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L^p-L^q boundedness of analytic families of fractional integrals

by

VALENTINA CASARINO and SILVIA SECCO (Torino)

Abstract. We consider a double analytic family of fractional integrals $S_z^{\gamma,\alpha}$ along the curve $t \mapsto |t|^\alpha$, introduced for $\alpha = 2$ by L. Grafakos in 1993 and defined by

$$(S_z^{\gamma,\alpha} f)(x_1, x_2) := \frac{1}{\Gamma(\frac{z+1}{2})} \iint |u-1|^z \psi(u-1) f(x_1 - t, x_2 - u|t|^\alpha)\, du\, |t|^\gamma \, \frac{dt}{t},$$

where ψ is a bump function on $\mathbb{R}$ supported near the origin, $f \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, $z, \gamma \in \mathbb{C}$, $\operatorname{Re}\gamma \geq 0$, $\alpha \in \mathbb{R}$, $\alpha \geq 2$.

We determine the set of all $(1/p, 1/q, \operatorname{Re} z)$ such that $S_z^{\gamma,\alpha}$ maps $L^p(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$ boundedly. Our proof is based on product-type kernel arguments. More precisely, we prove that the kernel $K_{-1+i\theta}^{i\varrho,\alpha}$ is a product kernel on $\mathbb{R}^2$, adapted to the curve $t \mapsto |t|^\alpha$; as a consequence, we show that the operator $S_{-1+i\theta}^{i\varrho,\alpha}$, $\theta, \varrho \in \mathbb{R}$, is bounded on $L^p(\mathbb{R}^2)$ for $1 < p < \infty$.

1. Introduction. In this paper, we discuss L^p-L^q boundedness for a double analytic family of fractional integrals, introduced, in the parabolic case, by L. Grafakos in 1993 [Gr2]. More precisely, we consider the analytic family of operators $S_z^{\gamma,\alpha}$ defined by

$$(S_z^{\gamma,\alpha} f)(x_1, x_2) := \frac{1}{\Gamma\left(\frac{z+1}{2}\right)} \iint |u-1|^z \psi(u-1) f(x_1 - t, x_2 - u|t|^\alpha)\, du\, |t|^\gamma \, \frac{dt}{t}$$

if $\operatorname{Re}\gamma \geq 0$, where the outer integral is interpreted in some appropriate sense if $\operatorname{Re}\gamma = 0$. Here ψ is a bump function supported near the origin on $\mathbb{R}$, $f \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, $z, \gamma \in \mathbb{C}$, $\operatorname{Re}\gamma \geq 0$, $\alpha \in \mathbb{R}$, $\alpha \geq 2$. Notice that for $z = -1$ and $\operatorname{Re}\gamma > 0$ the operator $S_z^{\gamma,\alpha}$ coincides with the fractional integration operator along the curve $t \mapsto |t|^\alpha$, which was studied in [RS] and in [Ch2].

Some authors recognized a product structure in this analytic family of singular integral operators with convolution kernels supported on curves in the plane. In particular, Grafakos pointed out that the failure of certain

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standard H^1-L^1 estimates when $\operatorname{Re} z = -1$ is due to this product structure, while A. Seeger and T. Tao used it to obtain sharp Lorentz space inequalities [Gr1, SeeT]. Anyway, product structures for these problems go back to Melrose and Greenleaf and Uhlmann [M, GU].

In this spirit, we prove here a strong endpoint bound for $S_z^{\gamma,\alpha}$ by using the theory of product-type kernels, which was introduced by R. Fefferman and E. Stein at the beginning of the eighties [FS]. For the precise definition of product kernels and a complete bibliography we refer the reader to a recent paper by A. Nagel, F. Ricci and E. M. Stein [NRS]; we only recall here that product kernels on $\mathbb{R}^2$ are singular distributions, satisfying differential inequalities and cancellation properties similar to those of the distribution $\mathrm{pv}\big(\frac{1}{x_1 x_2}\big)$.

More specifically, the question is for which $p, q, \operatorname{Re} z, \operatorname{Re}\gamma$ the operators $S_z^{\gamma,\alpha}$ are bounded from $L^p(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$. Since a necessary condition for $S_z^{\gamma,\alpha}$ to map L^p to L^q is that $1/p - 1/q = \operatorname{Re}\gamma/(\alpha+1)$, it suffices to consider the set Σ_α consisting of all $(1/p, 1/q, \operatorname{Re} z)$ such that $S_z^{\gamma,\alpha}$ is bounded from L^p to L^q. When $\alpha = 2$, Grafakos showed that the interior of the set Σ_2 of all $(1/p,\, 1/q,\, \operatorname{Re} z)$ for which $S_z^{\gamma} := S_z^{\gamma,2}$ maps $L^p(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$ boundedly coincides with the interior of the closed tetrahedron $ABCD$ with vertices $A = (0,0,-1)$, $B = (1/2, 1/2, -3/2)$, $C = (1,1,-1)$, $D = (1,0,0)$, and that Σ_2 contains, moreover, the open faces ABD, BCD, and the closed edge BD.

Furthermore, he established some weak-type inequalities. More precisely, he proved that no strong-type bound holds on the open segments CD and AD, that S_z^{γ} maps the parabolic real Hardy space H^1 (that is, the Hardy space on $\mathbb{R}^2$ defined with respect to the non-isotropic scaling $(x_1, x_2) \mapsto (tx_1, t^2 x_2)$ [CT1, CT2]) to weak L^p on CD and that, by duality, S_z^{γ} maps $L^{p',1}$ to parabolic BMO on AD.

Weak-type bounds on AB and CB are subtler. Grafakos proved $L^{p,p'}$ results, by using methods in [Ch1]. He showed, in particular, that S_z^{γ} maps L^p to $L^{p,p'}$ on the open segment BC and that, by duality, S_z^{γ} maps $L^{p',p}$ to $L^{p'}$ on AB. Then M. Christ showed the failure of endpoint L^p bounds on AB for $S_z^{\gamma,\alpha}$, $\alpha \geq 2$ [Ch3]; finally, Seeger and Tao proved the sharp $L^p \to L^{p,2}$ bound [SeeT].

Along the open segment AC (that is, when $\operatorname{Re} z = -1$ and $\operatorname{Re}\gamma = 0$) Grafakos proved that $S_z^{\gamma,2}$ maps the parabolic Hardy space H^1 to $L^{1,\infty}$ and that it is not of weak type $(1,1)$ when $\operatorname{Im} z \neq 0$.

Anyway, as a consequence of a real interpolation approach, it may be reasonably expected that the operators $S_z^{\gamma,\alpha}$ are bounded on L^p, for all $1 < p < \infty$, along AC. Here, we prove this fact in a direct way. More precisely, we prove that for all $\alpha \geq 2$ the operators $S_z^{\gamma,\alpha}$ are bounded on L^p, for $1 < p < \infty$, on the open diagonal AC and therefore on the open faces ACB and ACD as well; in this way we completely characterize the set Σ_α, which turns out to be independent of α. Our proof relies on the fact that the convolution kernel of $S_z^{\gamma,\alpha}$ along AC, $K_{-1+i\theta}^{i\varrho,\alpha}$, $\varrho := \operatorname{Im}\gamma$, is a product-type kernel adapted to the curve $x_1 \mapsto |x_1|^\alpha$ on $\mathbb{R}^2$. Such kernels, whose singularities are concentrated along the coordinate axis $x_1 = 0$ and the curve $x_2 = |x_1|^\alpha$, have been recently introduced and studied by one of the authors [Se], who proved in particular the L^p boundedness of the associated convolution operators for $1 < p < \infty$. More precisely, we first prove that the distribution $H^{\mu,\nu}$, where $\mu,\nu \in \mathbb{R}$, $\mu \neq 0$, defined by

$$\langle H^{\mu,\nu}, f\rangle := \lim_{\varepsilon\to 0^+} \iint |x_2|^{-1+i\mu}\psi\left(\frac{x_2}{|x_1|^\alpha}\right)|x_1|^{-1+\varepsilon+i\nu} \operatorname{sgn} x_1\, f(x_1,x_2)\, dx_1\, dx_2$$

for $f \in \mathcal{S}(\mathbb{R}^2)$, is a product kernel on $\mathbb{R}^2$; then we show that $K_{-1+i\theta}^{i\varrho,\alpha}$ essentially coincides with the kernel $H^{\mu,\nu}$, with $\mu = \theta$ and $\nu = \varrho - \alpha\theta$, adapted to the curve $x_2 = |x_1|^\alpha$, so that, as a consequence of Theorem 1.3 in [Se], $S_z^{\gamma,\alpha}$ is bounded on L^p, $1 < p < \infty$, along the diagonal AC.

It is worth noticing that if $S_{-1+i\theta}^{i\varrho}$ were exactly a product of a Hilbert transform on the parabola and a singular integral in the vertical direction, then the argument would be much simpler. Anyway, the $S_{-1+i\theta}^{i\varrho}$ turn out to be more general product-type operators.

As underlined in a recent paper of A. Seeger and S. Wainger [SeeW], the operators $S_z^{\gamma,\alpha}$ provide a model family of operators in the class $\mathcal{I}^{\varrho',-\sigma}$ defined by A. Greenleaf and G. Uhlmann in [GU], consisting of oscillatory integrals with singular symbols.

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2. A particular product kernel. Let ψ be an even smooth function on $\mathbb{R}$ such that $\psi = 1$ on $[0, 1/2]$ and $\psi = 0$ on $(1,\infty)$, with $0 \le \psi \le 1$ on $(1/2, 1)$ and such that ψ' changes sign only once.

Take $\alpha,\mu,\nu \in \mathbb{R}$, $\alpha \ge 2$, $\mu \neq 0$. Define

$$\begin{aligned}(2.1)\quad & \langle H^{\mu,\nu}, f\rangle \\ & := \lim_{\varepsilon\to 0^+} \iint |x_2|^{-1+i\mu}\psi\left(\frac{x_2}{|x_1|^\alpha}\right)|x_1|^{-1+\varepsilon+i\nu} \operatorname{sgn} x_1\, f(x_1,x_2)\, dx_1\, dx_2\end{aligned}$$

for every $f \in \mathcal{S}(\mathbb{R}^2)$.

We shall prove in Theorem 2.6 that $H^{\mu,\nu}$ defines a product kernel on $\mathbb{R}^2$ (see Def. 2.1.1 in [NRS]).

The proof of this result, which may be of independent interest, will be divided into some lemmata.

LEMMA 2.1. *$H^{\mu,\nu}$, defined by* (2.1), *is a tempered distribution.*

Proof. For any $f \in \mathcal{S}(\mathbb{R}^2)$ we have

$$\begin{aligned}
\langle H^{\mu,\nu}, f\rangle &= \lim_{\varepsilon\to 0^+} \iint |x_2|^{-1+i\mu}\psi\Big(\frac{x_2}{|x_1|^\alpha}\Big)|x_1|^{-1+\varepsilon+i\nu}\operatorname{sgn} x_1 f(x_1,x_2)\,dx_1\,dx_2 \\
&= \lim_{\varepsilon\to 0^+}\Big(\iint\limits_{|x_1|\le 1,\,|x_2|\le|x_1|^\alpha} + \iint\limits_{|x_1|>1,\,|x_2|\le|x_1|^\alpha}\Big)|x_2|^{-1+i\mu}\psi\Big(\frac{x_2}{|x_1|^\alpha}\Big) \\
&\qquad\qquad\qquad \times |x_1|^{-1+\varepsilon+i\nu}\operatorname{sgn} x_1\, f(x_1,x_2)\,dx_1\,dx_2 \\
&= \lim_{\varepsilon\to 0^+}(I_{1,\varepsilon}+I_{2,\varepsilon}).
\end{aligned}$$

Since ψ is even,

$$\iint\limits_{|x_1|\le 1,\,|x_2|\le|x_1|^\alpha} |x_2|^{-1+i\mu}\psi\Big(\frac{x_2}{|x_1|^\alpha}\Big)|x_1|^{-1+\varepsilon+i\nu}\operatorname{sgn} x_1\,dx_1\,dx_2 = 0,$$

so that, after integration by parts with respect to x_2, we may write

$$\begin{aligned}
I_{1,\varepsilon} = &\iint\limits_{|x_1|\le 1,\,|x_2|\le|x_1|^\alpha} |x_2|^{-1+i\mu}\psi\Big(\frac{x_2}{|x_1|^\alpha}\Big)|x_1|^{-1+\varepsilon+i\nu}\operatorname{sgn} x_1 \\
&\qquad\qquad\qquad \times (f(x_1,x_2)-f(0,0))\,dx_1\,dx_2 \\
= &-\iint\limits_{|x_1|\le 1,\,|x_2|\le|x_1|^\alpha} \operatorname{sgn} x_2\,\frac{|x_2|^{i\mu}}{i\mu}\,|x_1|^{-1-\alpha+\varepsilon+i\nu}\operatorname{sgn} x_1\,\psi'\Big(\frac{x_2}{|x_1|^\alpha}\Big) \\
&\qquad\qquad\qquad \times (f(x_1,x_2)-f(0,0))\,dx_1\,dx_2 \\
&-\iint\limits_{|x_1|\le 1,\,|x_2|\le|x_1|^\alpha} \operatorname{sgn} x_2\,\frac{|x_2|^{i\mu}}{i\mu}\,\psi\Big(\frac{x_2}{|x_1|^\alpha}\Big)(\partial_{x_2}f)(x_1,x_2)|x_1|^{-1+\varepsilon+i\nu} \\
&\qquad\qquad\qquad \times \operatorname{sgn} x_1\,dx_1\,dx_2.
\end{aligned}$$

As a consequence of the mean value theorem, we find that

$$|I_{1,\varepsilon}| \le \frac{4\sqrt{2}}{|\mu|}\,\|\psi'\|_\infty\|f\|_{(1)} + \frac{4}{\alpha|\mu|}\,\|\psi\|_\infty\|f\|_{(1)} \le \frac{c_1}{|\mu|}\,\|f\|_{(1)}$$

for some positive constant c_1, uniformly with respect to $\varepsilon \in (0,1)$. An analogous computation shows that

$$|I_{2,\varepsilon}| \le \frac{4}{|\mu|}\,\|\psi'\|_\infty\|f\|_{(3)} + \frac{4}{|\mu|}\,\|\psi\|_\infty\|f\|_{(2[\alpha]+5)} \le \frac{c_1}{|\mu|}\,\|f\|_{(2[\alpha]+5)},$$

yielding, together with the previous estimate,

$$|\langle H^{\mu,\nu}, f\rangle| \le \frac{C}{|\mu|}\,\|f\|_{(2[\alpha]+5)} = C_\mu \|f\|_{(2[\alpha]+5)},$$

where the constant C_μ grows at most exponentially in μ. ∎

It is not hard to show that the kernel $H^{\mu,\nu}$ defined by (2.1) coincides with the function

$$H^{\mu,\nu}(x_1,x_2) = |x_2|^{-1+i\mu}\psi\bigg(\frac{x_2}{|x_1|^\alpha}\bigg)|x_1|^{-1+i\nu}\operatorname{sgn} x_1 \tag{2.2}$$

on $\mathbb{R}^2 \setminus (\{x_1=0\}\cup\{x_2=0\})$, so that we shall now prove that $H^{\mu,\nu}$ satisfies the right differential inequalities. We will need the following lemma.

LEMMA 2.2. *For any positive integer β we have*

$$\begin{aligned}(2.3)\quad \partial_{x_1}^\beta\bigg(\psi\bigg(\frac{x_2}{|x_1|^\alpha}\bigg)\bigg) \\ = \sum_{l=1}^{\beta} c_{l,\beta,\alpha}(\partial_{x_1}^{\beta-l+1}\psi)\bigg(\frac{x_2}{|x_1|^\alpha}\bigg)\frac{x_2^{\beta-l+1}}{|x_1|^{(\alpha+1)\beta-\alpha l+\alpha}}(\operatorname{sgn} x_1)^\beta.\end{aligned}$$

Proof. The proof goes by induction on the order β of derivation and it is omitted. ∎

In the following the symbols C and C_σ will denote constants which may vary from one formula to the other and that grow at most exponentially in $|\sigma|$ when $|\sigma|$ tends to ∞. Here σ may denote a set of indices, like, e.g., $\sigma = (\beta,\mu,\nu)$; in this case we require that C_σ grows at most exponentially in $|\beta|$, $|\mu|,|\nu|$ when $|\beta|$, $|\mu|,|\nu|$ tend to ∞. Such constants will be called *of admissible growth*.

PROPOSITION 2.3. *For any multi-index $\beta=(\beta_1,\beta_2)$, $\beta_1,\beta_2\in\mathbb{N}$, there exists a constant $C_{\beta,\mu,\nu}$ of admissible growth such that*

$$\begin{aligned}(2.4)\quad &|\partial_{x_1}^{\beta_1}\partial_{x_2}^{\beta_2}H^{\mu,\nu}(x_1,x_2)| \\ &\le C_{\beta,\mu,\nu}|x_1|^{-1-\beta_1}|x_2|^{-1-\beta_2} \qquad \text{on } \mathbb{R}^2\setminus(\{x_1=0\}\cup\{x_2=0\}).\end{aligned}$$

Proof. Take $(x_1,x_2)\in\mathbb{R}^2\setminus(\{x_1=0\}\cup\{x_2=0\})$. If $(\beta_1,\beta_2)=(0,0)$, then

$$|H^{\mu,\nu}(x_1,x_2)| \le |x_1|^{-1}|x_2|^{-1}\|\psi\|_\infty \le |x_1|^{-1}|x_2|^{-1}.$$

If $\beta_1=0$ and $\beta_2\neq 0$, the Leibniz rule yields

$$\begin{aligned}\partial_{x_2}^{\beta_2}H^{\mu,\nu}(x_1,x_2) = |x_1|^{-1+i\nu}\operatorname{sgn} x_1 \sum_{\gamma_1+\gamma_2=\beta_2} c_{\gamma_1,\gamma_2,\mu}|x_2|^{-1-\gamma_1+i\mu}\operatorname{sgn} x_2 \\ \times |x_1|^{-\alpha\gamma_2}(\partial_{x_2}^{\gamma_2}\psi)\bigg(\frac{x_2}{|x_1|^\alpha}\bigg).\end{aligned}$$

On the set where ψ does not vanish we have $|x_1|^{-\alpha\gamma_2} \le |x_2|^{-\gamma_2}$, so that

$$\begin{aligned}|\partial_{x_2}^{\beta_2} H^{\mu,\nu}(x_1,x_2)| &\le |x_1|^{-1} \sum_{\gamma_1+\gamma_2=\beta_2} |c_{\gamma_1,\gamma_2,\mu}|\, |x_2|^{-1-\gamma_1-\gamma_2} \\ &\le C_{\beta_2,\mu}|x_1|^{-1}|x_2|^{-1-\beta_2}.\end{aligned}$$

When $\beta_1 \neq 0$ and $\beta_2 = 0$, by applying the Leibniz formula and Lemma 2.2 we obtain

$$\begin{aligned}\partial_{x_1}^{\beta_1} H^{\mu,\nu}(x_1,x_2) &= |x_2|^{-1+i\mu} \sum_{\gamma_1+\gamma_2=\beta_1} c_{\gamma_1,\gamma_2,\nu}|x_1|^{-1-\gamma_1+i\nu}(\operatorname{sgn} x_1)^{\gamma_1+1} \\ &\quad\times \sum_{l=1}^{\gamma_2} c_{l,\gamma_2,\alpha}\left((\partial_{x_1}^{\gamma_2-l+1}\psi)\left(\frac{x_2}{|x_1|^\alpha}\right)\right)\frac{x_2^{\gamma_2-l+1}}{|x_1|^{(\alpha+1)\gamma_2-\alpha l+\alpha}}\,(\operatorname{sgn} x_1)^{\gamma_2},\end{aligned}$$

so that

$$\begin{aligned}|\partial_{x_1}^{\beta_1} H^{\mu,\nu}(x_1,x_2)| &\le |x_2|^{-1} \sum_{\gamma_1+\gamma_2=\beta_1} |c_{\gamma_1,\gamma_2,\nu}|\, |x_1|^{-1-\gamma_1} \\ &\qquad\qquad \times \sum_{l=1}^{\gamma_2} c_{\gamma_2,l}|x_1|^{\alpha\gamma_2-\alpha l+\alpha-(\alpha+1)\gamma_2+\alpha l-\alpha} \\ &\le |x_2|^{-1}|x_1|^{-1-\beta_1} \sum_{\gamma_1+\gamma_2=\beta_1} |c_{\gamma_1,\gamma_2,\nu}| \sum_{l=1}^{\gamma_2} c_{\gamma_2,l} \\ &= C_{\mu,\nu,\beta}|x_2|^{-1}|x_1|^{-1-\beta_1},\end{aligned}$$

where we have used, in particular, the fact that $|x_2|^{\gamma_2-l+1} \le |x_1|^{\alpha\gamma_2-\alpha l+\alpha}$ and that $|(\partial_{x_1}^{\gamma_2-l+1}\psi)(x_2/|x_1|^\alpha)| \le c_{\gamma_2,l}$ for some positive constant $c_{\gamma_2,l}$ on the set where ψ is not vanishing.

Finally, consider the case $\beta_1 \neq 0 \neq \beta_2$. By applying the Leibniz rule and Lemma 2.2 we obtain

$$\begin{aligned}\partial_{x_1}^{\beta_1}\partial_{x_2}^{\beta_2} H^{\mu,\nu}(x_1,x_2) &= \sum_{\gamma_1+\gamma_2=\beta_2} c_{\gamma_1,\gamma_2,\mu}|x_2|^{-1-\gamma_1+i\mu}(\operatorname{sgn} x_2)^{\gamma_1} \\ &\times \sum_{\delta_1+\delta_2=\beta_1} c_{\delta_1,\delta_2,\nu}|x_1|^{-1-\alpha\gamma_2+i\nu-\delta_1} \sum_{l=1}^{\delta_2} c_{l,\delta_2,\alpha}\left((\partial_{x_1}^{\delta_2-l+1}\partial_{x_2}^{\gamma_2}\psi)\left(\frac{x_2}{|x_1|^\alpha}\right)\right) \\ &\qquad\qquad \times \frac{|x_2|^{\delta_2-l+1}}{|x_1|^{(\alpha+1)\delta_2-\alpha l+\alpha}}\,(\operatorname{sgn} x_1)^{1+\delta_1+\delta_2},\end{aligned}$$

so that

$$
\begin{aligned}
&|\partial_{x_1}^{\beta_1}\partial_{x_2}^{\beta_2} H^{\mu,\nu}(x_1,x_2)| \\
&\qquad \le |x_1|^{-1-\beta_1}|x_2|^{-1-\beta_2}\Big(\sum_{\gamma_1+\gamma_2=\beta_2}|c_{\gamma_1,\gamma_2,\mu}| \\
&\qquad\qquad \times\Big(\sum_{\delta_1+\delta_2=\beta_1}|c_{\delta_1,\delta_2,\nu}|\Big(\sum_{l=1}^{\delta_2}|c_{l,\delta_2,\alpha}|\cdot\|\psi\|_{(\delta_2-l+1+\gamma_2)}\Big)\Big)\Big) \\
&\qquad = C_{\beta,\mu,\nu}|x_1|^{-1-\beta_1}|x_2|^{-1-\beta_2},
\end{aligned}
$$

where we have used in particular the fact that

$$
\left|(\partial_{x_1}^{\delta_2-l+1}\partial_{x_2}^{\gamma_2}\psi)\left(\frac{x_2}{|x_1|^\alpha}\right)\right| \le \frac{\|\psi\|_{(\delta_2-l+1+\gamma_2)}}{(|x_2|\,|x_1|^{-\alpha})^{\delta_2-l+1+\gamma_2}}. \ \blacksquare
$$

We now have to prove some essential cancellation properties.

In the following, if φ_1 is a function of x_1 and φ_2 is a function of x_2, the symbol $\varphi_1\otimes\varphi_2$ will denote the function on $\mathbb{R}^2$ defined by $(\varphi_1\otimes\varphi_2)(x_1,x_2) := \varphi_1(x_1)\varphi_2(x_2)$.

Let $\varphi_1(x_1)$ be any normalized bump function in $\mathcal{C}^1(\mathbb{R}_{x_1})$ (that is, φ_1 is a $\mathcal{C}^1$ function on $\mathbb{R}$ supported on $(-1,1)$, with $\mathcal{C}^1$-norm bounded by 1). Take $R_1>0$ and put $\varphi_{1,R_1}(x_1)=\varphi_1(x_1/R_1)$. Then define the distribution $H^{\mu,\nu}_{\varphi_{1,R_1}}$ on $\mathbb{R}_{x_2}$ by

$$
\langle H^{\mu,\nu}_{\varphi_{1,R_1}},\varphi_2\rangle = \langle H^{\mu,\nu},\varphi_{1,R_1}\otimes\varphi_2\rangle
$$

for any test function φ_2 on $\mathbb{R}_{x_2}$.

The following result holds.

Proposition 2.4.

(i) *The distribution $H^{\mu,\nu}_{\varphi_{1,R_1}}$ coincides with the smooth function*

$$
H^{\mu,\nu}_{\varphi_{1,R_1}}(x_2) = |x_2|^{-1+i\mu}\int\psi\left(\frac{x_2}{|x_1|^\alpha}\right)|x_1|^{-1+i\nu}\operatorname{sgn}x_1\varphi_1\left(\frac{x_1}{R_1}\right)dx_1
$$

on $\mathbb{R}_{x_2}\setminus\{0\}$. Moreover, for any positive integer β there exists a constant $C_{\beta,\mu}$ of admissible growth such that

$$
|\partial_{x_2}^{\beta}H^{\mu,\nu}_{\varphi_{1,R_1}}(x_2)| \le C_{\beta,\mu}|x_2|^{-1-\beta} \quad \textit{for all } x_2\in\mathbb{R}\setminus\{0\}, \tag{2.5}
$$

uniformly in φ_1, R_1.

(ii) *For any normalized bump function φ_2 of class $\mathcal{C}^1(\mathbb{R}_{x_2})$ and any $R_2>0$ there exists a constant $C_{\mu,\alpha}$ of admissible growth such that*

$$
|\langle H^{\mu,\nu}_{\varphi_{1,R_1}},\varphi_{2,R_2}\rangle| \le C_{\mu,\alpha} \tag{2.6}
$$

independently of φ_1, R_1, φ_2, R_2, where $\varphi_{2,R_2}(x_2) := \varphi_2(x_2/R_2)$.

Proof. Since it is not difficult to show that $H^{\mu,\nu}_{\varphi_{1,R_1}}$ coincides with a $\mathcal{C}^\infty$ function on $\mathbb{R}_{x_2}\setminus\{0\}$, we will only prove that it satisfies the differential inequalities (2.5) and cancellation condition (2.6).

Take any positive integer β and $x_2 \in \mathbb{R} \setminus \{0\}$.

If $\beta = 0$, since the map $x_1 \mapsto \psi(x_2/|x_1|^\alpha)|x_1|^{-1+i\nu} \operatorname{sgn} x_1$ is integrable on the set $\{|x_1| \le R_1\}$ and is odd, a standard application of the mean value theorem yields

$$\begin{aligned}
&|H^{\mu,\nu}_{\varphi_1,R_1}(x_2)| \\
&\qquad = |x_2|^{-1} \Big| \int_{|x_1|\le R_1} \psi\Big(\frac{x_2}{|x_1|^\alpha}\Big)\, |x_1|^{-1+i\nu} \Big(\varphi_1\Big(\frac{x_1}{R_1}\Big) - \varphi_1(0)\Big) \operatorname{sgn} x_1 \, dx_1 \Big| \\
&\qquad \le \|\varphi_1\|_{\mathcal{C}^1} |x_2|^{-1} \int_{|x_1|\le R_1} \psi\Big(\frac{x_2}{|x_1|^\alpha}\Big)\, |x_1|^{-1}\, \frac{|x_1|}{R_1}\, dx_1 \\
&\qquad \le 2\|\psi\|_\infty |x_2|^{-1} = C|x_2|^{-1},
\end{aligned}$$

uniformly with respect to R_1 and φ_1. In the last inequality we have used in particular the fact that $\|\varphi_1\|_{\mathcal{C}^1} \le 1$.

Assume now $\beta \ne 0$. By applying the Leibniz formula we obtain

$$\begin{aligned}
|\partial^\beta_{x_2} H^{\mu,\nu}_{\varphi_1,R_1}(x_2)| &= \Big| \sum_{\beta_1+\beta_2=\beta} c_{\beta_1,\beta_2,\mu} |x_2|^{-1+i\mu-\beta_1} (\operatorname{sgn} x_2)^{\beta_1} \\
&\quad \times \int_{|x_1|\le R_1} (\partial^{\beta_2}_{x_2}\psi)\Big(\frac{x_2}{|x_1|^\alpha}\Big)|x_1|^{-\alpha\beta_2}\Big(\varphi_1\Big(\frac{x_1}{R_1}\Big) - \varphi_1(0)\Big)|x_1|^{-1+i\nu} \operatorname{sgn} x_1\, dx_1 \Big| \\
&\le \sum_{\beta_1+\beta_2=\beta} |c_{\beta_1,\beta_2,\mu}|\, |x_2|^{-1-\beta_1} \int_{|x_1|\le R_1} \Big|(\partial^{\beta_2}_{x_2}\psi)\Big(\frac{x_2}{|x_1|^\alpha}\Big)\Big| |x_1|^{-\alpha\beta_2-1} \\
&\qquad \times \Big|\varphi_1\Big(\frac{x_1}{R_1}\Big) - \varphi_1(0)\Big|\, dx_1.
\end{aligned}$$

Now observe that

$$\Big|(\partial^{\beta_2}_{x_2}\psi)\Big(\frac{x_2}{|x_1|^\alpha}\Big)\Big| \le \|\psi\|_{(\beta_2)}\, \frac{|x_1|^{\alpha\beta_2}}{|x_2|^{\beta_2}}$$

and $|\varphi_1(x_1/R_1) - \varphi_1(0)| \le |x_1|/R_1$, whence

$$\begin{aligned}
|\partial^\beta_{x_2} H^{\mu,\nu}_{\varphi_1,R_1}(x_2)| &\le \sum_{\beta_1+\beta_2=\beta} |c_{\beta_1,\beta_2,\mu}|\, |x_2|^{-1-\beta_1-\beta_2} \|\psi\|_{(\beta_2)} \int_{\{x_1\,:\,|x_1|\le R_1\}} \frac{dx_1}{R_1} \\
&\le 2C_{\beta,\mu}\, \|\psi\|_{(\beta_2)} |x_2|^{-1-\beta} = C_{\beta,\mu}|x_2|^{-1-\beta}
\end{aligned}$$

uniformly with respect to φ_1, R_1. This proves (2.5).

It will now be shown that $H^{\mu,\nu}_{\varphi_1,R_1}$ fulfills the right cancellation conditions as well. Choose any normalized bump function φ_2 of class $\mathcal{C}^1(\mathbb{R}_{x_2})$ and take $R_2 > 0$.

With a change of variables we find

$$\langle H^{\mu,\nu}_{\varphi_{1,R_1}}, \varphi_{2,R_2}\rangle = R_1^{i\nu} R_2^{i\mu} \lim_{\varepsilon\to 0^+} J_\varepsilon,$$

where

$$J_\varepsilon := \iint |x_2|^{-1+i\mu}\psi\biggl(\frac{R_2}{R_1^\alpha}\frac{x_2}{|x_1|^\alpha}\biggr)|x_1|^{-1+\varepsilon+i\nu}\,\operatorname{sgn} x_1\,\varphi_1(x_1)\varphi_2(x_2)\,dx_1\,dx_2.$$

To estimate J_ε, it is convenient to consider separately the cases $R_2 \ge R_1^\alpha$ and $R_2 < R_1^\alpha$.

If $R_2 \ge R_1^\alpha$, the proof is similar to the proof of (2.5). Set

$$A := \biggl\{(x_1,x_2)\in\mathbb{R}^2 \,:\, |x_1|\le 1,\, |x_2| \le \frac{R_1^\alpha}{R_2}\,|x_1|^\alpha\biggr\}.$$

Then

$$\begin{aligned} J_\varepsilon = \iint_A &|x_2|^{-1+i\mu}\,|x_1|^{-1+\varepsilon+i\nu}\,\operatorname{sgn} x_1\,\psi\biggl(\frac{R_2}{R_1^\alpha}\,\frac{x_2}{|x_1|^\alpha}\biggr) \\ &\times (\varphi_1(x_1)\varphi_2(x_2) - \varphi_1(0)\varphi_2(0))\,dx_2\,dx_1. \end{aligned}$$

Now an integration by parts with respect to x_2 yields

$$\begin{aligned} J_\varepsilon &= -\iint_A \frac{\operatorname{sgn} x_2}{i\mu}\,|x_2|^{i\mu}|x_1|^{-1-\alpha+\varepsilon+i\nu}(\partial_{x_2}\psi)\biggl(\frac{R_2}{R_1^\alpha}\,\frac{x_2}{|x_1|^\alpha}\biggr) \\ &\quad\times(\varphi_1(x_1)\varphi_2(x_2)-\varphi_1(0)\varphi_2(0))\,\operatorname{sgn} x_1\,dx_1\,dx_2 \\ &\quad-\iint_A \frac{\operatorname{sgn} x_2}{i\mu}\,|x_2|^{i\mu}\,\psi\biggl(\frac{R_2}{R_1^\alpha}\,\frac{x_2}{|x_1|^\alpha}\biggr)\varphi_2'(x_2)\varphi_1(x_1)|x_1|^{-1+\varepsilon+i\nu}\,\operatorname{sgn} x_1\,dx_1\,dx_2 \\ &=: J_\varepsilon' + J_\varepsilon''. \end{aligned}$$

In order to bound J_ε' we use the mean value theorem and observe that

$$\begin{aligned} |\varphi_1(x_1)&\varphi_2(x_2)-\varphi_1(0)\varphi_2(0)| \\ &\le |\varphi_1(x_1)-\varphi_1(0)|\,|\varphi_2(x_2)| + |\varphi_1(0)|\,|\varphi_2(x_2)-\varphi_2(0)| \\ &\le \biggl(|x_1| + \frac{R_1^\alpha}{R_2}\,|x_1|^\alpha\biggr)\|\varphi_1\|_{\mathcal{C}^1}\|\varphi_2\|_{\mathcal{C}^1} \le |x_1|\biggl(1+\frac{R_1^\alpha}{R_2}\biggr) \le 2|x_1|, \end{aligned}$$

where we have used in particular the fact that $0 \le R_1^\alpha/R_2 \le 1$ and that $\|\varphi_1\|_{\mathcal{C}^1}, \|\varphi_2\|_{\mathcal{C}^1} \le 1$. Thus

$$|J_\varepsilon'| \le \frac{C}{|\mu|}\,\|\psi'\|_\infty \int_{|x_1|\le 1}\biggl(\int_{|x_2|\le \frac{R_1^\alpha}{R_2}|x_1|^\alpha} dx_2\biggr)|x_1|^{-\alpha+\varepsilon}\,dx_1 \le \frac{C}{\mu}\,\|\psi'\|_\infty, \tag{2.7}$$

uniformly with respect to φ_1, φ_2, R_1, R_2, ε.

A similar estimate may be obtained for $|J''_\varepsilon|$, since

$$
\begin{aligned}
(2.8)\qquad |J''_\varepsilon| &\le \frac{\|\psi\|_\infty}{|\mu|}\,\|\varphi_1\|_{\mathcal{C}^1}\,\|\varphi_2\|_{\mathcal{C}^1}\int\limits_{|x_1|\le 1}\Bigg(\int\limits_{|x_2|\le\frac{R_1^\alpha}{R_2}|x_1|^\alpha} dx_2\Bigg)|x_1|^{-1+\varepsilon}\,dx_1\\
&\le \frac{C}{|\mu|}\,\|\psi\|_\infty\,\frac{R_1^\alpha}{R_2}\,\frac{1}{\alpha+\varepsilon}\le\frac{C}{\alpha|\mu|}
\end{aligned}
$$

uniformly with respect to φ_1, φ_2, R_1, R_2, ε. By combining (2.7) and (2.8) we finally get

$$|J_\varepsilon|\le|J'_\varepsilon|+|J''_\varepsilon|\le\frac{C}{|\mu|}.$$

Let us now consider the case $R_2 < R_1^\alpha$. Set

$$
\begin{aligned}
A_1 &:= \Big\{(x_1,x_2)\in\mathbb{R}^2 : |x_1|\le\frac{R_2^{1/\alpha}}{R_1},\ |x_2|\le\frac{R_1^\alpha}{R_2}\,|x_1|^\alpha\Big\},\\
A_2 &:= \Big\{(x_1,x_2)\in\mathbb{R}^2 : \frac{R_2^{1/\alpha}}{R_1}\le|x_1|\le 1,\ |x_2|\le 1\Big\}.
\end{aligned}
$$

We may now rewrite J_ε as

$$
\begin{aligned}
J_\varepsilon &= \Big(\iint\limits_{A_1}+\iint\limits_{A_2}\Big)|x_2|^{-1+i\mu}\psi\Big(\frac{x_2}{|x_1|^\alpha}\,\frac{R_2}{R_1^\alpha}\Big)|x_1|^{-1+\varepsilon+i\nu}\\
&\qquad\times\operatorname{sgn}x_1\,\varphi_1(x_1)\varphi_2(x_2)\,dx_1\,dx_2\\
&=: J_{\varepsilon,A_1}+J_{\varepsilon,A_2}.
\end{aligned}
$$

The estimate of $|J_{\varepsilon,A_1}|$ is obtained by integrating by parts with respect to x_2 and then repeatedly applying the mean value theorem. Since the reasoning is similar to the estimate of $|J'_\varepsilon|$, we omit it and we only state that

$$|J_{\varepsilon,A_1}|\le C/|\mu|,$$

uniformly with respect to φ_1, φ_2, R_1, R_2, ε.

In order to estimate J_{ε,A_2}, we rewrite it after an integration by parts with respect to x_2 as

$$
\begin{aligned}
J_{\varepsilon,A_2} &= -\iint\limits_{A_2}\frac{\operatorname{sgn}x_2}{i\mu}\,|x_2|^{i\mu}|x_1|^{-1-\alpha+\varepsilon+i\nu}(\partial_{x_2}\psi)\Big(\frac{x_2}{|x_1|^\alpha}\,\frac{R_2}{R_1^\alpha}\Big)\,\varphi_2(x_2)\varphi_1(x_1)\\
&\qquad\times\operatorname{sgn}x_1\,dx_1\,dx_2\\
&= -\iint\limits_{A_2}\frac{\operatorname{sgn}x_2}{i\mu}\,|x_2|^{i\mu}\psi\Big(\frac{x_2}{|x_1|^\alpha}\,\frac{R_2}{R_1^\alpha}\Big)\varphi_2'(x_2)\varphi_1(x_1)|x_1|^{-1+\varepsilon+i\nu}\\
&\qquad\times\operatorname{sgn}x_1\,dx_1\,dx_2.
\end{aligned}
$$

At this point it is easy to check that both integrals above are uniformly bounded by $C/|\mu|$, so that

$$|J_\varepsilon| \le |J_{\varepsilon,A_1}| + |J_{\varepsilon,A_2}| \le C/|\mu|,$$

where C may depend on α, independently of φ_1, φ_2, R_1, R_2, ε, and this concludes the proof. ∎

Let $\varphi_2(x_2)$ be, as in Proposition 2.4, any normalized bump function in $C^1(\mathbb{R}_{x_2})$. Take $R_2 > 0$ and set, as above, $\varphi_{2,R_2}(x_2) := \varphi_2(x_2/R_2)$. Then define the distribution $H^{\mu,\nu}_{\varphi_2,R_2}$ on $\mathbb{R}_{x_2}$ by

$$\begin{aligned}\langle H^{\mu,\nu}_{\varphi_2,R_2}, \varphi_1\rangle &:= \langle H^{\mu,\nu}, \varphi_1 \otimes \varphi_{2,R_2}\rangle \\ &= \lim_{\varepsilon\to 0} \iint |x_1|^{-1+\varepsilon+i\nu} \operatorname{sgn} x_1\, |x_2|^{-1+i\mu}\, \psi\biggl(\frac{x_2}{|x_1|^\alpha}\biggr) \\ &\qquad \times \varphi_1(x_1)\varphi_2\biggl(\frac{x_2}{R_2}\biggr)\, dx_1\, dx_2\end{aligned}$$

for any test function φ_1 on $\mathbb{R}_{x_1}$.

To conclude our proof that $H^{\mu,\nu}$ defines a product kernel on $\mathbb{R}^2$ we only need the following cancellation property.

Proposition 2.5.

(i) *The distribution $H^{\mu,\nu}_{\varphi_2,R_2}$ coincides with the function*

$$H^{\mu,\nu}_{\varphi_2,R_2}(x_1) = |x_1|^{-1+i\nu} \operatorname{sgn} x_1 \int |x_2|^{-1+i\mu} \psi\biggl(\frac{x_2}{|x_1|^\alpha}\biggr)\varphi_2\biggl(\frac{x_2}{R_2}\biggr)\, dx_2 \tag{2.9}$$

on $\mathbb{R}_{x_1} \setminus \{0\}$. Moreover, for any positive integer β there exists a constant $C_{\beta,\mu,\nu,\alpha}$ of admissible growth such that

$$|\partial^\beta_{x_1} H^{\mu,\nu}_{\varphi_2,R_2}(x_1)| \le C_{\beta,\mu,\nu,\alpha}|x_1|^{-1-\beta} \quad \textit{for all } x_1 \in \mathbb{R}\setminus\{0\}, \tag{2.10}$$

uniformly in φ_2, R_2.

(ii) *For any normalized bump function φ_1 of class $C^1(\mathbb{R}_{x_1})$ and any $R_1 > 0$ there exists a constant $C_{\mu,\alpha}$ of admissible growth such that*

$$|\langle H^{\mu,\nu}_{\varphi_2,R_2}, \varphi_{1,R_1}\rangle| \le C_{\mu,\alpha} \tag{2.11}$$

independently of φ_1, R_1, φ_2, R_2.

Proof. First of all, observe that the integral on the right-hand side of (2.9) is absolutely convergent. Thus (2.9) follows from a routine application of Fubini's theorem.

It will now be shown that $H^{\mu,\nu}_{\varphi_2,R_2}$ satisfies the right differential inequalities and cancellation conditions.

If $x_1 \in \mathbb{R} \setminus \{0\}$ and $\beta = 0$, integrating by parts we obtain

$$\begin{aligned}
|H^{\mu,\nu}_{\varphi_2,R_2}(x_1)| &= |x_1|^{-1}\bigg| \int\limits_{|x_2|\le|x_1|^\alpha} \psi\bigg(\frac{x_2}{|x_1|^\alpha}\bigg)|x_2|^{-1+i\mu}\varphi_2\bigg(\frac{x_2}{R_2}\bigg)\,dx_2\bigg| \\
&= |x_1|^{-1}\bigg| \int\limits_{|x_2|\le|x_1|^\alpha} \frac{|x_2|^{i\mu}}{i\mu}\operatorname{sgn} x_2 |x_1|^{-\alpha}(\partial_{x_2}\psi)\bigg(\frac{x_2}{|x_1|^\alpha}\bigg)\varphi_2\bigg(\frac{x_2}{R_2}\bigg)\,dx_2 \\
&\qquad + \int\limits_{|x_2|\le|x_1|^\alpha} \frac{|x_2|^{i\mu}}{i\mu}\operatorname{sgn} x_2 \psi\bigg(\frac{x_2}{|x_1|^\alpha}\bigg)\frac{1}{R_2}\,\varphi_2'\bigg(\frac{x_2}{R_2}\bigg)\,dx_2\bigg| \\
&\le \frac{|x_1|^{-1}}{|\mu|}\bigg(\int\limits_{|x_2|\le|x_1|^\alpha} \|\psi'\|_\infty |x_1|^{-\alpha}\|\varphi_2\|_{\mathcal{C}^2}\,dx_2 \\
&\qquad + \int\limits_{\{|x_2|\le|x_1|^\alpha\}\cap\{|x_2|\le R_2\}} \frac{\|\psi\|_\infty}{R_2}\,\|\varphi_2\|_{\mathcal{C}^2}\,dx_2\bigg) \\
&\le \frac{|x_1|^{-1}}{|\mu|}\bigg(2\|\psi'\|_\infty + \frac{\|\psi\|_\infty}{R_2}\int\limits_{|x_2|\le R_2} dx_2\bigg) \\
&\le \frac{2(\|\psi\|_\infty + \|\psi'\|_\infty)}{|\mu|}\,|x_1|^{-1} = C_{\beta,\mu}|x_1|^{-1}
\end{aligned}$$

uniformly with respect to R_2 and φ_2.

If $\beta \neq 0$, one first applies the Leibniz rule, (2.2) and an integration by parts to obtain

$$\begin{aligned}
\partial^\beta_{x_1} H^{\mu,\nu}_{\varphi_2,R_2}(x_1) &= \sum_{\beta_1+\beta_2=\beta} c_{\beta_1,\beta_2,\nu}|x_1|^{-1+i\nu-\beta_1}(\operatorname{sgn} x_1)^{1+\beta_1}\sum_{l=1}^{\beta_2} c_l \\
&\quad \times \bigg(- \int\limits_{|x_2|\le|x_1|^\alpha} |x_2|^{i\mu}\,\frac{\operatorname{sgn} x_2}{i\mu}\,(\partial_{x_2}\,\partial^{\beta_2-l+1}_{x_1}\psi)\bigg(\frac{x_2}{|x_1|^\alpha}\bigg) \\
&\qquad\qquad \times \frac{x_2^{\beta_2-l+1}}{|x_1|^{(\alpha+1)\beta_2-\alpha l+2\alpha}}\,\varphi_2\bigg(\frac{x_2}{R_2}\bigg)dx_2 \\
&- \int\limits_{|x_2|\le|x_1|^\alpha} |x_2|^{i\mu}\,\frac{\operatorname{sgn} x_2}{i\mu}\,(\partial^{\beta_2-l+1}_{x_1}\psi)\bigg(\frac{x_2}{|x_1|^\alpha}\bigg)\frac{(\beta_2-l+1)x_2^{\beta_2-l}}{|x_1|^{(\alpha+1)\beta_2-\alpha l+\alpha}}\,\varphi_2\bigg(\frac{x_2}{R_2}\bigg)dx_2 \\
&\quad - \int\limits_{|x_2|\le|x_1|^\alpha} |x_2|^{i\mu}\,\frac{\operatorname{sgn} x_2}{i\mu}\,(\partial^{\beta_2-l+1}_{x_1}\psi)\bigg(\frac{x_2}{|x_1|^\alpha}\bigg) \\
&\qquad\qquad \times \frac{x_2^{\beta_2-l+1}}{|x_1|^{(\alpha+1)\beta_2-\alpha l+\alpha}}\,\frac{1}{R_2}\,\varphi_2'\bigg(\frac{x_2}{R_2}\bigg)dx_2\bigg).
\end{aligned}$$

Set now

$$c_{\beta_2,l} := \sup_{|x_2|\le|x_1|^\alpha} \left|(\partial_{x_1}^{\beta_2-l+1}\psi)\left(\frac{x_2}{|x_1|^\alpha}\right)\right|,$$

$$c'_{\beta_2,l} := \sup_{|x_2|\le|x_1|^\alpha} \left|(\partial_{x_2}\partial_{x_1}^{\beta_2-l+1}\psi)\left(\frac{x_2}{|x_1|^\alpha}\right)\right|.$$

Then it is easy to check that

$$\begin{aligned}|\partial_{x_1}^\beta H^{\mu,\nu}_{\varphi_2,R_2}(x_1)| &\le \frac{1}{|\mu|}\sum_{\beta_1+\beta_2=\beta} |c_{\beta_1,\beta_2,\nu}|\,|x_1|^{-1-\beta_1}\sum_{l=1}^{\beta_2}|c_l| \\ &\quad\times\Bigg(c'_{\beta_2,l}\int\limits_{|x_2|\le|x_1|^\alpha}|x_1|^{-\beta_2-\alpha}\,dx_2 \\ &\quad + (\beta_2-l+1)c_{\beta_2,l}\int\limits_{|x_2|\le|x_1|^\alpha}|x_1|^{-\beta_2-\alpha}\,dx_2 \\ &\quad + \frac{c_{\beta_2,l}}{R_2}\int\limits_{|x_2|\le R_2}|x_1|^{-\beta_2}\,dx_2\Bigg) \\ &\le \frac{C_{\nu,\beta}}{|\mu|}\,|x_1|^{-1-\beta} = C_{\beta,\mu,\nu,\alpha}|x_1|^{-1-\beta},\end{aligned}$$

independently of R_2 and φ_2.

Finally, (ii) coincides essentially with (2.6), which has been proved in Proposition 2.4. ∎

As a consequence of the previous lemmata and propositions, we obtain

THEOREM 2.6. *The distribution $H^{\mu,\nu}$, defined by* (2.1), *is a product kernel on $\mathbb{R}^2$.*

3. A problem of fractional integration. Let ψ be a bump function as defined at the beginning of Section 2, that is, an even smooth function on $\mathbb{R}$ such that $\psi = 1$ on $[0,1/2]$ and $\psi = 0$ on $(1,\infty)$, with $0 \le \psi \le 1$ on $(1/2,1)$ and such that ψ' changes sign only once. Following [Gr1] we define a family of analytic distributions D_z, $\operatorname{Re} z > -1$, as

$$\langle D_z, f\rangle := \frac{1}{\Gamma\left(\frac{z+1}{2}\right)}\int |u-1|^z\psi(u-1)f(u)\,du$$

for all $f \in \mathcal{C}_c^\infty(\mathbb{R})$. It is straightforward to check that D_z may be extended to all $z \in \mathbb{C}$. Take now $\alpha \in \mathbb{R}$, $\alpha \ge 2$. Define an analytic family of distributions $K_z^{\gamma,\alpha}$, for γ and z in $\mathbb{C}$ with $\operatorname{Re}\gamma \ge 0$, in the following way:

$$\langle K_z^{\gamma,\alpha}, f\rangle := \int \langle D_z(u), f(t,u|t|^\alpha)\rangle |t|^\gamma\,\frac{dt}{t}. \tag{3.1}$$

We remark that, if $\operatorname{Re}\gamma = 0$, then

$$\langle K_z^{\gamma,\alpha}, f\rangle := \lim_{\varepsilon\to 0}\int \langle D_z(u), f(t, u|t|^\alpha)\rangle |t|^{i\varrho+\varepsilon}\,\frac{dt}{t},$$

where $\operatorname{Im}\gamma = \varrho$, for every $f \in \mathcal{C}_c^\infty(\mathbb{R}^2)$. Observe moreover that $K_z^{\gamma,\alpha}$ depends analytically on both γ and z.

At this point we may introduce the family of convolution operators with kernel $K_z^{\gamma,\alpha}$ defined by (3.1), that is,

$$\begin{aligned}(3.2)\qquad (S_z^{\gamma,\alpha} f)(x_1, x_2) &:= (K_z^{\gamma,\alpha} * f)(x_1, x_2)\\ &= \int \langle D_z(u), f(x_1 - t, x_2 - u|t|^\alpha)\rangle |t|^\gamma\,\frac{dt}{t}.\end{aligned}$$

A necessary condition for $S_z^{\gamma,\alpha}$ to be a bounded operator from $L^p(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$ is that $1/p - 1/q = \operatorname{Re}\gamma/(\alpha+1)$, so that we may define the set

$$\Sigma_\alpha := \left\{\left(\frac{1}{p}, \frac{1}{q}, \operatorname{Re} z\right) : S_z^{\gamma,\alpha} \text{ maps } L^p(\mathbb{R}^2) \text{ to } L^q(\mathbb{R}^2) \text{ boundedly}\right\}.$$

In order to determine Σ_α, we shall use the following lemma.

Lemma 3.1. *Let λ, ϱ, a and ε be real numbers, with $\varrho \neq 0$, $a > 0$, $|\lambda| > 1$ and $0 < \varepsilon < 1$. Then*

$$\left|\int_{|t|<a} e^{-i\lambda t}\,\frac{|t|^{i\varrho+\varepsilon}}{t}\,dt\right| \le C_\varrho,$$

where C_ϱ denotes a positive constant, of admissible growth in ϱ, independent of ε and λ.

Proof. First of all, assume $\lambda > 0$ (the other case is analogous) and observe that

$$\mathcal{I} := \int_{|t|<a} e^{-i\lambda t}\,\frac{|t|^{i\varrho+\varepsilon}}{t}\,dt = -2i\lambda^{-i\varrho-\varepsilon}\int_0^{\lambda a} \sin t \cdot t^{i\varrho+\varepsilon-1}\,dt.$$

Now, if $\lambda a < 2$, we obtain

$$|\mathcal{I}| \le 2\int_0^{\lambda a} t^\varepsilon\,dt \le C.$$

If $\lambda a \ge 2$, we integrate by parts twice to obtain $|\mathcal{I}| \le C_\varrho$. ∎

Proposition 3.2. *The operator $S_z^{\gamma,\alpha}$ maps L^1 to L^∞ if $\operatorname{Re}\gamma = \alpha + 1$ and $\operatorname{Re} z = 0$.*

Proof. The proof is similar to that of [Gr2, p. 655] and it is omitted. ∎

In order to obtain an L^2-L^2 estimate for $S_z^{\gamma,\alpha}$ at the height $\operatorname{Re} z = -3/2$, we compute the Fourier transform of the distribution $K_{-3/2+i\theta}^{i\varrho,\alpha}$. Here we use the methods of [Gr1] and [Gr2].

The most interesting situation occurs when $\varrho \neq 0$ and $\alpha > 2$. To treat this case, we introduce the distributions

$$G^{i\varrho+\varepsilon,\alpha}_{-3/2+i\theta,N}(\xi_1,\xi_2) := \int\limits_{|t|<N} L_{-3/2+i\theta}(\xi_2|t|^\alpha)e^{-i(\xi_1 t+\xi_2|t|^\alpha)}\,\frac{|t|^{i\varrho+\varepsilon}}{t}\,dt, \tag{3.3}$$

where $\varrho \neq 0$, $\theta \in \mathbb{R}$, $\varepsilon > 0$, $N > 0$ and L_r, $r \in \mathbb{C}$, is an even smooth function on the real line, defined as

$$L_r(v) := \frac{2^{r+1}\sqrt{\pi}}{\Gamma(-r/2)}(|\cdot|^{-r-1} * \widehat{\psi})(v), \qquad v \in \mathbb{R}.$$

Lemma 3.3.

(a) *For every $\varrho \neq 0$ and $\theta \in \mathbb{R}$ the limit* $\lim_{\varepsilon\to 0,\, N\to\infty} G^{i\varrho+\varepsilon,\alpha}_{-3/2+i\theta,N}$ *exists almost everywhere on* $\mathbb{R}^2$.

(b) *There exists an admissible constant $C_{\theta,\alpha,\varrho}$ such that*

$$|G^{i\varrho+\varepsilon,\alpha}_{-3/2+i\theta,N}(\xi_1,\xi_2)| \le C_{\theta,\alpha,\varrho}|\xi_2|^{-1/\alpha}(1+|\xi_1|\,|\xi_2|^{-1/\alpha}) \tag{3.4}$$

for almost all $(\xi_1,\xi_2) \in \mathbb{R}^2$, for $\alpha > 2$, uniformly with respect to $\varepsilon > 0$ and $N > 0$.

Proof. (a) Let $\xi_2 \neq 0$. By setting $N' = |\xi_2|^{1/\alpha}N$, $\lambda = |\xi_2|^{-1/\alpha}\xi_1$, $\varepsilon_2 = \operatorname{sgn}\xi_2$, we obtain

$$\begin{aligned}
&G^{i\varrho+\varepsilon,\alpha}_{-3/2+i\theta,N}(\xi_1,\xi_2)\\
&\qquad = \Big(\int\limits_{|t|<a} + \int\limits_{a\le|t|<N'}\Big)|\xi_2|^{-(i\varrho+\varepsilon)/\alpha}L_{-3/2+i\theta}(|t|^\alpha)e^{-i(\lambda t+\varepsilon_2|t|^\alpha)}\,\frac{|t|^{i\varrho+\varepsilon}}{t}\,dt\\
&\qquad =: I'_a + I''_{a,N'},
\end{aligned}$$

where the positive constant $a = a_{\theta,\varrho,\alpha}$ will be chosen later.

We shall now compute the limit of I'_a, which may be rewritten as

$$\begin{aligned}
I'_a = |\xi_2|^{-(i\varrho+\varepsilon)/\alpha}\Big(&\int\limits_{|t|<a}(L_{-3/2+i\theta}(|t|^\alpha)e^{-i\varepsilon_2|t|^\alpha} - L_{-3/2+i\theta}(0))e^{-i\lambda t}\,\frac{|t|^{i\varrho+\varepsilon}}{t}\,dt\\
&+ \int\limits_{|t|<a} L_{-3/2+i\theta}(0)\,\frac{e^{-i\lambda t}-1}{t}\,|t|^{i\varrho+\varepsilon}\,dt\Big).
\end{aligned}$$

Now, by applying the mean value theorem and the dominated convergence theorem it is easy to see that I'_a converges, when $\varepsilon \to 0$, to

$$\begin{aligned}
i\varrho|\xi_2|^{-i\varrho/\alpha}\Big(&\int\limits_{|t|<a}(L_{-3/2+i\theta}(|t|^\alpha)e^{-i\varepsilon_2|t|^\alpha} - L_{-3/2+i\theta}(0))e^{-i\lambda t}\,\frac{|t|^{i\varrho}}{t}\,dt\\
&+ \int\limits_{|t|<a} L_{-3/2+i\theta}(0)\,\frac{e^{-i\lambda t}-1}{t}\,|t|^{i\varrho}\,dt\Big).
\end{aligned} \tag{3.5}$$

In order to estimate $I''_{a,N'}$, we use Lemma 3.2 in [Gr1], stating that

$$L_r(v) = c_r|v|^{-r-1} + R(v),$$

where $R(v) = O(|v|^{-M})$ for all $M > 0$ as $|v| \to \infty$, and $\operatorname{Re} r < 0$. Thus

$$\begin{aligned}(3.6)\qquad &\int_{a\le|t|<N'} L_{-3/2+i\theta}(|t|^\alpha)e^{-i(\lambda t+\varepsilon_2|t|^\alpha)}\,\frac{|t|^{i\varrho+\varepsilon}}{t}\,dt \\ &= \int_{a\le|t|<N'} C_\theta|t|^{\alpha/2-i\theta\alpha}e^{-i(\lambda t+\varepsilon_2|t|^\alpha)}\,\frac{|t|^{i\varrho+\varepsilon}}{t}\,dt \\ &\quad+ \int_{a\le|t|<N'} R_{-3/2+i\theta}(|t|^\alpha)e^{-i(\lambda t+\varepsilon_2|t|^\alpha)}\,\frac{|t|^{i\varrho}}{t}\,dt.\end{aligned}$$

Since $R_{-3/2+i\theta}(|t|^\alpha) = O(|t|^{-\alpha M})$ for all $M > 0$, it is easy to check that the limit

$$\lim_{\varepsilon\to0,\,N\to\infty} |\xi_2|^{-(i\varrho+\varepsilon)/\alpha} \int_{a\le|t|<N'} R_{-3/2+i\theta}(|t|^\alpha)e^{-i(\lambda t+\varepsilon_2|t|^\alpha)}\,\frac{|t|^{i\varrho+\varepsilon}}{t}\,dt$$

exists and is equal to

$$|\xi_2|^{-i\varrho/\alpha} \int_{a\le|t|<\infty} R_{-3/2+i\theta}(|t|^\alpha)e^{-i(\lambda t+\varepsilon_2|t|^\alpha)}\,\frac{|t|^{i\varrho}}{t}\,dt.$$

To compute the limit of the main term in (3.6), note that

$$\begin{aligned}&\int_{a\le|t|<N'} |t|^{\alpha/2-i\theta\alpha}e^{-i(\lambda t+\varepsilon_2|t|^\alpha)}\,\frac{|t|^{i\varrho+\varepsilon}}{t}\,dt \\ &\qquad= \int_{a\le t<N'} t^{\alpha/2-1+\varepsilon+i(\varrho-\alpha\theta)}e^{-i\varepsilon_2\,t^\alpha}\big(e^{-i\lambda t}+e^{i\lambda t}\big)\,\frac{dt}{t}.\end{aligned}$$

After a routine integration by parts, it is not hard to conclude the proof of (a).

(b) It suffices to prove that both $|I'_a|$ and $|I''_{a,N'}|$ are bounded by $C_{\theta,\alpha,\varrho}|p(\xi_1,\xi_2)|$ for some function p satisfying

$$|p(\xi_1,\xi_2)| \le C_{\theta,\alpha,\varrho}|\xi_2|^{-1/\alpha}(1+|\xi_1|\cdot|\xi_2|^{-1/\alpha})$$

for almost all $(\xi_1,\xi_2)\in\mathbb{R}^2$. By arguing as in (a), we see that

$$|I'_a| \le C_{\theta,\alpha,\varrho}|\xi_2|^{-\varepsilon/\alpha}(1+|\lambda|).$$

Since $|\xi_2|^{-\varepsilon/\alpha} \le |\xi_2|^{-1/\alpha}$ for $|\xi_2|\le 1$, the right bound for $|I'_a|$ is proved.

In order to estimate $|I''_{a,N'}|$, observe that the main term on the right-hand side of (3.6) may be written as

$$\int_a^{N'} -t^{\alpha/2-1+\varepsilon+i(\varrho-\alpha\theta)} e^{i(\lambda t-\varepsilon_2 t^\alpha)}\,dt + \int_a^{N'} t^{\alpha/2-1+\varepsilon+i(\varrho-\alpha\theta)} e^{-i(\lambda t+\varepsilon_2 t^\alpha)}\,dt =: I_+ + I_-.$$

If $|\lambda| \le 2\alpha a^{\alpha-1}$, integrating by parts we obtain

$$\begin{aligned} |I_+| &= \Big|\int_a^{N'} t^{\alpha/2-1+\varepsilon+i(\varrho-\alpha\theta)} e^{i(\lambda t-\varepsilon_2 t^\alpha)}\,dt\Big| \\ &\le \frac{1}{\alpha}\Big(N'^{-\alpha/2+\varepsilon} + a^{-\alpha/2+\varepsilon} \\ &\quad + \int_a^{N'} (\sqrt{(-\alpha/2+\varepsilon)^2+(\varrho-\alpha\theta)^2}\cdot t^{-1} + |\lambda|)t^{-\alpha/2+\varepsilon}\,dt\Big). \end{aligned}$$

For $\alpha > 2$ the integrals above converge and

$$|I_+| \le C_{\theta,\varrho,\alpha,a}(1+|\lambda|),$$

uniformly with respect to ε and N'. An analogous estimate may be proved for $|I_-|$.

Consider now the case $|\lambda| > 2\alpha a^{\alpha-1}$. It is convenient to rewrite I_+ as

$$\begin{aligned} I_+ &= \Bigg(\int_a^{(\frac{|\lambda|}{2\alpha})^{\frac{1}{\alpha-1}}} + \int_{(\frac{|\lambda|}{2\alpha})^{\frac{1}{\alpha-1}}}^{(\frac{2|\lambda|}{\alpha})^{\frac{1}{\alpha-1}}} + \int_{(\frac{2|\lambda|}{\alpha})^{\frac{1}{\alpha-1}}}^{N'}\Bigg) t^{\alpha/2+\varepsilon-1+i(\varrho-\alpha\theta)} e^{i(\lambda t-\varepsilon_2 t^\alpha)}\,dt \\ &=: I_+^{(1)} + I_+^{(2)} + I_+^{(3)}. \end{aligned}$$

To estimate $I_+^{(1)}$, we integrate by parts to obtain

$$\begin{aligned} |I_+^{(1)}| \le \frac{\big(\frac{|\lambda|}{2\alpha}\big)^{\frac{1}{\alpha-1}(\alpha/2-1+\varepsilon)}}{|\lambda-|\lambda|/2|} + \frac{a^{\alpha/2-1+\varepsilon}}{|\lambda-\varepsilon_2\alpha a^{\alpha-1}|} + \int_a^{(\frac{|\lambda|}{2\alpha})^{\frac{1}{\alpha-1}}} \frac{t^{\alpha/2-2+\varepsilon}}{|\lambda-\varepsilon_2\alpha t^{\alpha-1}|^2} \\ \times (\sqrt{(\alpha/2-1+\varepsilon)^2+(\varrho-\alpha\theta)^2}\cdot|\lambda-\varepsilon_2\alpha t^{\alpha-1}| + \alpha(\alpha-1)t^{\alpha-1})\,dt. \end{aligned}$$

By using the inequalities

$$|\lambda-|\lambda|/2| \ge |\lambda|/2, \qquad |\lambda-\varepsilon_2\alpha a^{\alpha-1}| \ge \alpha a^{\alpha-1},$$
$$|\lambda|/2 \le |\lambda-\varepsilon_2\alpha t^{\alpha-1}| \le 3|\lambda|/2,$$

we finally get

$$|I_+^{(1)}| \le C_{\theta,\varrho,a,\alpha}. \tag{3.7}$$

An analogous estimate may be proved for the integral $I_+^{(3)}$. Indeed, integrating by parts and using the inequalities

$$|\lambda - 2|\lambda|\,| \geq |\lambda|, \qquad |\lambda - \varepsilon_2 \alpha N'^{\alpha-1}| \geq \frac{\alpha}{2}\, N'^{\alpha-1},$$
$$\frac{\alpha}{2}\, t^{\alpha-1} \leq |\lambda - \varepsilon_2 \alpha t^{\alpha-1}| \leq \frac{3}{2}\, \alpha t^{\alpha-1},$$

we obtain in a similar way

$$|I_+^{(3)}| \leq C_{\theta,\varrho,a,\alpha}, \tag{3.8}$$

independently of ε and N'.

In order to estimate $I_+^{(2)}$, we observe that it is an oscillatory integral with phase

$$\varphi(t) = (\varrho - \alpha\theta)\ln t + \lambda t - \varepsilon_2 t^{\alpha}.$$

By choosing the constant a such that $a \geq \max\{|\varrho - \alpha\theta|^{1/(\alpha-2)}, 1\}$, we have $|\varphi''(t)| \geq (|\lambda|/2\alpha)^{(\alpha-2)/(\alpha-1)}$, so that a routine application of van der Corput's lemma yields

$$|I_+^{(2)}| \leq C_{\theta,\varrho,a,\alpha}|\lambda|^{\varepsilon/2(\alpha-1)}. \tag{3.9}$$

By collecting (3.7), (3.8) and (3.9) we conclude that $|I_+|$ is bounded by $C_{\theta,\varrho,a,\alpha}|\lambda|^{\varepsilon/2(\alpha-1)}$ for some admissible constant $C_{\theta,\varrho,a,\alpha}$. Since similar estimates hold for $|I_-|$ and for the remainder in (3.6), we conclude that

$$|I''_{a,N'}| \leq C_{\theta,\alpha,\varrho}|\xi_2|^{-\varepsilon/\alpha}(1 + |\lambda|^{\varepsilon/2(\alpha-1)})$$

a.e. in $\mathbb{R}^2$. Since for $|\lambda| > 2\alpha a^{\alpha-1}$ we have $|\lambda|^{-\varepsilon/2(\alpha-1)} \leq |\lambda|$, it is easy to conclude that

$$|G^{i\varrho+\varepsilon,\alpha}_{-3/2+i\theta,N}(\xi_1,\xi_2)| \leq C_{\theta,\alpha,\varrho}|\xi_2|^{-1/\alpha}(1 + |\xi_1|\,|\xi_2|^{-1/\alpha})$$

for almost all $(\xi_1,\xi_2) \in \mathbb{R}^2$, uniformly with respect to $\varepsilon > 0$ and $N > 0$. ∎

PROPOSITION 3.4. *The operator $S_z^{\gamma,\alpha}$ maps L^2 to L^2 if* $\operatorname{Re}\gamma = 0$ *and* $\operatorname{Re} z = -3/2$.

Proof. We shall first consider the case $\gamma = i\varrho$, $\varrho,\theta \in \mathbb{R}$, $\varrho \neq 0$ and we will prove that the Fourier transform of the kernel $K^{i\varrho,\alpha}_{-3/2+i\theta}$ is given by $\lim_{\varepsilon\to 0,\, N\to\infty} G^{i\varrho+\varepsilon,\alpha}_{-3/2+i\theta,N}$, where the distributions $G^{i\varrho+\varepsilon,\alpha}_{-3/2+i\theta,N}$ have been studied in the previous lemma.

Let f be any Schwartz function on $\mathbb{R}^2$ and $\alpha > 2$. Thus

$$\begin{aligned}
\langle \widehat{K^{i\varrho,\alpha}_{-3/2+i\theta}}, f\rangle &= \langle K^{i\varrho,\alpha}_{-3/2+i\theta}, \widehat{f}\rangle \\
&= \lim_{\varepsilon\to 0}\int \langle D_{-3/2+i\theta}(u), \widehat{f}(t,u|t|^\alpha)\rangle \frac{|t|^{i\varrho+\varepsilon}}{t}\,dt \\
&= \lim_{\varepsilon\to 0}\lim_{N\to\infty}\int\limits_{|t|<N}\iint \widehat{D_{-3/2+i\theta}}(\xi_2|t|^\alpha)f(\xi_1,\xi_2)e^{-i\xi_1 t}\,d\xi_1\,d\xi_2\,\frac{|t|^{i\varrho+\varepsilon}}{t}\,dt \\
&= \lim_{\varepsilon\to 0}\lim_{N\to\infty}\int\limits_{|t|<N}\iint f(\xi_1,\xi_2)L_{-3/2+i\theta}(\xi_2|t|^\alpha)e^{-i\xi_1 t}e^{-i\xi_2|t|^\alpha}\,d\xi_1\,d\xi_2\,\frac{|t|^{i\varrho+\varepsilon}}{t}\,dt \\
&= \lim_{\varepsilon\to 0}\lim_{N\to\infty}\iint G^{i\varrho+\varepsilon,\alpha}_{-3/2+i\theta,N}(\xi_1,\xi_2)f(\xi_1,\xi_2)\,d\xi_1\,d\xi_2,
\end{aligned}$$

where we have used the fact that $\widehat{D}_z(v) = e^{-iv}L_z(v)$. Observe now that, as a consequence of Lemma 3.3,

$$\begin{aligned}
(3.10)\qquad &|G^{i\varrho+\varepsilon,\alpha}_{-3/2+i\theta,N}(\xi_1,\xi_2)f(\xi_1,\xi_2)| \\
&\qquad\le C_{\theta,\alpha,\varrho}|\xi_2|^{-1/\alpha}(1+|\xi_1|\,|\xi_2|^{-1/\alpha})|f(\xi_1,\xi_2)|
\end{aligned}$$

for almost all $(\xi_1,\xi_2)\in\mathbb{R}^2$, uniformly with respect to $\varepsilon > 0$ and $N > 0$. It is easy to check that the function on the right-hand side of (3.10) is integrable on $\mathbb{R}^2$, so that by the dominated convergence theorem we obtain

$$\langle \widehat{K^{i\varrho,\alpha}_{-3/2+i\theta}}, f\rangle = \iint \lim_{\varepsilon\to 0}\lim_{N\to\infty} G^{i\varrho+\varepsilon,\alpha}_{-3/2+i\theta,N}(\xi_1,\xi_2)f(\xi_1,\xi_2)\,d\xi_1\,d\xi_2.$$

For the case $\alpha = 2$ we refer the reader to [Gr2, p. 655].

We shall now prove that

$$(3.11)\qquad |\widehat{K^{i\varrho,\alpha}_{-3/2+i\theta}}(\xi_1,\xi_2)| \le C_{\alpha,\theta,\varrho}\quad \text{for almost all } (\xi_1,\xi_2)\in\mathbb{R}^2$$

for some constant $C_{\alpha,\theta,\varrho}$ of admissible growth.

As a consequence of Lemma 3.1 we have

$$|(3.5)| \le C_{\alpha,\theta,\varrho}$$

for a.a. $(\xi_1,\xi_2)\in\mathbb{R}^2$, so that we only have to show that the limit of the main term in (3.6) is bounded. This is not hard, after a standard integration by parts.

The case $\gamma = i\varrho$, $\varrho = 0$, is similar and easier and we do not treat it here. ∎

The following proposition yields boundedness of $S^{\gamma,\alpha}_z$ on the closed segment $AE\setminus\{A\}$ and, by duality, on the segment $EC\setminus\{C\}$ as well, where $E := (2/3, 1/3, -1)$. The proof is similar to the proof of Proposition in [Gr2, pp. 656–658], and therefore we omit it.

Proposition 3.5. *The operator* $S_z^{\gamma,\alpha}$ *maps* L^p *to* L^{2p} *for all* $3/2 \le p < \infty$ *if* $\operatorname{Re}\gamma = (\alpha+1)/2p$ *and* $\operatorname{Re} z = -1$.

Consider now the distribution $H^{\mu,\nu}$ defined by (2.1). In Theorem 2.6 we proved that $H^{\mu,\nu}$ is a product-type kernel on $\mathbb{R}^2$. Define now a distribution $\widetilde{H}$ by the formula

$$\int \widetilde{H}(x_1,x_2) f(x_1,x_2)\,dx_1\,dx_2 = \int H(x_1,x_2) f(x_1, x_2 + |x_1|^\alpha)\,dx_1\,dx_2$$

for every function $f \in \mathcal{S}(\mathbb{R}^2)$ and $\alpha \ge 2$. We say, in analogy to [Se, Def. 1.2], that $\widetilde{H}$ is a *product-type kernel adapted to the curve* $x_1 \mapsto |x_1|^\alpha$ on $\mathbb{R}^2$.

Since $f(x_1, x_2 + |x_1|^\alpha)$ belongs to $\mathcal{C}^2(\mathbb{R}^2)$, the kernel $\widetilde{H}$ is a well defined distribution, singular along the coordinate axis $x_1 = 0$ and the curve $x_2 = |x_1|^\alpha$.

S. Secco recently proved ([Se, Th. 1.3]) that the convolution operator $T : f \mapsto f * \widetilde{H}$ defined on the Schwartz class can be extended to a bounded operator on $L^p(\mathbb{R}^2)$ for all $1 < p < \infty$. Her result yields the boundedness of the operator $S_z^{\gamma,\alpha}$ on the open segment AC, as the following theorem shows.

Theorem 3.6. *The operator* $S_z^{\gamma,\alpha}$ *maps* L^p *to* L^p *for all* $1 < p < \infty$ *if* $\operatorname{Re}\gamma = 0$ *and* $\operatorname{Re} z = -1$.

Proof. If $\theta = 0$, then

$$S_{-1}^{i\varrho,\alpha} f(x_1,x_2) = \lim_{\varepsilon\to 0} \int f(x_1 - t, x_2 - |t|^\alpha)|t|^{i\varrho+\varepsilon}\,\frac{dt}{t}.$$

It is a well known result ([SW]) that $S_{-1}^{i\varrho,\alpha}$ maps L^p to L^p for all $1 < p < \infty$.

If $\theta \ne 0$, then the convolution kernel $K_{-1+i\theta}^{i\varrho,\alpha}$ may be written as

$$\langle K_{-1+i\theta}^{i\varrho,\alpha}, f\rangle = \frac{1}{\Gamma(i\theta/2)} \lim_{\varepsilon\to 0} \iint |x_2 - |x_1|^\alpha|^{-1+i\theta} \psi\left(\frac{x_2}{|x_1|^\alpha} - 1\right) \times |x_1|^{-1+\varepsilon+i(\varrho-\alpha\theta)} \operatorname{sgn} x_1 f(x_1,x_2)\,dx_1\,dx_2$$

and it essentially coincides with the kernel $H^{\mu,\nu}$, defined by (2.1), with $\mu = \theta$, $\nu = \varrho - \alpha\theta$, adapted to the curve $x_1 \mapsto |x_1|^\alpha$. Thus Theorem 2.6 and [Se, Theorem 1.3] imply that $S_{-1+i\theta}^{i\varrho,\alpha}$ maps L^p to L^p for all $1 < p < \infty$. ■

Finally, we completely characterize the set Σ_α defined above.

Theorem 3.7. *For* $\operatorname{Re}\gamma > 0$ *the analytic family of fractional integrals* $S_z^{\gamma,\alpha}$ *maps* L^p *to* L^q *if and only if* $(1/p, 1/q, \operatorname{Re} z)$ *belongs either to the interior of the closed tetrahedron* $ABCD$ *with vertices* $A = (0,0,-1)$, $B = (1/2, 1/2, -3/2)$, $C = (1,1,-1)$, $D = (1,0,0)$, *or to the open faces* ABD, BCD, ACD, *or to the closed edge* $BD \setminus \{B\}$.

For $\operatorname{Re}\gamma = 0$ *the integrals* $S_z^{\gamma,\alpha}$ *map* L^p *to* L^p *if and only if* $(1/p, 1/p, \operatorname{Re} z)$ *belongs to the open segment* AC *or to the open face* $ACB \cup \{B\}$.

Proof. Propositions 3.2 and 3.4 yield, respectively, boundedness at D and B. By interpolation, $S_z^{\gamma,\alpha}$ maps L^p to $L^{p'}$ on the closed edge BD.

As mentioned before, Proposition 3.5 implies boundedness along $AE \setminus \{A\}$, so that, by interpolating this segment with B and D, we prove boundedness on the open face ABD, and therefore, by duality, on BCD as well.

In the light of Theorem 3.6, $S_z^{\gamma,\alpha}$ maps L^p to L^p along the open segment AC. Thus, by interpolating with B and D, we obtain boundedness on the faces, respectively, ACB and ACD (the latter, in particular, was not covered by the results in [Gr2]). Moreover, interpolation between AC and E yields boundedness on the open face ACE, so that interpolating between ACE and B and D we finally fill the interior of the closed tetrahedron.

For the proof of the necessity, we refer the reader to [Gr2, p. 659] and to the Introduction above. ∎

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Dipartimento di Matematica
Politecnico di Torino
Corso Duca degli Abruzzi 24
10129 Torino, Italy
E-mail: casarino@calvino.polito.it
secco@calvino.polito.it

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