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Simultaneous solutions of operator Sylvester equations

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Abstract. We consider simultaneous solutions of operator Sylvester equations $A_iX - XB_i = C_i$ $(1 \le i \le k)$, where $(A_1, \dots, A_k)$ and $(B_1, \dots, B_k)$ are commuting k-tuples of bounded linear operators on Banach spaces $\mathcal{E}$ and $\mathcal{F}$, respectively, and $(C_1, \dots, C_k)$ is a (compatible) k-tuple of bounded linear operators from $\mathcal{F}$ to $\mathcal{E}$, and prove that if the joint Taylor spectra of $(A_1, \dots, A_k)$ and $(B_1, \dots, B_k)$ do not intersect, then this system of Sylvester equations has a unique simultaneous solution.

1. Introduction. It is well known that if A and B are bounded linear operators on Banach spaces $\mathcal{E}$ and $\mathcal{F}$, respectively, such that $\sigma(A) \cap \sigma(B) = \emptyset$, then for each bounded linear operator $C : \mathcal{F} \to \mathcal{E}$, there exists a unique bounded linear operator $X : \mathcal{F} \to \mathcal{E}$ which is the solution of the operator equation

$$AX - XB = C. \tag{1.1}$$

In the case of finite-dimensional spaces $\mathcal{E}$ and $\mathcal{F}$, equation (1.1) is known as the *Sylvester equation*, and the above result is the Sylvester theorem, a well known fact which can be found in many textbooks in matrix theory (see, e.g., [5]). For bounded linear operators, the above result was first obtained by M. G. Krein (see, e.g., [3]) and then, independently, by Rosenblum [7], who showed that the solution operator X has the form

$$X = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - A)^{-1} C (\lambda I - B)^{-1}\, d\lambda, \tag{1.2}$$

where Γ is a union of closed contours in the plane, with total winding numbers 1 around $\sigma(A)$ and 0 around $\sigma(B)$.

In [6], the present authors considered the question of simultaneous solutions of a system of Sylvester equations

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$$A_iX - XB_i = C_i \quad (1 \le i \le k), \tag{1.3}$$

where $\mathcal{A} = (A_1, \ldots, A_k)$ and $\mathcal{B} = (B_1, \ldots, B_k)$ are commuting k-tuples of matrices of dimensions $n \times n$ and $m \times m$, respectively, and proved that the system (1.3) has a unique simultaneous solution X for every k-tuple of $m \times n$ matrices $\mathcal{C} = (C_1, \ldots, C_k)$ which satisfy the compatibility condition

$$A_iC_j - C_jB_i = A_jC_i - C_iB_j \quad (\text{for all } i, j,\ 1 \le i, j \le k), \tag{1.4}$$

if and only if the joint spectra of $\mathcal{A}$ and $\mathcal{B}$ do not intersect.

Recall that the joint spectrum for commuting matrices $\mathcal{A} = (A_1, \ldots, A_k)$ is defined as the joint point spectrum, that is, it consists of elements $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$ in $\mathbb{C}^k$ such that there exists a common eigenvector $\mathbf{x} \ne 0$, $A_i\mathbf{x} = \lambda_i\mathbf{x}$ for all $i = 1, \ldots, k$.

The main idea in the proof in [6] is the observation that if the joint spectrum of a k-tuple of commuting matrices $\mathcal{T} = (T_1, \ldots, T_k)$ consists of two disjoint components K_1 and K_2, then there exists an idempotent matrix F which commutes with $T_1, \ldots, T_k$ such that the joint spectrum of the restrictions of the k-tuple $(T_1, \ldots, T_k)$ to the range of F is K_1, and the joint spectrum of the restrictions of $(T_1, \ldots, T_k)$ to the range of $I - F$ is K_2.

In this paper, we consider systems of operator Sylvester equations (1.3), where $\mathcal{A} = (A_1, \ldots, A_k)$ and $\mathcal{B} = (B_1, \ldots, B_k)$ are commuting k-tuples of bounded linear operators on Banach spaces $\mathcal{E}$ and $\mathcal{F}$, respectively, and we extend the main result of [6] to this case.

There are several notions of joint spectrum of commuting k-tuples of operators, which all coincide with the joint point spectrum in the case of operators on finite-dimensional spaces, but are different in the general case of infinite-dimensional Banach spaces. Note that any definition of spectrum depends on a definition of singularity of a commuting k-tuple $\mathcal{T} = (T_1, \ldots, T_k)$: if the notion of singularity is defined, then the spectrum of $\mathcal{T}$ consists of all $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$ such that the k-tuple $\mathcal{T} - \boldsymbol{\lambda} = (T_1 - \lambda_1 I, \ldots, T_k - \lambda_k I)$ is singular.

The classical notion of spectrum of $\mathcal{T}$, $\mathrm{Sp}_{\mathfrak{B}}(\mathcal{T})$, is defined relative to a commutative Banach algebra $\mathfrak{B}$ containing $\mathcal{T}$. Namely, $\mathcal{T}$ is called *non-singular* (in $\mathfrak{B}$) if there exist $S_1, \ldots, S_k \in \mathfrak{B}$ such that $\sum_{i=1}^k T_iS_i = I$. As $\mathfrak{B}$ one can take, for example, the algebra $\mathrm{Alg}(\mathcal{T})$ generated by $\mathcal{T}$, or the bicommutant $\mathcal{T}''$ of $\mathcal{T}$.

J .L. Taylor introduced the notion of joint spectrum, $\mathrm{Sp}(\mathcal{T})$, which does not depend on any commutative algebra containing $\mathcal{T}$. Namely, to each commuting k-tuple $\mathcal{T}$ is associated a complex, called the Koszul complex, and $\mathcal{T}$ is called non-singular if its Koszul complex is exact (see precise definition below). It turns out that $\mathrm{Sp}(\mathcal{T}) \subset \mathrm{Sp}_{\mathfrak{B}}(\mathcal{T})$ for any $\mathfrak{B}$ and the inclusion is, in general, strict. Thus, the functional calculus introduced in [10] for functions analytic on $\mathrm{Sp}(\mathcal{T})$ is richer than the functional calculus based on other notions of joint spectrum, developed in the classical papers by Shilov [8], Arens [1], Calderón [2], and Waelbröck [11].

In this paper we prove the following theorem, which is an extension of the above mentioned result of [6].

THEOREM 1.1. *Let $\mathcal{A} = (A_1, \ldots, A_k)$ and $\mathcal{B} = (B_1, \ldots, B_k)$ be commuting k-tuples of bounded linear operators on Banach spaces $\mathcal{E}$ and $\mathcal{F}$, respectively, such that* $\mathrm{Sp}(\mathcal{A}) \cap \mathrm{Sp}(\mathcal{B}) = \emptyset$. *Then for every k-tuple $(C_1, \ldots, C_k)$ of bounded linear operators from $\mathcal{F}$ to $\mathcal{E}$ which satisfy the condition* (1.4)*, there exists a unique bounded linear operator $X : \mathcal{F} \to \mathcal{E}$ which is a simultaneous solution of the Sylvester operator equations* (1.3)*.*

Note that, since the Taylor spectrum $\mathrm{Sp}(\mathcal{T})$ is contained in $\mathrm{Sp}_{\mathfrak{B}}(\mathcal{T})$, the condition $\mathrm{Sp}(\mathcal{A}) \cap \mathrm{Sp}(\mathcal{B}) = \emptyset$ in Theorem 1.1 is less restrictive than analogous conditions when the Taylor spectrum is replaced by other notions of joint spectrum of $\mathcal{A}$ and $\mathcal{B}$ relative to commutative Banach algebras containing $\mathcal{A}$ and $\mathcal{B}$, respectively.

The proof of Theorem 1.1 uses the functional calculus developed by Taylor for analytic functions on $\mathrm{Sp}(\mathcal{T})$ and, in particular, the Idempotent Theorem, which states that if $\mathrm{Sp}(\mathcal{T})$ is a disjoint union of two compact sets K_1 and K_2, then there exists an idempotent operator F such that $\mathrm{Sp}(\mathcal{T}|\mathrm{range}(F)) = K_1$ and $\mathrm{Sp}(\mathcal{T}|\ker(F)) = K_2$ (see [10, Theorem 4.9]). This is an analog of the celebrated Shilov Idempotent Theorem in the theory of commutative Banach algebras [8]. The solution X of the operator equations (1.3) can be obtained from the idempotent operator F as in the case of simultaneous Sylvester equations for matrices considered in [6].

Below, $\mathcal{X}$, $\mathcal{E}$ and $\mathcal{F}$ are Banach spaces, and "operator" always means "bounded linear operator". We denote by $L(\mathcal{E})$ the set of all operators on $\mathcal{E}$, and by $L(\mathcal{F}, \mathcal{E})$ the set of all operators from $\mathcal{F}$ to $\mathcal{E}$. If $\mathcal{T}$ is a family of operators on $\mathcal{X}$, then $\mathcal{T}'$ denotes its commutant, $\mathcal{T}' = \{S \in L(\mathcal{X}) : ST = TS \ \forall T \in \mathcal{T}\}$, and $\mathcal{T}''$ denotes its bicommutant (the commutant of the commutant). For a domain $\mathcal{U}$ in $\mathbb{C}^k$, we denote by $\mathfrak{A}(\mathcal{U})$ the algebra of analytic functions on $\mathcal{U}$, and if K is a compact set in $\mathbb{C}^k$, then $\mathfrak{A}(K)$ is the algebra of functions analytic on a domain containing K.

2. Preliminaries: the Taylor joint spectrum. Let E^k be the complex exterior algebra with identity 1 generated by k generators. In other words, if we denote by $e_1, \ldots, e_k$ the natural basis in $\mathbb{C}^k$, and $E_0^k = \mathbb{C}$, $E_m^k = \underbrace{\mathbb{C}^k \wedge \cdots \wedge \mathbb{C}^k}_{m \text{ times}}$ for $m = 1, \ldots, k$, where $\wedge$ is a multiplication such that $e_i \wedge e_j = -e_j \wedge e_i$, then $E^k = \bigoplus_{m=0}^{k} E_m^k$. Note that the elements $e_{i_1} \wedge \cdots \wedge e_{i_m}$,

$1 \le i_1 < \cdots < i_m \le k$, form a basis in E_m^k, so that $\dim E_m^k = \binom{k}{m}$ and $\dim E^k = 2^k$.

Let $\mathcal{X}$ be a complex Banach space, $\mathcal{T} = (T_1, \ldots, T_k)$ a k-tuple of pairwise commuting operators on $\mathcal{X}$, and

$$\mathcal{X}_m = \mathcal{X} \otimes E_m^k. \tag{2.1}$$

Then $\mathcal{X}_m$ is spanned by the elements $x \otimes e_{i_1} \wedge \cdots \wedge e_{i_m}$, where $(i_1, \ldots, i_m)$ is a multi-index with $1 \le i_1 < \cdots < i_m \le k$, $x \in \mathcal{X}$. In other words, $\mathcal{X}_m$ is a direct sum of $\binom{k}{m}$ copies of $\mathcal{X}$, multi-indexed by $1 \le i_1 < \cdots < i_m \le k$.

For $m = 1, \ldots, k$, let $d_m : \mathcal{X}_m \to \mathcal{X}_{m-1}$ be defined by

$$d_m(x \otimes e_{i_1} \wedge \cdots \wedge e_{i_m}) = \sum_{l=1}^{m} (-1)^{l+1} T_{i_l} x \otimes e_{i_1} \wedge \cdots \wedge \widehat{e_{i_l}} \wedge \cdots \wedge e_{i_m},$$

where $\widehat{\ }$ means deletion. Then one can directly verify that d_m satisfies the condition $d_m d_{m+1} = 0$ for all $m = 0, 1, \ldots, k$ (where, of course, $d_0 : \mathcal{X}_0 \to \{0\}$ and $d_{k+1} : \{0\} \to X_k$ are naturally added), which means that the sequence

$$0 \xleftarrow{d_0} \mathcal{X}_0 \xleftarrow{d_1} \mathcal{X}_1 \leftarrow \cdots \xleftarrow{d_k} \mathcal{X}_k \xleftarrow{d_{k+1}} 0 \tag{2.2}$$

is a chain complex. This complex is called the *Koszul complex* of the k-tuple $\mathcal{T}$ on $\mathcal{X}$ and is denoted by $\mathcal{K}(\mathcal{X}, \mathcal{T})$.

Definition 2.1. The k-tuple $\mathcal{T}$ is called *non-singular* if its Koszul complex $\mathcal{K}(\mathcal{X}, \mathcal{T})$ is exact, i.e., if in the sequence (2.2) we have $\ker(d_m) = \operatorname{ran}(d_{m+1})$ for all $m = 0, 1, \ldots, k$.

For $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$, we let $\mathcal{T} - \boldsymbol{\lambda} := (T_1 - \lambda_1 I, \ldots, T_k - \lambda_k I)$.

Definition 2.2. A point $\boldsymbol{\lambda} \in \mathbb{C}^k$ is called a *non-singular point* for $\mathcal{T}$ if $\mathcal{T} - \boldsymbol{\lambda}$ is non-singular. The set of all singular points of $\mathcal{T}$ is called the (*Taylor*) *joint spectrum* of $\mathcal{T}$ and denoted by $\operatorname{Sp}(\mathcal{T})$.

Taylor [9] has shown that for each commutative k-tuple $\mathcal{T}$ in $L(\mathcal{X})$ ($\mathcal{X} \ne \{0\}$), $\operatorname{Sp}(\mathcal{T})$ is a non-empty compact subset in $\mathbb{C}^k$. Moreover, $\operatorname{Sp}(\mathcal{T}) \subset \operatorname{Sp}_{\mathcal{T}'}(\mathcal{T})$ and the inclusion is, in general, proper. Since $\mathcal{T}'$ contains any commutative Banach algebra $\mathfrak{B}$ which contains $\mathcal{T}$, this implies that $\operatorname{Sp}(\mathcal{T})$ is, in general, smaller than $\operatorname{Sp}_{\mathfrak{B}}(\mathcal{T})$ for any such $\mathfrak{B}$.

Taylor [10] also developed a functional calculus of several commuting operators. Namely, if $\mathcal{U}$ is an open set containing $\operatorname{Sp}(\mathcal{T})$ and f is a function analytic in $\mathcal{U}$, then $f(\mathcal{T})$ is defined as a bounded linear operator on $\mathcal{X}$. The mapping $f \mapsto f(\mathcal{T})$ defines a homomorphism from the algebra $\mathfrak{A}(\operatorname{Sp}(\mathcal{T}))$ of functions analytic in a domain containing $\operatorname{Sp}(\mathcal{T})$ into the algebra $\mathcal{T}''$. Moreover, under this homomorphism we have $1(\mathcal{T}) = I$ and $z_i(\mathcal{T}) = T_i$ for $i = 1, \ldots, k$ [10, Theorem 4.3]. If $\operatorname{Sp}(\mathcal{T}) = K_1 \cup K_2$, where K_1 and K_2 are disjoint compact sets, and $F = \chi_{K_1}(\mathcal{T})$, where χ_{K_1} is the characteristic function of K_1, then F is an idempotent operator (that is, a projection) which belongs to $\mathcal{T}''$. If we set $\mathcal{X}_1 = \mathrm{range}(F)$, $\mathcal{X}_2 = \ker(F)$, then $\mathcal{X}_1$ and $\mathcal{X}_2$ satisfy: (i) $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$; (ii) $\mathcal{X}_1$ and $\mathcal{X}_2$ are invariant under any operator which commutes with each $T_i, i = 1, \ldots, k$; (iii) $\mathrm{Sp}(\mathcal{T}|\mathcal{X}_1) = K_1$, $\mathrm{Sp}(\mathcal{T}|\mathcal{X}_2) = K_2$ [10, Theorem 4.9].

3. A relation between simultaneous solutions of Sylvester equations, commutant and bicommutant. First we observe the following simple but useful fact which has a straightforward proof.

Proposition 3.1. *Let $\mathcal{A} = (A_1, \ldots, A_k)$ be a k-tuple in $L(\mathcal{E})$, $\mathcal{B} = (B_1, \ldots, B_k)$ a k-tuple in $L(\mathcal{F})$ and $\mathcal{C} = (C_1, \ldots, C_k)$ a k-tuple in $L(\mathcal{F}, \mathcal{E})$, and let $\mathcal{T} = (T_1, \ldots, T_k)$ be defined by* (3.6) *below. Then a bounded linear operator $X : \mathcal{F} \to \mathcal{E}$ is a simultaneous solution of the system* (1.3) *if and only if $F_X \in \mathcal{T}'$, where*

$$F_X = \begin{pmatrix} I & X \\ \mathrm{O} & \mathrm{O} \end{pmatrix}. \tag{3.1}$$

In the next theorem, we show that $F_X \in \mathcal{T}''$ if and only if the corresponding homogeneous Sylvester equations have only the trivial simultaneous solutions. We would like to emphasize that in Proposition 3.1, as well as in Theorem 3.2 below, neither the commutativity of $\mathcal{A}$ and $\mathcal{B}$, nor the compatibility of $\mathcal{C}$ is assumed.

Theorem 3.2. *Let $\mathcal{A} = (A_1, \ldots, A_k)$ and $\mathcal{B} = (B_1, \ldots, B_k)$ be k-tuples in $L(\mathcal{E})$ and $L(\mathcal{F})$, respectively, and $\mathcal{C} = (C_1, \ldots, C_k)$ be a k-tuple in $L(\mathcal{F}, \mathcal{E})$. Suppose that the system* (1.3) *has a simultaneous solution X. Then $F_X \in \mathcal{T}''$ if and only if the homogeneous systems of Sylvester equations $A_iY - YB_i = \mathrm{O}$, $ZA_i - B_iZ = \mathrm{O}$ have only the trivial simultaneous solutions.*

Proof. First, we prove the theorem for the case $C_i = \mathrm{O}$ for all $i = 1, \ldots, k$ and $X = \mathrm{O}$.

Suppose the homogeneous systems $A_iY - YB_i = \mathrm{O}$, $ZA_i - B_iZ = \mathrm{O}$ have only the trivial simultaneous solutions. Let $T_i^{(0)} = A_i \oplus B_i$ and $\mathcal{T}^{(0)} = (T_1^{(0)}, \ldots, T_k^{(0)})$ and $F = I \oplus \mathrm{O}$. We must show that $F \in (\mathcal{T}^{(0)})''$.

Suppose $S \in (\mathcal{T}^{(0)})'$ and let S have the following block form:

$$S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}.$$

From $ST_i^{(0)} = T_i^{(0)}S$ we have

$$A_iS_1 = S_1A_i, \tag{3.2}$$
$$A_iS_2 = S_2B_i, \tag{3.3}$$
$$B_iS_3 = S_3A_i, \tag{3.4}$$
$$B_iS_4 = S_4B_i, \tag{3.5}$$

for all $i = 1, \dots, k$. From (3.3) and the fact that the equations $A_iY - YB_i = \mathrm{O}$ have only the trivial simultaneous solution it follows that $S_2 = 0$. Analogously, from (3.4) and the fact that the equations $ZA_i - B_iZ = \mathrm{O}$ have only the trivial simultaneous solution it follows that $S_3 = 0$. Therefore, $S = S_1 \oplus S_4$, so that $SF = FS$, that is, $F \in (\mathcal{T}^{(0)})''$.

Conversely, suppose that $F \in (\mathcal{T}^{(0)})''$. Let $Y : \mathcal{F} \to \mathcal{E}$ and $Z : \mathcal{E} \to \mathcal{F}$ be such that $A_iY - YB_i = \mathrm{O}$ and $ZA_i - B_iZ = \mathrm{O}$ for all $i = 1, \dots, k$. To show that $Y = \mathrm{O}$ we consider the operator G_Y defined by

$$G_Y = \begin{pmatrix} \mathrm{O} & Y \\ \mathrm{O} & \mathrm{O} \end{pmatrix}.$$

It is easy to see that $G_Y \in (\mathcal{T}^{(0)})'$. Hence $G_YF = FG_Y$, which implies $Y = \mathrm{O}$. Analogously, consider the operator

$$H_Z = \begin{pmatrix} \mathrm{O} & \mathrm{O} \\ Z & \mathrm{O} \end{pmatrix}$$

and observe that $H_Z \in (\mathcal{T}^{(0)})'$. Hence $H_ZF = FH_Z$, which implies $Z = \mathrm{O}$.

Now to derive the general case observe that if X is a simultaneous solution of (1.3), then the operators T_i defined by

$$T_i = \begin{pmatrix} A_i & C_i \\ \mathrm{O} & B_i \end{pmatrix} \qquad (1 \le i \le k) \tag{3.6}$$

are simultaneously similar to $T_i^{(0)}$. Namely, if

$$V = \begin{pmatrix} I & X \\ \mathrm{O} & I \end{pmatrix},$$

then it can be directly verified that $VT_iV^{-1} = T_i^{(0)}$ for all $i = 1, \dots, k$. Since $(\mathcal{T}^{(0)})' = \{VSV^{-1} : S \in T'\}$, $(\mathcal{T}^{(0)})'' = \{VSV^{-1} : S \in \mathcal{T}''\}$ and $F = VF_XV^{-1}$, we obtain the statement for the general case. ∎

4. Proof of the main result. Let $\mathcal{A} = (A_1, \dots, A_k)$ and $\mathcal{B} = (B_1, \dots, B_k)$ be commuting k-tuples in $L(\mathcal{E})$ and $L(\mathcal{F})$, respectively, and $\mathcal{C} = (C_1, \dots, C_k)$ be a k-tuple in $L(\mathcal{F}, \mathcal{E})$. Define $\mathcal{S}_i \in L(L(\mathcal{F}, E))$ by

$$\mathcal{S}_iX := A_iX - XB_i \quad (X \in L(\mathcal{F}, \mathcal{E}),\ 1 \le i \le k). \tag{4.1}$$

Then the Sylvester equations (1.3) can be rewritten in the form

$$\mathcal{S}_i X = C_i \quad (1 \le i \le k). \tag{4.2}$$

Since the $\mathcal{S}_i$ are pairwise commuting, we have $\mathcal{S}_j\mathcal{S}_i X = \mathcal{S}_i\mathcal{S}_j X$ $(1 \le i, j \le k)$. Hence from (4.2) we have the following necessary condition for the existence of a simultaneous solution of (1.3):

$$\mathcal{S}_i C_j = \mathcal{S}_j C_i \quad (1 \le i, j \le k), \tag{4.3}$$

which is another form of the compatibility condition (1.4). Furthermore, if we define operators T_i on $\mathcal{X} = \mathcal{E} \oplus \mathcal{F}$ by (3.6), then either of the conditions (1.4), (4.3) is equivalent to $T_iT_j = T_jT_i$ $(1 \le i, j \le k)$, i.e. the k-tuple $\mathcal{T} = (T_1, \ldots, T_k)$ is commuting.

From the definition of the joint Taylor spectrum we have the following fact, which can be seen by looking at the Koszul complex of $\mathcal{T}$ and the canonical short exact sequence $0 \to \mathcal{E} \to \mathcal{X} \to \mathcal{F} \to 0$ (see [9, Lemma 1.2]).

Lemma 4.1. $\mathrm{Sp}(\mathcal{T}) \subset \mathrm{Sp}(\mathcal{A}) \cup \mathrm{Sp}(\mathcal{B})$.

Proposition 4.2. *If $\mathcal{T} = (T_1, \ldots, T_k)$ is a commuting k-tuple which has the block upper triangular form* (3.6), *and f is analytic on a domain containing* $\mathrm{Sp}(\mathcal{A}) \cup \mathrm{Sp}(\mathcal{B})$, *then $f(\mathcal{T})$ has the block upper triangular form*

$$f(\mathcal{T}) = \begin{pmatrix} f(\mathcal{A}) & Y \\ \mathrm{O} & f(\mathcal{B}) \end{pmatrix} \tag{4.4}$$

for some $Y \in L(\mathcal{F}, \mathcal{E})$.

Proof. Note that since $\mathcal{E}$ is invariant under T_i, one can define operators $\widehat{T}_i$ on the quotient space $\widehat{\mathcal{X}} := \mathcal{X}/\mathcal{E}$ by $\widehat{T}_i\hat{x} = \widehat{T_ix}$. From the decomposition $\mathcal{X} = \mathcal{E} \oplus \mathcal{F}$ and the block upper triangular form (3.6) of T_i, it follows that if we define a mapping $\pi : \widehat{\mathcal{X}} \to \mathcal{F}$ by $\pi(\hat{x}) = y_0$, where $x = x_0 + y_0$ is the decomposition of x according to the direct sum $\mathcal{X} = \mathcal{F} \oplus \mathcal{E}$, then π is a (natural) isomorphism between $\widehat{\mathcal{X}}$ and $\mathcal{F}$ and

$$(\pi\widehat{T}_i)(\hat{x}) = (B_i\pi)(\hat{x}) \quad \text{for all } x \in \mathcal{X} \quad (1 \le i \le k). \tag{4.5}$$

If f is analytic on a domain containing $\mathrm{Sp}(\mathcal{A}) \cup \mathrm{Sp}(\mathcal{B})$, then, in view of the inclusion $\mathrm{Sp}(\mathcal{T}) \subset \mathrm{Sp}(\mathcal{A}) \cup \mathrm{Sp}(\mathcal{B})$, $f(\mathcal{T})$, as well as $f(\mathcal{A})$ and $f(\mathcal{B})$, are well defined. It can be seen from the definition of the functional calculus in [10] that if $x \in \mathcal{E}$, then $f(\mathcal{T})x \in \mathcal{E}$ and $f(\mathcal{T})x = f(\mathcal{A})x$ and if $\hat{x} \in \widehat{\mathcal{X}}$, then $f(\widehat{\mathcal{T}})\hat{x} = \widehat{f(\mathcal{T})x}$. From (4.5) it follows that $\pi f(\widehat{\mathcal{T}}) = f(\mathcal{B})\pi$ (see [10, Proposition 4.5]). This implies that $f(\mathcal{T})$ has the form (4.4). ∎

Proposition 4.3. *If $\mathcal{T} = (T_1, \ldots, T_k)$ is a commuting k-tuple which has the block upper triangular form*

$$T_i = \begin{pmatrix} A_i & A_iX - XB_i \\ \mathrm{O} & B_i \end{pmatrix} \qquad (1 \le i \le k), \tag{4.6}$$

and f is analytic on a domain containing $\mathrm{Sp}(\mathcal{A}) \cup \mathrm{Sp}(\mathcal{B})$, *then $f(\mathcal{T})$ has the block upper triangular form*

$$f(\mathcal{T}) = \begin{pmatrix} f(\mathcal{A}) & f(\mathcal{A})X - Xf(\mathcal{B}) \\ \mathrm{O} & f(\mathcal{B}) \end{pmatrix}. \tag{4.7}$$

Proof. By Proposition 4.2, $f(\mathcal{T})$ has the form (4.4). Let $C_i = A_iX - XB_i$ and F_X be defined by (3.1). By Proposition 3.1, $F_X \in \mathcal{T}'$, hence $F_X f(\mathcal{T}) = f(\mathcal{T})F_X$, which implies $Y = f(\mathcal{A})X - Xf(\mathcal{B})$. ∎

Proposition 4.3 for $k = 1$ is contained in [4].

Proof of Theorem 1.1. To prove the existence of a simultaneous solution X of (1.3), we apply Taylor's functional calculus described in Section 2. Namely, by Lemma 4.1 we have $\mathrm{Sp}(\mathcal{T}) \subset K_1 \cup K_2$, where $K_1 = \mathrm{Sp}(\mathcal{A})$, $K_2 = \mathrm{Sp}(\mathcal{B})$ are disjoint compact sets. Therefore, if χ is the characteristic function of K_1, then $\chi \in \mathfrak{A}(\mathrm{Sp}(\mathcal{T}))$ and, by Proposition 4.2,

$$\chi(\mathcal{T}) = \begin{pmatrix} \chi(\mathcal{A}) & X \\ \mathrm{O} & \chi(\mathcal{B}) \end{pmatrix} = \begin{pmatrix} I & X \\ \mathrm{O} & \mathrm{O} \end{pmatrix}. \tag{4.8}$$

Since $\chi(\mathcal{T})$ commutes with $\mathcal{T}$, it follows, by Proposition 3.1, that X is a simultaneous solution of (1.3). The uniqueness follows from Theorem 3.2, since $F_X = \chi(\mathcal{T}) \in \mathcal{T}''$. ∎

From Theorem 1.1 we obtain the following results, which are extensions of well known results from the case of single operators to the multivariate case.

Corollary 4.4. *Suppose $\mathcal{T} = (T_1, \ldots, T_k)$ is a commuting k-tuple in $L(\mathcal{E} \oplus \mathcal{F})$ which has the form* (3.6) *such that* $\mathrm{Sp}(\mathcal{A}) \cap \mathrm{Sp}(\mathcal{B}) = \emptyset$. *Then there exists an invertible operator $V \in L(\mathcal{E} \oplus \mathcal{F})$ such that*

$$VT_iV^{-1} = \begin{pmatrix} A_i & \mathrm{O} \\ \mathrm{O} & B_i \end{pmatrix} \quad (1 \le i \le k). \tag{4.9}$$

Indeed, the operator V can be chosen in the form

$$V = \begin{pmatrix} I & X \\ \mathrm{O} & I \end{pmatrix}, \tag{4.10}$$

where X is the simultaneous solution of equations (1.3).

Corollary 4.5. *Suppose $\mathcal{T} = (T_1, \ldots, T_k)$ is a commuting k-tuple in $L(\mathcal{E} \oplus \mathcal{F})$ which has the form* (3.6) *such that* $\mathrm{Sp}(\mathcal{A}) \cap \mathrm{Sp}(\mathcal{B}) = \emptyset$. *Then $\mathcal{T}'$*

consists of operators S of the form

$$S = \begin{pmatrix} Q & X \\ \mathrm{O} & R \end{pmatrix}, \tag{4.11}$$

where $Q \in \mathcal{A}'$, $R \in \mathcal{B}'$ and X is uniquely determined by Q and R as the simultaneous solution of $A_iX - XB_i = QC_i - C_iR$ $(1 \le i \le k)$.

Proof. First assume that $C_i = \mathrm{O}$ for $i = 1, \dots, k$. We show that in this case $\mathcal{T}' = \{S = Q \oplus R : Q \in \mathcal{A}',\ R \in \mathcal{B}'\}$. In fact, if $S = \left(\begin{smallmatrix} Q & M \\ N & R \end{smallmatrix}\right) \in \mathcal{T}'$, then from $ST_i = T_iS$ we have $A_iM = MB_i$ and $NA_i = B_iN$ for $i = 1, \dots, k$, so, by Theorem 1.1, we have $M = \mathrm{O}$, $N = \mathrm{O}$. The general case is obtained from this particular case and Corollary 4.4. ∎

Corollary 4.6. *Let $\mathcal{A} = (A_1, \dots, A_k)$ be a commuting k-tuple in $L(\mathcal{E})$, $(B_1, \dots, B_k)$ a commuting k-tuple in $L(\mathcal{F})$, $\mathcal{C} = (C_1, \dots, C_k)$ a k-tuple in $L(\mathcal{F}, E)$ which satisfies the compatibility condition* (1.3)*, and X the simultaneous solution of* (1.3)*. Furthermore, let $\mathcal{T} = (T_1, \dots, T_k)$ and F_X be defined by* (3.6) *and* (3.1)*. Then $\mathrm{Sp}(\mathcal{A}) \cap \mathrm{Sp}(\mathcal{B}) = \emptyset$ if and only if there is an analytic function f on $\mathrm{Sp}(\mathcal{A}) \cup \mathrm{Sp}(\mathcal{B})$ such that $F_X = f(\mathcal{T})$.*

Proof. The "only if" part is already contained in the proof of Theorem 1.1. To show the "if" part, we note that if f is analytic on $\mathrm{Sp}(\mathcal{A}) \cup \mathrm{Sp}(\mathcal{B})$ and $f(\mathcal{T}) = F_X$, then, by Proposition 4.2, $f(\mathcal{A}) = I$, $f(\mathcal{B}) = \mathrm{O}$. Applying [10, Theorem 4.8], we have $f(\boldsymbol{\lambda}) = 1$ for all $\boldsymbol{\lambda} \in \mathrm{Sp}(\mathcal{A})$ and $f(\boldsymbol{\lambda}) = 0$ for all $\boldsymbol{\lambda} \in \mathrm{Sp}(\mathcal{B})$, hence $\mathrm{Sp}(\mathcal{A}) \cap \mathrm{Sp}(\mathcal{B}) = \emptyset$. ∎

Corollary 4.6 for the case of a single operator $(k = 1)$ is contained in [4].

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