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Fractional Hajłasz–Morrey–Sobolev spaces on quasi-metric measure spaces

by

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Abstract. In this article, via fractional Hajłasz gradients, the authors introduce a class of fractional Hajłasz–Morrey–Sobolev spaces, and investigate the relations among these spaces, (grand) Morrey–Triebel–Lizorkin spaces and Triebel–Lizorkin-type spaces on both Euclidean spaces and RD-spaces.

1. Introduction. Morrey spaces were originally introduced and studied by Morrey [22] within the theory of partial differential equations, and serve as a natural generalization of Lebesgue spaces. Recall that Morrey spaces on a space $\mathcal{X}$ of homogeneous type are defined as follows.

DEFINITION 1.1. Let $0 < p \le q \le \infty$. The *Morrey space* $\mathcal{M}_p^q(\mathcal{X})$ is defined to be the space of all measurable functions f on $\mathcal{X}$ such that

$$\|f\|_{\mathcal{M}_p^q(\mathcal{X})} := \sup_{B\subset\mathcal{X}} [\mu(B)]^{1/q-1/p}\Big[\int_B |f(x)|^p\, d\mu(x)\Big]^{1/p} < \infty, \tag{1.1}$$

where the supremum is taken over all balls in $\mathcal{X}$.

In recent years, due to application in partial differential equations, there is an increasing interest in function spaces based on Morrey spaces, such as Morrey–Sobolev spaces, Morrey-Besov spaces and Morrey–Triebel–Lizorkin spaces; see, for example, [18, 21, 30, 23, 33, 26, 27]. These spaces are defined via replacing the Lebesgue norm in the definitions of some classical spaces (for example, Sobolev spaces and Besov spaces) by the Morrey norm.

On the other hand, the study of Sobolev type spaces on metric measure spaces has achieved a great progress in the last two decades. Hajłasz [6] in 1996 introduced the notion of Hajłasz gradients, which became an effective tool to introduce Sobolev spaces on metric measure spaces. From then on,

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several different approaches to introduce Sobolev spaces on metric measure spaces were developed; see, for example, [14, 5, 25, 8, 7, 12, 31, 15, 11]. Using Hajłasz gradients, Hajłasz–Morrey–Sobolev spaces were recently introduced and investigated in [19].

In 2003, Yang [31] and Hu [12] introduced a fractional version of Hajłasz gradients, called s-Hajłasz gradients, and used these to introduce and investigate fractional Hajłasz–Sobolev spaces on metric measure spaces. Motivated by this, the main purpose of this article is to investigate the Morrey version of fractional Hajłasz–Sobolev spaces on metric measure spaces.

In this article, if there are no additional assumptions, $(\mathcal{X}, d, \mu)$ *always denotes* a metric measure space of homogeneous type. Recall that a triple $(\mathcal{X}, d, \mu)$ is called a *space of homogeneous type* in the sense of Coifman and Weiss [3, 4] if d is a *quasi-metric* on $\mathcal{X}$, that is,

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;
(iii) there exists a constant $K \in [1, \infty)$ such that, for all $x, y, z \in \mathcal{X}$,

$$d(x, y) \leq K[d(x, z) + d(z, y)], \tag{1.2}$$

and μ is a non-trivial Borel regular measure satisfying the following condition: there exists a constant $C_0 \in [1, \infty)$ such that, for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, 2r)) \leq C_0 \mu(B(x, r)) \quad (\textit{doubling property}). \tag{1.3}$$

It is known that any quasi-metric on $\mathcal{X}$ determines a topology on $\mathcal{X}$ and the class of all balls of $\mathcal{X}$ is a basis on $\mathcal{X}$, where a *ball* is defined to be the set

$$B(x, r) := \{y \in \mathcal{X} : d(x, y) < r\}$$

whenever $x \in \mathcal{X}$ and $r \in (0, \infty)$. If $K = 1$ in (1.2), then $(\mathcal{X}, d, \mu)$ is called a *metric measure space of homogeneous type.*

It is easy to see that, if μ is doubling, then, for any $\lambda \in (0, \infty)$, there exists a positive constant C_λ, depending only on λ and C_0 in (1.3), such that, for all $r \in (0, \infty)$ and $x \in \mathcal{X}$,

$$\mu(B(x, \lambda r)) \leq C_\lambda \mu(B(x, r)).$$

We now recall the notion of s-Hajłasz gradients from [31].

DEFINITION 1.2. Let $(\mathcal{X}, d, \mu)$ be a quasi-metric measure space, $s \in (0, \infty)$ and f be a measurable function on $\mathcal{X}$. A non-negative measurable function g is called an *s-Hajłasz gradient* of f if

$$|f(x) - f(y)| \leq [d(x, y)]^s [g(x) + g(y)] \tag{1.4}$$

for μ-almost all $x, y \in \mathcal{X}$. Moreover, denote by $D^s(f)$ the collection of all s-Hajłasz gradients of f.

Obviously, 1-Hajłasz gradients are just Hajłasz gradients introduced in [6].

Definition 1.3. Let $(\mathcal{X}, d, \mu)$ be a quasi-metric measure space, $s \in (0, \infty)$ and $0 < p \le q \le \infty$. The *homogeneous fractional Hajłasz–Morrey–Sobolev space* $H\dot{M}^s_{p,q}(\mathcal{X})$ is defined to be the set of all measurable functions f on $\mathcal{X}$ which have an s-Hajłasz gradient $g_f \in \mathcal{M}^q_p(\mathcal{X})$. Moreover, for all $f \in H\dot{M}^s_{p,q}(\mathcal{X})$, let

$$\|f\|_{H\dot{M}^s_{p,q}(\mathcal{X})} := \inf_{g_f \in D^s(f) \cap \mathcal{M}^q_p(\mathcal{X})} \|g_f\|_{\mathcal{M}^q_p(\mathcal{X})}. \tag{1.5}$$

The *inhomogeneous fractional Hajłasz–Morrey–Sobolev space* $HM^s_{p,q}(\mathcal{X})$ is then defined as $HM^s_{p,q}(\mathcal{X}) := H\dot{M}^s_{p,q}(\mathcal{X}) \cap \mathcal{M}^q_p(\mathcal{X})$ endowed with the quasi-norm

$$\|\cdot\|_{HM^s_{p,q}(\mathcal{X})} := \|\cdot\|_{\mathcal{M}^q_p(\mathcal{X})} + \|\cdot\|_{H\dot{M}^s_{p,q}(\mathcal{X})}.$$

The main purpose of this article is to investigate the relations among fractional Hajłasz–Morrey–Sobolev spaces, (grand) Morrey–Triebel–Lizorkin spaces and Triebel–Lizorkin-type spaces on both Euclidean spaces and RD-spaces. Recall that an RD-space is a metric space endowed with a measure satisfying both the doubling and the reverse doubling conditions, which was originally introduced in [10] (see also [35]).

First, in Section 2, in the Euclidean setting, we prove that fractional Hajłasz–Morrey–Sobolev spaces coincide with some Morrey–Triebel–Lizorkin spaces (see [30, 23]) and Triebel–Lizorkin-type spaces (see [32, 33]), and also their grand version (see Theorem 2.5). As a byproduct, the coincidence between Hajłasz–Morrey–Sobolev spaces and Hardy-Morrey–Sobolev spaces on $\mathbb{R}^n$ is also established (see Theorem 2.11).

Section 3 is devoted to establishing the coincidence between fractional Hajłasz–Morrey–Sobolev spaces and grand Morrey–Triebel–Lizorkin spaces on RD-spaces (see Theorem 3.3). This result generalizes [16, Theorem 5.2] to the level of Morrey spaces. Due to the difference between Lebesgue and Morrey norms, compared with the proof of [16, Theorem 5.2], the proof of Theorem 3.3 is much more complicated and needs some additional tools such as Christ's dyadic cubes and the construction of a partition of unity on metric measure spaces of homogeneous type (see Lemma 3.4).

To end this section, we make some conventions on notation. Throughout, we denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. The symbols $A \lesssim B$ and $A \gtrsim B$ mean $A \le CB$ and $A \ge CB$, respectively, where C is a positive constant. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$. For any ball B and $f \in L^1(B)$, $\fint_B f(x) d\mu(x)$ denotes the *integral mean* of f on B,

$$\fint_B f(x)\, d\mu(x) := \frac{1}{\mu(B)} \int_B f(x)\, d\mu(x).$$

2. Characterizations of Hajłasz–Morrey–Sobolev spaces on $\mathbb{R}^n$. In this section, we concentrate on $\mathbb{R}^n$ and consider relations among Hajłasz–Morrey–Sobolev spaces, Morrey–Triebel–Lizorkin spaces in [23, 24], and also Triebel–Lizorkin-type spaces in [36, 33, 24].

To introduce Morrey–Triebel–Lizorkin spaces on $\mathcal{X}$, we need the following notion of s-Hajłasz gradients at level $k \in \mathbb{Z}$, which was originally introduced in [17]. Compared with s-Hajłasz gradients, the s-Hajłasz gradients at level k have an additional restriction on the distance of the points x and y.

Definition 2.1. Let $(\mathcal{X}, d, \mu)$ be a quasi-metric measure space, $s \in (0, \infty)$, $k \in \mathbb{Z}$ and f be a measurable function on $\mathcal{X}$. A non-negative measurable function g on $\mathcal{X}$ is called an *s-Hajłasz gradient of f at level k* if (1.4) holds for μ-almost all $x, y \in \mathcal{X}$ satisfying $2^{-k-1} \le d(x, y) < 2^{-k}$. If, for each $k \in \mathbb{Z}$, g_k is an s-Hajłasz gradient of f at level k, then the sequence $\vec{g} := \{g_k\}_{k\in\mathbb{Z}}$ is called a *fractional s-Hajłasz gradient* of f. Let $\mathbb{D}^s(f)$ denote the collection of all fractional s-Hajłasz gradients of f.

With these gradients, we now introduce the following Morrey–Triebel–Lizorkin spaces.

Definition 2.2. Let $(\mathcal{X}, d, \mu)$ be a quasi-metric measure space, $s \in (0, \infty)$, $0 < p \le q \le \infty$ and $r \in (0, \infty]$. The *homogeneous Hajłasz–Morrey–Triebel–Lizorkin space* $\dot{M}^s_{p,q,r}(\mathcal{X})$ is defined to be the collection of all measurable functions f satisfying

$$\|f\|_{\dot{M}^s_{p,q,r}(\mathcal{X})} := \inf_{\vec{g}\in\mathbb{D}^s(f)} \|\vec{g}\|_{\mathcal{M}^q_p(\mathcal{X},l^r)} := \inf_{\vec{g}\in\mathbb{D}^s(f)} \Big\|\Big\{\sum_{k\in\mathbb{Z}} |g_k|^r\Big\}^{1/r}\Big\|_{\mathcal{M}^q_p(\mathcal{X})} < \infty. \tag{2.1}$$

The *inhomogeneous Hajłasz–Morrey–Triebel–Lizorkin space* $M^s_{p,q,r}(\mathcal{X})$ is then defined as $M^s_{p,q,r}(\mathcal{X}) := \dot{M}^s_{p,q,r}(\mathcal{X}) \cap \mathcal{M}^q_p(\mathcal{X})$ endowed with the quasi-norm

$$\|\cdot\|_{M^s_{p,q,r}(\mathcal{X})} := \|\cdot\|_{\mathcal{M}^q_p(\mathcal{X})} + \|\cdot\|_{\dot{M}^s_{p,q,r}(\mathcal{X})}.$$

The following conclusion is frequently used in this section.

Proposition 2.3. *Let $(\mathcal{X}, d, \mu)$ be a quasi-metric measure space. Then, for all $s \in (0, \infty)$ and $0 < p \le q \le \infty$,*

$$\dot{M}^s_{p,q,\infty}(\mathcal{X}) = H\dot{M}^s_{p,q}(\mathcal{X}) \quad \text{and} \quad M^s_{p,q,\infty}(\mathcal{X}) = HM^s_{p,q}(\mathcal{X})$$

with equivalent quasi-norms.

Proof. We only prove the homogeneous case. Let $f \in H\dot{M}^s_{p,q}(\mathcal{X})$ and $g \in D^s(f)$ be such that $\|g\|_{\mathcal{M}^q_p(\mathcal{X})} \lesssim \|f\|_{H\dot{M}^s_{p,q}(\mathcal{X})}$. For all $k \in \mathbb{Z}$, let $g_k := g$. Then $\vec{g} := \{g_k\}_{k\in\mathbb{Z}} \in \mathbb{D}^s(f)$ and $\|\sup_{k\in\mathbb{Z}} g_k\|_{\mathcal{M}^q_p(\mathcal{X})} = \|g\|_{\mathcal{M}^q_p(\mathcal{X})}$. This implies that $f \in \dot{M}^s_{p,q,\infty}(\mathcal{X})$ and

$$\|f\|_{\dot{M}^s_{p,q,\infty}(\mathcal{X})} \lesssim \|f\|_{H\dot{M}^s_{p,q}(\mathcal{X})}.$$

Therefore, $H\dot{M}^s_{p,q}(\mathcal{X}) \subset \dot{M}^s_{p,q,\infty}(\mathcal{X})$.

To see the converse embedding, let $f \in \dot{M}^s_{p,q,\infty}(\mathcal{X})$. Then there exists a sequence $\vec{g} := \{g_k\}_{k\in\mathbb{Z}} \in \mathbb{D}^s(f)$ such that

$$\Big\|\sup_{k\in\mathbb{Z}} g_k\Big\|_{\mathcal{M}^q_p(\mathcal{X})} \lesssim \|f\|_{\dot{M}^s_{p,q,\infty}(\mathcal{X})}.$$

Define $g := \sup_{k\in\mathbb{Z}} g_k$. Then $g \in D^s(f)$ and

$$\|f\|_{H\dot{M}^s_{p,q}(\mathcal{X})} \le \|\vec{g}\|_{\mathcal{M}^q_p(\mathcal{X},l^\infty)} \lesssim \|f\|_{\dot{M}^s_{p,q,\infty}(\mathcal{X})}.$$

Therefore, $f \in H\dot{M}^s_{p,q}(\mathcal{X})$ and hence $\dot{M}^s_{p,q,\infty}(\mathcal{X}) \subset H\dot{M}^s_{p,q}(\mathcal{X})$. ∎

Now we go back to the Euclidean setting and recall Morrey–Triebel–Lizorkin spaces [30, 23] and Triebel–Lizorkin-type spaces [32, 33] (see also [24, 36, 26, 27]). Let $\mathcal{S}(\mathbb{R}^n)$ denote the set of all Schwartz functions on $\mathbb{R}^n$, $\mathcal{S}_\infty(\mathbb{R}^n)$ the set of all $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \phi(x)x^\gamma\,dx = 0$ for all multi-indices $\gamma \in \mathbb{Z}^n_+$, and $\mathcal{S}'_\infty(\mathbb{R}^n)$ its topological dual; here and hereafter $\mathbb{Z}_+ := \{0,1,\ldots\}$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\operatorname{supp}\widehat{\varphi} \subset \{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\} \text{ and } |\widehat{\varphi}(\xi)| \ge c \text{ if } 3/5 \le |\xi| \le 5/3,$$

where c is a positive constant independent of ξ. Let $s \in \mathbb{R}$ and $r \in (0,\infty]$. The *Morrey–Triebel–Lizorkin space* $\dot{\mathcal{E}}^s_{p,q,r}(\mathbb{R}^n)$ with $0 < p \le q \le \infty$ is defined to be the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{\mathcal{E}}^s_{p,q,r}(\mathbb{R}^n)} := \Big\|\Big[\sum_{j\in\mathbb{Z}} 2^{jsr}|\varphi_j * f|^r\Big]^{1/r}\Big\|_{\mathcal{M}^q_p(\mathbb{R}^n)} < \infty,$$

while the *Triebel–Lizorkin-type space* $\dot{F}^{s,\tau}_{p,r}(\mathbb{R}^n)$ with $p \in (0,\infty)$ and $\tau \in [0,\infty)$ is defined to be the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}^{s,\tau}_{p,r}(\mathbb{R}^n)} := \sup_{\substack{x\in\mathbb{R}^n\\ k\in\mathbb{Z}}} 2^{kn\tau}\Big\{\int_{B(x,2^{-k})} \Big[\sum_{j=k}^\infty 2^{jsr}|\varphi_j * f(y)|^r\Big]^{p/r} dy\Big\}^{1/p} < \infty.$$

It was proved in [24, Theorem 1.1(ii)] that $\dot{\mathcal{E}}^s_{p,q,r}(\mathbb{R}^n) = \dot{F}^{s,1/p-1/q}_{p,r}(\mathbb{R}^n)$.

We also recall the "grand" counterparts of these spaces. For all $N \in \mathbb{Z}_+ \cup \{-1\}$ and $m,l \in \mathbb{Z}_+$, let

$$\mathcal{A}^l_{N,m}(\mathbb{R}^n) := \Big\{\phi \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \phi(x)x^\gamma\,dx = 0 \text{ if } |\gamma| \le N, \text{ and } \|\phi\|_{\mathcal{S}_{N+l+1,m}(\mathbb{R}^n)} \le 1\Big\},$$

where

$$\|\phi\|_{\mathcal{S}_{i,j}(\mathbb{R}^n)} := \sup_{|\gamma|\le i,\, x\in\mathbb{R}^n} (1+|x|)^j|\partial^\gamma\phi(x)|.$$

Then the *grand Morrey–Triebel–Lizorkin space* $\mathcal{A}^l_{N,m}\dot{\mathcal{E}}^s_{p,q,r}(\mathbb{R}^n)$ and the *grand Triebel–Lizorkin-type space* $\mathcal{A}^l_{N,m}\dot{F}^{s,\tau}_{p,r}(\mathbb{R}^n)$ are defined, respectively, in the same way as $\dot{\mathcal{E}}^s_{p,q,r}(\mathbb{R}^n)$ and $\dot{F}^{s,\tau}_{p,r}(\mathbb{R}^n)$ with $|\varphi_j * f|$ replaced by

$$\sup_{\phi\in\mathcal{A}^l_{N,m}(\mathbb{R}^n)} |\phi_j * f|.$$

Recall that the space $\mathcal{A}^l_{N,m}\dot{F}^{s,\tau}_{p,r}(\mathbb{R}^n)$ was first introduced by Soto [28], and it was proved therein that, if $\tau \in [0, 1/p]$,

$$N + 1 > \max\{s, n/\min\{1, p, r\} - n - s\}$$

and $m > \max\{n/\min\{1, p, r\}, n + N + 1\}$, then

$$\mathcal{A}^l_{N,m}\dot{F}^{s,\tau}_{p,r}(\mathbb{R}^n) = \dot{F}^{s,\tau}_{p,r}(\mathbb{R}^n).$$

From this, we easily deduce that

$$\mathcal{A}^l_{N,m}\dot{\mathcal{E}}^s_{p,q,r}(\mathbb{R}^n) = \mathcal{A}^l_{N,m}\dot{F}^{s,1/p-1/q}_{p,r}(\mathbb{R}^n) = \dot{F}^{s,1/p-1/q}_{p,r}(\mathbb{R}^n) = \dot{\mathcal{E}}^s_{p,q,r}(\mathbb{R}^n).$$

Via fractional s-Hajłasz gradients, the space $\dot{M}^{s,\tau}_{p,r}(\mathbb{R}^n)$ was also defined in [28] to be the set of all measurable functions f such that

$$\|f\|_{\dot{M}^{s,\tau}_{p,r}(\mathbb{R}^n)} := \inf_{\vec{g}\in\mathbb{D}^s(f)} \sup_{\substack{x\in\mathbb{R}^n\\ k\in\mathbb{Z}}} 2^{kn\tau}\left\{\int_{B(x,2^{-k})}\left(\sum_{j=k}^{\infty}[g_j(y)]^r\right)^{p/r} dy\right\}^{1/p} < \infty. \tag{2.2}$$

We have the following conclusion.

Proposition 2.4. *Let $s \in \mathbb{R}$, $0 < p \le q \le \infty$ and $r \in (0,\infty]$. Then $\dot{M}^{s,1/p-1/q}_{p,r}(\mathbb{R}^n) = \dot{M}^s_{p,q,r}(\mathbb{R}^n)$ with equivalent quasi-norms.*

Proof. It is easy to deduce from the definitions of the quasi-norms (2.1) and (2.2) that, for all $f \in \dot{M}^s_{p,q,r}(\mathbb{R}^n)$,

$$\|f\|_{\dot{M}^{s,1/p-1/q}_{p,r}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{M}^s_{p,q,r}(\mathbb{R}^n)} < \infty$$

and hence $f \in \dot{M}^{s,1/p-1/q}_{p,r}(\mathbb{R}^n)$. This shows that

$$\dot{M}^{s,1/p-1/q}_{p,r}(\mathbb{R}^n) \subset \dot{M}^s_{p,q,r}(\mathbb{R}^n).$$

We now prove the opposite inclusion. The case $p = q$ is obvious. For $p < q$, when $r = \infty$, we pick $f \in \dot{M}^s_{p,q,\infty}(\mathbb{R}^n)$ and $\vec{g} \in \mathbb{D}^s(f)$ such that

$$\left\|\sup_{j\in\mathbb{Z}} g_j\right\|_{\mathcal{M}^q_p(\mathbb{R}^n)} \lesssim \|f\|_{\dot{M}^s_{p,q,\infty}(\mathbb{R}^n)}.$$

Then, for any $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, we have

$$\begin{aligned} 2^{kn(1/p-1/q)}\Big\{\int_{B(x,2^{-k})} \sup_{j\ge k}\,[g_j(y)]^p\,dy\Big\}^{1/p} &\le 2^{kn(1/p-1/q)}\Big\{\int_{B(x,2^{-k})} \sup_{j\in\mathbb{Z}}\,[g_j(y)]^p\,dy\Big\}^{1/p} \\ &\le \Big\|\sup_{j\in\mathbb{Z}} g_j\Big\|_{\mathcal{M}_p^q(\mathbb{R}^n)} \lesssim \|f\|_{\dot M^s_{p,q,\infty}(\mathbb{R}^n)}, \end{aligned}$$

from which we deduce that $\|f\|_{\dot M^{s,1/p-1/q}_{p,\infty}(\mathbb{R}^n)} \lesssim \|f\|_{\dot M^s_{p,q,\infty}(\mathbb{R}^n)}$. Thus, in this case, we have $\dot M^s_{p,q,\infty}(\mathbb{R}^n) \subset \dot M^{s,1/p-1/q}_{p,\infty}(\mathbb{R}^n)$.

When $r < \infty$, by the triangle inequality, if $p \le r$, then

$$\begin{aligned} 2^{kn(1/p-1/q)}\Big\{\int_{B(x,2^{-k})} \Big(\sum_{j=-\infty}^{k-1}[g_j(y)]^r\Big)^{p/r} dy\Big\}^{1/p} &\le 2^{kn(1/p-1/q)}\Big\{\sum_{j=-\infty}^{k-1}\int_{B(x,2^{-k})}[g_j(y)]^p\,dy\Big\}^{1/p} \\ &\lesssim \sup_{i\in\mathbb{Z}} 2^{in(1/p-1/q)}\Big\{\int_{B(x,2^{-i})}[g_i(y)]^p\,dy\Big\}^{1/p}; \end{aligned} \tag{2.3}$$

and if $p > r$, then by the Minkowski inequality,

$$\begin{aligned} 2^{kn(1/p-1/q)}\Big\{\int_{B(x,2^{-k})} \Big(\sum_{j=-\infty}^{k-1}[g_j(y)]^r\Big)^{p/r} dy\Big\}^{1/p} &\le 2^{kn(1/p-1/q)}\Big\{\sum_{j=-\infty}^{k-1}\Big(\int_{B(x,2^{-k})}[g_j(y)]^p\,dy\Big)^{r/p}\Big\}^{1/r} \\ &\lesssim \sup_{i\in\mathbb{Z}} 2^{in(1/p-1/q)}\Big\{\int_{B(x,2^{-i})}[g_i(y)]^p\,dy\Big\}^{1/p}. \end{aligned} \tag{2.4}$$

Now, by the quasi-linearity of $\|\cdot\|_{\mathcal{M}_p^q(\mathbb{R}^n,l^r)}$, (2.3) and (2.4), we have

$$\|f\|_{\dot M^{s,1/p-1/q}_{p,r}(\mathbb{R}^n)} \lesssim \|f\|_{\dot M^s_{p,q,r}(\mathbb{R}^n)},$$

which implies that $\dot M^s_{p,q,r}(\mathbb{R}^n) \subset \dot M^{s,1/p-1/q}_{p,r}(\mathbb{R}^n)$. ∎

The main result of this section reads as follows.

Theorem 2.5.

(i) *Let $s \in (0,1)$ and $n/(n+s) < p \le q < \infty$. If $\mathcal{A} := \mathcal{A}^l_{0,m}(\mathbb{R}^n)$ with $l \in \mathbb{Z}_+$ and $m \in (n+1,\infty)$, then*
$$H\dot M^s_{p,q}(\mathbb{R}^n) = \mathcal{A}\dot F^{s,1/p-1/q}_{p,\infty}(\mathbb{R}^n) = \dot F^{s,1/p-1/q}_{p,\infty}(\mathbb{R}^n)$$
with equivalent quasi-norms.

(ii) *If* $n/(n+1) < p \le q < \infty$, *then* $H\dot{M}^1_{p,q}(\mathbb{R}^n) = \dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n)$ *with equivalent quasi-norms.*

To prove Theorem 2.5(ii), we need the following Hardy–Morrey spaces introduced by Jia and Wang [13].

Definition 2.6. Let $0 < p \le q \le \infty$, $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi(x)\,dx = 1$ and $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^n : |x| \le 1\}$. The *Hardy–Morrey space* $\mathcal{H}M^q_p(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\mathcal{M}_\psi f \in \mathcal{M}^q_p(\mathbb{R}^n)$, where, for all $x \in \mathbb{R}^n$,

$$\mathcal{M}_\psi f(x) := \sup_{t>0} |f * \psi_t(x)|$$

and, for all $t \in (0,\infty)$, $\psi_t(\cdot) := t^{-n}\psi(\cdot/t)$. Moreover, let

$$\|f\|_{\mathcal{H}M^q_p(\mathbb{R}^n)} := \|\mathcal{M}_\psi f\|_{\mathcal{M}^q_p(\mathbb{R}^n)}.$$

Remark 2.7. (i) From [29, p. 57] and the boundedness of the Hardy–Littlewood maximal operator on $\mathcal{M}^q_p(\mathbb{R}^n)$, it follows that, when $1 < p \le q \le \infty$, we have $\mathcal{M}^q_p(\mathbb{R}^n) = \mathcal{H}M^q_p(\mathbb{R}^n)$ with equivalent norms.

(ii) By the results in [13, Section 2], $\mathcal{H}M^q_p(\mathbb{R}^n)$ is independent of the choice of ψ as in Definition 2.6.

(iii) It was proved in [24, Corollary 3.2] that, for all $0 < p \le q < \infty$, the Hardy–Morrey space $\mathcal{H}M^q_p(\mathbb{R}^n)$ and the space $\dot{F}^{0,1/p-1/q}_{p,2}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.

The Hardy–Morrey–Sobolev space is then defined as follows.

Definition 2.8. Let $0 < p \le q \le \infty$. The *homogeneous Hardy–Morrey–Sobolev space* $\mathcal{H}\dot{M}^1_{p,q}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that $D_j f \in \mathcal{H}M^q_p(\mathbb{R}^n)$ for all $j \in \{1,\dots,n\}$, where $D_j f$ denotes the jth distributional derivative of f. Moreover,

$$\|f\|_{\mathcal{H}\dot{M}^1_{p,q}(\mathbb{R}^n)} := \sum_{j=1}^n \|D_j f\|_{\mathcal{H}M^q_p(\mathbb{R}^n)}.$$

Proposition 2.9. *Let* $0<p\le q<\infty$. *Then* $\mathcal{H}\dot{M}^1_{p,q}(\mathbb{R}^n) = \dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n)$ *with equivalent quasi-norms.*

Proof. We first show $\mathcal{H}\dot{M}^1_{p,q}(\mathbb{R}^n) \subset \dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n)$. Let $f \in \mathcal{H}\dot{M}^1_{p,q}(\mathbb{R}^n)$. Then, for all $j \in \{1,\dots,n\}$, $D_j f \in \mathcal{H}M^q_p(\mathbb{R}^n)$ and

$$\|D_j f\|_{\mathcal{H}M^q_p(\mathbb{R}^n)} \le \|f\|_{\mathcal{H}\dot{M}^1_{p,q}(\mathbb{R}^n)}.$$

Recall that, by [24, Corollary 3.2], $\mathcal{H}M^q_p(\mathbb{R}^n) = \dot{F}^{0,1/p-1/q}_{p,2}(\mathbb{R}^n)$ with equivalent quasi-norms. Thus $D_j f \in \dot{F}^{0,1/p-1/q}_{p,2}(\mathbb{R}^n)$ and

$$\|D_j f\|_{\dot{F}^{0,1/p-1/q}_{p,2}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{H}\dot{M}^1_{p,q}(\mathbb{R}^n)}. \tag{2.5}$$

For $j \in \{1, \ldots, n\}$, let R_j be the *Riesz transform* defined by setting, for all $g \in \mathcal{S}'_\infty(\mathbb{R}^n)$,

$$(R_j g)^\wedge(\xi) := -i\frac{\xi_j}{|\xi|}\widehat{g}(\xi), \quad \xi \in \mathbb{R}^n \setminus \{0\},$$

and I_σ with $\sigma \in \mathbb{R}$ the *Riesz potential operator* defined by setting, for all $g \in \mathcal{S}'_\infty(\mathbb{R}^n)$,

$$(I_\sigma g)^\wedge(\xi) := |\xi|^\sigma \widehat{g}(\xi), \quad \xi \in \mathbb{R}^n \setminus \{0\}.$$

Then it is well known that

$$D_j f = -I_1 R_j f \quad \text{in } \mathcal{S}'_\infty(\mathbb{R}^n). \tag{2.6}$$

Recall that, by [33, Proposition 3.5], $f \in \dot{F}^{s+\sigma,\tau}_{p,r}(\mathbb{R}^n)$ if and only if $I_\sigma f \in \dot{F}^{s,\tau}_{p,r}(\mathbb{R}^n)$, and $\|f\|_{\dot{F}^{s+\sigma,\tau}_{p,r}(\mathbb{R}^n)} \approx \|I_\sigma f\|_{\dot{F}^{s,\tau}_{p,r}(\mathbb{R}^n)}$. Hence, by $D_j f \in \dot{F}^{0,1/p-1/q}_{p,2}(\mathbb{R}^n)$ and (2.6), together with (2.5), we know that $R_j f \in \dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n)$ and

$$\|R_j f\|_{\dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n)} \approx \|D_j f\|_{\dot{F}^{0,1/p-1/q}_{p,2}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{H}\dot{M}^1_{p,q}(\mathbb{R}^n)}$$

for all $j \in \{1, \ldots, n\}$.

On the other hand, by the mapping properties of Fourier multipliers on Triebel–Lizorkin-type spaces [34, Theorem 1.5], R_j is a bounded operator on $\dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n)$ and hence $R_j R_j f \in \dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n)$, which, combined with the fact that

$$f = \sum_{j=1}^n R_j R_j f \quad \text{in } \mathcal{S}'_\infty(\mathbb{R}^n),$$

further implies that $f \in \dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n)$ and $\|f\|_{\dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{H}\dot{M}^1_{p,q}(\mathbb{R}^n)}$. This finishes the proof of $\mathcal{H}\dot{M}^1_{p,q}(\mathbb{R}^n) \subset \dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n)$.

To prove the opposite inclusion, let $f \in \dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n)$. Then, by the mapping properties of pseudo-differential operators on Triebel–Lizorkin-type spaces from [24, Theorem 1.5], for all $j \in \{1, \ldots, n\}$ we have $D_j f \in \dot{F}^{0,1/p-1/q}_{p,2}(\mathbb{R}^n) = \mathcal{H}M^q_p(\mathbb{R}^n)$ and

$$\|D_j f\|_{\mathcal{H}M^q_p(\mathbb{R}^n)} \approx \|D_j f\|_{\dot{F}^{0,1/p-1/q}_{p,2}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n)}.$$

This implies that $\dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n) \subset \mathcal{H}\dot{M}^1_{p,q}(\mathbb{R}^n)$. ■

In particular, when $p = q$, Proposition 2.9 reduces to the known coincidence between Hardy-Sobolev spaces and Triebel–Lizorkin spaces. By [36, Proposition 8.2], we know that $\dot{F}^{0,1/p-1/q}_{p,2}(\mathbb{R}^n) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$ in the sense of $\mathcal{S}'_\infty(\mathbb{R}^n)$, which, together with Proposition 2.9, implies the following conclusion, the details being omitted.

Corollary 2.10. *Let $0 < p \le q < \infty$. Then $\mathcal{H}\dot{M}^1_{p,q}(\mathbb{R}^n) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$ in the sense of $\mathcal{S}'_\infty(\mathbb{R}^n)$.*

As an application of Corollary 2.10, we have the following conclusion.

Theorem 2.11. *Let $n/(n+1) < p \le q \le \infty$. Then $\mathcal{H}\dot{M}^1_{p,q}(\mathbb{R}^n) = H\dot{M}^1_{p,q}(\mathbb{R}^n)$ with equivalent quasi-norms.*

Proof. The case $q = \infty$ is a special case of [6, Theorem 1], so we only need to consider $q < \infty$. Assume first that $f \in \mathcal{H}\dot{M}^1_{p,q}(\mathbb{R}^n)$. Then, by Definition 2.8, $D_j f \in \mathcal{H}M^q_p(\mathbb{R}^n)$ for each $j \in \{1, \dots, n\}$. By Corollary 2.10 and [15, Theorem 7], for each ball $B \subset \mathbb{R}^n$ there exists a set $E \subset \mathbb{R}^n$ of measure zero such that, for all $x, y \in B \setminus E$,

$$|f(x) - f(y)| \lesssim |x - y|[M_1(Df)(x) + M_1(Df)(y)],$$

where we used the notation of [15]: for all $x \in \mathbb{R}^n$,

$$M_1(Df)(x) := \max_{j\in\{1,\dots,n\}} M_1(D_j f)(x) := \max_{j\in\{1,\dots,n\}} \sup |\langle D_j f, \varphi\rangle|,$$

with the supremum taken over all compactly supported smooth functions φ such that, for some $r \in (0,\infty)$ and all $j \in \{1, \dots, n\}$,

$$\operatorname{supp}\varphi \subset B(x,r), \quad \|\varphi\|_{L^\infty(\mathbb{R}^n)} \le r^{-n} \text{ and } \|D_j\varphi\|_{L^\infty(\mathbb{R}^n)} \le r^{-n-1}. \tag{2.7}$$

Therefore, by the definition of Hajłasz gradients, $g := M_1(Df)$ is a Hajłasz gradient of f modulo constants.

Let ψ satisfy (2.7) and $\int_{\mathbb{R}^n} \psi(x)\,dx = 1$. Using Remark 2.7(ii) and repeating the proofs of [13, Lemmas 2.1 and 2.4], we find that, if $n/(n+1) < 1 \le q < \infty$, then

$$\|g\|_{\mathcal{M}^q_p(\mathbb{R}^n)} \approx \sum_{j=1}^n \|\mathcal{M}_\psi(D_j f)\|_{\mathcal{M}^q_p(\mathbb{R}^n)},$$

which, combined with Definitions 2.6 and 2.8, implies that

$$\|g\|_{\mathcal{M}^q_p(\mathbb{R}^n)} \approx \sum_{j=1}^n \|\mathcal{M}_\psi(D_j f)\|_{\mathcal{M}^q_p(\mathbb{R}^n)} \approx \sum_{j=1}^n \|D_j f\|_{\mathcal{H}M^q_p(\mathbb{R}^n)} \approx \|f\|_{\mathcal{H}\dot{M}^1_{p,q}(\mathbb{R}^n)}.$$

Thus, $f \in H\dot{M}^1_{p,q}(\mathbb{R}^n)$ and $\|f\|_{H\dot{M}^1_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{H}\dot{M}^1_{p,q}(\mathbb{R}^n)}$. This proves that $\mathcal{H}\dot{M}^1_{p,q}(\mathbb{R}^n) \subset H\dot{M}^1_{p,q}(\mathbb{R}^n)$.

Conversely, let $f \in H\dot{M}^1_{p,q}(\mathbb{R}^n)$. Then, by Definition 1.3, there exists a Hajłasz gradient $g \in \mathcal{M}^q_p(\mathbb{R}^n)$ of f such that $\|g\|_{\mathcal{M}^q_p(\mathbb{R}^n)} \le 2\|f\|_{H\dot{M}^1_{p,q}(\mathbb{R}^n)}$. Notice that, for any ball $B(a,r) \subset \mathbb{R}^n$ with center $a \in \mathbb{R}^n$ and radius $r > 0$, by the Hölder inequality, we have $g \in L^p_{\mathrm{loc}}(B(a,2r)) \subset L^{n/(n+1)}_{\mathrm{loc}}(B(a,2r))$, since $p > n/(n+1)$. From this and [15, Proposition 5], it follows that

$$\inf_{c\in\mathbb{R}} \frac{1}{|B(a,r)|} \int_{B(a,r)} |f(x)-c|\,dx \lesssim r\Big\{\frac{1}{|B(a,2r)|}\int_{B(a,2r)} [g(x)]^{n/(n+1)}\,dx\Big\}^{(n+1)/n},$$

which implies $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\frac{1}{|B(a,r)|} \int_{B(a,r)} |f(x)-f_{B(a,r)}|\,dx \lesssim r\Big\{\frac{1}{|B(a,2r)|}\int_{B(a,2r)} [g(x)]^{n/(n+1)}\,dx\Big\}^{(n+1)/n}. \tag{2.8}$$

By this and the Hölder inequality, we conclude that, for all $\phi \in \mathcal{S}_\infty(\mathbb{R}^n)$,

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n} f(x)\phi(x)\,dx\Big| \\
&\quad = \Big|\int_{\mathbb{R}^n} [f(x)-f_{B(0,1)}]\phi(x)\,dx\Big| \\
&\quad \lesssim \int_{B(0,1)} \frac{|f(x)-f_{B(0,1)}|}{(1+|x|)^N}\,dx + \sum_{i=1}^\infty \int_{B(0,2^i)\setminus B(0,2^{i-1})} \frac{|f(x)-f_{B(0,1)}|}{(1+|x|)^N}\,dx \\
&\quad \lesssim \Big\{\int_{B(0,2)} [g(x)]^{n/(n+1)}\,dx\Big\}^{(n+1)/n} + \sum_{i=1}^\infty \frac{2^{-i(N-n)}}{|B(0,2^i)|}\int_{B(0,2^i)} |f(x)-f_{B(0,2^i)}|\,dx \\
&\quad \lesssim \sum_{i=0}^\infty 2^{-i(N-n-1)}\Big\{\frac{1}{|B(0,2^{i+1})|}\int_{B(0,2^{i+1})} [g(x)]^p\,dx\Big\}^{1/p} \\
&\quad \lesssim \sum_{i=0}^\infty 2^{-i(N-n-1+n/q)}\|g\|_{\mathcal{M}^q_p(\mathbb{R}^n)} \lesssim \|f\|_{H\dot{M}^1_{p,q}(\mathbb{R}^n)},
\end{aligned}$$

where we have chosen $N > n+1-n/q$. Thus, $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$.

Moreover, similar to the proof of [15, Theorem 1], by (2.8), we conclude that

$$\mathcal{M}_\psi(D_j f)(x) \lesssim [M(g^{n/(n+1)})(x)]^{(n+1)/n}, \qquad x \in \mathbb{R}^n, \tag{2.9}$$

where M denotes the *Hardy–Littlewood maximal operator*, namely, for any locally integrable function f on $\mathbb{R}^n$ and $x \in \mathbb{R}^n$,

$$Mf(x) := \sup_{B\ni x} \fint_B f(y)\,dy, \tag{2.10}$$

where the supremum is taken over all balls B containing x. By the boundedness of M on $\mathcal{M}^{(n+1)q/n}_{(n+1)p/n}(\mathbb{R}^n)$, together with $1 < \frac{n+1}{n}p \le \frac{n+1}{n}q \le \infty$, we know that

$$M(g^{n/(n+1)}) \in \mathcal{M}^{(n+1)q/n}_{(n+1)p/n}(\mathbb{R}^n),$$

which, combined with (2.9) and (1.1), implies that $\mathcal{M}_\psi(D_j f) \in \mathcal{M}^q_p(\mathbb{R}^n)$ and $\|\mathcal{M}_\psi(D_j f)\|_{\mathcal{M}^q_p(\mathbb{R}^n)} \lesssim \|g\|_{\mathcal{M}^q_p(\mathbb{R}^n)}$. Therefore, by Definition 2.8 and the choice of g, we find that $f \in \mathcal{H}\dot{M}^1_{p,q}(\mathbb{R}^n)$ and

$$\|f\|_{\mathcal{H}\dot{M}^1_{p,q}(\mathbb{R}^n)} \lesssim \|g\|_{\mathcal{M}^q_p(\mathbb{R}^n)} \lesssim \|f\|_{H\dot{M}^1_{p,q}(\mathbb{R}^n)},$$

which completes the proof of Theorem 2.11. ∎

Theorem 2.11 generalizes [15, Theorem 1] by taking $p = q$.

We are now ready to prove Theorem 2.5.

Proof of Theorem 2.5. Clause (i) is deduced from the following equalities: for $s \in (0,1)$ and $n/(n+s) < p \le q < \infty$,

$$\begin{aligned} H\dot{M}^s_{p,q}(\mathbb{R}^n) &= \dot{M}^s_{p,q,\infty}(\mathbb{R}^n) = \dot{M}^{s,1/p-1/q}_{p,\infty}(\mathbb{R}^n) \\ &= \mathcal{A}\dot{F}^{s,1/p-1/q}_{p,\infty}(\mathbb{R}^n) = \dot{F}^{s,1/p-1/q}_{p,\infty}(\mathbb{R}^n) \end{aligned}$$

with equivalent quasi-norms, where the first equality is Proposition 2.3, the second follows from Proposition 2.4, and the third and fourth ones come from [28, Theorems 1.1(i) and 1.2(i)].

By combining Proposition 2.9 and Theorem 2.11, we obtain (ii). ∎

3. Characterizations of Hajłasz–Morrey–Sobolev spaces on RD-spaces. In this section, we focus on the corresponding conclusion of Theorem 2.5 on *RD-spaces*, that is, metric measure spaces of homogeneous type satisfying also the following *inverse doubling condition*: there exists a constant $c_0 \in (1,\infty)$ such that, for all $x \in \mathcal{X}$ and $r \in (0, \mathrm{diam}(\mathcal{X})/2)$,

$$\mu(B(x,2r)) \ge c_0 \mu(B(x,r)),$$

here and hereafter, for any subset E of $\mathcal{X}$, $\mathrm{diam}(E)$ denotes its diameter,

$$\mathrm{diam}(E) := \sup_{x,\,y \in E} d(x,y).$$

See also [10, 35] for several equivalent definitions of RD-spaces.

To show the coincidence of Hajłasz–Morrey–Sobolev spaces and grand Morrey–Triebel–Lizorkin spaces, we need first to recall certain test functions and approximations of the identity on metric measure spaces of homogeneous type. For any $r \in (0,\infty)$ and $x, y \in \mathcal{X}$, let

$$V(x,y) := \mu(B(x, d(x,y))) \quad \text{and} \quad V_r(x) := \mu(B(x,r)).$$

It is easy to see that $V(x,y) \approx V(y,x)$ for all $x, y \in \mathcal{X}$.

The following test functions were originally introduced in [9, Definition 2.2] (see also [10, Definition 2.8]).

DEFINITION 3.1. Let $(\mathcal{X}, d, \mu)$ be a metric measure space of homogeneous type, $x_1 \in \mathcal{X}$, $r \in (0, \infty)$, $\beta \in (0, 1]$ and $\gamma \in (0, \infty)$. A function φ on $\mathcal{X}$ is said to be in the space $\mathcal{G}(x_1, r, \beta, \gamma)$ if there exists a non-negative constant $\widetilde{C}$ such that

(i) $|\varphi(x)| \le \widetilde{C} \dfrac{1}{V_r(x_1) + V(x_1, x)} \left[\dfrac{r}{r + d(x_1, x)}\right]^\gamma$ for all $x \in \mathcal{X}$;

(ii) $|\varphi(x) - \varphi(y)| \le \widetilde{C} \left[\dfrac{d(x, y)}{r + d(x_1, x)}\right]^\beta \dfrac{1}{V_r(x_1) + V(x_1, x)} \left[\dfrac{r}{r + d(x_1, x)}\right]^\gamma$ for all $x, y \in \mathcal{X}$ satisfying that $d(x, y) \le [r + d(x_1, x)]/2$.

Moreover, for any $\varphi \in \mathcal{G}(x_1, r, \beta, \gamma)$, its norm is defined by

$$\|\varphi\|_{\mathcal{G}(x_1, r, \beta, \gamma)} := \inf\{\widetilde{C} : \text{(i) and (ii) hold true}\}.$$

Fixing $x_1 \in \mathcal{X}$, let $\mathcal{G}(\beta, \gamma) := \mathcal{G}(x_1, 1, \beta, \gamma)$ and

$$\mathring{\mathcal{G}}(\beta, \gamma) := \Big\{ f \in \mathcal{G}(\beta, \gamma) : \int_{\mathcal{X}} f(x)\, d\mu(x) = 0 \Big\}.$$

Denote by $(\mathcal{G}(\beta, \gamma))'$ and $(\mathring{\mathcal{G}}(\beta, \gamma))'$ the respective dual spaces of $\mathcal{G}(\beta, \gamma)$ and $\mathring{\mathcal{G}}(\beta, \gamma)$. Obviously, by the definition of $\mathring{\mathcal{G}}(\beta, \gamma)$,

$$(\mathring{\mathcal{G}}(\beta, \gamma))' = (\mathcal{G}(\beta, \gamma))'/\mathbb{C}.$$

Let $\mathcal{A} := \{\mathcal{A}_k(x)\}_{x \in \mathcal{X},\, k \in \mathbb{Z}}$ with

$$\mathcal{A}_k(x) := \{\phi \in \mathring{\mathcal{G}}(1, 2) : \|\phi\|_{\mathcal{G}(x, 2^{-k}, 1, 2)} \le 1\} \tag{3.1}$$

for all $x \in \mathcal{X}$ (see [16, Definition 5.2]).

The following notion of approximations of the identity with bounded supports was first introduced in [10, Definition 2.3].

DEFINITION 3.2. Let $(\mathcal{X}, d, \mu)$ be a metric measure space of homogeneous type. A sequence $\{S_k\}_{k \in \mathbb{Z}}$ of bounded linear integral operators is called an *approximation of the identity of order* 1 (for short, 1-AOTI) with bounded support if there exist positive constants C_3 and C_4 such that, for all $k \in \mathbb{Z}$ and $x, \tilde{x}, y, \tilde{y} \in \mathcal{X}$, $S_k(x, y)$, the integral kernel of S_k, is a measurable function from $\mathcal{X} \times \mathcal{X}$ into $\mathbb{C}$ satisfying

(i) $S_k(x, y) = 0$ if $d(x, y) > C_4 2^{-k}$, and

$$|S_k(x, y)| \le C_3 \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)};$$

(ii) if $d(x, \tilde{x}) \le \max\{C_4, 1\} 2^{1-k}$, then

$$|S_k(x, y) - S_k(\tilde{x}, y)| \le C_3 2^k d(x, \tilde{x}) \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)};$$

(iii) if $d(x,\tilde{x}) \le \max\{C_4,1\}2^{1-k}$, then

$$|S_k(y,x) - S_k(y,\tilde{x})| \le C_3 2^k d(x,\tilde{x}) \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)};$$

(iv) if $d(x,\tilde{x}) \le \max\{C_4,1\}2^{1-k}$ and $d(y,\tilde{y}) \le \max\{C_4,1\}2^{1-k}$, then

$$|[S_k(x,y) - S_k(x,\tilde{y})] - [S_k(\tilde{x},y) - S_k(\tilde{x},\tilde{y})]| \le C_3 2^{2k} \frac{d(x,\tilde{x})d(y,\tilde{y})}{V_{2^{-k}}(x) + V_{2^{-k}}(y)};$$

(v) $\int_{\mathcal{X}} S_k(x,y)\,d\mu(y) = 1 = \int_{\mathcal{X}} S_k(x,y)\,d\mu(x)$.

It is known that there always exists a 1-AOTI with bounded support on a space of homogeneous type (see [10, Theorem 2.6]).

Let $(\mathcal{X},d,\mu)$ be a metric measure space of homogeneous type, $s \in (0,1]$, $p \in (0,\infty)$, $q \in [p,\infty]$ and $r \in (0,\infty]$. The *homogeneous grand Morrey–Triebel–Lizorkin space* $\mathcal{A}\dot{F}^s_{p,q,r}(\mathcal{X})$ is defined to be the set of all $f \in (\mathcal{G}(1,2))'$ such that

$$\|f\|_{\mathcal{A}\dot{F}^s_{p,q,r}(\mathcal{X})} := \Big\| \Big\{ \sum_{k\in\mathbb{Z}} 2^{ksr} \sup_{\phi \in \mathcal{A}_k(\cdot)} |\langle f,\phi\rangle|^r \Big\}^{1/r} \Big\|_{\mathcal{M}^q_p(\mathcal{X})} < \infty$$

with the usual modification when $r = \infty$.

THEOREM 3.3. *Let $(\mathcal{X},d,\mu)$ be an RD-space. If $s \in (0,1]$, $p \in (\frac{n}{n+s},\infty)$ and $q \in [p,\infty]$, then $\mathcal{A}\dot{F}^s_{p,q,\infty}(\mathcal{X}) = H\dot{M}^s_{p,q}(\mathcal{X})$ with equivalent quasi-norms.*

To prove this theorem, we need the following partition of unity for $\mathcal{X}$, which is obtained by using Christ's dyadic cube decomposition for spaces of homogeneous type in [2, Theorem 11], and the construction of a partition of unity by Macías and Segovia [20, Lemmas (2.9) and (2.16)].

LEMMA 3.4. *Let $(\mathcal{X},d,\mu)$ be a metric measure space of homogeneous type. Then there exist a sequence $\{B_j\}_j$ of open balls with the finite intersection property and a sequence $\{\phi_j\}_j$ of non-negative functions in $\mathcal{G}(1,2)$ such that $\mu(\mathcal{X} \setminus \bigcup_j B_j) = 0$, $\operatorname{supp}\phi_j \subset B_j$ for all j and $\sum_j \phi_j(x) = 1$ for almost every $x \in \mathcal{X}$.*

Proof. By Christ's dyadic cube decomposition [2, Theorem 11], there exists a (possibly finite) sequence $\{Q_j\}_j$ of open sets such that $\mu(\mathcal{X}\setminus\bigcup_j Q_j)=0$, $Q_j \cap Q_k = \emptyset$ if $j \ne k$ and $\operatorname{diam}(Q_j) \approx \delta$ for all j and some constant $\delta \in (0,1)$. Furthermore, there exist positive constants C_1, C_2 and points $\{x_j\}_j$ such that $B_{j,1} \subset Q_j \subset B_{j,2}$, where $B_{j,1} := B(x_j, C_1\delta)$ and $B_{j,2} := B(x_j, C_2\delta)$. From this, we easily deduce that $\mu(\mathcal{X} \setminus \bigcup_j B_{j,2}) = 0$ and there exists $N \in \mathbb{N}$ such that each $B_{j,2}$ intersects at most N balls from $\{B_{k,2}\}_k$.

To obtain the functions ϕ_j, we apply Macías–Segovia's method of constructing a partition of unity (see [20, Lemma (2.16)]). Let η be an infinitely differentiable function on $[0,\infty)$ such that $\eta(x) = 1$ on $[0,1]$ and $\eta(x) = 0$

on $[3/2, \infty)$. For every j and all $x \in \mathcal{X}$, let

$$\psi_j(x) := \eta\left(2\frac{d(x, x_j)}{C_2\delta}\right).$$

These functions are obviously non-negative and supported in $B_{j,2}$. From the finite intersection property of $\{B_{j,2}\}_j$ and the definition of $\{\psi_j\}_j$, we see that $1 \le \sum_j \psi_j(x) \le N$ for almost every $x \in \mathcal{X}$. For all j and $x \in \mathcal{X}$, let

$$\phi_j(x) := \frac{\psi_j(x)}{\sum_j \psi_j(x)}.$$

Then, for almost every $x \in \mathcal{X}$, $\sum_j \phi_j(x) = 1$. Moreover, similar to the proof of [20, Lemma (2.16)], we see that $\phi_j \in \mathcal{G}(1, 2)$. Therefore, $\{\phi_j\}_j$ and $\{B_{j,2}\}_j$ are as desired. ∎

We also need the following two technical lemmas.

LEMMA 3.5. *Let $(\mathcal{X}, d, \mu)$ be a metric measure space of homogeneous type, $x, y \in \mathcal{X}$ and $k_0 \in \mathbb{Z}$ such that $2^{-k_0-1} < d(x, y) \le 2^{-k_0}$. Assume that $k \le k_0$. Then there exists a positive constant $\widetilde{C}$, independent of k_0, k, x and y, such that*

$$\widetilde{C}2^{-k}[d(x, y)]^{-1}[S_k(x, \cdot) - S_k(y, \cdot)] \in \mathcal{A}_k(x),$$

where $\mathcal{A}_k(x)$ is as in (3.1).

Proof. For $x, y \in \mathcal{X}$ and $k \in \mathbb{Z}$ as in Lemma 3.5, let

$$\phi_k^{(x,y)}(z) := 2^{-k}[d(x, y)]^{-1}[S_k(x, z) - S_k(y, z)], \qquad z \in \mathcal{X}.$$

By Definition 3.2(v), it is easy to see that $\int_{\mathcal{X}} \phi_k^{(x,y)}(z)\, d\mu(z) = 0$. We now prove that $\phi_k^{(x,y)}$ satisfies condition (i) of Definition 3.1 with $\gamma = 2$ and $r = 2^{-k}$, that is, for all $z \in \mathcal{X}$,

$$|\phi_k^{(x,y)}(z)| \lesssim \frac{1}{V_{2^{-k}}(x) + V(x, z)}\left[\frac{2^{-k}}{2^{-k} + d(x, z)}\right]^2. \tag{3.2}$$

We consider two cases.

CASE 1: $d(x, z) > C_4 2^{-k}$ and $d(y, z) > C_4 2^{-k}$. In this case, (3.2) holds automatically by Definition 3.2(i).

CASE 2: $d(x, z) \le C_4 2^{-k}$ or $d(y, z) \le C_4 2^{-k}$. In this case, as $k \le k_0$ and hence

$$d(x, y) \le 2^{-k_0} \le 2^{-k} < \max\{C_4, 1\}2^{1-k}, \tag{3.3}$$

from Definition 3.2(ii) we see that

$$|S_k(x, z) - S_k(y, z)| \lesssim 2^k d(x, y)\frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(z)}. \tag{3.4}$$

When $d(x,z) \le C_4 2^{-k}$, by (1.3) we see that

$$\frac{2^{-k}}{2^{-k}+d(x,z)} \approx 1 \quad \text{and} \quad V_{2^{-k}}(x)+V_{2^{-k}}(z) \approx V_{2^{-k}}(x)+V(x,z). \tag{3.5}$$

When $d(y,z) \le C_4 2^{-k}$, since $k \le k_0$, we know that

$$d(x,z) \le d(x,y)+d(y,z) \le 2^{-k_0} + C_4 2^{-k} \le (1+C_4)2^{-k}.$$

Thus, in this case, (3.5) also holds. By (3.5) and (3.4), we obtain (3.2).

We now turn to proving that $\phi_k^{(x,y)}$ satisfies condition (ii) of Definition 3.1 with $\beta = 1$, $\gamma = 2$ and $r = 2^{-k}$, that is, for all $z, w \in \mathcal{X}$ with $d(z,w) \le [2^{-k}+d(x,z)]/2$,

$$\begin{aligned} &|\phi_k^{(x,y)}(z) - \phi_k^{(x,y)}(w)| \\ &\qquad \lesssim \frac{d(z,w)}{2^{-k}+d(x,z)} \frac{1}{V_{2^{-k}}(x)+V(x,z)} \left[\frac{2^{-k}}{2^{-k}+d(x,z)}\right]^2. \end{aligned} \tag{3.6}$$

Write

$$\begin{aligned} \mathrm{I} &:= |\phi_k^{(x,y)}(z) - \phi_k^{(x,y)}(w)| \\ &= 2^{-k}[d(x,y)]^{-1} |[S_k(x,z) - S_k(x,w)] - [S_k(y,z) - S_k(y,w)]|. \end{aligned}$$

We consider three cases.

Case a: $d(x,z) \le \max\{C_4, 1\} 2^{1-k}$. In this case, (3.5) holds and

$$d(z,w) \le [2^{-k}+d(x,z)]/2 \le \max\{C_4,1\}2^{1-k}.$$

Thus, by (3.3) and Definition 3.2(iv), we find that

$$\mathrm{I} \lesssim \frac{2^k d(z,w)}{V_{2^{-k}}(x)+V_{2^{-k}}(z)}.$$

From this and (3.5), we deduce (3.6).

Case b: $d(x,z) > \max\{C_4, 1\} 2^{2-k}$. Then, by Definition 3.2(i), we know that $S_k(x,z) = 0$. Observing that

$$d(y,z) \ge d(x,z) - d(x,y) > \max\{C_4,1\}2^{2-k} - 2^{-k} > C_4 2^{-k},$$

by Definition 3.2(i) again, we also have $S_k(y,z) = 0$.

Subcase b1: $d(z,w) \le d(x,z)/4$. Then

$$d(x,w) \ge d(x,z) - d(z,w) \ge \frac{3}{4} d(x,z) > C_4 2^{-k}$$

and

$$d(y,w) \ge d(x,z) - d(x,y) - d(z,w) \ge \frac{3}{4} d(x,z) - d(x,y) > C_4 2^{-k}.$$

Thus, $S_k(x,w) = S_k(y,w) = 0$, and hence $\mathrm{I} = 0$.

SUBCASE b2: $d(z,w) > d(x,z)/4$. Recall that, since $d(x,y) \le 2^{-k_0} \le 2^{-k}$, we have $2^{-k} + d(y,w) \approx 2^{-k} + d(x,w)$ and

$$V_{2^{-k}}(y) + V(y,w) \approx V_{2^{-k}}(x) + V(x,w).$$

Combining this with $S_k(y,z) = S_k(x,z) = 0$ and Definition 3.2(i) yields

$$\begin{aligned} \mathrm{I} &= 2^{-k}[d(x,y)]^{-1}|S_k(x,w) - S_k(y,w)| \\ &\lesssim \frac{1}{V_{2^{-k}}(x) + V(x,w)}\left[\frac{2^{-k}}{2^{-k} + d(x,w)}\right]^2. \end{aligned} \tag{3.7}$$

Observe that, in the present subcase,

$$d(x,z)/4 < d(z,w) \le [2^{-k} + d(x,z)]/2$$

and $d(x,z) \ge 2^{-k}$, which implies that

$$2^{-k} + d(x,w) \approx 2^{-k} + d(x,z). \tag{3.8}$$

From these estimates, we deduce that

$$V_{2^{-k}}(x) + V(x,w) \approx V_{2^{-k}}(x) + V(x,z) \tag{3.9}$$

and

$$1 \lesssim \frac{d(z,w)}{d(x,z)} \lesssim \frac{d(z,w)}{2^{-k} + d(x,z)}. \tag{3.10}$$

Combining (3.7)–(3.10), we obtain (3.6).

CASE c: $\max\{C_4,1\}2^{1-k} < d(x,z) \le \max\{C_4,1\}2^{2-k}$. Then $S_k(x,z)=0$ and

$$d(z,w) \le \frac{2^{-k} + d(x,z)}{2} < \max\{C_4,1\}2^{2-k}.$$

If $d(z,w) \le \max\{C_4,1\}2^{1-k}$, the proof is the same as that for Case a. If

$$\max\{C_4,1\}2^{1-k} < d(z,w) \le \max\{C_4,1\}2^{2-k},$$

then $d(z,w) \approx d(x,z) \approx 2^{-k}$ and

$$d(y,z) \ge d(x,z) - d(x,y) > \max\{C_4,1\}2^{1-k} - 2^{-k_0} > C_4 2^{-k}.$$

From this and Definition 3.2(i), we see that $S_k(y,z) = 0$ and, in this case,

$$\mathrm{I} = 2^{-k}[d(x,y)]^{-1}|S_k(x,w) - S_k(y,w)|.$$

Then, repeating the proof of Case b, we also obtain (3.6). ∎

This lemma implies the following corollary, the details being omitted.

COROLLARY 3.6. *Let $(\mathcal{X}, d, \mu)$ be a metric measure space of homogeneous type, $x, y \in \mathcal{X}$ and $k_0 \in \mathbb{Z}$ such that $2^{-k_0-1} < d(x,y) \le 2^{-k_0}$. Assume that $k \le k_0$. Then there exists a positive constant $\widetilde{C}$, independent of k_0, k, x and y, such that, for all $\sigma \in [0,1]$,*

$$\widetilde{C}2^{-k\sigma}[d(x,y)]^{-\sigma}[S_k(x,\cdot) - S_k(y,\cdot)] \in \mathcal{A}_k(x),$$

where $\mathcal{A}_k(x)$ is as in (3.1).

LEMMA 3.7. *Let $(\mathcal{X}, d, \mu)$ be a metric measure space of homogeneous type, $x \in \mathcal{X}$ and $k \in \mathbb{Z}$. Then there exists a positive constant $\widetilde{C}$, independent of k and x, such that*

$$\widetilde{C}[S_{k+1}(x,\cdot) - S_k(x,\cdot)] \in \mathcal{A}_k(x).$$

Proof. For any $x \in \mathcal{X}$ and $k \in \mathbb{Z}$, let

$$\phi_k^x(y) := S_{k+1}(x,y) - S_k(x,y)$$

for all $y \in \mathcal{X}$. Then $\int_{\mathcal{X}} \phi_k^x(y)\,d\mu(y) = 0$ by Definition 3.2(v). Write

$$\mathrm{I}_1 := |S_{k+1}(x,y) - S_k(x,y)|,$$
$$\mathrm{I}_2 := |[S_{k+1}(x,y) - S_k(x,y)] - [S_{k+1}(x,z) - S_k(x,z)]|.$$

It suffices to show that

$$\mathrm{I}_1[V_{2^{-k}}(x) + V(x,y)]\left[\frac{d(x,y)+2^{-k}}{2^{-k}}\right]^2 \lesssim 1 \tag{3.11}$$

and, for any $d(y,z) \le [2^{-k} + d(x,y)]/2$,

$$\mathrm{I}_2 \frac{[2^{-k} + d(x,y)]^3[V_{2^{-k}}(x) + V(x,y)]}{2^{-2k} d(y,z)} \lesssim 1. \tag{3.12}$$

To prove (3.11), we consider two cases.

CASE 1: $d(x,y) > C_4 2^{-k}$. In this case, from Definition 3.2(i), we deduce that $\mathrm{I}_1 = 0$ and hence (3.11) holds true.

CASE 2: $d(x,y) \le C_4 2^{-k}$. Then, by Definition 3.2(i),

$$\begin{aligned} \mathrm{I}_1[V_{2^{-k}}(x) + V(x,y)]\left[\frac{d(x,y)+2^{-k}}{2^{-k}}\right]^2 \\ \lesssim \frac{V_{2^{-k}}(x) + V(x,y)}{V_{2^{-k-1}}(x) + V_{2^{-k-1}}(y)} + \frac{V_{2^{-k}}(x) + V(x,y)}{V_{2^{-k}}(x) + V_{2^{-k}}(y)}. \end{aligned} \tag{3.13}$$

From (1.3), it follows that

$$V(x,y) \lesssim V_{2^{-k}}(y),\ \ V_{2^{-k}}(x) \lesssim V_{2^{-k-1}}(x) \text{ and } V_{2^{-k}}(y) \lesssim V_{2^{-k-1}}(y). \tag{3.14}$$

Combining (3.13) and (3.14), we obtain (3.11).

To prove (3.12), we let $d(y,z) \le [2^{-k} + d(x,y)]/2$ and consider three cases.

CASE a: $d(y,z) \le \max\{C_4, 1\}2^{-k}$. Then, by (3.14) and Definition 3.2(iii),

$$\begin{aligned} \mathrm{I}_2 &\le |S_{k+1}(x,y) - S_{k+1}(x,z)| + |S_k(x,y) - S_k(x,z)| \\ &\lesssim \frac{2^k d(y,z)}{V_{2^{-k}}(x) + V_{2^{-k}}(y)}. \end{aligned} \tag{3.15}$$

If $d(x,y) \le C_4 2^{-k}$, then (3.15) implies (3.12). If $d(x,y) > C_4 2^{-k}$ and $d(x,z) > C_4 2^{-k}$, then (3.12) is trivially true, since, by Definition 3.2(i), $\mathrm{I}_2 = 0$ in this subcase. If $d(x,y) > C_4 2^{-k}$ and $d(x,z) \le C_4 2^{-k}$, then $d(x,y) \lesssim 2^{-k}$ since $d(y,z) \le [2^{-k} + d(x,y)]/2$ and $d(x,y) \le d(x,z) + d(y,z)$; thus, in this subcase, (3.12) also holds true.

CASE b: $\max\{C_4, 1\}2^{-k} < d(y,z) \le \max\{C_4, 1\}2^{-k+1}$. In this case, by (1.3),

$$\begin{aligned} V_{2^{-k-1}}(x) + V(x,y) &\approx V_{2^{-k-1}}(x) + V(x,z) \\ &\approx V_{2^{-k}}(x) + V(x,y) \approx V_{2^{-k}}(x) + V(x,z). \end{aligned}$$

From this and Definition 3.2(i), we deduce that

$$\mathrm{I}_2 \lesssim \frac{1}{V_{2^{-k}}(x) + V(x,y)}. \tag{3.16}$$

If $d(x,y) \le C_4 2^{-k}$ or $d(x,z) \le C_4 2^{-k}$, then, by $d(y,z) \lesssim 2^{-k}$, we always have $d(x,y) \lesssim 2^{-k}$, which further implies that $2^{-k} + d(x,y) \approx 1$; from this and (3.16), we conclude that (3.12) holds true. If $d(x,y) > C_4 2^{-k}$ and $d(x,z) > C_4 2^{-k}$, then, in this subcase, $\mathrm{I}_2 = 0$ and (3.12) is trivially true. Thus, (3.12) always holds in Case b.

CASE c: $d(y,z) > \max\{C_4, 1\}2^{-k+1}$. In this case, from

$$d(y,z) \le [2^{-k} + d(x,y)]/2,$$

we deduce that

$$d(x,y) \ge 2d(y,z) - 2^{-k} > \max\{C_4, 1\}2^{-k+2} - 2^{-k} > C_4 2^{-k+1}. \tag{3.17}$$

If $d(x,z) > C_4 2^{-k}$, then, in this subcase, $\mathrm{I}_2 = 0$ and (3.12) is trivially true. If $d(x,z) \le C_4 2^{-k}$, then

$$d(x,y) \le d(x,z) + d(y,z) \le C_4 2^{-k} + [2^{-k} + d(x,y)]/2$$

and hence $d(x,y) \lesssim 2^{-k}$, which, together with (3.17), (1.3) and

$$d(y,z) \le [2^{-k} + d(x,y)]/2,$$

implies that $d(x,y) \approx 2^{-k}$, $V(x,y) \lesssim V_{2^{-k}}(x)$, $d(y,z) \lesssim 2^{-k}$ and hence $d(y,z) \approx 2^{-k}$; from these estimates and Definition 3.2(i), we finally deduce that

$$\begin{aligned} \mathrm{I}_2 &\lesssim \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(z)} \lesssim \frac{1}{V_{2^{-k}}(x) + V(x,y)} \\ &\lesssim \frac{2^{-2k} d(y,z)}{[2^{-k} + d(x,y)]^3 [V_{2^{-k}}(x) + V(x,y)]}. \end{aligned}$$

This finishes the proof of (3.12). ∎

To prove Theorem 3.3, we also need the following result, whose proof is similar to those of [7, Theorem 8.1] and [16, Lemma 4.1], the details being omitted.

LEMMA 3.8. *Let $(\mathcal{X}, d, \mu)$ be an RD-space, $s \in (0,1]$, $p \in (0, n/s)$ and*

$$p^* = np/(n - sp).$$

Then there exists a positive constant C such that $u \in L^{p^*}(B_0)$ and

$$\inf_{c\in\mathbb{R}}\Big[\fint_{B_0} |u(x) - c|^{p^*}\, d\mu(x)\Big]^{1/p^*} \le C r_0^s \Big\{\fint_{2B_0} [g(x)]^p\, d\mu(x)\Big\}^{1/p}$$

for all $u \in H\dot{M}^s_{p,q}(\mathcal{X})$, $g \in D^s(u)$ and all balls B_0 with radius r_0.

As a consequence of Lemma 3.8, we have the following conclusion.

LEMMA 3.9. *Let $(\mathcal{X}, d, \mu)$ be an RD-space, $s \in (0,1]$, $p \in [n/(n+s), n/s)$ and $p^* := np/(n-sp)$. Then, for each $u \in H\dot{M}^s_{p,q}(\mathcal{X})$, there exists a constant C such that*

$$u - C \in L^{p^*}(\mathcal{X}) \quad \textit{and} \quad \|u - C\|_{L^{p^*}(\mathcal{X})} \le \widetilde{C}\|u\|_{H\dot{M}^s_{p,q}(\mathcal{X})},$$

where $\widetilde{C}$ is a positive constant independent of u and C.

We are now ready to show Theorem 3.3.

Proof of Theorem 3.3. The proof of $H\dot{M}^s_{p,q}(\mathcal{X}) \subset \mathcal{A}\dot{F}^s_{p,q,\infty}(\mathcal{X})$ is similar to that of [16, Theorem 1.1]. For completeness, we give some details with the aid of the last two lemmas.

Let $f \in H\dot{M}^s_{p,q}(\mathcal{X})$. Choose $g \in D^s(f)$ such that

$$\|g\|_{\mathcal{M}^q_p(\mathcal{X})} \le 2\|f\|_{H\dot{M}^s_{p,q}(\mathcal{X})}.$$

Then, for all $x \in \mathcal{X}$, $k \in \mathbb{Z}$ and $\phi \in \mathcal{A}_k(x_1)$, by the moment condition of ϕ,

$$\begin{aligned}
\text{(3.18)} \qquad \mathrm{I} &:= \Big|\int_{\mathcal{X}} f(x)\phi(x)\, d\mu(x)\Big| \\
&= \Big|\int_{\mathcal{X}} \Big[f(x) - \fint_{B(x_1,2^{-k})} f(z)\, d\mu(z)\Big]\phi(x)\, d\mu(x)\Big| \\
&\le \sum_{i=0}^{\infty} \int_{B(x_1,2^{i+1-k})\setminus B(x_1,2^{i-k})} \Big|f(x) - \fint_{B(x_1,2^{-k})} f(z)\, d\mu(z)\Big| |\phi(x)|\, d\mu(x) \\
&\quad + \int_{B(x_1,2^{-k})} \Big|f(x) - \fint_{B(x_1,2^{-k})} f(z)\, d\mu(z)\Big|\, |\phi(x)|\, d\mu(x) \\
&=: \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}$$

By (3.1) and Definition 3.1(i) with $r = 2^{-k}$ and $\gamma = 2$, with the aid of Lemma 3.8, we know that

$$
\begin{aligned}
\text{(3.19)}\quad \mathrm{I}_1 &\lesssim \sum_{i=0}^{\infty} \int_{B(x_1,2^{i+1-k})\setminus B(x_1,2^{i-k})} \left| f(x) - \fint_{B(x_1,2^{-k})} f(z)\,d\mu(z) \right| \\
&\qquad \times \frac{1}{V_{2^{-k}}(x_1) + V(x,x_1)} \left[\frac{1}{1 + 2^k d(x,x_1)} \right]^2 d\mu(x) \\
&\lesssim \sum_{i=0}^{\infty} 2^{-2i} \fint_{B(x_1,2^{i+1-k})} \left| f(x) - \fint_{B(x_1,2^{-k})} f(z)\,d\mu(z) \right| d\mu(x) \\
&\lesssim \sum_{i=0}^{\infty} 2^{-2i} \sum_{j=0}^{i+1} \fint_{B(x_1,2^{-k+j})} \left| f(x) - \fint_{B(x_1,2^{-k+j})} f(z)\,d\mu(z) \right| d\mu(x) \\
&\lesssim \sum_{i=0}^{\infty} 2^{-2i} \sum_{j=0}^{i+1} 2^{-ks+js} \left\{ \fint_{B(x_1,2^{-k+j+1})} [g(x)]^{n/(n+s)}\,d\mu(x) \right\}^{(n+s)/n} \\
&\lesssim \sum_{j=0}^{\infty} 2^{-ks} 2^{-j(2-s)} [M(g^{n/(n+s)})(x_1)]^{(n+s)/n} \\
&\lesssim 2^{-ks} [M(g^{n/(n+s)})(x_1)]^{(n+s)/n};
\end{aligned}
$$

here and hereafter, M denotes the *Hardy–Littlewood maximal operator* on $\mathcal{X}$, which is defined as in (2.10) with $\mathbb{R}^n$ and the Lebesgue measure replaced by $\mathcal{X}$ and the Borel measure μ, respectively. Similarly, due to $\phi \in \mathcal{A}_k(x_1)$ and Definition 3.1, with the aid of Lemma 3.8 again, we have

$$
\begin{aligned}
\text{(3.20)}\quad \mathrm{I}_2 &= \int_{B(x_1,2^{-k})} \left| f(x) - \fint_{B(x_1,2^{-k})} f(z)\,d\mu(z) \right| |\phi(x)|\,d\mu(x) \\
&\lesssim \int_{B(x_1,2^{-k})} \left| f(x) - \fint_{B(x_1,2^{-k})} f(z)\,d\mu(z) \right| \frac{1}{V_{2^{-k}}(x_1) + V(x,x_1)} \\
&\qquad \times \left[\frac{1}{1 + 2^k d(x,x_1)} \right]^2 d\mu(x) \\
&\lesssim \fint_{B(x_1,2^{-k})} \left| f(x) - \fint_{B(x_1,2^{-k})} f(z)\,d\mu(z) \right| d\mu(x) \\
&\lesssim 2^{-ks} [M(g^{n/(n+s)})(x_1)]^{(n+s)/n}.
\end{aligned}
$$

Combining (3.18)–(3.20), we have

$$
\text{(3.21)}\qquad \mathrm{I} \lesssim 2^{-ks} [M(g^{n/(n+s)})(x_1)]^{(n+s)/n};
$$

since $x_1 \in \mathcal{X}$, $k \in \mathbb{Z}$ and $\phi \in \mathcal{A}_k(x_1)$ are arbitrary and M is bounded on

$\mathcal{M}^{q(n+s)/n}_{p(n+s)/n}(\mathcal{X})$ for $p \in (n/(n+s), \infty]$ and $q \in [p, \infty]$ (see [1] or [19, Lemma 3.4]), this implies that

$$\begin{aligned}\|f\|_{\mathcal{A}\dot{F}^s_{p,q,\infty}(\mathcal{X})} &= \Big\|\sup_{k\in\mathbb{Z}} 2^{ks} \sup_{\phi\in\mathcal{A}_k(\cdot)} |\langle f, \phi\rangle|\Big\|_{\mathcal{M}^q_p(\mathcal{X})} \\ &\lesssim \|[M(g^{n/(n+s)})]^{(n+s)/n}\|_{\mathcal{M}^q_p(\mathcal{X})} \\ &\approx \|M(g^{n/(n+s)})\|^{(n+s)/n}_{\mathcal{M}^{q(n+s)/n}_{p(n+s)/n}(\mathcal{X})} \lesssim \|g\|_{\mathcal{M}^q_p(\mathcal{X})} < \infty.\end{aligned}$$

It remains to show that $f \in (\mathcal{G}(1,2))'$. We may assume that

$$M(g^{n/(n+s)})(x_1) < \infty$$

as in the proof of [16, Theorem 1.3]. Then, by borrowing some tricks from the proofs of [16, Theorem 1.3], Lemma 3.9 and the estimate (3.21) above, we conclude that $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ and, for all $\psi \in \mathcal{G}(1,2)$,

$$\Big|\int_{\mathcal{X}} f(x)\psi(x)\,d\mu(x)\Big| \lesssim \|\psi\|_{\mathcal{G}(1,2)},$$

which implies $f \in (\mathcal{G}(1,2))'$. Thus, $f \in \mathcal{A}\dot{F}^s_{p,q,\infty}(\mathcal{X})$ and

$$\|f\|_{\mathcal{A}\dot{F}^s_{p,q,\infty}(\mathcal{X})} \lesssim \|f\|_{H\dot{M}^s_{p,q}(\mathcal{X})}.$$

We now prove

$$\mathcal{A}\dot{F}^s_{p,q,\infty}(\mathcal{X}) \subset H\dot{M}^s_{p,q}(\mathcal{X}).$$

Let $f \in \mathcal{A}\dot{F}^s_{p,q,\infty}(\mathcal{X})$ and $\{S_k\}_{k\in\mathbb{Z}}$ be a 1-AOTI with bounded support. We first assume that f is a locally integrable function. In this case, applying [10, Proposition 2.7], we know that, for almost every $x \in \mathcal{X}$,

$$\lim_{k\to\infty} S_k(f)(x) = f(x),$$

from which we deduce that, for almost all $x, y \in \mathcal{X}$,

$$\begin{aligned}(3.22)\qquad &|f(x) - f(y)| \\ &\quad\le |S_{k_0}(f)(x) - S_{k_0}(f)(y)| \\ &\qquad + \sum_{k\ge k_0} [|S_{k+1}(f)(x) - S_k(f)(x)| + |S_{k+1}(f)(y) - S_k(f)(y)|],\end{aligned}$$

where $k_0 \in \mathbb{Z}$ such that $2^{-k_0-1} < d(x,y) \le 2^{-k_0}$. For all $k\in\mathbb{Z}$ and $x,y,z\in\mathcal{X}$, let

$$\begin{aligned}\phi^{(x,y)}_{k_0}(z) &:= S_{k_0}(x,z) - S_{k_0}(y,z), \\ \phi^x_k(y) &:= S_{k+1}(x,y) - S_k(x,y).\end{aligned}$$

Then, by Lemmas 3.5 and 3.7, there exists a positive constant $\widetilde{C}$ such that

$\widetilde{C}\phi_{k_0}^{(x,y)} \in \mathcal{A}_{k_0}(x)$ and $\widetilde{C}\phi_k^x \in \mathcal{A}_k(x)$. For every $x \in \mathcal{X}$, let

$$g(x) := \sup_{k\in\mathbb{Z}} \sup_{\phi\in\mathcal{A}_k(x)} 2^{ks}|\langle f, \phi\rangle|, \tag{3.23}$$

where $\mathcal{A}_k(x)$ is as in (3.1). Then, as $f \in \mathcal{A}\dot{F}^s_{p,q,\infty}(\mathcal{X})$, we see that $g \in \mathcal{M}^q_p(\mathcal{X})$ and, by (3.22), (3.23) and the choice of k_0, we conclude that

$$\begin{aligned} &|f(x) - f(y)| \\ &\quad\lesssim \sup_{\phi\in\mathcal{A}_{k_0}(x)} |\langle f, \phi\rangle| + \sum_{k\ge k_0} \Big[\sup_{\phi\in\mathcal{A}_k(x)} |\langle f, \phi\rangle| + \sup_{\phi\in\mathcal{A}_k(y)} |\langle f, \phi\rangle|\Big] \\ &\quad\lesssim \sum_{k\ge k_0} 2^{-ks}[g(x) + g(y)] \lesssim [d(x, y)]^s[g(x) + g(y)]. \end{aligned} \tag{3.24}$$

Therefore, from (1.5) and $g \in \mathcal{M}^q_p(\mathcal{X})$, we deduce that $f \in H\dot{M}^s_{p,q}(\mathcal{X})$ and

$$\|f\|_{H\dot{M}^s_{p,q}(\mathcal{X})} \lesssim \|g\|_{\mathcal{M}^q_p(\mathcal{X})} \approx \|f\|_{\mathcal{A}\dot{F}^s_{p,q,\infty}(\mathcal{X})},$$

which implies the desired conclusion. Thus, to complete the proof, we only need to show that, for every $f \in \mathcal{A}\dot{F}^s_{p,q,\infty}(\mathcal{X})$, there exists a locally integrable function $\widetilde{f}$ which coincides with f in $(\mathcal{G}(1, 2))'$. For $p \in (1,\infty)$, the proof is similar to that of [16, Theorem 1.1], the details being omitted. We now assume that $p \in (n/(n + s), 1]$. Let $x, y \in \mathcal{X}$. We pick $k_0 \in \mathbb{Z}$ such that $2^{-k_0-1} < d(x, y) \le 2^{-k_0}$. If $k > k_0$, then, by the same reasoning as in the proof of (3.24), using Corollary 3.6 with $\sigma = 0$ and Lemma 3.7, we see that

$$\begin{aligned} &|S_k(f)(x) - S_k(f)(y)| \\ &\quad\lesssim \sum_{j=k_0}^{k-1} [|S_{j+1}(f)(x) - S_j(f)(x)| + |S_{j+1}(f)(y) - S_j(f)(y)|] \\ &\qquad + |S_{k_0}(f)(x) - S_{k_0}(f)(y)| \\ &\quad\lesssim \sum_{j=k_0}^{\infty} 2^{-js}[g(x) + g(y)] + 2^{-k_0 s}g(x) \lesssim 2^{-k_0 s}[g(x) + g(y)] \\ &\quad\approx [d(x, y)]^s[g(x) + g(y)]. \end{aligned} \tag{3.25}$$

If $k \le k_0$, then, by Corollary 3.6 with $\sigma = s$, there exists a positive constant $\widetilde{C}$ such that

$$\widetilde{C}2^{-ks}[d(x, y)]^{-s}[S_k(x, \cdot) - S_k(y, \cdot)] \in \mathcal{A}_k(x).$$

From this, we deduce that

$$2^{-ks}[d(x, y)]^{-s}\Big|\int_{\mathcal{X}} [S_k(x, z) - S_k(y, z)]f(z)\, d\mu(z)\Big| \lesssim 2^{-ks}g(x)$$

and hence

$$
\begin{aligned}
(3.26)\qquad &|S_k(f)(x)-S_k(f)(y)| \\
&= 2^{ks}[d(x,y)]^s\Big\{2^{-ks}[d(x,y)]^{-s}\Big|\int\limits_{\mathcal{X}}[S_k(x,z)-S_k(y,z)]f(z)\,d\mu(z)\Big|\Big\} \\
&\lesssim [d(x,y)]^s g(x) \lesssim [d(x,y)]^s[g(x)+g(y)].
\end{aligned}
$$

Thus, by (3.25) and (3.26), g is an s-Hajłasz gradient of $S_k(f)$ for all $k\in\mathbb{Z}$.

Fix a bounded set $B\subset\mathcal{X}$. By [16, Lemma 4.1], for any $k\in\mathbb{Z}$ there exists $C_k\in\mathbb{R}$ such that $S_k(f)-C_k\in L^{p^*}(B)$ and

(3.27)

$$
\bigg[\frac{1}{\mu(B)}\int\limits_B |S_k(f)(x)-C_k|^{p^*}\,d\mu(x)\bigg]^{1/p^*} \lesssim r_B\bigg\{\frac{1}{\mu(2B)}\int\limits_{2B}[g(x)]^p\,d\mu(x)\bigg\}^{1/p},
$$

where $p^*:=np/(n-p)>1$, since $p>n/(n+s)$ and $s\in(0,1]$. From the weak compactness of $L^{p^*}(B)$, it follows that $\{S_k(f)-C_k\}_{k\in\mathbb{Z}}$ has a subsequence, denoted by $\{S_k(f)-C_k\}_{k\in\mathbb{Z}}$ again, which converges weakly in $L^{p^*}(B)$ and hence almost everywhere in B to a certain function $\widetilde{f}^B\in L^{p^*}(B)$. Moreover, notice that, for all $x\in\mathcal{X}$, $k\in\mathbb{Z}$ and $i\in\mathbb{N}$,

$$
\begin{aligned}
(3.28)\qquad |S_k(f)(x)-S_{k+i}(f)(x)| &\le \sum_{j=0}^{i-1}|S_{k+j}(f)(x)-S_{k+j+1}(f)(x)| \\
&\lesssim 2^{-ks}g(x).
\end{aligned}
$$

Thus, as $g\in\mathcal{M}_p^q(\mathcal{X})$, we see that $S_k(f)-S_{k'}(f)\in\mathcal{M}_p^q(\mathcal{X})$ for all $k,k'\in\mathbb{Z}$. By the definition of $\mathcal{M}_p^q(\mathcal{X})$, we further know that $S_k(f)-S_{k'}(f)\in L^p(B)$. On the other hand, from the Hölder inequality, (3.28) and (3.27), we deduce that, for all $k,k'\in\mathbb{Z}_+$,

$$
\begin{aligned}
|C_k-C_{k'}| &= \frac{1}{\mu(B)}\int\limits_B |C_k-C_{k'}|\,d\mu(x) \\
&\le \frac{1}{\mu(B)}\int\limits_B |S_k(f)(x)-C_k-S_{k'}(f)(x)+C_{k'}|\,d\mu(x) \\
&\quad+\frac{1}{\mu(B)}\int\limits_B |S_k(f)(x)-S_{k'}(f)(x)|\,d\mu(x) \\
&\le \frac{1}{[\mu(B)]^{1/p^*}}[\|S_k(f)-C_k\|_{L^{p^*}(B)}+\|S_{k'}(f)-C_{k'}\|_{L^{p^*}(B)}] \\
&\quad+\frac{1}{[\mu(B)]^{1/p}}\|S_k(f)-S_{k'}(f)\|_{L^p(B)} \\
&\lesssim r_B\bigg\{\frac{1}{\mu(2B)}\int\limits_{2B}[g(x)]^p\,d\mu(x)\bigg\}^{1/p}+2^{-\min(ks,k's)}\frac{1}{[\mu(B)]^{1/p}}\|g\|_{L^p(B)} \\
&\lesssim (r_B+1)[\mu(B)]^{-1/q}\|g\|_{\mathcal{M}_p^q(\mathcal{X})}.
\end{aligned}
$$

Thus, $|C_k - C_{k'}|$ is dominated by a positive number depending on B but not on $k, k' \in \mathbb{Z}_+$. This implies that we can choose a subsequence $\{C_{k_j}\}_{j\in\mathbb{N}}$ that converges to a positive constant $\widetilde{C}^B$, which depends on B, as $j \to \infty$. Since $S_{k_j}(f) - C_{k_j}$ converges almost everywhere in B to $\widetilde{f}^B$ as $j \to \infty$, and $S_k(f)$ converges to f almost everywhere as $k \to \infty$, it follows that f coincides with $\widetilde{f}^B + \widetilde{C}^B$ almost everywhere in B, and hence in $(\mathcal{G}(1,2) \cap \Phi(B))'$, where $\Phi(B)$ is the set of all functions on $\mathcal{X}$ with supports in B. Write $f^B := \widetilde{f}^B + \widetilde{C}^B$. Then $f = f^B$ in $(\mathcal{G}(1,2) \cap \Phi(B))'$ and, since $\widetilde{f}^B \in L^{p^*}(B)$, f^B is also a locally integrable function.

We still need to show that there exists $\widetilde{f} \in L^1_{\mathrm{loc}}(\mathcal{X})$ such that

$$\langle f, \psi\rangle = \langle \widetilde{f}, \psi\rangle$$

for all $\psi \in \mathcal{G}(1,2)$. To this end, by Lemma 3.4, choose a partition of unity, $\{\phi_j\}_j \subset \mathcal{G}(1,2)$, on $\mathcal{X}$ and a sequence $\{B_j\}_j$ of open balls, with the finite intersection property, such that $\mu(\mathcal{X} \setminus (\bigcup_j B_j)) = 0$, $\operatorname{supp}\phi_j \subset B_j$, ϕ_j is non-negative and $\sum_j \phi_j(x) = 1$ for almost every $x \in \mathcal{X}$.

Let f^{B_j} be the locally integrable representation of f in $(\mathcal{G}(1,2) \cap \Phi(B_j))'$ obtained in the previous way, and define $\widetilde{f} := f^{B_j}$ pointwise on B_j for all j. Notice that, by the construction of f^{B_j}, for almost every $x \in B_i \cap B_j$,

$$f^{B_i}(x) = f(x) = f^{B_j}(x).$$

Thus, $\widetilde{f}$ is well defined. Moreover, $\sum_j \psi\phi_j$ converges in $\mathcal{G}(1,2)$ for all ψ in $\mathcal{G}(1,2)$. Indeed, for any $\varepsilon \in (0,\infty)$, by the construction of $\{B_j\}_j$ and ϕ_j in $\mathcal{G}(1,2)$, in particular the finite intersection property, there exists $L \in \mathbb{N}$ such that $B_j \cap B(x_1, 1/\varepsilon) = \emptyset$ for all $j \geq L$. It follows that, for $x \in B(x_1, 1/\varepsilon)$,

$$\sum_{j\geq L} |\psi(x)\phi_j(x)| = 0.$$

For $x \notin B(x_1, 1/\varepsilon)$, since $\sum_{j\geq L} \phi_j \leq 1$ and $\psi \in \mathcal{G}(1,2)$, we see that

$$\begin{aligned}
\sum_{j\geq L} |\psi(x)\phi_j(x)| &\lesssim \frac{1}{[V_1(x_1) + V(x_1,x)]^2} \left[\frac{1}{1 + d(x_1,x)}\right]^4 \\
&\lesssim \frac{1}{V_1(x_1)} \frac{1}{V_1(x_1) + V(x_1,x)} \left(\frac{1}{1 + 1/\varepsilon}\right)^2 \left[\frac{1}{1 + d(x_1,x)}\right]^2 \\
&\lesssim \varepsilon^2 \frac{1}{V_1(x_1) + V(x_1,x)} \left[\frac{1}{1 + d(x_1,x)}\right]^2.
\end{aligned}$$

Therefore, for all $x \in \mathcal{X}$,

$$\sum_{j\geq L} |\psi(x)\phi_j(x)| \lesssim \varepsilon^2 \frac{1}{V_1(x_1) + V(x_1,x)} \left[\frac{1}{1 + d(x_1,x)}\right]^2.$$

Moreover, for all $x,y\in\mathcal{X}$ such that $d(x,y)\le[1+d(x_1,x)]/2$, we have

$$\sum_{j\ge L}|\psi(x)\phi_j(x)-\psi(y)\phi_j(y)| \lesssim \varepsilon^2\left[\frac{d(x,y)}{1+d(x_1,x)}\right]\frac{1}{V_1(x_1)+V(x_1,x)}\left[\frac{1}{1+d(x_1,x)}\right]^2.$$

Indeed, if $x,y\in B(x_1,1/\varepsilon)$, then

$$\sum_{j\ge L}|\psi(x)\phi_j(x)-\psi(y)\phi_j(y)|=0;$$

if $y\in B(x_1,1/\varepsilon)$ but $x\notin B(x_1,1/\varepsilon)$, then $\phi_j(y)=0$ for all $j\ge L$, and hence

$$\begin{aligned}\sum_{j\ge L}&|\psi(x)\phi_j(x)-\psi(y)\phi_j(y)|\\ &=\sum_{j\ge L}|\psi(x)\phi_j(x)-\psi(x)\phi_j(y)|=\sum_{j\ge L}|\psi(x)|\,|\phi_j(x)-\phi_j(y)|\\ &\lesssim \frac{d(x,y)}{1+d(x_1,x)}\,\frac{1}{[V_1(x_1)+V(x_1,x)]^2}\left[\frac{1}{1+d(x_1,x)}\right]^4\\ &\lesssim \varepsilon^2\frac{d(x,y)}{1+d(x_1,x)}\,\frac{1}{V_1(x_1)+V(x_1,x)}\left[\frac{1}{1+d(x_1,x)}\right]^2;\end{aligned}$$

if $x,y\notin B(x_1,1/\varepsilon)$, by the finite intersection property of $\{B_j\}_{j\in\mathbb{N}}$ and an argument similar to that used above, we also have the desired inequality. Thus, $\sum_{j\in\mathbb{N}}\psi\phi_j$ converges in $\mathcal{G}(1,2)$, and therefore

$$\begin{aligned}\langle f,\psi\rangle&=\left\langle f,\psi\sum_{j\in\mathbb{N}}\phi_j\right\rangle=\sum_{j\in\mathbb{N}}\langle f,\psi\phi_j\rangle\\ &=\sum_{j\in\mathbb{N}}\langle f^{B_j},\psi\phi_j\rangle=\left\langle \widetilde{f},\sum_{j=1}^{\infty}\psi\phi_j\right\rangle=\langle\widetilde{f},\psi\rangle. \quad\blacksquare\end{aligned}$$

We remark that Theorem 3.3 for $p=q$ goes back to [16, Theorem 5.2]. However, the proof of Theorem 3.3 is different from that in [16]: it needs several localized arguments, in which a partition of unity on spaces of homogeneous type plays a key role (see Lemma 3.4).

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