}, \tag{5.9}$$

where $p' = p/(p-1)$ denotes the conjugate Hölder exponent to p.

Case 1: $n \in T_m^{(1)} = \{n \in T_m : J_n \subset [\widetilde{x}, x]\}$. Let $\widetilde{T}_m^{(1)} := T_m^{(1)} \setminus \{\min T_m^{(1)}\}$. By definition, the interval J_n is at most $k-1$ grid points in $\mathcal{T}_n$ away from t_n. Since the number m of grid points between $\widetilde{x}$ and x is constant for all $n \in T_m$, there are only $2(k-1)$ possibilities for J_n with $n \in \widetilde{T}_m^{(1)}$. By Lemma 4.2 applied with $\beta = 0$, every J_n is a characteristic interval of at most F_k points t_m, and thus

$$\mathrm{card}\, T_m^{(1)} \leq 2(k-1)F_k + 1. \tag{5.10}$$

By Lemmata 3.7 and 3.6 respectively,

$$\int_0^{\widetilde{x}} |f_n(t)|^p\, dt \lesssim \gamma^{p d_n(\widetilde{x})} \|f_n\|_p^p \quad \text{and} \quad \int_V |f_n(t)|^{p'}\, dt \lesssim \gamma^{p' d_n(V)} \|f_n\|_{p'}^{p'} \tag{5.11}$$

for $n \in T_m^{(1)}$. Furthermore, $d_n(\widetilde{x}) + d_n(V) = m$ by definition of d_n, the location of J_n and the fact that $n \in T_m^{(1)}$. So, using (5.9), (5.11) and Lemma 3.6, we get

$$\sum_{n\in T_m^{(1)}} R^{pd_n(V)}|a_n|^p \int_0^{\widetilde{x}} |f_n(t)|^p\,dt$$

$$\begin{aligned}
&\le \sum_{n\in T_m^{(1)}} R^{pd_n(V)} \int_V |f(t)|^p\,dt \cdot \Big(\int_V |f_n(t)|^{p'}\,dt\Big)^{p-1} \int_0^{\widetilde{x}} |f_n(t)|^p\,dt \\
&\lesssim \sum_{n\in T_m^{(1)}} R^{pd_n(V)} \gamma^{p(d_n(\widetilde{x})+d_n(V))} \|f_n\|_p^p \|f_n\|_{p'}^p \int_V |f(t)|^p\,dt \\
&\lesssim \sum_{n\in T_m^{(1)}} (R\gamma)^{pm} \int_V |f(t)|^p\,dt.
\end{aligned}$$

Finally, we employ (5.10) to obtain

$$\sum_{n\in T_m^{(1)}} R^{pd_n(V)}|a_n|^p \int_0^{\widetilde{x}} |f_n(t)|^p\,dt \lesssim (R\gamma)^{pm} \int_V |f(t)|^p\,dt, \tag{5.12}$$

which concludes the proof of Case 1.

CASE 2: $n \in T_m^{(2)} = \{n \in T_m : \widetilde{x} \in J_n,\ |J_n \cap [\widetilde{x}, x]| \ge |V|,\ J_n \not\subset [\widetilde{x}, x]\}$. In this case we have $d_n(V) = m$, and thus Lemma 3.7 implies

$$\int_V |f_n(t)|^{p'}\,dt \le \|f_n\|_{L^\infty(V)}^{p'} |V| \lesssim \gamma^{p'm} |J_n|^{-p'/2} |V|.$$

We use (5.9) and this estimate to obtain

$$\begin{aligned}
|a_n|^p \|f_n\|_p^p &\le \int_V |f(t)|^p\,dt \cdot \Big(\int_V |f_n(t)|^{p'}\,dt\Big)^{p-1} \|f_n\|_p^p \\
&\lesssim \int_V |f(t)|^p\,dt \cdot \gamma^{pm} |J_n|^{-p/2} |V|^{p-1} \|f_n\|_p^p.
\end{aligned}$$

Lemma 3.6 further yields

$$\begin{aligned}
|a_n|^p \|f_n\|_p^p &\lesssim \gamma^{pm} |J_n|^{-p/2+1-p/2} |V|^{p-1} \int_V |f(t)|^p\,dt \\
&\le \gamma^{pm} |J_n|^{1-p} |V|^{p-1} \|f\|_p^p.
\end{aligned} \tag{5.13}$$

If $n_0 < n_1 < \cdots < n_s$ is an enumeration of all elements in $T_m^{(2)}$, by definition of $T_m^{(2)}$ we have

$$J_{n_0} \supset J_{n_1} \supset \cdots \supset J_{n_s} \quad \text{and} \quad |J_{n_s}| \ge |V|.$$

Thus, Lemma 4.2 and the fact that $1 < p < 2$ imply

$$\sum_{n\in T_m^{(2)}} |J_n|^{1-p} \sim_p |J_{n_s}|^{1-p} \le |V|^{1-p}. \tag{5.14}$$

We finally use (5.13) and (5.14) to conclude that

$$\begin{aligned}\sum_{n\in T_m^{(2)}} R^{pd_n(V)}|a_n|^p\|f_n\|_p^p &\lesssim (R\gamma)^{pm}|V|^{p-1}\|f\|_p^p \sum_{n\in T_m^{(2)}} |J_n|^{1-p} \\ &\lesssim_p (R\gamma)^{pm}\|f\|_p^p.\end{aligned} \tag{5.15}$$

Case 3: $n \in T_m^{(3)} = \{n \in T_m : J_n \subset [0, \widetilde{x}]$ or $(\widetilde{x} \in J_n$ with $|J_n \cap [\widetilde{x}, x]| \le |V|$ and $J_n \not\subset [\widetilde{x}, x])\}$. For $n \in T_m^{(3)}$, we denote by $(x_i)_{i=1}^m$ the finite sequence of points in $\mathcal{T}_n \cap [\widetilde{x}, x]$ in increasing order and counting multiplicities. If there exists $n \in T_m^{(3)}$ such that x_1 is the right endpoint of J_n and $\widetilde{x} \in J_n$, we define $x^* := x_1$. If not, we set $x^* := \widetilde{x}$. By definition of $T_m^{(3)}$ and x^*, we have

$$|V| \le |[x^*, x]| \le 2|V|. \tag{5.16}$$

Furthermore, for all $n \in T_m^{(3)}$,

$$J_n \subset [0, x^*] \quad \text{and} \quad |[x^*, x] \cap \mathcal{T}_n| = m.$$

Moreover,

$$m + d_n(x^*) - k \le d_n(V) \le m + d_n(x^*), \tag{5.17}$$

where the exact value of $d_n(V)$ depends on the multiplicity of x^* in $\mathcal{T}_n$ (which cannot exceed k). By Lemma 3.7 and (5.17) we have

$$\sup_{t\in V} |f_n(t)| \lesssim \gamma^{m+d_n(x^*)} \frac{|J_n|^{1/2}}{|J_n| + \operatorname{dist}(x, J_n)}.$$

Hence

$$\int_V |f_n(t)|^{p'}\, dt \lesssim |V| \cdot \gamma^{p'(m+d_n(x^*))} \frac{|J_n|^{p'/2}}{(|J_n| + \operatorname{dist}(x, J_n))^{p'}}. \tag{5.18}$$

Employing (5.9), (5.18) and Lemma 3.6 gives

$$\begin{aligned} R^{pd_n(V)}|a_n|^p\|f_n\|_p^p &\le R^{pd_n(V)} \int_V |f(t)|^p\, dt \cdot \Big(\int_V |f_n(t)|^{p'}\, dt\Big)^{p-1} \|f_n\|_p^p \\ &\lesssim R^{pd_n(V)}\|f\|_p^p|V|^{p-1}\gamma^{p(m+d_n(x^*))} \frac{|J_n|^{p/2}}{(|J_n| + \operatorname{dist}(x, J_n))^p}\|f_n\|_p^p \\ &\lesssim R^{pd_n(V)}\|f\|_p^p|V|^{p-1}\gamma^{p(m+d_n(x^*))} \frac{|J_n|}{(|J_n| + \operatorname{dist}(x, J_n))^p}. \end{aligned}$$

Inequality (5.17) then yields

$$R^{pd_n(V)}|a_n|^p\|f_n\|_p^p \le (R\gamma)^{p(m+d_n(x^*))}\|f\|_p^p|V|^{p-1}\frac{|J_n|}{(|J_n|+\operatorname{dist}(x,J_n))^p}. \tag{5.19}$$

We now have to sum this inequality. In order to do this we split our analysis depending on the value of $d_n(x^*)$. For fixed $j \in \mathbb{N}_0$ we consider $n \in T_m^{(3)}$ with $d_n(x^*) = j$. Let $\beta = 1/4$. Then, by Lemma 4.2, each point t (which is not a grid point) belongs to at most F_k intervals J_n^β with $n \in T_m^{(3)}$ and $d_n(x^*) = j$. Here J_n^β is the unique closed interval with

$$|J_n^\beta| = \beta|J_n| \quad \text{and} \quad \inf J_n^\beta = \inf J_n.$$

Furthermore, for $t \in J_n$, we have

$$|J_n| + \operatorname{dist}(x,J_n) \ge x - t.$$

Hence

$$\sum_{\substack{n\in T_m^{(3)}\\ d_n(x^*)=j}} \frac{|J_n|\,|V|^{p-1}}{(|J_n|+\operatorname{dist}(x,J_n))^p} \le \beta^{-1} \sum_{\substack{n\in T_m^{(3)}\\ d_n(x^*)=j}} \int_{J_n^\beta} \frac{|V|^{p-1}}{(x-t)^p}\,dt$$

$$\le \frac{F_k}{\beta}|V|^{p-1}\int_{-\infty}^{x^*}(x-t)^{-p}\,dt \lesssim_p \frac{|V|^{p-1}}{(x-x^*)^{p-1}} \le 1,$$

where in the last step we used (5.16). Combining (5.19) and the last inequality and summing over j (here we use the fact that $R\gamma < 1$), we arrive at

$$\sum_{n\in T_m^{(3)}} R^{pd_n(V)}|a_n|^p\|f_n\|_p^p \lesssim_{p,R} (R\gamma)^{pm}\|f\|_p^p. \tag{5.20}$$

Case 4: $n \in T_m^{(4)} = \{n \in T_m : x \in J_n, |J_n \cap [\widetilde{x},x]| \ge |V|, J_n \not\subset [\widetilde{x},x]\}$. We can ignore the cases $m = 0$ and ($m = 1$ and $[\widetilde{x},x] \cap \mathcal{T}_n = \{x\}$) since these are settled in Case 2. We define $\widetilde{T}_m^{(4)}$ to be the set of all remaining indices from $T_m^{(4)}$. Let $n \in \widetilde{T}_m^{(4)}$. Then the definition of $T_m^{(4)}$ implies

$$d_n(V) = d_n([x,y]) = 0. \tag{5.21}$$

Moreover, there exists at least one point of $\mathcal{T}_n$ in V (since $n \ge \mathrm{n}(V)$ for $n \in T_m$) and at least one point of $\mathcal{T}_n$ in $[\widetilde{x},x]$ (since $m \ge 1$). Thus we have

$$|V| \le |J_n| \le 3|V|. \tag{5.22}$$

Since $x \in J_n$ for all $n \in \widetilde{T}_m^{(4)}$, the family $\{J_n : n \in \widetilde{T}_m^{(4)}\}$ is a decreasing collection of sets. Inequality (5.22) and a multiple application of Lemma 4.2 with sufficiently large β gives a constant c_k depending only on k such that

$$\operatorname{card}\widetilde{T}_m^{(4)} \le c_k. \tag{5.23}$$

We employ Lemmata 3.7 and 3.6 to get

$$\int_0^{\widetilde{x}} |f_n(t)|^p\,dt \lesssim \gamma^{pm}|J|^{p/2-p+1} = \gamma^{pm}|J|^{1-p/2} \lesssim \gamma^{pm}\|f_n\|_p^p. \tag{5.24}$$

Hence

$$\begin{aligned}\sum_{n\in\widetilde{T}_m^{(4)}} & R^{pd_n(V)}|a_n|^p \int_0^{\widetilde{x}} |f_n(t)|^p\,dt \\ &\lesssim \sum_{n\in\widetilde{T}_m^{(4)}} \int_V |f(t)|^p\,dt\cdot\Big(\int_V |f_n(t)|^{p'}\,dt\Big)^{p-1}\int_0^{\widetilde{x}} |f_n(t)|^p\,dt \\ &\lesssim \sum_{n\in\widetilde{T}_m^{(4)}} \int_V |f(t)|^p\,dt\cdot \|f_n\|_{p'}^p\gamma^{pm}\|f_n\|_p^p \le \sum_{n\in\widetilde{T}_m^{(4)}} \gamma^{pm}\|f\|_p^p,\end{aligned}$$

where we used (5.21) and (5.9) in the first inequality, (5.24) in the second and Lemma 3.6 in the last one. Consequently, considering (5.23), the last display implies

$$\sum_{n\in\widetilde{T}_m^{(4)}} R^{pd_n(V)}|a_n|^p \int_0^{\widetilde{x}} |f_n(t)|^p\,dt \lesssim \gamma^{pm}\|f\|_p^p. \tag{5.25}$$

Case 5: $n\in T_m^{(5)} = \{n\in T_m : J_n\subset[x,\widetilde{y}]$ or $(x\in J_n$ with $|J_n\cap[\widetilde{x},x]|\le |V|$ and $J_n\not\subset[\widetilde{x},x])\}$. If there exists $n\in T_m^{(5)}$ with $x_m=\inf J_n$, then we define $x'=x_m$. If there exists no such index, we set $x'=x$. We now fix $n\in T_m^{(5)}$. By definition of x' and $\widetilde{x}$,

$$m+d_n(x')-k\le d_n(\widetilde{x})\le m+d_n(x'). \tag{5.26}$$

The exact relation between $d_n(\widetilde{x})$ and $d_n(x')$ depends on the multiplicity of the point x' in the grid $\mathcal{T}_n$. By definition of $T_m^{(5)}$,

$$\operatorname{dist}(\widetilde{x},J_n)\le 5|V| \quad\text{and}\quad |V|\le\operatorname{dist}(\widetilde{x},J_n).$$

Moreover,

$$|J_n|\le|[x',\widetilde{y}]|\le 4|V| \quad\text{and}\quad d_n(V)\le d_n(x'). \tag{5.27}$$

The last two displays now imply

$$|J_n|+\operatorname{dist}(\widetilde{x},J_n)\sim|V|.$$

Lemma 3.7, together with the former observation, yields

$$\int_0^{\widetilde{x}} |f_n(t)|^p\,dt \lesssim \gamma^{pd_n(\widetilde{x})}\frac{|J_n|^{p/2}}{(|J_n|+\operatorname{dist}(\widetilde{x},J_n))^{p-1}} \lesssim \gamma^{pd_n(\widetilde{x})}\frac{|J_n|^{p/2}}{|V|^{p-1}}.$$

Inserting (5.26) in this inequality, we get

$$\int_0^{\widetilde{x}} |f_n(t)|^p\,dt \lesssim \gamma^{p(d_n(x')+m)} \frac{|J_n|^{p/2}}{|V|^{p-1}}. \tag{5.28}$$

For each $n \in T_m^{(5)}$, we split $[x', \widetilde{y}]$ into three disjoint subintervals I_ℓ, $1 \le \ell \le 3$, defined by

$$I_1 := [x', \inf J_n], \qquad I_2 := J_n, \qquad I_3 := [\sup J_n, \widetilde{y}].$$

Correspondingly, we set

$$a_{n,\ell} := \int_{I_\ell \cap V} f(t) f_n(t)\,dt, \qquad \ell = 1,2,3.$$

We start by analyzing the choice $\ell = 2$ and first observe that by definition of I_2,

$$|a_{n,2}|^p \le \|f_n\|_{p'}^p \int_{J_n} |f(t)|^p\,dt. \tag{5.29}$$

We split the index set $T_m^{(5)}$ further and look at the set of those $n \in T_m^{(5)}$ such that $d_n(x') = j$ for fixed $j \in \mathbb{N}_0$. These indices n may be arranged in packets such that the intervals J_n from one packet have the same left endpoint and the maximal intervals of different packets are disjoint. Observe that the intervals J_n from one packet form a decreasing collection of sets. Let J_{n_0} be the maximal interval of one packet. Define $\mathcal{I}_j := \{n \in T_m^{(5)} : d_n(x') = j, J_n \subset J_{n_0}\}$. Then we use (5.27) and (5.29) to estimate

$$\begin{aligned} E_{2,j} &:= \sum_{n \in \mathcal{I}_j} R^{p d_n(V)} |a_{n,2}|^p \int_0^{\widetilde{x}} |f_n(t)|^p\,dt \\ &\le \sum_{n \in \mathcal{I}_j} R^{pj} \|f_n\|_{p'}^p \int_{J_n} |f(t)|^p\,dt \cdot \int_0^{\widetilde{x}} |f_n(t)|^p\,dt. \end{aligned}$$

Continuing, we use (5.28) to get

$$E_{2,j} \lesssim R^{pj} \int_{J_{n_0}} |f(t)|^p\,dt \cdot \sum_{n \in \mathcal{I}_j} \|f_n\|_{p'}^p \gamma^{p(d_n(x')+m)} \frac{|J_n|^{p/2}}{|V|^{p-1}}.$$

By Lemma 3.6, $\|f_n\|_{p'} \sim |J|^{1/p'-1/2}$, and thus

$$E_{2,j} \lesssim (R\gamma)^{pj} \gamma^{pm} \int_{J_{n_0}} |f(t)|^p\,dt \cdot \sum_{n \in \mathcal{I}_j} \frac{|J_n|^{p-1}}{|V|^{p-1}}.$$

We apply Lemma 4.2 to the above sum to conclude that

$$E_{2,j} \lesssim_p (R\gamma)^{pj}\gamma^{pm} \int\limits_{J_{n_0}} |f(t)|^p\,dt \cdot \frac{|J_{n_0}|^{p-1}}{|V|^{p-1}} \lesssim (R\gamma)^{pj}\gamma^{pm} \int\limits_{J_{n_0}} |f(t)|^p\,dt,$$

where in the last inequality we used (5.27). Now, summing over all maximal intervals J_{n_0} and over j finally yields (note that $R\gamma < 1$)

$$\sum_{n\in T_m^{(5)}} R^{pd_n(V)}|a_{n,2}|^p \int\limits_0^{\widetilde{x}} |f_n(t)|^p\,dt \lesssim_{p,R} \gamma^{pm}\|f\|_p^p. \tag{5.30}$$

This completes the proof of the case $\ell = 2$.

Now consider $\ell = 3$. Fix $j \in \mathbb{N}_0$ and let $(n_{j,r})_{r=1}^\infty$ be the subsequence of all $n \in T_m^{(5)}$ with $d_n(x') = j$. For two such indices $n_1 < n_2$ we have either

$$(\inf J_{n_1} = \inf J_{n_2} \text{ and } J_{n_2} \subset J_{n_1}) \quad\text{or}\quad \sup J_{n_2} \le \inf J_{n_1}.$$

Observe that $J_{n_2} = J_{n_1}$ is possible, but by Lemma 4.2 (with $\beta = 0$) only F_k times, with F_k only depending on k. Therefore, with $\beta_{n_{j,r}} := \sup J_{n_{j,r}}$ for $r \ge 1$ and $\beta_{n_{j,0}} := \widetilde{y}$,

$$d_{n_{j,s}}(\beta_{n_{j,r}}) \ge \frac{s-r}{F_k} - 1, \qquad s \ge r \ge 1.$$

Thus for $s \ge r \ge 1$ by Lemmata 3.7 and 3.6 we obtain

$$\int\limits_{\beta_{n_{j,r}}}^{\beta_{n_{j,r-1}}} |f_{n_{j,s}}(t)|^{p'}\,dt \lesssim \gamma^{p' d_{n_{j,s}}(\beta_{n_{j,r}})}\|f_{n_{j,s}}\|_{p'}^{p'} \lesssim \gamma^{p'\frac{s-r}{F_k}}\|f_{n_{j,s}}\|_{p'}^{p'}, \tag{5.31}$$

and similarly, using also (5.26),

$$\int\limits_0^{\widetilde{x}} |f_{n_{j,s}}|^p\,dt \lesssim \gamma^{pd_{n_{j,s}}(\widetilde{x})}\|f_{n_{j,s}}\|_p^p \lesssim \gamma^{p(m+d_{n_{j,s}}(x'))}\|f_{n_{j,s}}\|_p^p. \tag{5.32}$$

Choosing $\kappa := \gamma^{1/(2F_k)} < 1$, we deduce that

$$\begin{aligned}
|a_{n_{j,s},3}|^p &= \Big|\int\limits_{\beta_{n_{j,s}}}^{\widetilde{y}} f(t)f_{n_{j,s}}(t)\,dt\Big|^p = \Big|\sum_{r=1}^s \kappa^{s-r}\kappa^{r-s}\int\limits_{\beta_{n_{j,r}}}^{\beta_{n_{j,r-1}}} f(t)f_{n_{j,s}}(t)\,dt\Big|^p \\
&\le \Big(\sum_{r=1}^s \kappa^{p'(s-r)}\Big)^{p/p'}\sum_{r=1}^s \kappa^{p(r-s)}\Big|\int\limits_{\beta_{n_{j,r}}}^{\beta_{n_{j,r-1}}} f(t)f_{n_{j,s}}(t)\,dt\Big|^p \\
&\lesssim \sum_{r=1}^s \kappa^{p(r-s)}\int\limits_{\beta_{n_{j,r}}}^{\beta_{n_{j,r-1}}} |f(t)|^p\,dt \cdot \Big(\int\limits_{\beta_{n_{j,r}}}^{\beta_{n_{j,r-1}}} |f_{n_{j,s}}(t)|^{p'}\,dt\Big)^{p/p'}.
\end{aligned}$$

We now use inequality (5.31) to obtain

$$(5.33)\qquad |a_{n_{j,s},3}|^p \lesssim \sum_{r=1}^{s} \gamma^{p\frac{s-r}{2F_k}} \int_{\beta_{n_{j,r}}}^{\beta_{n_{j,r-1}}} |f(t)|^p\,dt \cdot \|f_{n_{j,s}}\|_{p'}^p.$$

Combining (5.33) and (5.32) yields

$$\begin{aligned} E_{3,j} := \sum_{\substack{n\in T_m^{(5)}\\ d_n(x')=j}} R^{pd_n(V)}|a_{n,3}|^p\|f\|^p_{L^p(0,\widetilde{x})} &= \sum_{s\geq 1} R^{pj}|a_{n_{j,s},3}|^p\|f_{n_{j,s}}\|^p_{L^p(0,\widetilde{x})} \\ &\lesssim \sum_{s\geq 1} R^{pj} \sum_{r=1}^{s} \gamma^{p\frac{s-r}{2F_k}} \|f_{n_{j,s}}\|_{p'}^p \int_{\beta_{n_{j,r}}}^{\beta_{n_{j,r-1}}} |f(t)|^p\,dt \cdot \gamma^{p(m+j)}\|f_{n_{j,s}}\|_p^p. \end{aligned}$$

Using again Lemma 3.6 gives

$$E_{3,j} \lesssim \gamma^{pm}(R\gamma)^{pj} \sum_{r\geq 1} \int_{\beta_{n_{j,r}}}^{\beta_{n_{j,r-1}}} |f(t)|^p\,dt \cdot \sum_{s\geq r} \gamma^{p\frac{s-r}{2F_k}} \lesssim \gamma^{pm}(R\gamma)^{pj}\|f\|_p^p.$$

Summing over j finally yields

$$(5.34)\qquad \sum_{n\in T_m^{(5)}} R^{pd_n(V)}|a_{n,3}|^p\|f\|^p_{L^p(0,\widetilde{x})} \lesssim_{p,R} \gamma^{pm}\|f\|_p^p,$$

since $R\gamma < 1$. This finishes the proof of the case $\ell = 3$.

We now come to the final part, $\ell = 1$. Fix j and n such that $d_n(x') = j$ and let $L_{1,n}, \ldots, L_{j,n}$ be the grid intervals in the grid $\mathcal{T}_n$ between x' and J_n, from left to right. Observe that f_n is a polynomial on each $L_{i,n}$. We define

$$b_{i,n} := \int_{L_{i,n}} f(t)f_n(t)\,dt, \qquad 1 \leq i \leq j.$$

For n with $d_n(x') = j$, we clearly have $a_{n,1} = \sum_{i=1}^{j} b_{i,n}$, and Hölder's inequality implies

$$(5.35)\qquad |b_{i,n}|^p \leq \int_{L_{i,n}} |f(t)|^p\,dt \cdot \Big(\int_{L_{i,n}} |f_n(t)|^{p'}\,dt\Big)^{p/p'}.$$

Remark 3.8 yields the bound

$$\sup_{t\in L_{i,n}} |f_n(t)| \lesssim \gamma^{j-i}\frac{|J_n|^{1/2}}{|J_n| + \operatorname{dist}(J_n, L_{i,n}) + |L_{i,n}|},$$

and inserting this in (5.35) gives

$$(5.36)\qquad |b_{i,n}|^p \leq \int_{L_{i,n}} |f(t)|^p\,dt \cdot \gamma^{p(j-i)}\frac{|J_n|^{p/2}|L_{i,n}|^{p-1}}{(|J_n| + \operatorname{dist}(J_n, L_{i,n}) + |L_{i,n}|)^p}.$$

Observe that we have the elementary inequality

$$\frac{|J_n|^{p/2}|L_{i,n}|^{p-1}}{(|J_n|+\operatorname{dist}(J_n,L_{i,n})+|L_{i,n}|)^p}\,\frac{|J_n|^{p/2}}{|V|^{p-1}} \le \frac{|J_n|}{|V|^{p-1}}(|J_n|+\operatorname{dist}(J_n,L_{i,n})+|L_{i,n}|)^{p-2}. \tag{5.37}$$

Combining (5.36), (5.37) and (5.28) allows us to estimate (recall that we have assumed that $d_n(x')=j$)

$$\begin{aligned} &R^{pd_n(V)}|b_{i,n}|^p\cdot\int_0^{\widetilde{x}}|f_n(t)|^p\,dt \\ &\lesssim R^{pj}\gamma^{p(j-i)}\int_{L_{i,n}}|f(t)|^p\,dt\cdot\frac{|J_n|^{p/2}|L_{i,n}|^{p-1}}{(|J_n|+\operatorname{dist}(J_n,L_{i,n})+|L_{i,n}|)^p}\cdot\gamma^{p(j+m)}\frac{|J_n|^{p/2}}{|V|^{p-1}} \\ &\lesssim R^{pj}\gamma^{p(2j+m-i)}\frac{|J_n|}{|V|^{p-1}}(|J_n|+\operatorname{dist}(J_n,L_{i,n})+|L_{i,n}|)^{p-2}\int_{L_{i,n}}|f(t)|^p\,dt. \end{aligned}\tag{5.38}$$

For fixed j and i we consider those indices n such that $d_n(x')=j$, and the corresponding intervals $L_{i,n}$. These intervals can be collected in packets such that all intervals from one packet have the same left endpoint and the maximal intervals of different packets are disjoint. For $\beta=1/4$, we denote by J_n^β the unique interval that has the same right endpoint as J_n and length $\beta|J_n|$. The intervals J_n corresponding to $L_{i,n}$'s from one packet can now be grouped in the same way as the $L_{i,n}$'s, and thus Lemma 4.2 implies the existence of a constant F_k depending only on k such that every point $t\in[0,1]$ belongs to at most F_k intervals J_n^β corresponding to the intervals $L_{i,n}$ from one packet.

We now consider one such packet and denote by u^* the left endpoint of (all) intervals $L_{i,n}$ in the packet. Then for $t\in J_n^\beta$ we have

$$|J_n|+\operatorname{dist}(L_{i,n},J_n)+|L_{i,n}|\ge|t-u^*|. \tag{5.39}$$

If L_i^* is the maximal interval of the packet, (5.38) and (5.39) yield

$$\begin{aligned} &\sum_{n\colon L_{i,n}\text{ in one packet}} R^{pd_n(V)}|b_{i,n}|^p\|f_n\|^p_{L^p(0,\widetilde{x})} \\ &\lesssim \frac{R^{pj}\gamma^{p(2j+m-i)}}{|V|^{p-1}}\sum_n|J_n|(|J_n|+\operatorname{dist}(L_{i,n},J_n)+|L_{i,n}|)^{p-2}\int_{L_{i,n}}|f(t)|^p\,dt \\ &\lesssim \frac{R^{pj}\gamma^{p(2j+m-i)}}{|V|^{p-1}}\int_{L_i^*}|f(t)|^p\,dt\cdot\sum_n\int_{J_n^\beta}|t-u^*|^{p-2}\,dt. \end{aligned}$$

Since every point t belongs to at most F_k intervals J_n^β in one packet of $L_{i,n}$'s, by using $J_n \subset [x', \widetilde{y}]$ and $p < 2$ we get

$$\sum_{n\colon L_{i,n} \text{ in one packet}} R^{pd_n(V)} |b_{i,n}|^p \|f_n\|^p_{L^p(0,\widetilde{x})} \lesssim \frac{R^{pj}\gamma^{p(2j+m-i)}}{|V|^{p-1}} \int_{L_i^*} |f(t)|^p\,dt \cdot \int_{u^*}^{\widetilde{y}} |t-u^*|^{p-2}\,dt \lesssim R^{pj}\gamma^{p(2j+m-i)} \int_{L_i^*} |f(t)|^p\,dt,$$

where in the last inequality we used (5.27). Since the maximal intervals L_i^* of different packets are disjoint, we can sum over all packets (for fixed j and i) to obtain

$$\sum_{\substack{n\in T_m^{(5)}\\ d_n(x')=j}} R^{pd_n(V)} |b_{i,n}|^p \|f_n\|^p_{L^p(0,\widetilde{x})} \lesssim R^{pj}\gamma^{p(2j+m-i)} \|f\|_p^p. \tag{5.40}$$

Let $\kappa := \gamma^{1/2} < 1$. Then for n such that $d_n(x') = j$ we have

$$|a_{n,1}|^p = \Big|\sum_{i=1}^{j} b_{i,n}\Big|^p = \Big|\sum_{i=1}^{j} \kappa^{j-i}\kappa^{i-j} b_{i,n}\Big|^p \lesssim_p \sum_{i=1}^{j} \kappa^{p(i-j)} |b_{i,n}|^p. \tag{5.41}$$

Combining (5.41) with (5.40) we get

$$\sum_{\substack{n\in T_m^{(5)}\\ d_n(x')=j}} R^{pd_n(V)} |a_{1,n}|^p \|f_n\|^p_{L^p(0,\widetilde{x})} \lesssim_p \sum_{i=1}^{j} \kappa^{p(i-j)} \sum_{\substack{n\in T_m^{(5)}\\ d_n(x')=j}} R^{pd_n(V)} |b_{i,n}|^p \|f_n\|^p_{L^p(0,\widetilde{x})} \lesssim \sum_{i=1}^{j} \kappa^{p(i-j)} R^{pj}\gamma^{p(2j+m-i)} \|f\|_p^p \lesssim (R\gamma)^{pj}\gamma^{pm}\|f\|_p^p.$$

Since $R\gamma < 1$, we sum over j to conclude that finally

$$\sum_{n\in T_m^{(5)}} R^{pd_n(V)} |a_{n,1}|^p \|f_n\|^p_{L^p(0,\widetilde{x})} \lesssim_{p,R} \gamma^{pm}\|f\|_p^p. \tag{5.42}$$

This finishes the proof of the case $\ell = 1$.

We can now combine the inequalities for $\ell = 1, 2, 3$, that is, (5.42), (5.30) and (5.34), to complete the analysis of Case 5 with the estimate

$$\sum_{n\in T_m^{(5)}} R^{pd_n(V)} |a_n|^p \|f_n\|^p_{L^p(0,\widetilde{x})} \lesssim_{p,R} \gamma^{pm}\|f\|_p^p. \tag{5.43}$$

CASE 6: $n \in T_m^{(6)} = \{n \in T_m : J_n \subset [\widetilde{y}, 1] \text{ or } (\widetilde{y} \in J_n \text{ with } J_n \not\subset [x, \widetilde{y}])\}$. Similarly to (5.8), we use the symmetric splitting of the indices n into

$$T_{\mathrm{r},s} := \{n \geq \mathrm{n}(V) : |[y, \widetilde{y}] \cap \mathcal{T}_n| = s\},$$

where r stands for "right". These collections are again split into six subcollections $T_{\mathrm{r},s}^{(i)}$, $1 \leq i \leq 6$, where the two of interest are

$$\begin{aligned} T_{\mathrm{r},s}^{(2)} &= \{n \in T_{r,s} : \widetilde{y} \in J_n, |J_n \cap [y, \widetilde{y}]| \geq |V|, J_n \not\subset [y, \widetilde{y}]\}, \\ T_{\mathrm{r},s}^{(3)} &= \big\{n \in T_{r,s} : J_n \subset [\widetilde{y}, 1] \text{ or} \\ &\qquad\qquad (\widetilde{y} \in J_n \text{ with } |J_n \cap [y, \widetilde{y}]| \leq |V| \text{ and } J_n \not\subset [y, \widetilde{y}])\big\}. \end{aligned}$$

The results (5.15) and (5.20) for $T_m^{(2)}$ and $T_m^{(3)}$ respectively had the form

$$\sum_{n \in T_m^{(2)} \cup T_m^{(3)}} R^{pd_n(V)} |a_n|^p \|f_n\|_p^p \lesssim_{p,R} (R\gamma)^{pm} \|f\|_p^p.$$

Observe that the p-norm of f_n on the left hand side is over the whole interval $[0, 1]$. The same argument as for $T_m^{(2)}$ and $T_m^{(3)}$ yields

$$\sum_{n \in T_{\mathrm{r},s}^{(2)} \cup T_{\mathrm{r},s}^{(3)}} R^{pd_n(V)} |a_n|^p \|f_n\|_p^p \lesssim_{p,R} (R\gamma)^{ps} \|f\|_p^p. \tag{5.44}$$

Now, since

$$\bigcup_{m \geq 0} T_m^{(6)} \subset \bigcup_{s \geq 0} T_{\mathrm{r},s}^{(2)} \cup T_{\mathrm{r},s}^{(3)},$$

inequality (5.44) implies

$$\begin{aligned} \sum_{m=0}^{\infty} \sum_{n \in T_m^{(6)}} & R^{pd_n(V)} |a_n|^p \|f_n\|_p^p \\ &\leq \sum_{s=0}^{\infty} \sum_{n \in T_{r,s}^{(2)} \cup T_{r,s}^{(3)}} R^{pd_n(V)} |a_n|^p \|f_n\|_p^p \lesssim_{p,R} \|f\|_p^p. \end{aligned} \tag{5.45}$$

After summing (5.12), (5.15), (5.20), (5.25) and (5.43) over m, we add inequality (5.45) to obtain finally

$$\sum_{n \geq \mathrm{n}(V)} R^{pd_n(V)} |a_n|^p \|f_n\|_{L^p(0,\widetilde{x})}^p \lesssim_{p,R} \|f\|_p^p.$$

The symmetric inequality

$$\sum_{n \geq \mathrm{n}(V)} R^{pd_n(V)} |a_n|^p \|f_n\|_{L^p(\widetilde{y},1)}^p \lesssim_{p,R} \|f\|_p^p$$

is proved analogously, and thus the proof of the lemma is complete. ∎

6. Proof of the main theorem. In this section, we prove our main result, Theorem 1.1, that is, unconditionality of orthonormal spline systems corresponding to an arbitrary admissible point sequence $(t_n)_{n\ge 0}$ in reflexive L^p.

Proof of Theorem 1.1. We recall the notation

$$Sf(t) = \Big(\sum_{n=-k+2}^{\infty} |a_n f_n(t)|^2\Big)^{1/2}, \quad Mf(t) = \sup_{m\ge -k+2}\Big|\sum_{n=-k+2}^{m} a_n f_n(t)\Big|$$

when

$$f = \sum_{n=-k+2}^{\infty} a_n f_n.$$

Since $(f_n)_{n=-k+2}^{\infty}$ is a basis in $L^p[0,1]$, $1\le p<\infty$, Khintchine's inequality implies that a necessary and sufficient condition for $(f_n)_{n=-k+2}^{\infty}$ to be an unconditional basis in $L^p[0,1]$ for $1<p<\infty$ is

$$\|Sf\|_p \sim_p \|f\|_p, \quad f\in L^p[0,1]. \tag{6.1}$$

We will prove (6.1) for $1<p<2$ since the case $p>2$ then follows by a duality argument.

We first prove the inequality

$$\|f\|_p \lesssim_p \|Sf\|_p. \tag{6.2}$$

Let $f\in L^p[0,1]$ with $f=\sum_{n=-k+2}^{\infty} a_n f_n$. We may assume that the sequence $(a_n)_{n\ge -k+2}$ has only finitely many nonzero entries. We will prove (6.2) by showing that $\|Mf\|_p \lesssim_p \|Sf\|_p$.

We first observe that

$$\|Mf\|_p^p = p\int_0^\infty \lambda^{p-1}\psi(\lambda)\,d\lambda \tag{6.3}$$

with $\psi(\lambda) := [Mf>\lambda]$. We will decompose f into two parts φ_1,φ_2 and estimate the distribution functions $\psi_i(\lambda) := [M\varphi_i > \lambda/2]$, $i\in\{1,2\}$, separately. To define φ_i, for $\lambda>0$ we set

$$E_\lambda := [Sf>\lambda], \qquad B_\lambda := [\mathcal{M}\mathbb{1}_{E_\lambda} > 1/2],$$
$$\Gamma := \{n : J_n\subset B_\lambda, -k+2\le n<\infty\}, \qquad \Lambda := \Gamma^c;$$

recall that J_n is the characteristic interval corresponding to the grid point t_n and the function f_n. Then, let

$$\varphi_1 := \sum_{n\in\Gamma} a_n f_n \quad\text{and}\quad \varphi_2 := \sum_{n\in\Lambda} a_n f_n.$$

Now we estimate $\psi_1 = [M\varphi_1 > \lambda/2]$:

$$\begin{aligned}\psi_1(\lambda) &= |\{t \in B_\lambda : M\varphi_1(t) > \lambda/2\}| + |\{t \notin B_\lambda : M\varphi_1(t) > \lambda/2\}| \\ &\le |B_\lambda| + \frac{2}{\lambda}\int_{B_\lambda^c} M\varphi_1(t)\,dt \le |B_\lambda| + \frac{2}{\lambda}\int_{B_\lambda^c}\sum_{n\in\Gamma}|a_n f_n(t)|\,dt.\end{aligned}$$

We decompose B_λ into a disjoint collection of open subintervals of $[0,1]$ and apply Lemma 5.1 to each of those intervals to deduce that

$$\begin{aligned}\psi_1(\lambda) &\lesssim |B_\lambda| + \frac{1}{\lambda}\int_{B_\lambda} Sf(t)\,dt = |B_\lambda| + \frac{1}{\lambda}\int_{B_\lambda\setminus E_\lambda} Sf(t)\,dt + \frac{1}{\lambda}\int_{E_\lambda\cap B_\lambda} Sf(t)\,dt \\ &\le |B_\lambda| + |B_\lambda \setminus E_\lambda| + \frac{1}{\lambda}\int_{E_\lambda} Sf(t)\,dt,\end{aligned}$$

where in the last inequality we simply used the definition of E_λ. Since the Hardy–Littlewood maximal function operator $\mathcal{M}$ is of weak type $(1,1)$, we have $|B_\lambda| \lesssim |E_\lambda|$, and thus finally

$$\psi_1(\lambda) \lesssim |E_\lambda| + \frac{1}{\lambda}\int_{E_\lambda} Sf(t)\,dt. \tag{6.4}$$

We now estimate $\psi_2(\lambda)$. From Theorem 2.8 and the fact that $\mathcal{M}$ is a bounded operator on $L^2[0,1]$ we obtain

$$\begin{aligned}\psi_2(\lambda) &\lesssim \frac{1}{\lambda^2}\|\mathcal{M}\varphi_2\|_2^2 \lesssim \frac{1}{\lambda^2}\|\varphi_2\|_2^2 = \frac{1}{\lambda^2}\|S\varphi_2\|_2^2 \\ &= \frac{1}{\lambda^2}\Big(\int_{E_\lambda} S\varphi_2(t)^2\,dt + \int_{E_\lambda^c} S\varphi_2(t)^2\,dt\Big).\end{aligned}$$

We apply Lemma 5.3 to the first summand to get

$$\psi_2(\lambda) \lesssim \frac{1}{\lambda^2}\int_{E_\lambda^c} S\varphi_2(t)^2\,dt. \tag{6.5}$$

Thus, combining (6.4) and (6.5) gives

$$\psi(\lambda) \le \psi_1(\lambda) + \psi_2(\lambda) \lesssim |E_\lambda| + \frac{1}{\lambda}\int_{E_\lambda} Sf(t)\,dt + \frac{1}{\lambda^2}\int_{E_\lambda^c} Sf(t)^2\,dt.$$

Inserting this into (6.3), we obtain

$$\begin{aligned}\|Mf\|_p^p &\lesssim p\int_0^\infty \lambda^{p-1}|E_\lambda|\,d\lambda + p\int_0^\infty \lambda^{p-2}\int_{E_\lambda} Sf(t)\,dt\,d\lambda \\ &\quad + p\int_0^\infty \lambda^{p-3}\int_{E_\lambda^c} Sf(t)^2\,dt\,d\lambda \\ &= \|Sf\|_p^p + p\int_0^1 Sf(t)\int_0^{Sf(t)} \lambda^{p-2}\,d\lambda\,dt + p\int_0^1 Sf(t)^2\int_{Sf(t)}^\infty \lambda^{p-3}\,d\lambda\,dt,\end{aligned}$$

and thus, since $1 < p < 2$,

$$\|Mf\|_p \lesssim_p \|Sf\|_p.$$

So, the inequality $\|f\|_p \lesssim_p \|Sf\|_p$ is proved.

We now turn to the proof of

$$\|Sf\|_p \lesssim_p \|f\|_p, \qquad 1 < p < 2. \tag{6.6}$$

It is enough to show that S is of weak type (p,p) whenever $1 < p < 2$. This is because S is (clearly) also of strong type 2 and we can use the Marcinkiewicz interpolation theorem to obtain (6.6). Thus we have to show

$$|[Sf > \lambda]| \lesssim_p \|f\|_p^p/\lambda^p, \qquad f \in L^p[0,1],\ \lambda > 0. \tag{6.7}$$

We fix f and $\lambda > 0$, define $G_\lambda := [\mathcal{M}f > \lambda]$ and observe that

$$|G_\lambda| \lesssim_p \|f\|_p^p/\lambda^p, \tag{6.8}$$

since $\mathcal{M}$ is of weak type (p,p), and, by the Lebesgue differentiation theorem,

$$|f| \le \lambda \quad \text{a.e. on } G_\lambda^c. \tag{6.9}$$

We decompose the open set $G_\lambda \subset [0,1]$ into a collection $(V_j)_{j=1}^\infty$ of disjoint open subintervals of $[0,1]$ and split f into

$$h := f \cdot \mathbb{1}_{G_\lambda^c} + \sum_{j=1}^\infty T_{V_j} f, \qquad g := f - h,$$

where for fixed index j, $T_{V_j} f$ is the projection of $f \cdot \mathbb{1}_{V_j}$ onto the space of polynomials of order k on the interval V_j.

We treat the functions h, g separately. The definition of h implies

$$\|h\|_2^2 = \int_{G_\lambda^c} |f(t)|^2\,dt + \sum_{j=1}^\infty \int_{V_j} (T_{V_j} f)(t)^2\,dt,$$

since the intervals V_j are disjoint. We apply (6.9) to the first summand and (2.1) to the second to obtain

$$\|h\|_2^2 \lesssim \lambda^{2-p} \int_{G_\lambda^c} |f(t)|^p\,dt + \lambda^2 |G_\lambda|,$$

and thus, in view of (6.8),

$$\|h\|_2^2 \lesssim_p \lambda^{2-p} \|f\|_p^p.$$

Hence

$$|[Sh > \lambda/2]| \le \frac{4}{\lambda^2}\|Sh\|_2^2 = \frac{4}{\lambda^2}\|h\|_2^2 \lesssim_p \frac{\|f\|_p^p}{\lambda^p},$$

which concludes the proof of (6.7) for h.

We turn to the proof of (6.7) for g. Since $p<2$, we have

$$Sg(t)^p = \Big(\sum_{n=-k+2}^{\infty} |\langle g, f_n\rangle|^2 f_n(t)^2\Big)^{p/2} \le \sum_{n=-k+2}^{\infty} |\langle g, f_n\rangle|^p |f_n(t)|^p. \tag{6.10}$$

For each j, we define $\widetilde{V}_j$ to be the open interval with the same center as V_j but with 5 times its length. Then set $\widetilde{G}_\lambda := \bigcup_{j=1}^\infty \widetilde{V}_j \cap [0,1]$ and observe that $|\widetilde{G}_\lambda| \le 5|G_\lambda|$. We get

$$|[Sg > \lambda/2]| \le |\widetilde{G}_\lambda| + \frac{2^p}{\lambda^p} \int_{\widetilde{G}_\lambda^c} Sg(t)^p\,dt.$$

By (6.8) and (6.10), this becomes

$$|[Sg > \lambda/2]| \lesssim_p \lambda^{-p}\Big(\|f\|_p^p + \sum_{n=-k+2}^{\infty} \int_{\widetilde{G}_\lambda^c} |\langle g, f_n\rangle|^p |f_n(t)|^p\,dt\Big).$$

But by definition of g and (2.2),

$$\|g\|_p^p = \sum_j \int_{V_j} |f(t) - T_{V_j} f(t)|^p\,dt \lesssim_p \sum_j \int_{V_j} |f(t)|^p\,dt \lesssim \|f\|_p^p,$$

so to prove $|[Sg > \lambda/2]| \le \lambda^{-p}\|f\|_p^p$, it is enough to show that

$$\sum_{n=-k+2}^{\infty} \int_{\widetilde{G}_\lambda^c} |\langle g, f_n\rangle|^p |f_n(t)|^p\,dt \lesssim \|g\|_p^p. \tag{6.11}$$

We let $g_j := g \cdot \mathbb{1}_{V_j}$. The supports of g_j are disjoint and we have $\|g\|_p^p = \sum_{j=1}^\infty \|g_j\|_p^p$. Furthermore $g = \sum_{j=1}^\infty g_j$ with convergence in L^p. Thus for each n,

$$\langle g, f_n\rangle = \sum_{j=1}^{\infty} \langle g_j, f_n\rangle,$$

and it follows from the definition of g_j that

$$\int_{V_j} g_j(t) p(t)\,dt = 0$$

for each polynomial p on V_j of order k. This implies that $\langle g_j, f_n\rangle = 0$ for $n < \mathrm{n}(V_j)$, where

$$\mathrm{n}(V) := \min\{n : \mathcal{T}_n \cap V \neq \emptyset\}.$$

Thus, for all $R > 1$ and every n,

$$
\begin{aligned}
(6.12)\qquad |\langle g, f_n\rangle|^p &= \Big|\sum_{j:\, n\ge \mathrm{n}(V_j)} \langle g_j, f_n\rangle\Big|^p \\
&\le \Big(\sum_{j:\, n\ge \mathrm{n}(V_j)} R^{d_n(V_j)}|\langle g_j, f_n\rangle| R^{-d_n(V_j)}\Big)^p \\
&\le \Big(\sum_{j:\, n\ge \mathrm{n}(V_j)} R^{pd_n(V_j)}|\langle g_j, f_n\rangle|^p\Big)\Big(\sum_{j:\, n\ge \mathrm{n}(V_j)} R^{-p'd_n(V_j)}\Big)^{p/p'},
\end{aligned}
$$

where $p' = p/(p-1)$. If we fix $n \ge \mathrm{n}(V_j)$, there is at least one point of the partition $\mathcal{T}_n$ contained in V_j. This implies that for each fixed $s \ge 0$, there are at most two indices j such that $n \ge \mathrm{n}(V_j)$ and $d_n(V_j) = s$. Therefore,

$$
\Big(\sum_{j:\, n\ge \mathrm{n}(V_j)} R^{-p'd_n(V_j)}\Big)^{p/p'} \lesssim_p 1,
$$

and from (6.12) we obtain

$$
|\langle g, f_n\rangle|^p \lesssim_p \sum_{j:\, n\ge \mathrm{n}(V_j)} R^{pd_n(V_j)}|\langle g_j, f_n\rangle|^p.
$$

Now we insert this inequality in (6.11) to get

$$
\begin{aligned}
&\sum_{n=-k+2}^{\infty} \int_{\widetilde{G}_\lambda^c} |\langle g, f_n\rangle|^p |f_n(t)|^p\,dt \\
&\qquad\lesssim_p \sum_{n=-k+2}^{\infty} \sum_{j:\, n\ge \mathrm{n}(V_j)} R^{pd_n(V_j)}|\langle g_j, f_n\rangle|^p \int_{\widetilde{G}_\lambda^c} |f_n(t)|^p\,dt \\
&\qquad\le \sum_{n=-k+2}^{\infty} \sum_{j:\, n\ge \mathrm{n}(V_j)} R^{pd_n(V_j)}|\langle g_j, f_n\rangle|^p \int_{\widetilde{V}_j^c} |f_n(t)|^p\,dt \\
&\qquad\le \sum_{j=1}^{\infty} \sum_{n\ge \mathrm{n}(V_j)} R^{pd_n(V_j)}|\langle g_j, f_n\rangle|^p \int_{\widetilde{V}_j^c} |f_n(t)|^p\,dt.
\end{aligned}
$$

We choose $R > 1$ such that $R\gamma < 1$ for $\gamma < 1$ from Theorem 2.7 and apply Lemma 5.4 to obtain

$$
\sum_{n=-k+2}^{\infty} \int_{\widetilde{G}_\lambda^c} |\langle g, f_n\rangle|^p |f_n(t)|^p\,dt \lesssim_p \sum_{j=1}^{\infty} \|g_j\|_p^p = \|g\|_p^p,
$$

proving (6.11) and hence $\|Sf\|_p^p \lesssim_p \|f\|_p^p$. Thus the proof of Theorem 1.1 is complete. ∎

Acknowledgements. During the development of this paper, the author visited repeatedly the IMPAN in Sopot/Gdańsk. It is his pleasure to thank this institution for its hospitality and for providing excellent working conditions. He is grateful to Anna Kamont, who suggested the problem considered in this paper and contributed many valuable remarks towards its solution, and to Zbigniew Ciesielski for many useful discussions.

The author is supported by the Austrian Science Fund, FWF project P 23987-N18 and the stays in Sopot/Gdańsk were supported by MNiSW grant N N201 607840.

References

[1] S. V. Bočkarev, *Some inequalities for Franklin series*, Anal. Math. 1 (1975), 249–257.

[2] W. Böhm, *Inserting new knots into B-spline curves*, Computer-Aided Design 12 (1980), 199–201.

[3] Z. Ciesielski, *Equivalence, unconditionality and convergence a.e. of the spline bases in L_p spaces*, in: Approximation Theory (Warszawa, 1975), Banach Center Publ. 4, PWN, Warszawa, 1979, 55–68.

[4] Z. Ciesielski, *Orthogonal projections onto spline spaces with arbitrary knots*, in: Function Spaces (Poznań, 1998), Lecture Notes in Pure Appl. Math. 213, Dekker, New York, 2000, 133–140.

[5] C. de Boor, *On the convergence of odd-degree spline interpolation*, J. Approx. Theory 1 (1968), 452–463.

[6] R. A. DeVore and G. G. Lorentz, *Constructive Approximation*, Grundlehren Math. Wiss. 303, Springer, Berlin, 1993.

[7] G. G. Gevorkyan and A. Kamont, *On general Franklin systems*, Dissertationes Math. 374 (1998), 59 pp.

[8] G. G. Gevorkyan and A. Kamont, *Unconditionality of general Franklin systems in* $L^p[0,1], 1 < p < \infty$, Studia Math. 164 (2004), 161–204.

[9] G. G. Gevorkyan and A. A. Sahakian, *Unconditional basis property of general Franklin systems*, Izv. Nats. Akad. Nauk Armenii Mat. 35 (2000), no. 4, 7–25.

[10] S. Karlin, *Total Positivity. Vol. I*, Stanford Univ. Press, Stanford, CA, 1968.

[11] M. Passenbrunner and A. Shadrin, *On almost everywhere convergence of orthogonal spline projections with arbitrary knots*, J. Approx. Theory 180 (2014), 77–89.

[12] A. Shadrin, *The L_∞-norm of the L_2-spline projector is bounded independently of the knot sequence: a proof of de Boor's conjecture*, Acta Math. 187 (2001), 59–137.

Markus Passenbrunner
Institute of Analysis
Johannes Kepler University Linz
Altenberger Strasse 69
4040 Linz, Austria
E-mail: markus.passenbrunner@jku.at

Received September 2, 2013
Revised version January 27, 2014 (7841)