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Representations of modules over a *-algebra and related seminorms

by

Camillo Trapani and Salvatore Triolo (Palermo)

Abstract. Representations of a module $\mathfrak{X}$ over a *-algebra $\mathfrak{A}_\#$ are considered and some related seminorms are constructed and studied, with the aim of finding bounded *-representations of $\mathfrak{A}_\#$.

1. Introduction. As is known, if $\mathfrak{A}_\#$ is an involutive algebra and π is a *-representation of $\mathfrak{A}_\#$ with domain $\mathcal{D}(\pi)$ in a Hilbert space $\mathcal{H}$, then $\mathcal{D}(\pi)$ may be viewed as a left $\mathfrak{A}_\#$-module with module operation defined by

$$a\cdot\xi=\pi(a)\xi,\qquad a\in\mathfrak{A}_\#,\ \xi\in\mathcal{D}(\pi).$$

From the reverse point of view, one can ask if every $\mathfrak{A}_\#$-module $\mathfrak{X}$ admits a *representation* that reproduces the situation of the above example. Such a representation, to be called *modular*, consists of a couple (Φ,π) where Φ is a linear map of $\mathfrak{X}$ into some Hilbert space, π is a *-representation defined on $\mathcal{D}(\pi)=\Phi(\mathfrak{X})$, and Φ and π are coupled by the relation

$$\pi(a)\Phi(x)=\Phi(ax),\qquad a\in\mathfrak{A}_\#,\ x\in\mathfrak{X}.$$

The existence of a modular representation and its possible continuity were examined in [8] in the case where $\mathfrak{X}$ is a Banach module over the C^*-algebra $\mathfrak{A}_\#$ and it was proved that the existence of a modular representation is equivalent to the possibility of performing a sort of Gelfand–Naimark–Segal (GNS) representation starting from certain (in general, not everywhere defined) positive sesquilinear forms, called *modular biweights* for the close analogy they exhibit with *biweights* on a partial *-algebra [1, 2]. These existence results will be restated (mostly without proofs) in Section 2 for the general case where $\mathfrak{X}$ is a left $\mathfrak{A}_\#$-module. In the case considered in [8], $\mathfrak{A}_\#$ was taken as a C^*-algebra, hence there was no room for a possibly *unbounded* representation of $\mathfrak{A}_\#$. In more general situations (for instance, if $\mathfrak{A}_\#$ is simply a *-algebra), *-representations of $\mathfrak{A}_\#$ take values in the *-algebra $\mathcal{L}^\dagger(\mathcal{D}(\pi))$

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of all weakly continuous endomorphisms of a pre-Hilbert space $\mathcal{D}(\pi)$, and these are often unbounded operators.

The problem we want to investigate here originates from the very well-known fact that a *-algebra $\mathfrak{A}_\#$ admits a bounded representation if, and only if, there is a C^*-seminorm defined on $\mathfrak{A}_\#$ [5]. A similar approach is suggested here by the following simple example.

Let (Φ, π) be a modular representation of $\mathfrak{A}_\#$, with π a bounded *-representation of $\mathfrak{A}_\#$ in Hilbert space $\mathcal{H}$. If we put

$$p^\Phi(x) = \|\Phi(x)\|, \qquad x \in \mathfrak{X},$$

then p^Φ is a seminorm on $\mathfrak{X}$ enjoying the following properties:

(i) $p^\Phi(ax) \leq \|\pi(a)\| p^\Phi(x)$, $\forall a \in \mathfrak{A}_\#$, $x \in \mathfrak{X}$,
(ii) $p_0^\Phi(a) := \sup_{p^\Phi(x)=1} p^\Phi(ax) = \|\pi(a)\|$, $\forall a \in \mathfrak{A}_\#$.

Thus the *reduced* seminorm p_0^Φ of p^Φ is a C^*-seminorm on $\mathfrak{A}_\#$. This example suggests considering seminorms p on $\mathfrak{X}$ for which the map $x \mapsto ax$ is p-continuous for every $a \in \mathfrak{A}_\#$ (we name them M-seminorms) and the corresponding reduced seminorm p_0 is a C^*-seminorm (in this case p is called an MC^*-seminorm)

On the other hand, if $\mathfrak{X}$ admits a nontrivial MC^*-seminorm, then $\mathfrak{A}_\#$ certainly possesses bounded *-representations, but in general we cannot say that a modular representation (Φ, π) of $\mathfrak{X}$ with π bounded does really exist.

The aim of this paper is to characterize the existence of a modular representation (Φ, π) such that π is bounded and (Φ, π) satisfies prescribed conditions of continuity. More precisely, assuming that an M-seminorm p is defined on $\mathfrak{X}$, we look for a modular representation (Φ, π) with π bounded and such that

$$\begin{cases} \|\Phi(x)\| \leq p(x), & \forall x \in \mathfrak{X}, \\ \|\pi(a)\| \leq p_0(a), & \forall a \in \mathfrak{A}_\#. \end{cases}$$

We prove that a necessary and sufficient condition for this to hold is that the family $\mathcal{S}_p(\mathfrak{X})$ of all p-bounded modular invariant forms (i.e. everywhere defined modular biweights) is nontrivial. This characterization relies on the fact that if $\mathfrak{X}$ carries an M-seminorm p, then starting from $\mathcal{S}_p(\mathfrak{X})$, it is possible to construct a nontrivial MC^*-seminorm $\mathfrak{s}^p$ on $\mathfrak{X}$.

As a second step, coming back to the example discussed above, we consider the following stronger question: given an M-seminorm p on $\mathfrak{X}$, does there exist a modular representation (Φ, π) such that $p(x) = \|\Phi(x)\|$ for every $x \in \mathfrak{X}$ and $p_0(a) = \|\pi(a)\|$ for every $a \in \mathfrak{A}_\#$? The answer is that a necessary and sufficient condition for a representation (Φ, π) of this type to exist is that p is an MC^*-seminorm satisfying the parallelogram law. The latter condition is, of course, quite strong, because it forces $\mathfrak{X}$ to be contained (up to a quotient) in a Hilbert space. Thus, going one step further, we refer the same question to the MC^*-seminorm $\mathfrak{s}^p$, which is, in general, weaker than p. The outcome is that a necessary and sufficient condition for the existence of a modular representation (Φ, π) with π bounded and having the properties

$$\begin{cases} \|\Phi(x)\| = \mathfrak{s}^p(x), & \forall x \in \mathfrak{X}, \\ \|\pi(a)\| = \mathfrak{s}_0^p(a), & \forall a \in \mathfrak{A}_\#, \end{cases}$$

is that $\mathcal{S}_p(\mathfrak{X})$ is rich enough and has a maximum.

2. Modules and representations. In this section we collect some definitions and preliminary results that are needed in what follows. We also give some examples that, as we shall see, play a crucial role for representations.

Let $\mathfrak{A}_\#$ be a *-algebra, with involution $^\#$, and $\mathfrak{X}$ a vector space. We say that $\mathfrak{X}$ is a *left* $\mathfrak{A}_\#$*-module* if there is a bilinear map

$$(a, x) \mapsto ax$$

from $\mathfrak{A}_\# \times \mathfrak{X}$ into $\mathfrak{X}$ such that

$$(a_1 a_2)x = a_1(a_2 x), \qquad \forall a_1, a_2 \in \mathfrak{A}_\#,\ x \in \mathfrak{X}.$$

If $\mathfrak{A}_\#$ has no unit, we can consider its unitization $\mathfrak{A}_\#^e := \mathfrak{A}_\# \oplus \mathbb{C}$; then $\mathfrak{X}$ is also an $\mathfrak{A}_\#^e$-module with module multiplication defined by

$$(a, \lambda)x := ax + \lambda x, \qquad x \in \mathfrak{X}, a \in \mathfrak{A}_\#, \lambda \in \mathbb{C}.$$

Thus there is no loss of generality in assuming that $\mathfrak{A}_\#$ has a unit.

We shall always suppose that the module action of $\mathfrak{A}_\#$ on $\mathfrak{X}$ is *nontrivial*, i.e., if $a \in \mathfrak{A}_\#$ and $ax = 0$ for every $x \in \mathfrak{X}$, then $a = 0$.

DEFINITION 2.1. Let $\mathfrak{X}$ be a left $\mathfrak{A}_\#$-module. A *modular representation* of $\mathfrak{X}$ in a Hilbert space $\mathcal{H}$ consists of a linear map $\Phi : \mathfrak{X} \to \mathcal{H}$, with $\Phi(\mathfrak{X})$ dense in $\mathcal{H}$, and a $^\#$-representation of $\mathfrak{A}_\#$, $\pi : \mathfrak{A}_\# \to \mathcal{L}^\dagger(\mathcal{D}(\pi))$, with $\mathcal{D}(\pi) = \Phi(\mathfrak{X})$, such that

$$\Phi(ax) = \pi(a)\Phi(x), \qquad \forall a \in \mathfrak{A}_\#,\ x \in \mathfrak{X}.$$

A modular representation as above will be denoted (Φ, π).

Let $\mathfrak{X}$ be a left $\mathfrak{A}_\#$-module and φ a positive sesquilinear form on $\mathfrak{X} \times \mathfrak{X}$. Positivity implies that the Cauchy–Schwarz inequality holds and that φ is hermitian; i.e.,

- $|\varphi(x, y)| \leq \varphi(x, x)^{1/2} \varphi(y, y)^{1/2},\ \forall x, y \in \mathfrak{X}$,
- $\varphi(x, y) = \overline{\varphi(y, x)},\ \forall x, y \in \mathfrak{X}$.

DEFINITION 2.2. Let $\mathfrak{X}$ be a left $\mathfrak{A}_\#$-module. A positive sesquilinear form φ on $\mathfrak{X} \times \mathfrak{X}$ is said to be *modular invariant* if

$$\varphi(ax, y) = \varphi(x, a^\# y), \qquad \forall a \in \mathfrak{A}_\#,\ x, y \in \mathfrak{X}.$$

The set of all modular invariant forms of $\mathfrak{X}$ is denoted by $\mathcal{MI}(\mathfrak{X})$.

REMARK 2.3. A modular invariant sesquilinear form is an everywhere defined modular biweight. Modular biweights were introduced in [8] for studying modular representations of Banach C^*-modules. They are, in general, defined only on a submodule of $\mathfrak{X}$.

We will show that every $\varphi \in \mathcal{MI}(\mathfrak{X})$ can be used to construct a modular representation of $\mathfrak{X}$.

Let $\mathfrak{X}$ be a left $\mathfrak{A}_\#$-module. Assume that there exists a linear map $\Phi : \mathcal{D}(\Phi) \to \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space, such that $\Phi(\mathfrak{X})$ is dense in $\mathcal{H}$ and

$$\langle \Phi(ax) | \Phi(y) \rangle = \langle \Phi(x) | \Phi(a^\# y) \rangle, \qquad \forall a \in \mathfrak{A}_\#,\ x, y \in \mathfrak{X}.$$

Then a $^\#$-representation of $\mathfrak{A}_\#$ can be easily defined by putting

$$\begin{cases} \mathcal{D}(\pi) := \Phi(\mathfrak{X}), \\ \pi(a)\Phi(x) = \Phi(ax), \quad a \in \mathfrak{A}_\#,\ x \in \mathfrak{X}. \end{cases}$$

It is easily seen that (Φ, π) is a modular representation of $\mathfrak{X}$.

Moreover, if we define

$$\varphi_\Phi(x, y) := \langle \Phi(x) | \Phi(y) \rangle, \qquad x, y \in \mathfrak{X},$$

then φ is a modular invariant form in the sense of Definition 2.2. Conversely, we will show that any modular invariant form defines a modular representation.

THEOREM 2.4. *For each $\varphi \in \mathcal{MI}(\mathfrak{X})$, there exist a Hilbert space $\mathcal{H}_\varphi$, a linear map $\Phi_\varphi : \mathfrak{X} \to \mathcal{H}_\varphi$ and a closed *-representation π_φ of $\mathfrak{A}_\#$ into $\mathcal{H}_\varphi$ such that*:

- $\varphi(x, y) = \langle \Phi_\varphi(x) | \Phi_\varphi(y) \rangle,\ \forall x, y \in \mathfrak{X}$,
- $\varphi(ax, y) = \langle \pi_\varphi(a)\Phi_\varphi(x) | \Phi_\varphi(y) \rangle,\ \forall a \in \mathfrak{A}_\#,\ x, y \in \mathfrak{X}$.

Proof. We put

$$\mathfrak{N}_\varphi = \{x \in \mathfrak{X} : \varphi(x, x) = 0\} = \{x \in \mathfrak{X} : \varphi(x, y) = 0,\ \forall y \in \mathfrak{X}\}.$$

Let $\mathfrak{X}_\varphi := \mathfrak{X}/\mathfrak{N}_\varphi$ and put $\lambda_\varphi(x) := x + \mathfrak{N}_\varphi$, $x \in \mathfrak{X}$. Then $\mathfrak{X}_\varphi$ is a pre-Hilbert space with respect to the inner product

$$\langle \lambda_\varphi(x) | \lambda_\varphi(y) \rangle = \varphi(x, y), \qquad x, y \in \mathfrak{X}.$$

Let $\mathcal{H}_\varphi$ be the Hilbert space completion of $\mathfrak{X}_\varphi$. The map

$$\Phi_\varphi : x \in \mathfrak{X} \mapsto \lambda_\varphi(x) \in \mathfrak{X}_\varphi \subset \mathcal{H}_\varphi$$

is, clearly, linear. Moreover, $\Phi_\varphi(\mathfrak{X})$ is, by the definition itself, dense in $\mathcal{H}_\varphi$.

Since if $x \in \mathfrak{N}_\varphi$ then $ax \in \mathfrak{N}_\varphi$ for every $a \in \mathfrak{A}_\#$, the map

$$\pi^0_\varphi(a)\lambda_\varphi(x) = \lambda_\varphi(ax), \qquad x \in \mathfrak{X},$$

is a well-defined operator in $\mathfrak{X}_\varphi$. It is easy to prove that π^0_φ is a *-representation of $\mathfrak{A}_\#$. The closure π_φ of π^0_φ is the desired *-representation of $\mathfrak{A}_\#$. ∎

DEFINITION 2.5. The triple $(\Phi_\varphi, \pi_\varphi, \mathcal{H}_\varphi)$ is called the *GNS construction* for the modular invariant form φ of $\mathfrak{X}$.

The previous discussion can be summarized in the following

PROPOSITION 2.6. *Let $\mathfrak{X}$ be a left $\mathfrak{A}_\#$-module. The following statements are equivalent.*

(i) *There exists a nontrivial modular representation (Φ, π) of $\mathfrak{X}$.*
(ii) *There exists a linear map $\Phi : \mathfrak{X} \to \mathcal{H}$, with $\mathcal{H}$ a Hilbert space and $\Phi(\mathfrak{X})$ dense in $\mathcal{H}$, with the property*

$$\langle \Phi(ax)|\Phi(y)\rangle = \langle \Phi(x)|\Phi(a^\# y)\rangle, \quad \forall a \in \mathfrak{A}_\#,\ x, y \in \mathfrak{X}.$$

(iii) *There exists a nonzero modular invariant sesquilinear form φ on $\mathfrak{X}$.*

PROPOSITION 2.7. *Let $\mathfrak{X}$ be a left $\mathfrak{A}_\#$-module. The following statements are equivalent.*

(i) *There exists a modular representation (Φ, π) of $\mathfrak{X}$ with π bounded.*
(ii) *There exists $\varphi \in \mathcal{MI}(\mathfrak{X})$ such that*

$$\forall a \in \mathfrak{A}_\#,\ \exists \gamma_a > 0 : \quad \varphi(ax, ax) \le \gamma_a \varphi(x, x), \quad \forall x \in \mathfrak{X}.$$

Proof. If $\varphi \in \mathcal{MI}(\mathfrak{X})$, then the *-representation π_φ is bounded if, and only if, the condition stated in (ii) is fulfilled, as is readily checked. On the other hand, if (Φ, π) is a modular representation of $\mathfrak{X}$ with π bounded, it is easy to see that the modular invariant form φ defined by $\varphi(x, y) = \langle \Phi(x)|\Phi(y)\rangle$, $x, y \in \mathfrak{X}$, satisfies the condition given in (ii). ∎

3. Bounded *-representations and MC^*-seminorms. We now introduce some classes of seminorms on $\mathfrak{X}$ which will help us analyse the existence of bounded *-representations of $\mathfrak{A}_\#$.

Let $\mathfrak{X}$ be a left $\mathfrak{A}_\#$-module and p a seminorm on $\mathfrak{X}$. We say that p is an *M-seminorm* if, for each $a \in \mathfrak{A}_\#$, there exists $\gamma_a > 0$ such that

$$p(ax) \le \gamma_a p(x), \quad \forall x \in \mathfrak{X}.$$

In this case, we can define the *reduced* seminorm p_0 by

$$p_0(a) = \sup_{p(x)=1} p(ax), \quad a \in \mathfrak{A}_\#.$$

With this definition one has

$$p(ax) \le p_0(a)p(x), \quad \forall a \in \mathfrak{A}_\#,\ x \in \mathfrak{X}. \tag{3.1}$$

Moreover,

$$p_0(ab) \le p_0(a)p_0(b), \quad \forall a, b \in \mathfrak{A}_\#. \tag{3.2}$$

If p_0 is a C^*-seminorm on $\mathfrak{A}_\#$, i.e. if it satisfies the C^*-condition $p_0(a^\# a) = p_0(a)^2$ for every $a \in \mathfrak{A}_\#$, then we say that p is an *MC^*-seminorm.* We notice that the C^*-condition implies that p_0 is submultiplicative [7].

Let (Φ,π) be a modular representation of $\mathfrak{X}$. We put

$$p^{\Phi}(x)=\|\Phi(x)\|, \qquad x\in\mathfrak{X}.$$

Then p^{Φ} is a *Hilbert seminorm*, i.e. it satisfies the parallelogram law

$$p^{\Phi}(x+y)^2+p^{\Phi}(x-y)^2=2p^{\Phi}(x)^2+2p^{\Phi}(y)^2, \qquad \forall x,y\in\mathfrak{X}.$$

Moreover,

Proposition 3.1. *The following statements are equivalent.*

(i) *p^{Φ} is an M-seminorm.*
(ii) *π is bounded on $\mathcal{D}(\pi)=\Phi(\mathfrak{X})$.*
(iii) *p^{Φ} is an MC^*-seminorm.*

Proof. (i)⇒(ii): If p^{Φ} is an M-seminorm, then, for every $a\in\mathfrak{A}_{\#}$, there exists $\gamma_a>0$ such that

$$p^{\Phi}(ax)\le\gamma_a p^{\Phi}(x), \qquad \forall x\in\mathfrak{X}.$$

Then we have

$$\|\pi(a)\Phi(x)\|=\|\Phi(ax)\|=p^{\Phi}(ax)\le\gamma_a p^{\Phi}(x)=\gamma_a\|\Phi(x)\|.$$

Therefore the restriction of π to $\mathcal{D}(\pi)$ is bounded.

(ii)⇒(iii): We have

$$p_0^{\Phi}(a)=\sup_{p^{\Phi}(x)=1}p^{\Phi}(ax)=\sup_{\|\Phi(x)\|=1}\|\Phi(ax)\|=\sup_{\|\Phi(x)\|=1}\|\pi(a)\Phi(x)\|=\|\overline{\pi(a)}\|.$$

Therefore p_0^{Φ} is a C^*-seminorm on $\mathfrak{A}_{\#}$.

(iii)⇒(i): This is trivial. ■

Let $\mathfrak{X}$ be a left $\mathfrak{A}_{\#}$-module and p an M-seminorm on $\mathfrak{X}$. We denote by $\mathcal{C}_p(\mathfrak{X})$ the family of modular invariant sesquilinear forms φ that are *p-bounded*, i.e.

$$|\varphi(x,y)|\le\gamma p(x)p(y) \quad \text{for some } \gamma>0 \text{ and all } x,y\in\mathfrak{X}.$$

We denote by $\|\varphi\|_p$ the infimum of all γ's for which the above inequality holds. Finally, let

$$\mathcal{S}_p(\mathfrak{X})=\{\varphi\in\mathcal{C}_p(\mathfrak{X}):\ \|\varphi\|_p\le1\}.$$

We put

$$\mathfrak{s}^p(x)=\sup_{\varphi\in\mathcal{S}_p(\mathfrak{X})}\varphi(x,x)^{1/2}, \qquad x\in\mathfrak{X},$$

and

$$N(\mathfrak{s}^p)=\{x\in\mathfrak{X}:\mathfrak{s}^p(x)=0\}.$$

Then, as is easily seen, $\mathfrak{s}^p$ is a seminorm on $\mathfrak{X}$ satisfying $\mathfrak{s}^p(x)\le p(x)$ for every $x\in\mathfrak{X}$, and $N(\mathfrak{s}^p)$ is an $\mathfrak{A}_{\#}$-submodule of $\mathfrak{X}$.

Proposition 3.2. *For every M-seminorm p, $\mathfrak{s}^p$ is an MC^*-seminorm on $\mathfrak{X}$.*

Proof. For every $\varphi \in \mathcal{MI}(\mathfrak{X})$, we put

$$\omega_\varphi^x(a) = \varphi(ax, x), \qquad a \in \mathfrak{A}_\#.$$

Then ω_φ^x is a positive linear functional on $\mathfrak{A}_\#$, and if $\varphi \in \mathcal{C}_p(\mathfrak{X})$, it is p_0-continuous, since

$$|\omega_\varphi^x(a)| \leq \|\varphi\|_p \, p_0(a) p(x)^2, \qquad \forall a \in \mathfrak{A}_\#.$$

The family

$$\mathcal{F} = \{\omega_\varphi^x : \varphi \in \mathcal{C}_p(\mathfrak{X}),\, x \in \mathfrak{X}\}$$

is balanced in the sense of Yood [9]. Therefore, if we put

$$|a|_\mathcal{F} = \sup\{\omega_\varphi^x(a^\# a)^{1/2} : \varphi \in \mathcal{C}_p(\mathfrak{X}),\, x \in \mathfrak{X},\, \varphi(x,x) = 1\},$$

then

$$\mathcal{D}(\mathcal{F}) = \{a \in \mathfrak{A}_\# : |a|_\mathcal{F} < \infty\} = \mathfrak{A}_\#$$

and $|\cdot|_\mathcal{F}$ is a C^*-seminorm on $\mathfrak{A}_\#$.

Since, for every $\varphi \in \mathcal{F}$ and $x \in \mathfrak{X}$, the form ω_φ^x is $|\cdot|_\mathcal{F}$-continuous, we get, for every $n \in \mathbb{N}$,

$$\varphi(ax, ax) \leq \varphi(x,x)^{1-2^{-n}} (\|\omega_\varphi^x\|_\mathcal{F} \, |(a^\# a)^{2^n}|_\mathcal{F})^{2^{-n}}, \qquad \forall a \in \mathfrak{A}_\#,$$

where $\|\omega_\varphi^x\|_\mathcal{F} = \sup\{|\omega_\varphi^x(a)| : |a|_\mathcal{F} = 1\}$. Letting $n \to \infty$, we have

$$\varphi(ax, ax) \leq |a^\# a|_\mathcal{F} \, \varphi(x,x). \tag{3.3}$$

This in turn implies that

$$\mathfrak{s}^p(ax) \leq |a|_\mathcal{F}^2 \mathfrak{s}^p(x), \qquad \forall x \in \mathfrak{X},\, a \in \mathfrak{A}_\#.$$

Thus $\mathfrak{s}^p$ is an M-seminorm on $\mathfrak{X}$. From this estimate it also follows that

$$\mathfrak{s}_0^p(a) \leq |a|_\mathcal{F}, \qquad a \in \mathfrak{A}_\#. \tag{3.4}$$

To complete the proof we only need to prove the converse inequality. For this, making use of the definition of $\mathfrak{s}^p$ and of (3.1), for every $\varphi \in \mathcal{C}_p(\mathfrak{X})$, one has

$$|\varphi(ax, x)| \leq \|\varphi\|_p \, \mathfrak{s}^p(ax)\mathfrak{s}^p(x) \leq \|\varphi\|_p \mathfrak{s}_0^p(a)\mathfrak{s}^p(x)^2, \quad \forall a \in \mathfrak{A}_\#,\, x \in \mathfrak{X}.$$

Therefore, every ω_φ^x is $\mathfrak{s}_0^p$-continuous. Then, proceeding as we did for getting the inequality (3.3), we can prove

$$\omega_\varphi^x(a^\# a) = \varphi(ax, ax) \leq \mathfrak{s}_0^p(a^\# a)\varphi(x,x).$$

This implies that

$$|a^\# a|_\mathcal{F} = |a|_\mathcal{F}^2 \leq \mathfrak{s}_0^p(a^\# a), \qquad \forall a \in \mathfrak{A}_\#. \tag{3.5}$$

Hence, by (3.4),

$$\mathfrak{s}_0^p(a)^2 \leq |a|_\mathcal{F}^2 = |a^\# a|_\mathcal{F} \leq \mathfrak{s}_0^p(a^\# a) \leq \mathfrak{s}_0^p(a^\#)\mathfrak{s}_0^p(a), \qquad \forall a \in \mathfrak{A}_\#.$$

Thus, $\mathfrak{s}_0^p(a) \leq \mathfrak{s}_0^p(a^\#)$, which in turn implies $\mathfrak{s}_0^p(a) = \mathfrak{s}_0^p(a^\#)$ for every $a \in \mathfrak{A}_\#$. Coming back to (3.5), one finally obtains

$$|a|_{\mathcal{F}} \leq \mathfrak{s}_0^p(a), \quad \forall a \in \mathfrak{A}_\#.$$

Then $|a|_{\mathcal{F}} = \mathfrak{s}_0^p(a)$ for all $a \in \mathfrak{A}_\#$, and thus $\mathfrak{s}_0^p$ is a C^*-seminorm on $\mathfrak{A}_\#$. ■

REMARK 3.3. Since $|\omega_\varphi^x(a)| \leq p_0(a)p(x)^2$ for every $a \in \mathfrak{A}_\#$ and $x \in \mathfrak{X}$, we have

$$|\omega_\varphi^x(a^\# a)| \leq p_0(a)^2 \varphi(x,x), \quad \forall a \in \mathfrak{A}_\#,\ x \in \mathfrak{X}.$$

This implies that, in general, $\mathfrak{s}_0^p(a) \leq p_0(a)$ for every $a \in \mathfrak{A}_\#$.

REMARK 3.4. Given a left $\mathfrak{A}_\#$-module $\mathfrak{X}$, it may well happen that $\mathcal{S}_p(\mathfrak{X}) = \{0\}$. If this occurs, one clearly has $\mathfrak{s}^p(x) = 0$ for every $x \in \mathfrak{X}$. This is quite a singular case, since it implies that there are no nontrivial modular representations of $\mathfrak{X}$. For this reason, we will suppose that $\mathcal{S}_p(\mathfrak{X})$ is nontrivial.

DEFINITION 3.5. An MC^*-seminorm p on $\mathfrak{X}$ is called *regular* if $p(x) = \mathfrak{s}^p(x)$ for every $x \in \mathfrak{X}$.

As we have seen before, to every $\varphi \in \mathcal{S}_p(\mathfrak{X})$ there corresponds a GNS construction $(\Phi_\varphi, \pi_\varphi, \mathcal{H}_\varphi)$. The p-boundedness of φ implies the p-continuity of Φ_φ and $\|\Phi_\varphi(x)\| \leq p(x)$ for every $x \in \mathfrak{X}$. Conversely, to every linear map Φ from $\mathfrak{X}$ into some Hilbert space $\mathcal{H}$ with the property

$$\langle \Phi(ax)|\Phi(y)\rangle = \langle \Phi(x)|\Phi(a^\# y)\rangle, \quad \forall a \in \mathfrak{A}_\#,\ x,y \in \mathfrak{X},$$

and such that

$$\|\Phi(x)\| \leq p(x), \quad \forall x \in \mathfrak{X},$$

there corresponds a sesquilinear form $\varphi_\Phi \in \mathcal{S}_p(\mathfrak{X})$ with

$$\varphi_\Phi(x,y) = \langle \Phi(x)|\Phi(y)\rangle, \quad \forall x,y \in \mathfrak{X}.$$

Thus we have

PROPOSITION 3.6. *$N(\mathfrak{s}^p)$ coincides with the intersection of the kernels of all the maps Φ, where (Φ,π) is a modular representation of $\mathfrak{X}$ with $\|\Phi(x)\| \leq p(x)$ for every $x \in \mathfrak{X}$. $N(\mathfrak{s}^p)$ is a p-closed $\mathfrak{A}_\#$-submodule of $\mathfrak{X}$ (i.e. if $\{x_n\} \subset N(\mathfrak{s}^p)$ and $p(x_n - x) \to 0$, then $x \in N(\mathfrak{s}^p)$).*

As a consequence, the existence of an M-seminorm on $\mathfrak{X}$ such that $\mathcal{S}_p(\mathfrak{X})$ is nontrivial implies that $\mathfrak{s}_0^p$ is a nonzero C^*-seminorm on $\mathfrak{A}_\#$. Therefore, $\mathfrak{A}_\#$ admits a bounded *-representation π such that $\|\pi(a)\| = \mathfrak{s}_0^p(a)$ for every $a \in \mathfrak{A}_\#$. But we can say more.

PROPOSITION 3.7. *Let $\mathfrak{X}$ be a left $\mathfrak{A}_\#$-module and p an M-seminorm on $\mathfrak{X}$. The following conditions are equivalent.*

(i) *There exists a modular representation* (Φ,π) *with the properties*

$$\begin{cases} \|\Phi(x)\| \le p(x), & \forall x \in \mathfrak{X}, \\ \|\pi(a)\| \le p_0(a), & \forall a \in \mathfrak{A}_\#. \end{cases}$$

(ii) $\mathcal{S}_p(\mathfrak{X}) \neq \{0\}$.

Proof. (i)⇒(ii): Define

$$\varphi(x,y) = \langle \Phi(x)|\Phi(y)\rangle, \qquad x,y \in \mathfrak{X}.$$

Then it is easy to see that $\varphi \in \mathcal{S}_p(\mathfrak{X})$.

(ii)⇒(i): Assume that $\varphi \in \mathcal{S}_p(\mathfrak{X})$ and let $(\lambda_\varphi, \pi_\varphi, \mathcal{H}_\varphi)$ be the corresponding GNS construction. Then, putting as before $\Phi_\varphi(x) = \lambda_\varphi(x)$, $x \in \mathfrak{X}$, we have

$$\|\Phi_\varphi(x)\|^2 = \|\lambda_\varphi(x)\|^2 = \varphi(x,x) \le p(x)^2, \qquad \forall x \in \mathfrak{X},$$

and

$$\|\pi_\varphi(a)\lambda_\varphi(x)\|^2 = \varphi(ax,ax) \le \mathfrak{s}_0^p(a)^2\varphi(x,x) = \mathfrak{s}_0^p(a)^2\|\lambda_\varphi(x)^2\|, \qquad \forall a \in \mathfrak{A}_\#.$$

Hence π_φ is bounded and

$$\|\pi_\varphi(a)\| \le \mathfrak{s}_0^p(a) \le p_0(a), \qquad \forall a \in \mathfrak{A}_\#. \ \blacksquare$$

As we have seen, if an M-seminorm p on $\mathfrak{X}$ is defined, then, if $\mathcal{S}_p(\mathfrak{X}) \neq \{0\}$ there exists a nontrivial MC^*-seminorm on $\mathfrak{X}$, namely $\mathfrak{s}^p$. Since $\mathfrak{s}_0^p$ is a C^*-seminorm, it is then natural to pose the following

Question 1. Given an M-seminorm p on $\mathfrak{X}$, does there exist a modular representation (Φ,π) such that Φ is p-bounded and $\mathfrak{s}_0^p(a) = \|\overline{\pi(a)}\|$ for every $a \in \mathfrak{A}_\#$?

In order to answer this question, we first state the following stronger one:

Question 2. Given an M-seminorm p on $\mathfrak{X}$, does there exist a representation (Φ,π) of $\mathfrak{X}$ such that $p(x) = \|\Phi(x)\|$ for every $x \in \mathfrak{X}$ and $p_0(a) = \|\overline{\pi(a)}\|$ for every $a \in \mathfrak{A}_\#$?

If the answer to Question 2 is affirmative, then, by Proposition 3.1, p is automatically an MC^*-seminorm. Some additional properties of p and $\mathfrak{s}^p$ are given in the following

Proposition 3.8. *Let* $\mathfrak{X}$ *be a left* $\mathfrak{A}_\#$*-module and* p *an* M*-seminorm on* $\mathfrak{X}$*. Assume that there exists a modular representation* (Φ,π) *such that* $p(x) = \|\Phi(x)\|$ *for every* $x \in \mathfrak{X}$*. Then the following statements hold.*

(i) p *is a Hilbert seminorm.*

(ii) p *is a regular* MC^**-seminorm.*

(iii) $p_0(a) = \mathfrak{s}_0^p(a) = \|\overline{\pi(a)}\|$ *for every* $a \in \mathfrak{A}_\#$.

(iv) *The set* $\mathcal{S}_p(\mathfrak{X})$ *has a maximum, i.e. there exists* $\overline{\varphi} \in \mathcal{S}_p(\mathfrak{X})$ *such that*

$$\overline{\varphi}(x,x) = \sup_{\varphi \in \mathcal{S}_p(\mathfrak{X})} \varphi(x,x) = \mathfrak{s}^p(x)^2 = p(x)^2, \qquad \forall x \in \mathfrak{X}.$$

Proof. (i) Since $p(\cdot) = \|\Phi(\cdot)\|$ and $\|\Phi(\cdot)\|$ is a Hilbert seminorm, p must obey the parallelogram law.

(ii) We put, as before, $\varphi_\Phi(x,y) = \langle \Phi(x)|\Phi(y)\rangle$, $x, y \in \mathfrak{X}$. Then

$$|\varphi_\Phi(x,y)| = |\langle \Phi(x)|\Phi(y)\rangle| \le \|\Phi(x)\|\,\|\Phi(y)\| = p(x)p(y), \quad \forall x, y \in \mathfrak{X}.$$

Thus, $\varphi_\Phi \in \mathcal{S}_p(\mathfrak{X})$. Then we have

$$p(x)^2 = \|\Phi(x)\|^2 \le \sup_{\varphi\in\mathcal{S}_p(\mathfrak{X})} \varphi(x,x) = \mathfrak{s}^p(x)^2.$$

Hence $p(x) = \mathfrak{s}^p(x)$ for every $x \in \mathfrak{X}$.

(iii) The equality $p(\cdot) = \mathfrak{s}^p(\cdot)$ also implies that $\mathfrak{s}_0^p(a) = p_0(a)$ for every $a \in \mathfrak{A}_\#$. Moreover,

$$\begin{aligned} p_0(a) &= \sup_{p(x)=1} p(ax) = \sup_{\|\Phi(x)\|=1} \|\Phi(ax)\| \\ &= \sup_{\|\Phi(x)\|=1} \|\pi(a)\Phi(x)\| = \|\overline{\pi(a)}\|, \quad \forall a \in \mathfrak{A}_\#. \end{aligned}$$

(iv) The form φ_Φ is indeed a maximum for $\mathcal{S}_p(\mathfrak{X})$. We have, in fact, for any $\varphi \in \mathcal{S}_p(\mathfrak{X})$,

$$\varphi(x,x) \le p(x)^2 = \|\Phi(x)\|^2 = \langle \Phi(x)|\Phi(x)\rangle = \varphi_\Phi(x,x), \quad \forall x \in \mathfrak{X}. \ \blacksquare$$

In order to prove the converse of the previous proposition, we need the following

LEMMA 3.9. *Let $\mathfrak{A}$ be a C^*-algebra with unit e, with norm $\|\cdot\|$ and involution *. Let $\mathfrak{B}$ be a closed subalgebra of $\mathfrak{A}$ which is a C^*-algebra, with respect to the same norm $\|\cdot\|$ and the involution $^\#$, and such that $e \in \mathfrak{B}$ and $e^\# = e$. Then $x^\# = x^*$ for every $x \in \mathfrak{B}$.*

Proof. Let F be a positive linear functional on $\mathfrak{A}$. Then F is bounded and $\|F\| = F(e)$. Let F_0 denote the restriction of F to $\mathfrak{B}$. Then

$$F(e) \le \|F_0\| \le \|F\| = F(e).$$

Hence, F_0 is positive on $\mathfrak{B}$, i.e., $F(x^\# x) \ge 0$ for every $x \in \mathfrak{B}$. Let now $y \in \mathfrak{B}$ with $y^\# = y$. Then $F_0(y)$ is real and, since F is hermitian, we get

$$F(y^*) = \overline{F(y)} = F(y).$$

Hence $F(y^* - y) = 0$ and, from the arbitrariness of F, $y = y^*$.

Let now $x \in \mathfrak{B}$. Then $x = z + iw$ where $z = (x + x^\#)/2$ and $w = (x - x^\#)/2i$. Then, since $z = z^\#$ and $w = w^\#$, one has $z = z^*$ and $w = w^*$. These imply that

$$x + x^\# = x^* + x^{\#*} \quad \text{and} \quad x - x^\# = x^{\#*} - x^*,$$

whence it follows that $x = x^{\#*}$. We conclude that $x^* = x^\#$. $\blacksquare$

PROPOSITION 3.10. *Let $\mathfrak{X}$ be a left $\mathfrak{A}_\#$-module and p an M-seminorm on $\mathfrak{X}$. The following statements are equivalent.*

(i) *p is an MC^*-seminorm and a Hilbert seminorm.*
(ii) *There exists a modular representation (Φ, π) such that $\|\Phi(x)\| = p(x)$ for every $x \in \mathfrak{X}$ and $\|\pi(a)\| = p_0(a)$ for every $a \in \mathfrak{A}_\#$.*

Proof. We need only prove that (i)⇒(ii). Since p satisfies the parallelogram law, if we put

$$\varphi_p(x, y) = \frac{1}{4}\sum_{k=0}^{3} i^k p(x + i^k y)^2, \qquad x, y \in \mathfrak{X},$$

then φ_p is a positive sesquilinear form on $\mathfrak{X}$ and

$$\{x \in \mathfrak{X} : \varphi_p(x, x) = 0\} = \{x \in \mathfrak{X} : p(x) = 0\} =: N(p).$$

Then $\mathfrak{X}/N(p)$ is a pre-Hilbert space with inner product

$$\langle \lambda_p(x) | \lambda_p(y)\rangle_p = \varphi_p(x, y), \qquad x, y \in \mathfrak{X},$$

where $\lambda_p(x) := x + N(p)$. Let $\mathcal{H}_p$ denote the Hilbert space completion of $\mathfrak{X}/N(p)$. We put $\Phi(x) = \lambda_p(x)$, $x \in \mathfrak{X}$. Then Φ is a linear map of $\mathfrak{X}$ into $\mathcal{H}_p$. By the definition itself, $\Phi(\mathfrak{X})$ is dense in $\mathfrak{X}$ and $\|\Phi(x)\| = p(x)$ for every $x \in \mathfrak{X}$.

For every $a \in \mathfrak{A}_\#$, we define a linear map $\pi(a)$ on $\mathfrak{X}/N(p)$ by

$$\pi(a)\lambda_p(x) = \lambda_p(ax), \qquad x \in \mathfrak{X}.$$

This map is well-defined, since if $a \in \mathfrak{A}_\#$ and $x \in N(p)$, then $ax \in N(p)$. Moreover, $\pi(a)$ is bounded. Indeed,

$$\begin{aligned}\|\pi(a)\lambda_p(x)\|_p^2 &= \|\lambda_p(ax)\|_p^2 = \varphi_p(ax, ax) = p(ax)^2 \\ &\le p_0(a)^2 p(x)^2 = p_0(a)^2\|\lambda_p(x)\|_p^2.\end{aligned}$$

Therefore $\pi(a)$ extends to a bounded operator on $\mathcal{H}_p$, denoted by the same symbol. It is easily seen that π preserves the algebraic operations of $\mathfrak{A}_\#$. For $a \in \mathfrak{A}_\#$, let $\pi(a)^*$ denote the Hilbert adjoint of $\pi(a)$. It remains to prove that $\pi(a^\#) = \pi(a)^*$ for every $a \in \mathfrak{A}_\#$.

For every $a \in \mathfrak{A}_\#$, we have

$$\begin{aligned}p_0(a) &= \sup_{p(x)=1} p(ax) = \sup_{\varphi_p(x,x)=1} \varphi_p(ax, ax)^{1/2} \\ &= \sup_{\|\lambda_p(x)\|_p=1} \|\lambda_p(ax)\|_p = \|\pi(a)\|.\end{aligned}$$

Since p_0 is a C^*-seminorm, we have

$$\|\pi(a)\|^2 = p_0(a)^2 = p_0(a^\# a) = \|\pi(a^\# a)\|. \tag{3.6}$$

Let $\mathfrak{N}_0$ be the norm closure of the algebra $\{\pi(a) : a \in \mathfrak{A}_\#\}$. By (3.6), $\mathfrak{N}_0$ is a C^*-algebra with respect to the norm $\|\cdot\|$ of bounded operators in $\mathcal{H}_p$ and the involution $\pi(a) \mapsto \pi(a^\#)$, which is well-defined since (3.6) implies that $\|\pi(a^\#)\| = \|\pi(a)\|$ for every $a \in \mathfrak{A}_\#$. Let $\mathfrak{N}$ be the C^*-subalgebra of $\mathcal{B}(\mathcal{H}_p)$ generated by $\mathfrak{N}_0$. Since $\pi(e)^* = \pi(e^\#) = \mathbb{I}$, the identity of $\mathcal{H}_p$, Lemma 3.9 implies that $\pi(a^\#) = \pi(a)^*$ for every $a \in \mathfrak{A}_\#$. Therefore π is a *-representation of $\mathfrak{A}_\#$. ∎

As is apparent from Proposition 3.8, the condition $\|\Phi(x)\| = p(x)$ for every $x \in \mathfrak{X}$ seems to be a really strong one, essentially because it forces p to be a Hilbert seminorm. The analysis of this situation, however, is of some help for answering Question 1.

If the set $\mathcal{S}_p(\mathfrak{X})$ has a maximum $\overline{\varphi}$, in the sense of (iv) of Proposition 3.8, then

$$\overline{\varphi}(x,x) = \sup_{\varphi \in \mathcal{S}_p(\mathfrak{X})} \varphi(x,x) = \mathfrak{s}^p(x)^2, \qquad \forall x \in \mathfrak{X}.$$

This implies that

$$\overline{\varphi}(x,y) = \frac{1}{4}\sum_{k=0}^{3} i^k \sup_{\varphi \in \mathcal{S}_p(\mathfrak{X})} \varphi(x+i^k y, x+i^k y).$$

Therefore, the right hand side of this equality must be a sesquilinear form on $\mathfrak{X}\times\mathfrak{X}$, which is not true in general. As we shall see below, a necessary and sufficient condition for this to hold is provided by the so-called *net property* (see [1, Sec. 9.3]).

Definition 3.11. We say that $\mathcal{S}_p(\mathfrak{X})$ has the *net property* if, for any finite subset $\{x_1,\dots,x_m\}$ of $\mathfrak{X}$, there exists a sequence $\{\varphi_n\}$ in $\mathcal{S}_p(\mathfrak{X})$ such that

$$\lim_{n\to\infty} \varphi_n(x_k,x_k) = \sup_{\varphi \in \mathcal{S}_p(\mathfrak{X})} \varphi(x_k,x_k)$$

for $k = 1,\dots,m$.

Theorem 3.12. *Let $\mathfrak{X}$ be a left $\mathfrak{A}_\#$-module and p an M-seminorm on $\mathfrak{X}$. The following statements are equivalent.*

- (i) *There exists an MC^*-seminorm q satisfying the parallelogram law and such that*
  - (i.a) $q(x) \le p(x)$ *for every* $x \in \mathfrak{X}$;
  - (i.b) $\mathcal{C}_q(\mathfrak{X}) = \mathcal{C}_p(X)$.
- (ii) *There exists a modular representation (Φ,π) of $\mathfrak{X}$ such that $\|\Phi(x)\| = \mathfrak{s}^p(x)$ for every $x \in \mathfrak{X}$ and $\|\pi(a)\| = \mathfrak{s}_0^p(a)$ for every $a \in \mathfrak{A}_\#$.*
- (iii) *$\mathcal{S}_p(\mathfrak{X})$ has a maximum.*
- (iv) *$\mathcal{S}_p(\mathfrak{X})$ has the net property.*

Proof. (i)⇒(ii): The assumption implies, by Proposition 3.10, that there exists a modular representation (Φ,π) of $\mathfrak{X}$ such that $\|\Phi(x)\| = q(x)$ for every $x \in \mathfrak{X}$ and $\|\pi(a)\| = q_0(a)$ for every $a \in \mathfrak{A}_\#$. By (ii) and (iii) of

Proposition 3.8 one has

$$q(x) = \mathfrak{s}^q(x) = \mathfrak{s}^p(x), \quad \forall x \in \mathfrak{X},$$

and hence

$$q_0(a) = \mathfrak{s}^q_0(a) = \mathfrak{s}^p_0(a), \quad \forall a \in \mathfrak{A}_\#.$$

The equality $\mathfrak{s}^p(\cdot) = \mathfrak{s}^q(\cdot)$ is due to (i.b).

(ii)⇒(iii): Put $\overline{\varphi}(x,y) = \langle \Phi(x)|\Phi(y)\rangle$ for every $x, y \in \mathfrak{X}$. Then it is easily seen that $\overline{\varphi} \in \mathcal{S}_p(\mathfrak{X})$. If $\varphi \in \mathcal{S}_p(\mathfrak{X})$ we have

$$\varphi(x,x) \le \mathfrak{s}^p(x)^2 = \|\Phi(x)\|^2 = \overline{\varphi}(x,x), \quad \forall x \in \mathfrak{X}.$$

Hence $\overline{\varphi}$ is a maximum of $\mathcal{S}_p(\mathfrak{X})$.

(iii)⇒(iv): Let $\overline{\varphi}$ be the maximum of $\mathfrak{X}$. It is clear that the constant sequence $\{\varphi_k\}$ with $\varphi_k = \overline{\varphi}$ satisfies the requirements of Definition 3.11.

(iv)⇒(iii): Since $\mathcal{S}_p(\mathfrak{X})$ has the net property, if we put

$$\overline{\varphi}(x,y) = \frac{1}{4}\sum_{k=0}^{3} i^k \sup_{\varphi \in \mathcal{S}_p(\mathfrak{X})} \varphi(x + i^k y, x + i^k y), \quad x, y \in \mathfrak{X},$$

then $\overline{\varphi}$ is a positive sesquilinear form on $\mathfrak{X} \times \mathfrak{X}$ satisfying the conditions of Definition 2.2, thus it is a modular invariant sesquilinear form on $\mathfrak{X}$. One has

$$\overline{\varphi}(x,x) = \sup_{\varphi \in \mathcal{S}_p(\mathfrak{X})} \varphi(x,x) \le p(x)^2.$$

Hence $\overline{\varphi} \in \mathcal{S}_p(\mathfrak{X})$ and it is the maximum.

(iii)⇒(i): Let $\overline{\varphi}$ be the maximum of $\mathcal{S}_p(\mathfrak{X})$ and define $q(x) = \overline{\varphi}(x,x)^{1/2}$, $x \in \mathfrak{X}$. Then, clearly, $q(x) \le p(x)$ for every $x \in \mathfrak{X}$. Moreover, if $\varphi \in \mathcal{S}_p(\mathfrak{X})$, then

$$\varphi(x,x) \le \overline{\varphi}(x,x) = q(x)^2, \quad \forall x \in \mathfrak{X}.$$

Hence, $\varphi \in \mathcal{S}_q$. This easily implies the equality $\mathcal{C}_q(\mathfrak{X}) = \mathcal{C}_p(X)$. ∎

Remark 3.13. We conclude by noticing that the existence of an MC^*-seminorm p on $\mathfrak{X}$ has other profitable features that are worth mentioning: these are due to the fact that a natural Banach C^*-module is defined by p. Indeed, let, as before,

$$N(p) = \{x \in \mathfrak{X} : p(x) = 0\}, \quad N(p_0) = \{a \in \mathfrak{A}_\# : p_0(a) = 0\}.$$

Now, let $\mathfrak{X}^p$ denote the completion of $\mathfrak{X}/N(p)$ with respect to the norm $\|x + N(p)\|_p = p(x)$, and $\mathfrak{A}^p_\#$ the completion of $\mathfrak{A}_\#/N(p_0)$ with respect to the norm $\|a + N(p_0)\|_{p_0} = p_0(a)$. Then $\mathfrak{A}^p_\#$ is a C^*-algebra and $\mathfrak{X}^p$ is a Banach $\mathfrak{A}^p_\#$-module. Let $(\widetilde{\Phi}, \widetilde{\pi})$ be a modular representation of $\mathfrak{X}^p$. Then we define a representation of $\mathfrak{X}$ by

$$\Phi(x) = \widetilde{\Phi}(x + N(p)), \quad x \in \mathfrak{X},$$

and a *-representation π of $\mathfrak{A}_\#$ by

$$\pi(a) = \widetilde{\pi}(a + N(p_0)), \qquad a \in \mathfrak{A}_\#.$$

Then (Φ, π) is a modular representation of $\mathfrak{X}$. Indeed, since $aN(p) \subseteq N(p)$ for every $a \in \mathfrak{A}_\#$ and $N(p_0)x \subseteq N(p)$ for every $x \in \mathfrak{X}$, we get

$$\begin{aligned}\Phi(ax) &= \widetilde{\Phi}(ax + N(p)) = \widetilde{\Phi}((a + N(p_0))(x + N(p))) \\ &= \widetilde{\pi}(a + N(p_0))\widetilde{\Phi}(x + N(p)) = \pi(a)\Phi(x).\end{aligned}$$

The *-representation π of $\mathfrak{A}_\#$ is automatically bounded and p_0-continuous. One has indeed

$$\|\pi(a)\| = \|\widetilde{\pi}(a + N(p_0))\| \le p_0(a), \qquad \forall a \in \mathfrak{A}_\#.$$

The p-continuity of Φ can also be checked by verifying one of the characterizations of the continuity of modular representations of Banach C^*-modules discussed in [8]. There are, of course, other situations where properties of $(\widetilde{\Phi}, \widetilde{\pi})$ can be pulled back to obtain properties of (Φ, π). For instance, if we prove that there exists a representation $\widetilde{\Phi}$ of $\mathfrak{X}^p$ satisfying

$$\|\widetilde{\Phi}(x + N(p))\| = \|x + N(p)\|_p = p(x),$$

then also a representation of $\mathfrak{X}$ with the same property is found.

4. Examples. In this final section we give some examples and applications of the ideas developed so far.

EXAMPLE 4.1. Let $\mathfrak{X}$ be a left Hilbert $\mathfrak{A}_\#$-module in the sense of [4]. Then $\mathfrak{X}$ is at once a left $\mathfrak{A}_\#$-module and a Hilbert space with inner product $\langle\cdot|\cdot\rangle$ such that

$$\langle ax|y\rangle = \langle x|a^\# y\rangle, \qquad \forall a \in \mathfrak{A}_\#,\ x, y \in \mathfrak{X}.$$

Then $\varphi(x, y) = \langle x|y\rangle$, $x, y \in \mathfrak{X}$, is a modular invariant form and it is, obviously, bounded with respect to the norm $p(\cdot) = \langle\cdot|\cdot\rangle^{1/2}$. If $\varphi \in \mathcal{C}_p(\mathfrak{X})$, then there exists a bounded operator T_φ in $\mathfrak{X}$ such that

$$\varphi(x, y) = \langle T_\varphi x|y\rangle, \qquad \forall x, y \in \mathfrak{X}.$$

From the properties of φ one deduces that $T_\varphi \ge 0$ and that $T_\varphi L_a = L_a T_\varphi$ for every $a \in \mathfrak{A}_\#$, where L_a denotes the operator of left multiplication by a.

Now $\varphi \in \mathcal{S}_p(\mathfrak{X})$ if, and only if, $\|T_\varphi\| \le 1$. Indeed, we have

$$\varphi \in \mathcal{S}_p(\mathfrak{X}) \;\Leftrightarrow\; \sup \frac{\varphi(x, x)}{p(x)^2} \le 1 \;\Leftrightarrow\; \sup \frac{\langle T_\varphi x|x\rangle}{p(x)^2} \le 1 \;\Leftrightarrow\; \|T_\varphi\| \le 1,$$

taking into account that T_φ is self-adjoint. Finally, it is clear that $\mathcal{S}_p(\mathfrak{X})$ has a maximum. Indeed, for any $\varphi \in \mathcal{S}_p(\mathfrak{X})$,

$$|\varphi(x, x)| \le p(x)^2 = \langle x|x\rangle.$$

The norm p of $\mathfrak{X}$ is clearly regular.

EXAMPLE 4.2. Let I be an interval of the real line. We consider $L^r(I)$, $r \geq 1$, as a Banach $L^\infty(I)$-module (if I has finite Lebesgue measure, then $L^\infty(I) \subset L^r(I)$ and we speak in this case of a CQ^*-algebra). Of course we take p to be the usual norm of $L^r(I)$ and we simply write $\mathcal{S}(\mathfrak{X})$ instead of $\mathcal{S}_p(\mathfrak{X})$. It is not difficult to see that, if $r \geq 2$, then $\mathcal{S}(L^r(I))$ is quite rich [3]; indeed,

$$\mathcal{S}(L^r(I)) = \{\varphi_w : w \in L^{r/(r-2)}(I),\ \|w\|_{r/(r-2)} = 1,\ w \geq 0\},$$

where

$$\varphi_w(x, y) = \int_I x(t)\overline{y(t)}w(t)\,dt, \qquad x, y \in L^r(I).$$

If $1 \leq r < 2$ then, as in [3], one can prove that $\mathcal{S}(L^r(I)) = \emptyset$. If $r \geq 2$, then

$$\sup\{\varphi_w(x, x) : w \in L^{r/(r-2)}(I),\ \|w\|_{r/(r-2)} = 1,\ w \geq 0\} = \|x\|_r$$

for all $x \in L^r(I)$. Then $\mathcal{S}(L^r(I))$ may have a maximum if it satisfies the parallelogram law. But this happens only if $r = 2$ (the maximum being the inner product itself).

If I is a bounded interval (we take $I = [0, 1]$), then, according to Proposition 3.7, a modular representation (Φ, π) of $L^r(I)$ with π bounded exists for any $r \geq 2$. Indeed, it suffices to define, for $x \in L^r(I)$, $\Phi(x) = x \in L^2(I)$ and, for every $v \in L^\infty(I)$, $(\pi(v)x)(t) = v(t)x(t)$, $x \in L^r(I)$.

EXAMPLE 4.3. Any *-algebra $\mathfrak{A}_\#$ may be viewed, in the obvious way, as a left $\mathfrak{A}_\#$-module. If ω is a positive linear functional on $\mathfrak{A}_\#$ then putting $\varphi_\omega(a, b) = \omega(b^\# a)$, one obtains a modular invariant form. Assume that there exists an M-seminorm on $\mathfrak{A}_\#$ such that the set of positive linear functionals ω on $\mathfrak{A}_\#$ for which φ_ω is p-bounded is nontrivial. This, of course, implies that $\mathcal{S}(\mathfrak{A}_\#)$ is nontrivial. Then $\mathfrak{A}_\#$ admits a nonzero C^*-seminorm, namely $\mathfrak{s}_0^p$. Hence $\mathfrak{A}_\#$ admits bounded *-representations.

EXAMPLE 4.4. Let $\mathfrak{A}_\#$ be a *-algebra (possibly without unit) and $\mathfrak{X}$ a left $\mathfrak{A}_\#$-module. Assume that $\mathfrak{A}_\#$ contains two elements a, b such that

$$abx - bax = x, \qquad \forall x \in \mathfrak{X}. \tag{4.1}$$

Then there cannot exist any modular representation (Φ, π), with π bounded, since in this case,

$$\pi(a)\pi(b)\Phi(x) - \pi(b)\pi(a)\Phi(x) = \Phi(x), \qquad \forall x \in \mathfrak{X}.$$

The density of $\Phi(\mathfrak{X})$ would then imply that $\pi(a)\pi(b) - \pi(b)\pi(a) = \mathbb{I}$, and this is impossible because of the Wiener–Wielandt theorem (see, e.g., [6, Sect. 2.2]). If $\mathfrak{A}_\#$ has a unit e, then from (4.1) it follows that $ab - ba = e$; if $\mathfrak{X}$ admits an M-seminorm p, then necessarily $\mathcal{S}_p(\mathfrak{X}) = \{0\}$, since otherwise $\mathfrak{s}_0^p$ would be a C^*-seminorm on $\mathfrak{A}_\#$, and $\mathfrak{A}_\#$ would have a bounded *-representation π such that $\pi(a)\pi(b) - \pi(b)\pi(a) = \mathbb{I}$.

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Dipartimento di Matematica ed Applicazioni
Università di Palermo
I-90123 Palermo, Italy
E-mail: trapani@unipa.it
salvo@math.unipa.it

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