STUDIA MATHEMATICA 213 (2) (2012)

Painlevé null sets, dimension and compact embedding of weighted holomorphic spaces

by

Alexander V. Abanin and
Pham Trong Tien (Rostov-na-Donu and Vladikavkaz)

Abstract. We obtain, in terms of associated weights, natural criteria for compact embedding of weighted Banach spaces of holomorphic functions on a wide class of domains in the complex plane. Our study is based on a complete characterization of finite-dimensional weighted spaces and canonical weights for them. In particular, we show that for a domain whose complement is not a Painlevé null set each nontrivial space of holomorphic functions with O-growth condition is infinite-dimensional.

1. Introduction. For a domain (i.e., an open connected set) G in $\mathbb{C}$, $H(G)$ denotes the space of all holomorphic functions on G equipped with the compact-open topology *co*. Let $w : G \to \mathbb{R}$ be a continuous and strictly positive function on G, here called a *weight*. Define the following weighted spaces of holomorphic functions:

$$H_w(G) := \left\{ f \in H(G) : \|f\|_w := \sup_{z\in G} \frac{|f(z)|}{w(z)} < \infty \right\},$$
$$H_{w0}(G) := \left\{ f \in H(G) : \frac{f(z)}{w(z)} \text{ vanishes at infinity on } G \right\},$$

endowed with the norm $\|\cdot\|_w$. As usual, we say that a function g *vanishes at infinity on* G if for every $\varepsilon > 0$ there exists a compact subset K of G such that $|g(z)| < \varepsilon$ for all $z \in G \setminus K$.

The best known space of the first type is the space $H^\infty(G)$ of all holomorphic bounded functions in G (see, e.g., [Gar], [H]). Spaces of the above two types arise in the growth theory of holomorphic functions, Fourier analysis, convolution and partial differential equations, etc. Therefore they were studied intensively in different directions in many papers (see, for instance,

2010 *Mathematics Subject Classification*: Primary 32K05; Secondary 32C18, 32C22, 46B20.
Key words and phrases: weighted Banach spaces of holomorphic functions, compact embedding, Painlevé null set.

DOI: 10.4064/sm213-2-4

[AD], [BB], [BS], [BDL], [BDLT], [BFJ], [BV], [BW], [L], [RS]). In [BBT] a systematic study of these spaces and some properties of inductive spectra composed from them was undertaken. It should be noted that in some of the above mentioned papers functions of several variables were considered.

One of the most important problems relating to $H_w(G)$ and $H_{w0}(G)$ is to characterize properties of these spaces and operators in them in terms of the relevant weights. As is well known, for general weights this may be impossible, but there is a chance to do this by using associated weights which contain exact and complete information on the holomorphic functions estimated by w. We now recall the notion of associated weights and some facts about them.

Let $B_w(G)$ denote the unit ball in $H_w(G)$. Following Bierstedt–Bonet–Taskinen [BBT], the function

$$\widetilde{w}(z) := \sup\{|f(z)| : f \in B_w(G)\}, \qquad z \in G,$$

is called the *associated* weight (with w). Note that $\log \widetilde{w}$ is a subharmonic function on G, $\widetilde{w} \le w$ on G, and $H_{\widetilde{w}}(G) = H_w(G)$ isometrically. Thus, when considering spaces of holomorphic functions with O-growth conditions we can use associated weights only. But so far nobody knows a complete description of associated weights. Moreover, even for a concrete weighted Banach space it might be a difficult problem to evaluate the associated weight. For these reasons it is more convenient to use the notion of canonical weight, which we now recall (see, e.g., [AT]).

We say that a weight w_1 is *dominated* by a weight w_2 ($w_1 \prec w_2$) on G if there is $C > 0$ such that $w_1(z) \le C w_2(z)$ for all $z \in G$. If $w_1 \prec w_2$ and $w_2 \prec w_1$, then w_1 and w_2 are called *equivalent* ($w_1 \sim w_2$). Obviously, $H_{w_1}(G) \hookrightarrow H_{w_2}(G)$ whenever $w_1 \prec w_2$ (here and below, $\hookrightarrow$ denotes a continuous embedding), and $H_{w_1}(G) = H_{w_2}(G)$ whenever $w_1 \sim w_2$. A weight w is called *canonical* if $w \sim \widetilde{w}$. It is clear that for every weight w on G we have a whole family $w(G)$ of canonical weights containing as a subclass all functions $C\widetilde{w}$, $C > 0$. Note that for any $\overline{w} \in w(G)$,

$$H_{\overline{w}}(G) = H_w(G) \quad \text{and} \quad \frac{1}{C}\|f\|_{\overline{w}} \le \|f\|_w \le C\|f\|_{\overline{w}}, \qquad f \in H(G),$$

with some constant $C \ge 1$ depending only on $\overline{w}$.

Although associated, as well as canonical, weights $\widetilde{w}$ carry the whole and exact information on $H_w(G)$, they rarely provide answers in a complete and simple form. In the present paper we develop a new, elementary, approach to this, which we illustrate on the problem of compact embedding of one space $H_{w_1}(G)$ into another $H_{w_2}(G)$. Everybody knows that the condition

$$\frac{w_1(z)}{w_2(z)} \text{ vanishes at infinity on } G$$

is sufficient for compact embedding of $H_{w_1}(G)$ into $H_{w_2}(G)$, while the converse is not true in general even if we replace w_1 by the associated weight $\widetilde{w}_1$. Our main idea is that the "wild" spaces, for which the converse fails, are mostly finite-dimensional. Thus, in Section 2 we undertake a detailed study of finite-dimensional weighted spaces and obtain several criteria for them. Starting from these criteria, in the next section we give the following answer to the problem of compact embedding.

THEOREM 1.1. *Let G be either the whole complex plane or a domain in $\mathbb{C}$ whose complement has no one-point component, and let w_1, w_2 be weights on G such that $H_{w_1}(G) \subset H_{w_2}(G)$. Suppose that $H_{w_1}(G)$ is infinite-dimensional whenever $G = \mathbb{C}$, and nontrivial in the other case. Then the inclusion of $H_{w_1}(G)$ into $H_{w_2}(G)$ is compact if and only if*

$$\text{(1.1)} \qquad \frac{\widetilde{w}_1(z)}{w_2(z)} \text{ vanishes at infinity on } G.$$

Note that the restrictions on G are essential in this theorem. To show this, we construct several examples. In Section 4 we give some applications of the above theorem. All our results can be trivially reformulated for open sets in $\mathbb{C}$.

2. Finite-dimensional weighted spaces. In this section we give a characterization of finite-dimensional spaces and answer the following questions: 1) what canonical weights define such spaces? 2) for which domains in $\mathbb{C}$ is there at least one finite-dimensional weighted space? In what follows, G denotes a domain in $\mathbb{C}$; $\mathbb{C}^*$ is the extended complex plane and $G^c := \mathbb{C}^* \setminus G$ the complement of G; $K \Subset G$ means that K is a compact set in G; for a continuous function f on a compact set K, $\|f\|_K := \max_{z\in K} |f(z)|$. Let us agree to consider the trivial space as a space of dimension zero. We start with the following simple functional criterion.

PROPOSITION 2.1. *The following assertions are equivalent:*

(i) *$H_w(G)$ has a finite dimension.*
(ii) *co induces on $H_w(G)$ its original normed topology.*
(iii) *$\exists K \Subset G\ \exists A > 0 : \|f\|_w \le A\|f\|_K,\ \forall f \in H_w(G)$.*

Proof. Obviously, (ii)$\Leftrightarrow$(iii).

(i)$\Rightarrow$(ii), since there is a unique separated locally convex topology on a finite-dimensional linear space.

Let now (ii) hold. Note that the unit ball $B_w(G)$ of $H_w(G)$ is always a compact set in $(H(G), co)$. By (ii), $B_w(G)$ is compact in $H_w(G)$, too. This implies that $H_w(G)$ is finite-dimensional, so (i) holds. ■

The next theorem gives a complete description of a finite-dimensional weighted space structure. We denote by $n(f)$ the number of zeros of a nontrivial function $f \in H(G)$ counted with multiplicities. Throughout the present paper we keep this agreement to count zeros with their multiplicities. Clearly, $n(f)$ is a nonnegative integer or $+\infty$.

THEOREM 2.2. *The following assertions are equivalent:*

(i) $H_w(G)$ *has dimension* $p \in \mathbb{N}$.
(ii) $H_w(G) = \operatorname{span}\{z^k f_0(z) : 0 \le k \le p-1\}$, *where* f_0 *is a holomorphic function in* G *having no zeros in* G.
(iii) $n(g) \le p-1$ *for every nontrivial function* $g \in H_w(G)$ *and there is a function* $f \in H_w(G)$ *with* $n(f) = p-1$.

Proof. Obviously, (ii) implies (i) and (iii).

To continue the proof, we need the following simple lemma.

LEMMA 2.3. *Let* $R(z) = P(z)/Q(z)$ *be a rational function, where* $\deg P \le \deg Q$ *and the polynomials* P *and* Q *have no common zeros. If* $Q(z) \neq 0$ *for* $z \in \partial G$ *and a function* f *in* $H_w(G)$ *or* $H_{w0}(G)$ *is such that* $fR \in H(G)$, *then* $fR \in H_w(G)$ *or* $fR \in H_{w0}(G)$, *respectively.*

Proof. Since $Q(z) \neq 0$ for $z \in \partial G$ and $\deg P \le \deg Q$, there exists a compact set K in G such that R has no singularities in $G \setminus \operatorname{int} K$ and

$$\sup_{z \in G \setminus K} |R(z)| =: A < \infty. \tag{2.1}$$

Then for every $f \in H_w(G)$ with $fR \in H(G)$ we have

$$\sup_{z \in K} \frac{|f(z)R(z)|}{w(z)} < \infty$$

and

$$\sup_{z \in G \setminus K} \frac{|f(z)R(z)|}{w(z)} \le A \sup_{z \in G \setminus K} \frac{|f(z)|}{w(z)} \le A\|f\|_w.$$

Hence, $fR \in H_w(G)$ whenever $fR \in H(G)$ and $f \in H_w(G)$. If, additionally, f/w vanishes at infinity on G, then so does fR/w in view of (2.1). ∎

Now we continue the proof of the theorem. Consider a nontrivial function $g \in H_w(G)$ and suppose that it has zeros $z_1, \dots, z_n$ with multiplicities $s_1, \dots, s_n$ (g may have other zeros). Then, by Lemma 2.3, the functions

$$g_{k,m}(z) := \frac{g(z)}{(z - z_k)^{m+1}} \qquad (1 \le k \le n;\ 0 \le m \le s_k - 1)$$

are in $H_w(G)$. Put $g_{0,0}(z) := g(z)$ and $s_0 := 1$. It is easy to see that the system $\{g_{k,m} : 0 \le k \le n,\ 0 \le m \le s_k - 1\}$ is linearly independent. Thus, (i) implies that

$$q := \max\{n(g) : g \text{ is a nontrivial function in } H_w(G)\} \le p - 1.$$

It remains to check that if $q < \infty$, then $H_w(G) = \operatorname{span}\{z^k f_0(z) : 0 \le k \le q\}$, where f_0 is a holomorphic function in G having no zeros in G. Indeed, in this case (i)$\Rightarrow$(iii)$\Rightarrow$(ii).

So, let $q < \infty$. Choose a function $f \in H_w(G)$ with $n(f) = q$. It has the form $f(z) = P_0(z) f_0(z)$, where P_0 is a polynomial of degree q and $f_0 \in H(G)$ has no zeros in G. Fixing $a \in G$, we can assume that $f_0(a) = 1$. Similarly, every nontrivial function $g \in H_w(G)$ has the form $g = Q g_0$, where Q is a polynomial with $\deg Q \le q$ and $g_0 \in H(G)$ has no zeros in G and $g_0(a) = 1$. Consider the function

$$h(z) := P(z) f_0(z) - g_0(z),$$

where the polynomial P of degree $\le q$ is uniquely defined by the following recurrence relation:

$$P(a) = 1, \qquad P^{(k)}(a) = g_0^{(k)}(a) - \sum_{j=0}^{k-1} C_k^j P^{(j)}(a) f_0^{(k-j)}(a) \qquad (1 \le k \le q).$$

By Lemma 2.3, g_0 and $P f_0$ are in $H_w(G)$. Then the function h belongs to $H_w(G)$ and $n(h) \ge q+1$, since $z = a$ is a zero of h of multiplicity $\ge q+1$. Hence, by the definition of q, we have $h \equiv 0$. Since g_0 has no zeros, it follows that $P(z) \equiv 1$ and $g_0 = f_0$. This completes the proof. ∎

Remark 2.4. From Theorem 2.2(ii) it follows that each nontrivial finite-dimensional weighted space $H_w(G)$ is generated by some function $f_0 \in H(G)$ having no zeros. Moreover, from the proof of this theorem we see that for a given space this function is uniquely defined by the condition $f_0(a) = 1$, where a is some fixed point in G.

From Theorem 2.2 and Lemma 2.3 we immediately deduce the following description of infinite-dimensional weighted spaces $H_w(G)$.

Corollary 2.5. *A space $H_w(G)$ is infinite-dimensional if and only if for every $k \in \mathbb{N}$ there exists a nontrivial function $f \in H_w(G)$ with $n(f) \ge k$. The latter trivially holds when there exists a nontrivial function $f \in H_w(G)$ having a countable* (*infinite*) *set of zeros. Additionally, every finite family $a_1, \ldots, a_n$ of points in G consists of zeros* (*not necessarily all*) *of orders $k_1, \ldots, k_n$ of some nontrivial function in $H_w(G)$.*

Remark 2.6. From Corollary 2.5 and Lemma 2.3 it follows that each infinite-dimensional weighted space $H_w(G)$ either contains a nontrivial function with a countable set of zeros or for every $n \in \mathbb{N}_0$ there always exists a function $f \in H_w(G)$ with exactly n zeros.

Theorem 2.2 gives a description of finite-dimensional spaces $H_w(G)$ under the assumption that they exist. As follows from [BDL, Corollary 2], for the unit disk $\mathbb{D} := \{z : |z| < 1\}$ each nontrivial weighted space $H_w(\mathbb{D})$ contains a subspace isomorphic to l_∞, and consequently is infinite-dimensional. Below we obtain a complete characterization of those domains G in $\mathbb{C}$ that admit nontrivial finite-dimensional spaces $H_w(G)$. Moreover, we describe a form of canonical weights for such spaces.

Recall (see, e.g., [Gam]) that a compact set K in $\mathbb{C}$ is called a *Painlevé null set* if every bounded holomorphic function on $\mathbb{C} \setminus K$ is a constant. We will also use the same definition for unbounded closed subsets E in $\mathbb{C}$.

Theorem 2.7.

(1) *If the complement $G^c := \mathbb{C}^* \setminus G$ is not a Painlevé null set, then each nontrivial space $H_w(G)$ is infinite-dimensional.*
(2) *If G^c is a Painlevé null set, then there exist nontrivial finite-dimensional spaces $H_w(G)$. In this case, the space is p-dimensional if and only if it can be given by a canonical weight having the form $(1+|z|)^{p-1}|f_0(z)|$, where f_0 is some holomorphic function in G having no zeros.*

Proof. (1) Suppose that $H_w(G)$ is nontrivial and G^c is not a Painlevé null set. Then there are a nontrivial function $f \in H_w(G)$ and a bounded function $h \in H(G)$ which is not a constant. Consider the nontrivial functions $g_k(z) := f(z)(h(z) - h(z_0))^k$, where z_0 is an arbitrary fixed point in G and $k \in \mathbb{N}$. They are in $H_w(G)$ and $n(g_k) \geq k$ for every $k \in \mathbb{N}$. From this and Corollary 2.5 it follows that $H_w(G)$ is infinite-dimensional.

(2) Let now G^c be a Painlevé null set. Consider a weight w of the form $(1 + |z|)^{p-1}|f_0(z)|$, where f_0 is a holomorphic function in G without zeros. It is clear that each function of the form Pf_0, with P being a polynomial of degree $\leq p - 1$, belongs to $H_w(G)$.

To prove that $H_w(G)$ is p-dimensional, it is enough to check, by Theorem 2.2(iii), that $n(f) \leq p-1$ for every nontrivial $f \in H_w(G)$. Suppose that $f \in H_w(G)$ has at least p zeros $z_1, \dots, z_p$ (with possibly $z_k = z_\ell$ for some $k \neq \ell$). Then, similarly to the proof of Lemma 2.3, the function

$$g(z) := \frac{f(z)}{(z - z_1) \dots (z - z_p) f_0(z)}$$

is holomorphic and bounded in G. Moreover, g satisfies the estimate

$$|g(z)| \leq \|f\|_w \frac{(1 + |z|)^{p-1}}{|z - z_1| \dots |z - z_p|}, \qquad z \in G \setminus \{z_1, \dots, z_p\}.$$

Since G^c is a Painlevé null set, this implies that g is trivial, and consequently f is trivial too. Thus, $H_w(G)$ is p-dimensional.

Let now $H_w(G)$ be p-dimensional, where w is some general weight. Then, by Theorem 2.2(ii),

$$H_w(G) = \operatorname{span}\{z^k f_0(z) : 0 \le k \le p-1\}, \tag{2.2}$$

where f_0 is some function in $H_w(G)$ having no zeros. Since G^c is a Painlevé null set, it has no interior points. Using a continuity argument, we then deduce from (2.2) that $(1+|z|)^{p-1}|f_0(z)|$ is a canonical weight for $H_w(G)$. This completes the proof. ∎

In connection with Theorem 2.7 and some of our further results, note that X. Tolsa has recently given a complete description of Painlevé null sets (see [T, Theorem 1.3]). Next, by L. Ahlfors [A], a compact set K is a Painlevé null set if and only if its analytic capacity $\gamma(K)$ is zero, where $\gamma(K) := \sup|f'(\infty)|$, the supremum being taken over all holomorphic functions f in $\mathbb{C} \setminus K$ with $|f(z)| \le 1$ for all $z \in \mathbb{C} \setminus K$. Obviously, an unbounded closed subset E in $\mathbb{C}$ is a Painlevé null set if and only if

$$\gamma\left(\left\{\frac{1}{z-a} : z \in E\right\} \cup \{0\}\right) = 0,$$

where a is an arbitrary point from $\mathbb{C} \setminus E$. Thus, the reader can easily reformulate Theorem 2.7 in terms of analytic capacity.

Tolsa's description of Painlevé null sets is too complicated to be used here, so we state several evident consequences of Theorem 2.7 based on some classical facts on Painlevé null sets.

Corollary 2.8. *Let G^c have at least one component containing more than one point. Then every nontrivial space $H_w(G)$ is infinite-dimensional.*

Proof. Follows immediately from Theorem 2.7 and the well-known fact (see, e.g., [Gam, p. 198]) that no compact set having a component with more than one point is a Painlevé null set. ∎

In other words, Corollary 2.8 states that if G admits nontrivial finite-dimensional weighted spaces, then G^c is totally disconnected.

For some domains G with G^c being a Painlevé null set, the structure of finite-dimensional weighted spaces and their canonical weights can be refined. The proofs of the two corollaries below are elementary and we omit them.

Corollary 2.9. *Functions of the form $f_0(z) = e^{u(z)}$, where u is an entire function, generate finite-dimensional weighted spaces of entire functions. Canonical weights for these spaces have the representation $(1+|z|)^{p-1}e^{h(z)}$ with some harmonic function h in the complex plane and $p \in \mathbb{N}$.*

Corollary 2.10. *Let $G = \mathbb{C} \setminus \{a_1, \ldots, a_n\}$ and suppose a function $f_0 \in H(G)$ defines a finite-dimensional weighted space $H_w(G)$ (see Theo-*

rem 2.2(ii)). If f_0 has a pole at a_k of order s_k $(k = 1, \dots, n)$, then

$$f_0(z) = \frac{e^{u(z)}}{(z-a_1)^{s_1} \dots (z-a_n)^{s_n}}, \qquad u \in H(\mathbb{C}),$$

and a canonical weight for $H_w(G)$ has the form

$$\frac{(1+|z|)^{p-1} e^{h(z)}}{|z-a_1|^{s_1} \dots |z-a_n|^{s_n}},$$

where h is some harmonic function in $\mathbb{C}$.

It is clear that if $H_w(G)$ is finite-dimensional, then so is $H_{w0}(G)$ (in this case the latter space may be trivial). The next theorem gives a complete answer to the question about the dimension of weighted spaces $H_{w0}(G)$.

Theorem 2.11.

(1) *Suppose that G^c is not a Painlevé null set. Then $H_{w0}(G)$ is either trivial or infinite-dimensional.*

(2) *Let G^c be a Painlevé null set. Then the following statements hold:*

  (a) *If G^c is finite and $H_w(G)$ is infinite-dimensional, then $H_{w0}(G)$ is infinite-dimensional too.*

  (b) *If $G^c \setminus \{\infty\}$ consists of $n \in \mathbb{N}_0$ points, $H_w(G)$ is p-dimensional and w is a canonical weight, then $H_{w0}(G)$ is $(p-n-1)^+$-dimensional (here $x^+ := \max(x,0)$).*

  (c) *If G^c contains infinitely many points and $H_w(G)$ is finite-dimensional, then $H_{w0}(G)$ is trivial.*

  (d) *If G^c contains infinitely many points and $H_w(G)$ is infinite-dimensional, then $H_{w0}(G)$ may be trivial, finite- or infinite-dimensional.*

Proof. (1) Suppose that $H_{w0}(G)$ contains a nontrivial function f. As in the proof of Theorem 2.7(1), take a bounded function $h \in H(G)$ which is not a constant. It is easy to see that the sequence $(fh^k)_{k=1}^{\infty}$ is contained in $H_{w0}(G)$ and linearly independent.

(2) (a) Let $G = \mathbb{C} \setminus \{z_1, \dots, z_n\}$ or $G = \mathbb{C}$ (then we put $n = 0$). Denote $\Pi_n(z) := (z-z_1)\dots(z-z_n)$ for $n \geq 1$, and $\Pi_0(z) \equiv 1$. Take $a \in G$. By Corollary 2.5, for each fixed $k \in \mathbb{N}$ there is a nontrivial $f_k \in H_w(G)$ having at $z = a$ a zero of order $n+k$. Then, by Lemma 2.3, the functions

$$g_{k,j}(z) := \frac{f_k(z)}{(z-a)^{n+j}} \Pi_n(z), \qquad j = 1, \dots, k,$$

are in $H_w(G)$. Since

$$\frac{|g_{k,j}(z)|}{w(z)} \leq \|f_k\|_w \frac{|\Pi_n(z)|}{|z-a|^{n+j}}, \qquad z \in G \setminus \{a\},$$

these functions vanish at infinity on G ($j = 1, \ldots, k$). It is easy to see that for each $k \in \mathbb{N}$ the functions $g_{k,j}(z)$, $j = 1, \ldots, k$, form a linearly independent system. Hence, $H_{w0}(G)$ is infinite-dimensional.

(b) and (c) follow immediately from the structure of finite-dimensional spaces $H_w(G)$ and their canonical weights given in Theorems 2.2(ii) and 2.7(2).

(d) We should give examples showing that $H_{w0}(G)$ can be trivial, finite- or infinite-dimensional, although the corresponding spaces $H_w(G)$ are all infinite-dimensional. To do this, consider $G = \mathbb{C}\backslash\{z_1, z_2, \ldots\}$, where $|z_{n+1}| > |z_n| + 2$. Denote $U_n := \{z : |z - z_n| < 1\}$ and $U := \bigcup_{n=1}^{\infty} U_n$. Consider the following weight on G:

$$w_1(z) := \begin{cases} 1/|z - z_n|, & z \in U_n \text{ for some } n \in \mathbb{N}, \\ 1, & z \notin U. \end{cases}$$

The space $H_{w_1}(G)$ contains the functions $f_n(z) := 1/(z - z_n)$ which form a linearly independent system. Hence, it is infinite-dimensional. On the other hand, every function $f \in H_{w_1 0}(G)$ has removable singularities at z_n ($n \in \mathbb{N}$) and at infinity. Therefore f is constant, and

$$f(\infty) = \lim_{z\to\infty,\, z\notin U} \frac{f(z)}{w_1(z)} = 0$$

implies that $f \equiv 0$. Thus, $H_{w_1 0}(G)$ is trivial.

Arguing as above, we see that, for the weight $w_2(z) := (1 + |z|)^p w_1(z)$, $H_{w_2}(G)$ is infinite-dimensional and $H_{w_2 0}(G)$ consists of all polynomials of degree $\leq p - 1$ and therefore is p-dimensional.

Finally, for the weight $w_3(z) := e^{|z|} w_1(z)$, $H_{w_3 0}(G)$ contains all polynomials and hence infinite-dimensional. ∎

Remark 2.12. (1) In connection with statement (1) of Theorem 2.11 we note the following. For G^c not being a Painlevé null set, the weighted space $H^{\infty}(G)$ ($= H_w(G)$ with $w(z) \equiv 1$) is infinite-dimensional, while the corresponding space $H_{w0}(G)$ is trivial in view of the maximum principle. Thus, there exist infinite-dimensional spaces $H_w(G)$ for which the corresponding spaces $H_{w0}(G)$ are trivial.

(2) We use the example from [BBT]. For the weight $w(z) = (1+|z|)^{p-1+\alpha}$ ($p \in \mathbb{N},\ 0 < \alpha < 1$), $H_{w0}(\mathbb{C})$ and $H_w(\mathbb{C})$ both coincide with the family of all polynomials of degree $\leq p - 1$ and have dimension p. But for the canonical weight $\overline{w}(z) = (1 + |z|)^{p-1}$, $\dim H_{\overline{w}}(\mathbb{C}) = p$, while $\dim H_{\overline{w}0}(\mathbb{C}) = p - 1$. Consequently, statement (2)(b) of Theorem 2.11 is always true only for canonical weights.

3. Compact embeddings. In what follows, w_1 and w_2 are weights on G. As is well known and mentioned above, if (1.1) holds, then $H_{w_1}(G) \subset H_{w_2}(G)$ and the inclusion is compact. From the standard example (see, e.g., [BBT, p. 149]) concerning finite-dimensional spaces of entire functions it follows that the converse is not true in the class of all weights. The following criterion was actually obtained in [BBT, Theorem 2.1(a)].

THEOREM 3.1. *For the inclusion $H_{w_1}(G) \subset H_{w_2}(G)$ to be compact it is necessary and sufficient that for every $\varepsilon > 0$ there exists a continuous function φ with compact support in G such that*

$$(\min(w_1, 1/\varphi))^{\sim} \le \varepsilon w_2 \quad \text{on } G.$$

It is clear that this result is rather complicated to apply because of using functions with compact support. From Bonet–Friz–Jorda [BFJ, Theorem 8] it follows that the natural equivalence between compact embedding of $H_{w_1}(G)$ into $H_{w_2}(G)$ and (1.1) holds if we assume additionally that $\overline{B_{w_1 0}(G)}^{co} = B_{w_1}(G)$. Here $B_{w_1 0}(G)$ and $B_{w_1}(G)$ are the unit balls in $H_{w_1 0}(G)$ and $H_{w_1}(G)$, respectively.

In this section we prove Theorem 1.1 containing, for some domains, a natural criterion for compact embedding of weighted spaces without additional assumptions. Furthermore, we construct some examples showing that our restrictions on domains cannot be weakened. We start with two simple results stated here for the reader's convenience.

PROPOSITION 3.2 ([BFJ, Proposition 5]). *Let w_1, w_2 be some weights on G. The following conditions are equivalent:*

(i) $H_{w_1}(G) \subset H_{w_2}(G)$,
(ii) $H_{w_1}(G) \hookrightarrow H_{w_2}(G)$,
(iii) $\widetilde{w}_1 \prec w_2$,
(iv) $\widetilde{w}_1 \prec \widetilde{w}_2$.

PROPOSITION 3.3. *The following statements are equivalent:*

(i) $H_{w_1}(G) \subset H_{w_2}(G)$ *and the inclusion is compact.*
(ii) *There exist a compact set K in G and a positive constant A such that*

$$\|f\|_{w_2} \le A\|f\|_K, \quad \forall f \in H_{w_1}(G).$$

Proof. We use the method in the proof of [BBT, Theorem 2.1(a)]. Since the normed topology in $H_{w_2}(G)$ is finer than the topology of pointwise convergence, (i) holds if and only if the unit ball $B_{w_1}(G)$ is compact in $H_{w_2}(G)$. Moreover, Montel's theorem implies that $B_{w_1}(G)$ is a compact in $(H(G), co)$. Hence, for $B_{w_1}(G)$ to be compact in $H_{w_2}(G)$ it is necessary and sufficient that the normed topology $\|\cdot\|_{w_2}$ and co coincide on $B_{w_1}(G)$, which is equivalent to (ii). ∎

From Proposition 3.2 one can easily obtain the following criterion for compact embedding of a finite-dimensional weighted space.

Proposition 3.4. *Let $H_{w_1}(G)$ be finite-dimensional. For the inclusion $H_{w_1}(G) \subset H_{w_2}(G)$ to be compact it is necessary and sufficient that $\widetilde{w}_1 \prec w_2$.*

Corollary 3.5. *Let $H_{w_1}(G)$ be finite-dimensional and w_1 canonical (by Theorem 2.7(ii), w_1 is equivalent to $(1+|z|)^p|f_0(z)|$ with some $p \in \mathbb{N}_0$ and holomorphic function f_0 in G without zeros). For the inclusion $H_{w_1}(G) \subset H_{w_2}(G)$ to be compact it is necessary and sufficient that $w_1 \prec w_2$.*

Remark 3.6. In the necessity part of Proposition 3.4 the associated weight $\widetilde{w}_1$ cannot be replaced by w_1; also, the assumption in Corollary 3.5 that w_1 is canonical cannot be weakened. To see this, it is enough to use the example from [BBT]: $G = \mathbb{C}$, $w_1(z) = (1+|z|)^{p+\alpha}$, $w_2(z) = (1+|z|)^p$, where $p \in \mathbb{N}$ and $0 < \alpha < 1$.

To prove Theorem 1.1 for entire functions, we need the following auxiliary statement of independent interest.

Proposition 3.7. *Suppose $H_w(\mathbb{C})$ is infinite-dimensional. Then $H_{w0}(\mathbb{C})$ is dense in $H_w(\mathbb{C})$ in the co topology.*

Proof. Fix $f \in H_w(\mathbb{C}) \setminus H_{w0}(\mathbb{C})$. We divide the argument into several steps.

Case 1. Suppose f has infinitely many zeros z_n ($n = 1, 2, \ldots$). Put $f_n(z) := z_n f(z)/(z_n - z)$. Clearly, $f_n \in H_{w0}(\mathbb{C})$ for all $n \in \mathbb{N}$. Given $R > 0$, for all $|z| \le R$ and $|z_n| > R$, we have

$$|f(z) - f_n(z)| = |f(z)|\frac{|z|}{|z_n - z|} \le \|f\|_R \frac{R}{|z_n| - R},$$

where $\|f\|_R := \max\{|f(z)| : |z| \le R\}$. It follows that f_n converges to f uniformly on $|z| \le R$. Hence, $f_n \to f$ in $(H(\mathbb{C}), co)$.

Case 2. Suppose f has a finite number of zeros in $\mathbb{C}$, say p $(0 \le p < \infty)$. Then it has the form $f(z) = P(z)e^{u(z)}$, where P is a polynomial of degree p and u is an entire function. The further proof is divided into two subcases covering all possible situations (see Remark 2.6).

Subcase 2a. Suppose that $H_w(\mathbb{C})$ contains a function g with exactly $p+1$ zeros $z_1, \ldots, z_{p+1}$. Then $g(z) = Q(z)e^{v(z)}$, where Q is a polynomial of degree $p+1$ and v is an entire function. As follows from Lemma 2.3 (we have already used this observation above), $\widetilde{Q}(z)e^{v(z)} \in H_w(\mathbb{C})$ for any polynomial $\widetilde{Q}$ with $\deg \widetilde{Q} \le p+1$. Moreover, $\widetilde{Q}(z)e^{v(z)} \in H_{w0}(\mathbb{C})$ for $\deg \widetilde{Q} \le p$. Since $f \notin H_{w0}(\mathbb{C})$, this implies that the entire function $v - u$ is not a constant. By Picard's theorem, this function takes all values of $\mathbb{C}$ except at most one. Thus, there exists n_0 such that for every $n \ge n_0$ there is $z_n \in \mathbb{C}$ with $v(z_n) - u(z_n) = n$. Clearly, $z_n \to \infty$ as $n \to \infty$. Note that the function $g_1(z) := P(z)e^{v(z)}$ is in $H_w(\mathbb{C})$ (and in $H_{w0}(\mathbb{C})$), and the entire function

$e^{u(z)} - e^{-n+v(z)}$ vanishes at $z = z_n$ ($n \geq n_0$). Hence, the function

$$f(z) - e^{-n} g_1(z) = P(z)(e^{u(z)} - e^{-n+v(z)})$$

is in $H_w(\mathbb{C})$ and vanishes at z_n. Therefore, the function

$$f_n(z) := \frac{z_n(f(z) - e^{-n} g_1(z))}{z_n - z}$$

belongs to $H_{w0}(\mathbb{C})$. Furthermore, for fixed $R > 0$, for all $|z| \leq R$ and $|z_n| > R$, we have

$$|f(z) - f_n(z)| = \left| f(z) - \frac{z_n f(z)}{z_n - z} + e^{-n} \frac{z_n g_1(z)}{z_n - z} \right|$$
$$\leq |f(z)| \frac{|z|}{|z_n| - |z|} + |g_1(z)| \frac{e^{-n}|z_n|}{|z_n| - |z|} \leq \|f\|_R \frac{R}{|z_n| - R} + \|g_1\|_R \frac{e^{-n}|z_n|}{|z_n| - R}.$$

This yields $f_n \to f$ in $(H(\mathbb{C}), co)$.

SUBCASE 2b. Suppose $H_w(\mathbb{C})$ contains a nontrivial function g having infinitely many zeros $(z_n)_{n=1}^{\infty}$. Take $a \in \mathbb{C}$ with $g(a) \neq 0$ and put

$$\lambda := \frac{(a - z_1) \ldots (a - z_{p+1})}{g(a)} e^{u(a)}.$$

Consider the functions

$$g_1(z) := \lambda \frac{g(z)}{(z - z_1) \ldots (z - z_{p+1})},$$
$$g_2(z) := P(z) g_1(z), \qquad h(z) := P(z)(e^{u(z)} - g_1(z)).$$

It is clear that $g_1, g_2 \in H_{w0}(\mathbb{C})$ and $f = g_2 + h$. In addition, h has at least $p+1$ zeros (at the zeros of P and at $z = a$).

If h has a finite number of zeros, then $H_w(\mathbb{C})$ contains a function with exactly $p+1$ zeros (as above, this is a consequence of Lemma 2.3). But this situation has already been considered in Subcase 2a.

If h has infinitely many zeros, by Case 1, there is a sequence $(h_n)_{n=1}^{\infty}$ consisting of functions from $H_{w0}(\mathbb{C})$ such that $h_n \to h$ in $(H(\mathbb{C}), co)$. Then the functions $g_2 + h_n$ are contained in $H_{w0}(\mathbb{C})$ and $g_2 + h_n \to f$ in $(H(\mathbb{C}), co)$. This completes the proof. ∎

Propositions 3.3 and 3.7 imply immediately the next statement playing a crucial role in the proof of our main theorem for entire functions.

LEMMA 3.8. *Suppose $H_{w_1}(\mathbb{C})$ is infinite-dimensional. If $H_{w_1}(\mathbb{C}) \subset H_{w_2}(\mathbb{C})$ and the inclusion is compact, then $H_{w_1 0}(\mathbb{C})$ is dense in $H_{w_1}(\mathbb{C})$ with respect to the norm $\|\cdot\|_{w_2}$.*

THEOREM 3.9. *Suppose $H_{w_1}(\mathbb{C})$ is infinite-dimensional. For the inclusion $H_{w_1}(\mathbb{C}) \subset H_{w_2}(\mathbb{C})$ to be compact it is necessary and sufficient that*

$$\lim_{z\to\infty} \frac{\widetilde{w}_1(z)}{w_2(z)} = 0. \tag{3.1}$$

Proof. Suppose that $H_{w_1}(\mathbb{C}) \subset H_{w_2}(\mathbb{C})$ is compact. By Proposition 3.2, there is $C > 0$ such that

$$\widetilde{w}_1(z) \le C w_2(z) \quad \text{for all } z \in \mathbb{C}. \tag{3.2}$$

Fix any $\varepsilon > 0$. Since the inclusion $H_{w_1}(\mathbb{C}) \subset H_{w_2}(\mathbb{C})$ is compact, we can find a finite ε-covering $(f_1, \dots, f_n)$ of the unit ball $B_{\widetilde{w}_1}(\mathbb{C})$ in $H_{w_2}(\mathbb{C})$. Next, by Lemma 3.8, for every f_k there exists f_{k0} in $H_{\widetilde{w}_1 0}(\mathbb{C})$ such that

$$\|f_k - f_{k0}\|_{w_2} \le \varepsilon, \quad 1 \le k \le n. \tag{3.3}$$

From (3.2) we have $f_{k0} \in H_{w_2 0}(\mathbb{C})$ $(1 \le k \le n)$. Hence, there exists a compact set K in $\mathbb{C}$ so that

$$\frac{|f_{k0}(z)|}{w_2(z)} \le \varepsilon \quad \text{for all } z \notin K \text{ and } k = 1, \dots, n. \tag{3.4}$$

Let now f be an arbitrary function from $B_{\widetilde{w}_1}(\mathbb{C})$. Find f_k from the ε-covering so that $\|f - f_k\|_{w_2} \le \varepsilon$. Then, applying (3.3) and (3.4), for $z \notin K$, we get

$$\frac{|f(z)|}{w_2(z)} \le \frac{|f(z) - f_k(z)|}{w_2(z)} + \frac{|f_k(z) - f_{k0}(z)|}{w_2(z)} + \frac{|f_{k0}(z)|}{w_2(z)} \le 3\varepsilon.$$

Consequently, $\widetilde{w}_1(z)/w_2(z) \le 3\varepsilon$ whenever $z \notin K$, and this completes the proof. ∎

COROLLARY 3.10. *Suppose $H_{w_1}(\mathbb{C})$ is infinite-dimensional and w_1 is canonical. For the inclusion $H_{w_1}(\mathbb{C}) \subset H_{w_2}(\mathbb{C})$ to be compact it is necessary and sufficient that*

$$\lim_{z\to\infty} \frac{w_1(z)}{w_2(z)} = 0.$$

REMARK 3.11. As above for finite-dimensional spaces, the use of the associated weight $\widetilde{w}_1$ in condition (3.1) of Theorem 3.9, as well as of a canonical weight w_1 in Corollary 3.10, is essential.

To see this, consider the following example.

EXAMPLE 3.12. Take an increasing sequence $(r_n)_{n=1}^{\infty}$ of positive numbers so that $r_1 > 2$ and $r_{n+1} > 4nr_n$ $(n \ge 1)$ and choose a constant σ with

$$\sigma > \log 2 \quad \text{and} \quad \frac{e^{2\sigma} - 1}{2\sigma} \log \frac{e^{2\sigma} - 1}{4e\sigma} + 1 < 0.$$

Then construct a function $\theta : \mathbb{R} \to [0; \infty)$ in the following way. Put

- $\theta(x) := e^x$ for $x \in (-\infty, \log r_1]$ and $x \in [\log r_n + 2\sigma, \log r_{n+1}]$;
- $\theta(x)$ is linear on $[\log r_n, \log r_n + \sigma]$ and $[\log r_n + \sigma, \log r_n + 2\sigma]$ with

$$\theta(\log r_n) = r_n, \quad \theta(\log r_n + \sigma) = 2e^\sigma r_n, \quad \theta(\log r_n + 2\sigma) = e^{2\sigma} r_n.$$

Using the inequality $\sigma > \log 2$, we easily deduce that θ is an increasing continuous function on $\mathbb{R}$ and $\theta(x) \geq e^x$ for all $x \in \mathbb{R}$.

Set $w_1(z) := e^{\theta(\log |z|)}$, $w_2(z) := e^{2|z|}$ $(z \in \mathbb{C})$. We show that the space $H_{w_1}(\mathbb{C})$ is infinite-dimensional and the inclusion $H_{w_1}(\mathbb{C}) \subset H_{w_2}(\mathbb{C})$ is compact.

Since $\theta(x) \geq e^x$ on $\mathbb{R}$, $w_1(z) \geq e^{|z|}$ on $\mathbb{C}$. Therefore, the functions $f_\lambda(z) := e^{\lambda z}$ $(\lambda \in \mathbb{C},\ |\lambda| = 1)$ are in $H_{w_1}(\mathbb{C})$. In particular, this space is infinite-dimensional.

By [BBT, Observation 1.5], the associated weight $\widetilde{w}_1$ is radial and

$$\widetilde{w}_1(z) = \max\{M_f(z) : f \in B_{w_1}(\mathbb{C})\},$$

where $M_f(z) := \max\{|f(\lambda z)| : |\lambda| = 1\}$. Consequently,

$$\widetilde{w}_1(z) \geq \max\{M_{f_\lambda}(z) : |\lambda| = 1\} = e^{|z|}, \qquad \forall z \in \mathbb{C},$$

or, what is the same, $e^r \leq \widetilde{w}_1(r)$ on $[0, \infty)$. Hence, $\widetilde{w}_1(r) = e^r$ for $r \in [e^{2\sigma} r_n, r_{n+1}]$ and $n \in \mathbb{N}$. From this it follows that $\gamma(x) - \log w_2(e^x) = -e^x$ on the intervals $[\log r_n + 2\sigma, \log r_{n+1}]$, where $\gamma(x) := \log \widetilde{w}_1(e^x), x \in \mathbb{R}$. Moreover, since $\gamma(x)$ is an increasing convex function on $\mathbb{R}$ (see, e.g., [BBT, p. 142]), for $x \in [\log r_n, \log r_n + 2\sigma]$,

$$\begin{aligned}\gamma(x) - \log w_2(e^x) &\leq \frac{e^{2\sigma} r_n - r_n}{2\sigma}(x - \log r_n) + r_n - 2e^x \\ &= r_n\left(\frac{e^{2\sigma} - 1}{2\sigma}(x - \log r_n) + 1 - 2e^{x - \log r_n}\right).\end{aligned}$$

Consider the function

$$g(t) := \frac{e^{2\sigma} - 1}{2\sigma} t + 1 - 2e^t, \qquad t \in [0, 2\sigma].$$

It is easy to see that

$$M := \max_{x \in [0, 2\sigma]} g(t) = g\left(\log \frac{e^{2\sigma} - 1}{4\sigma}\right) = \frac{e^{2\sigma} - 1}{2\sigma} \log \frac{e^{2\sigma} - 1}{4e\sigma} + 1.$$

Hence, $\gamma(x) - \log w_2(e^x) \leq M r_n$ on $[\log r_n, \log r_n + 2\sigma]$. Note that $M < 0$, because of the choice of σ. Then $\gamma(x) - \log w_2(e^x) \to -\infty$ as $x \to +\infty$, and consequently

$$\lim_{z \to \infty} \frac{\widetilde{w}_1(z)}{w_2(z)} = 0.$$

By Theorem 3.9, this implies that the inclusion $H_{w_1}(\mathbb{C}) \subset H_{w_2}(\mathbb{C})$ is compact.

On the other hand,

$$\frac{w_1(e^{(\log r_n+\sigma)})}{w_2(e^{(\log r_n+\sigma)})} = \frac{e^{\theta(\log r_n+\sigma)}}{w_2(e^{\sigma} r_n)} = 1, \qquad \forall n \geq 1,$$

which gives

$$\limsup_{z\to\infty} \frac{w_1(z)}{w_2(z)} \geq 1.$$

Now we consider the case when G is a domain in $\mathbb{C}$ whose complement has no one-point component. By Corollary 2.8, in this case each nontrivial weighted space $H_w(G)$ is infinite-dimensional.

THEOREM 3.13. *Let $G \neq \mathbb{C}$ be a domain in $\mathbb{C}$ whose complement has no one-point component and suppose $H_{w_1}(G)$ is nontrivial. For the inclusion $H_{w_1}(G) \subset H_{w_2}(G)$ to be compact it is necessary and sufficient that condition* (1.1) *holds.*

Proof. We will use some ideas of Bonet–Domański–Lindström (see [BDL, proof of Theorem 1]). Assume that the inclusion $H_{w_1}(G) \subset H_{w_2}(G)$ is compact but (1.1) does not hold. Then there is a sequence $(z_n)_{n=1}^{\infty}$ in G so that

$$\frac{\widetilde{w}_1(z_n)}{w_2(z_n)} \geq c > 0 \qquad (n = 1, 2, \ldots).$$

Without loss of generaligty we will assume that $z_n \to z_0 \in \partial G \cup \{\infty\}$. Let $\mathcal{L}$ denote the component of G^c containing z_0. Since $\mathcal{L}$ contains more than one point, by Riemann's mapping theorem, the domain $D := \mathbb{C}^* \setminus \mathcal{L}$ is conformally equivalent to the unit disk $\mathbb{D} := \{z : |z| < 1\}$. From [H, p. 204, Corollary] it then follows that there is a subsequence $(z_{n_k})_{k=1}^{\infty}$ which is interpolating for $H^{\infty}(D)$. Using [W, Theorem III. E. 4], we can find a sequence $(h_k)_{k=1}^{\infty}$ in $H^{\infty}(D)$ so that

$$h_j(z_{n_k}) = \begin{cases} 1, & k = j, \\ 0, & k \neq j \end{cases}$$

and, for some $M > 0$,

$$\sum_{k=1}^{\infty} |h_k(z)| \leq M, \qquad \forall z \in D.$$

Next, by [BBT, Property 1.2(iv)], for each $k \in \mathbb{N}$ there is a function $f_k \in B_{w_1}(G)$ with $|f_k(z_{n_k})| = \widetilde{w}_1(z_{n_k})$. Define $g_k := f_k h_k$ for $k \in \mathbb{N}$. Clearly, $g_k \in H_{w_1}(G)$ for all $k \geq 1$, and g_k tends to 0 in $(H(G), co)$. Applying Proposition 3.3, we derive that g_k tends to 0 with respect to the norm $\|\cdot\|_{w_2}$. On the other hand, for every $k \geq 1$,

$$\|g_k\|_{w_2} = \sup_{z\in G} \frac{|g_k(z)|}{w_2(z)} \geq \frac{|g_k(z_{n_k})|}{w_2(z_{n_k})} = \frac{|f_k(z_{n_k})|}{w_2(z_{n_k})} = \frac{\widetilde{w}_1(z_{n_k})}{w_2(z_{n_k})} \geq c > 0.$$

This contradiction completes the proof. ∎

COROLLARY 3.14. *Let $G \neq \mathbb{C}$ be a domain in $\mathbb{C}$ whose complement has no one-point component and suppose $H_{w_1}(G)$ is nontrivial and the weight w_1 is canonical. Then for the inclusion $H_{w_1}(G) \subset H_{w_2}(G)$ to be compact it is necessary and sufficient that w_1/w_2 vanishes at infinity on G.*

The assumption that the complement of G in $\mathbb{C}^*$ has no one-point component is essential in Theorem 3.13 and Corollary 3.14. Indeed, consider the following example (see also the example in Bonet–Vogt [BV, p. 95]).

EXAMPLE 3.15. Let $G = \mathbb{D} \setminus \{0\}$ (as above, $\mathbb{D}$ denotes the unit disk in $\mathbb{C}$). Consider the weights

$$w_1(z) = \frac{1}{|z|(1-|z|)} \quad \text{and} \quad w_2(z) = \frac{1}{|z|(1-|z|)^2}, \qquad z \in G.$$

It is easy to see that w_k is a canonical weight for $H_{w_k}(G)$ and each function from $H_{w_k}(G)$ has either a removable singularity or a simple pole at $z = 0$ ($k = 1, 2$). This implies that the operator $\Phi : f(z) \mapsto zf(z)$ is an isomorphism between $H_{w_k}(G)$ and $H_{v_k}(\mathbb{D})$, where $v_k(z) = 1/(1-|z|)^k$ ($k = 1, 2$). Since v_1/v_2 vanishes at infinity on $\mathbb{D}$, the inclusion $H_{v_1}(\mathbb{D}) \subset H_{v_2}(\mathbb{D})$ is compact. Thus, the inclusion $H_{w_1}(G) \subset H_{w_2}(G)$ is compact, too. On the other hand, $\lim_{z\to 0} w_1(z)/w_2(z) = 1$.

Finally, note that Theorem 3.13 and Corollary 3.14 might fail if we use a general type weight w_1 in (1.1) or a noncanonical weight in Corollary 3.14. To see this, it is enough to consider the following example. In this case the explanation is similar to the one in Example 3.12, so we omit it.

EXAMPLE 3.16. Let $G = \mathbb{D}$. Take a constant $\sigma \in (\sqrt{2}-1, 1/2)$ and an increasing sequence $(r_n)_{n=1}^{\infty}$ of positive numbers such that $(r_n)^{\sigma^2} < r_{n+1} < 1$, $n \geq 1$. Then construct a function $\theta : (-\infty, 0] \to [0, +\infty)$ in the following way. Put

- $\theta(x) := -1/x$ for $x \in (-\infty, \log r_1]$ and $x \in [\sigma^2 \log r_n, \log r_{n+1}]$, $n \geq 1$;
- $\theta(x)$ is linear on $[\log r_n, \sigma \log r_n]$ and $[\sigma \log r_n, \sigma^2 \log r_n]$ with

$$\theta(\log r_n) = -\frac{1}{\log r_n}, \;\theta(\sigma \log r_n) = -\frac{2}{\sigma \log r_n}, \;\theta(\sigma^2 \log r_n) = -\frac{1}{\sigma^2 \log r_n}.$$

Then, for the weights on $\mathbb{D}$ given by

$$w_1(z) := e^{\theta(\log|z|)} \quad \text{and} \quad w_2(z) := e^{-2/\log|z|},$$

we have

$$\limsup_{|z|\to 1-} \frac{w_1(z)}{w_2(z)} > 0,$$

in spite that the inclusion $H_{w_1}(G) \subset H_{w_2}(G)$ is compact.

To end this section, note that our results, Theorem 3.9 and Corollary 3.10, on compact embedding of spaces of entire functions cannot be extended automatically to functions of several variables. To see this, it is enough to consider the following canonical weights:

$$w_1(z) := e^{|z_1|}, \quad w_2(z) := (1+|z_1|)e^{|z_1|},\ z = (z_1, z_2) \in \mathbb{C}^2.$$

4. Some applications. In this section we demonstrate that the results obtained above may be used to refine some known answers to several problems.

By J. Bonet and E. Wolf [BW, Corollary 2], the spaces $H_w(G)$ and $H_{w0}(G)$ are not reflexive whenever $H_{w0}(G)$ is infinite-dimensional. Using this, we have the following consequence of Theorem 2.11.

PROPOSITION 4.1. *Suppose one of the following conditions holds:*

(a) *G^c is not a Painlevé null set and $H_{w0}(G)$ is nontrivial.*
(b) *G^c is finite and $H_w(G)$ is infinite-dimensional.*

Then $H_w(G)$ and $H_{w0}(G)$ are both nonreflexive.

REMARK 4.2. In view of Theorem 2.7(2), one can reformulate assertion (b) of Proposition 4.1 in the following way:

Suppose G^c is finite. A nontrivial space $H_w(G)$ is reflexive if and only if it can be defined by a weight of the form $(1+|z|)^{p-1}|f_0(z)|$, where $p \in \mathbb{N}$ and f_0 is some function in $H(G)$ having no zeros.

Let now $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ be an increasing sequence of weights on a domain G. Define the space $\mathcal{V}H(G) := \operatorname{ind}_n H_{v_n}(G)$, endowed with the natural inductive limit topology. Note that $\mathcal{V}H(G)$ is regular (i.e., each of its bounded sets is contained and bounded in some $H_{v_n}(G)$) and complete. Put

$$\overline{V} = \overline{V}(\mathcal{V}) := \left\{\overline{v} \text{ weight on } G : \sup_{z\in G} \frac{v_n(z)}{\overline{v}(z)} < \infty,\ \forall n\right\};$$

$$H\overline{V}(G) := \left\{f \in H(G) : \|f\|_{\overline{v}} = \sup_{z\in G} \frac{|f(z)|}{\overline{v}(z)} < \infty,\ \forall \overline{v} \in \overline{V}\right\}.$$

As is known, $H\overline{V}(G)$ and $\mathcal{V}H(G)$ coincide as sets and have the same bounded sets. The space $H\overline{V}(G)$ is called *the projective hull* of $\mathcal{V}H(G)$.

In [BBT, Theorem 2.1(a), (b)] criteria were obtained for the space $\mathcal{V}H(G)$ to be (DFS) and, respectively, for $H\overline{V}(G)$ to be semi-Montel. But these criteria seem very complicated to use. In this connection, Bonet and Vogt [BV, Theorem 3] improved [BBT, Theorem 2.1(a)] under the additional assumption that $\bigcup_{n=1}^{\infty} H_{v_n0}(G) = \bigcup_{n=1}^{\infty} H_{\widetilde{v}_n0}(G)$. They proved that in this case the space $\mathcal{V}H(G)$ is (DFS) if and only if the sequence $(\widetilde{v}_n)_{n=1}^{\infty}$ of associated weights satisfies *condition* (S): for each n there is $m > n$ such that $\widetilde{v}_n/\widetilde{v}_m$ vanishes at infinity on G. In the case of radial weights the above assumption can be removed for the unit disk (see [BBT, Proposition 3.5]) or simplified for balanced domains (see [BV, Corollary 6]). Also, Bierstedt and Bonet [BB, Theorem 15] gave, under some additional conditions, a characterization of the semi-Montel property of $H\overline{V}(G)$, which is simpler than [BBT, Theorem 2.1(b)]. Applying our results from the previous sections, we obtain the following new criteria in this direction.

THEOREM 4.3. *Let G be either the whole complex plane $\mathbb{C}$ or a domain in $\mathbb{C}$ whose complement has no one-point component. The space $\mathcal{V}H(G)$ is (DFS) if and only if at least one of the following conditions holds:*

(i) *$\mathcal{V}H(G)$ is finite-dimensional.*
(ii) *$\mathcal{V}H(G)$ is infinite-dimensional, while all $H_{v_n}(G)$ are finite-dimensional.*
(iii) *Some space $H_{v_n}(G)$ is infinite-dimensional and*

$$\forall n \in \mathbb{N}\ \exists m > n : \widetilde{v}_n(z)/v_m(z) \text{ vanishes at infinity on } G.$$

Proof. Immediate from Proposition 3.4 and Theorems 3.9 and 3.13. ∎

To prove a similar criterion establishing that the space $H\overline{V}(G)$ is semi-Montel, we need the next lemma.

LEMMA 4.4. *The space $H\overline{V}(G)$ is semi-Montel if and only if for all $n \in \mathbb{N}$ and $\overline{v} \in \overline{V}$ the inclusions $H_{v_n}(G) \subset H_{\overline{v}}(G)$ are compact.*

Proof. The above mentioned properties of $H\overline{V}(G)$ and $\mathcal{V}H(G)$ imply that $H\overline{V}(G)$ is semi-Montel if and only if for every $n \in \mathbb{N}$ the unit ball B_n of $H_{v_n}(G)$ is compact in $H\overline{V}(G)$. This is equivalent to $H\overline{V}(G)$ and *co* defining the same topology on B_n for every $n \in \mathbb{N}$, i.e.,

$$\forall n \in \mathbb{N}\ \forall \overline{v} \in \overline{V}\ \exists K \Subset G\ \exists C > 0 : \quad \|f\|_{\overline{v}} \le C\|f\|_K,\ \forall f \in H_{v_n}(G).$$

Thus, by Proposition 3.3, $H\overline{V}(G)$ is semi-Montel if and only if for every $n \in \mathbb{N}$ and $\overline{v} \in \overline{V}$ the inclusion $H_{v_n}(G) \subset H_{\overline{v}}(G)$ is compact. ∎

THEOREM 4.5. *Let G be either the whole complex plane $\mathbb{C}$ or a domain in $\mathbb{C}$ whose complement has no one-point component. The space $H\overline{V}(G)$ is semi-Montel if and only if at least one of the following conditions holds:*

(i) *$\mathcal{V}H(G)$ is finite-dimensional.*
(ii) *$\mathcal{V}H(G)$ is infinite-dimensional, while all $H_{v_n}(G)$ are finite-dimensional.*
(iii) *Some space $H_{v_n}(G)$ is infinite-dimensional and*

$$\widetilde{v}_n(z)/\overline{v}(z) \text{ vanishes at infinity on } G \text{ for all } n \in \mathbb{N} \text{ and } \overline{v} \in \overline{V}.$$

Acknowledgments. The authors thank the referee for useful remarks and comments that led to improvement of this paper.

This research was supported by Ministry of Education and Science of Russia, projects 14.A18.21.0356 and 8210.

References

[AT] A. V. Abanin and P. T. Tien, *Continuation of holomorphic functions and some of its applications*, Studia Math. 200 (2010), 279–295.

[A] L. Ahlfors, *Bounded analytic function*, Duke Math. J. 14 (1947), 1–11.

[AD] J. M. Anderson and J. Duncan, *Duals of Banach spaces of entire functions*, Glasgow Math. J. 32 (1990), 215–220.

[BB] K. D. Bierstedt and J. Bonet, *Weighted (LB)-spaces of holomorphic functions: $\mathcal{V}H(G) = \mathcal{V}_0H(G)$ and completeness of $\mathcal{V}_0H(G)$*, J. Math. Anal. Appl. 323 (2006), 747–767.

[BBT] K. D. Bierstedt, J. Bonet, and J. Taskinen, *Associated weights and spaces of holomorphic functions*, Studia Math. 127 (1998), 137–168.

[BS] K. D. Bierstedt and W. H. Summers, *Biduals of weighted Banach spaces of analytic functions*, J. Austral. Math. Soc. Ser. A 54 (1993), 70–79.

[BDL] J. Bonet, P. Domański, and M. Lindström, *Essential norm and weak compactness of composition operators on weighted Banach spaces of analytic functions*, Canad. Math. Bull. 42 (1999), 139–148.

[BDLT] J. Bonet, P. Domański, M. Lindström, and J. Taskinen, *Composition operators between weighted Banach spaces of analytic functions*, J. Austral. Math. Soc. 64 (1998), 101–118.

[BFJ] J. Bonet, M. Friz, and E. Jorda, *Composition operators between weighted inductive limits of spaces of holomorphic functions*, Publ. Math. Debrecen 67 (2005), 333–348.

[BV] J. Bonet and D. Vogt, *Weighted spaces of holomorphic functions and sequence spaces*, Note Mat. 17 (1997), 87–97.

[BW] J. Bonet and E. Wolf, *A note on weighted Banach spaces of holomorphic functions*, Arch. Math. (Basel) 81 (2003), 650–654.

[Gam] T. W. Gamelin, *Uniform Algebras*, Prentice-Hall, 1969.

[Gar] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, 1981.

[H] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, NJ, 1967.

[L] W. Lusky, *On the isomorphism classes of weighted spaces of harmonic and holomorphic functions*, Studia Math. 160 (2006), 19–45.

[RS] L. A. Rubel and A. L. Shields, *The second duals of certain spaces of analytic functions*, J. Austral. Math. Soc. 11 (1970), 276–280.

[T] X. Tolsa, *Painlevé's problem and the semiadditivity of analytic capacity*, Acta Math. 190 (2003), 105–149.

[W] P. Wojtaszczyk, *Banach Spaces for Analysts*, Cambridge Univ. Press, 1991.

Alexander V. Abanin, Pham Trong Tien
Southern Federal University (SFU)
Rostov-na-Donu 344090, The Russian Federation
and
Southern Institute of Mathematics (SIM)
Vladikavkaz 362027, The Russian Federation
E-mail: abanin@math.rsu.ru
phamtien@mail.ru

Received July 1, 2012
Revised version November 2, 2012 (7564)