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Quotients with a shrinking basis

by

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Dedicated to the memory of Professor Aleksander Pełczyński

Abstract. We present a simple proof of a theorem that yields as a corollary a result of Valdivia that sharpens an old result of Johnson and Rosenthal.

Let E denote a Banach space, and let E' denote the dual of E. If $A \subset E$, let $[A]$ denote the vector subspace of E generated by A. If $N \subset E'$, let $N^\top$ denote the subspace

$$N^\top = \{x \in E : x'(x) = 0 \text{ for every } x' \in N\}.$$

Theorem 1. *Let F be a Banach space with a shrinking Schauder basis $(y_n)_{n=1}^\infty$, and let $(y'_n)_{n=1}^\infty \subset F'$ denote the sequence of coordinate functionals. Let E be a Banach space, and suppose there exists a basic sequence $(x'_n)_{n=1}^\infty \subset E'$ with the following properties:*

(i) *$(x'_n)_{n=1}^\infty$ is equivalent to $(y'_n)_{n=1}^\infty$.*
(ii) *The series $\sum_{n=1}^\infty x'_n(x) y_n$ converges in F for every $x \in E$.*

Let $N = \overline{[x'_n : n \in \mathbb{N}]} \subset E'$. Then the quotient $E/N^\top$ is isomorphic to F. In particular $E/N^\top$ has a shrinking Schauder basis.

Proof. Let $T : E \to F$ be defined by

$$Tx = \sum_{n=1}^\infty x'_n(x) y_n \quad \text{for every } x \in E.$$

Then T is well defined, by (ii), and T is clearly linear. It follows from the definition of T that $y'_n(Tx) = x'_n(x)$ for every $x \in E$ and $n \in \mathbb{N}$. Using this we can readily verify that T has a closed graph, and is therefore continuous. To show that T is surjective it suffices to show that the dual mapping T' is an isomorphic embedding (see [2, Theorem 4.7-C]). The dual mapping

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$T' : F' \to E'$ is given by

$$T'y'(x) = y'(Tx) = \sum_{n=1}^{\infty} y'(y_n)x'_n(x) \quad \text{for every } y' \in F' \text{ and } x \in E.$$

In particular $T'y'_n = x'_n$ for every $n \in \mathbb{N}$. Since $(y_n)_{n=1}^{\infty}$ is a shrinking basis for F, $(y'_n)_{n=1}^{\infty}$ is a basis for F'. It follows from (i) that $T' : F' \to N$ is an isomorphism, and therefore $T' : F' \to E'$ is an isomorphic embedding. Since

$$\operatorname{Ker} T = \{x \in E : x'_n(x) = 0 \text{ for every } n \in \mathbb{N}\} = N^{\top},$$

it follows that $E/N^{\top}$ is isomorphic to F. ∎

By applying the theorem with $F = c_0$ we immediately obtain the following result of Valdivia [3, Proposition 4].

Corollary 2. *Let E be a Banach space and suppose there exists a basic sequence $(x'_n)_{n=1}^{\infty} \subset E'$ with the following properties:*

(i) $(x'_n)_{n=1}^{\infty}$ *is equivalent to the usual Schauder basis of* ℓ_1.
(ii) $(x'_n(x))_{n=1}^{\infty} \in c_0$ *for every* $x \in E$.

Let $N = \overline{[x'_n : n \in \mathbb{N}]} \subset E'$. Then the quotient $E/N^{\top}$ is isomorphic to c_0.

Johnson and Rosenthal have shown that if E is a separable Banach space whose dual contains a subspace isomorphic to ℓ_1, then E has a quotient isomorphic to c_0 (see [1, Theorem IV.3]). Their proof shows that if E is a Banach space whose dual contains a basic sequence $(x'_n)_{n=1}^{\infty}$ which satisfies conditions (i) and (ii) in the corollary, then $(x'_n)_{n=1}^{\infty}$ has a subsequence $(x'_{n_j})_{j=1}^{\infty}$ such that, if we set $N = \overline{[x'_{n_j} : j \in \mathbb{N}]}$, then $E/N^{\top}$ is isomorphic to c_0. Valdivia's result is a refinement of this.

References

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