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Unconditionality of orthogonal spline systems in H^1

by

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Abstract. We give a simple geometric characterization of knot sequences for which the corresponding orthonormal spline system of arbitrary order k is an unconditional basis in the atomic Hardy space $H^1[0,1]$.

1. Introduction. This paper belongs to a series of papers studying properties of orthonormal spline systems with arbitrary knots. The detailed study of such systems started in the 1960's with Z. Ciesielski's papers [2, 3] on properties of the Franklin system, which is an orthonormal system consisting of continuous piecewise linear functions with dyadic knots. Next, the 1972 results by J. Domsta [11] made it possible to extend the study to orthonormal spline systems of higher order—and higher smoothness—with dyadic knots. These systems occurred to be bases or unconditional bases in several function spaces like $L^p[0,1]$, $1 \le p < \infty$, $C[0,1]$, $H^p[0,1]$, $0 < p \le 1$, Sobolev spaces $W^{p,k}[0,1]$; they also give characterizations of BMO and VMO spaces, and various spaces of smooth functions (Hölder functions, Zygmund class, Besov spaces). One should mention here the work of Z. Ciesielski, J. Domsta, S. V. Bochkarev, P. Wojtaszczyk, S.-Y. A. Chang, P. Sjölin, J.-O. Strömberg (for more detailed references see e.g. [13], [15], [16]). Nowadays, results of this kind are known for wavelets.

The extension of these results to orthonormal spline systems with arbitrary knots began with the case of piecewise linear systems, i.e. general Franklin systems, or orthonormal spline systems of order 2. This was possible due to precise estimates of the inverse of the Gram matrix of piecewise linear B-spline bases with arbitrary knots, as presented in [19]. First results in this direction were obtained in [5] and [13]. We would like to mention here two results by G. G. Gevorkyan and A. Kamont. First, each general Franklin system is an unconditional basis in $L^p[0,1]$ for $1 < p < \infty$ (see [14]).

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Second, there is a simple geometric characterization of knot sequences for which the corresponding general Franklin system is a basis or an unconditional basis in $H^1[0,1]$ (see [15]). For both of these results, an essential tool is the association of a so called characteristic interval to each general Franklin function f_n.

The case of splines of higher order is much more difficult. The existence of a uniform bound for L^∞-norms of orthogonal projections on spline spaces of order k with arbitrary order (i.e. a bound depending on k, but independent of the sequence of knots)—was a long-standing problem known as C. de Boor's conjecture (1973) (cf. [8]). The case of $k = 2$ was settled earlier by Z. Ciesielski [2], the cases $k = 3, 4$ were solved by C. de Boor himself (1968, 1981) in [7, 9], but the positive answer in the general case was given by A. Yu. Shadrin [22] only in 2001. A much simplified and shorter proof was recently obtained by M. v. Golitschek (2014) in [24]. An immediate consequence of A.Yu. Shadrin's result is that if a sequence of knots is dense in $[0,1]$, then the corresponding orthonormal spline system of order k is a basis in $L^p[0,1]$, $1 \le p < \infty$, and in $C[0,1]$. Moreover, Z. Ciesielski [4] obtained several consequences of Shadrin's result, one of them being an estimate for the inverse of the B-spline Gram matrix. Using this estimate, G. G. Gevorkyan and A. Kamont [16] extended a part of their result from [15] to orthonormal spline systems of arbitrary order and obtained a characterization of knot sequences for which the corresponding orthonormal spline system of order k is a basis in $H^1[0,1]$. Further extension required more precise estimates for the inverse of B-spline Gram matrices, of the type known for the piecewise linear case. Such estimates were obtained recently by M. Passenbrunner and A. Yu. Shadrin [21]. Using these estimates, M. Passenbrunner [20] proved that for each sequence of knots, the corresponding orthonormal spline system of order k is an unconditional basis in $L^p[0,1]$, $1 < p < \infty$.

The main result of the present paper is a characterization of those knot sequences for which the corresponding orthonormal spline system of order k is an unconditional basis in $H^1[0,1]$.

The paper is organized as follows. In Section 2 we give the necessary definitions and we formulate the main result of this paper, Theorem 2.4. In Sections 3 and 4 we recall or prove several facts needed to establish our results. In particular, in Section 4 we recall precise pointwise estimates for orthonormal spline systems with arbitrary knots, the associated characteristic intervals and some combinatorial facts for characteristic intervals. Then Section 5 contains some auxiliary results, and the proof of Theorem 2.4 is given in Section 6.

The results contained in this paper were obtained independently by two teams, G. Gevorkyan & K. Keryan and A. Kamont & M. Passenbrunner at the same time, so we have decided to produce a joint paper.

2. Definitions and the main result. Let $k \geq 2$ be an integer. In this work, we are concerned with orthonormal spline systems of order k with arbitrary partitions. We let $\mathcal{T} = (t_n)_{n=2}^{\infty}$ be a dense sequence of points in the open unit interval $(0,1)$ such that each point occurs at most k times. Moreover, define $t_0 := 0$ and $t_1 := 1$. Such point sequences are called *k-admissible*. For $-k+2 \leq n \leq 1$, let $\mathcal{S}_n^{(k)}$ be the space of polynomials of order $n+k-1$ (or degree $n+k-2$) on the interval $[0,1]$ and $(f_n^{(k)})_{n=-k+2}^{1}$ be the collection of orthonormal polynomials in $L^2 \equiv L^2[0,1]$ such that the degree of $f_n^{(k)}$ is $n+k-2$. For $n \geq 2$, let $\mathcal{T}_n$ be the ordered sequence of points consisting of the grid points $(t_j)_{j=0}^{n}$ repeated according to their multiplicities and where the knots 0 and 1 have multiplicity k, i.e.,

$$\begin{aligned}\mathcal{T}_n = (0 = \tau_{n,1} = \cdots = \tau_{n,k} < \tau_{n,k+1} \\ \leq \cdots \leq \tau_{n,n+k-1} < \tau_{n,n+k} = \cdots = \tau_{n,n+2k-1} = 1).\end{aligned}$$

In that case, we also define $\mathcal{S}_n^{(k)}$ to be the space of polynomial splines of order k with grid points $\mathcal{T}_n$. For each $n \geq 2$, the space $\mathcal{S}_{n-1}^{(k)}$ has codimension 1 in $\mathcal{S}_n^{(k)}$, and therefore there exists $f_n^{(k)} \in \mathcal{S}_n^{(k)}$ orthonormal to $\mathcal{S}_{n-1}^{(k)}$. Observe that $f_n^{(k)}$ is unique up to sign.

DEFINITION 2.1. The system of functions $(f_n^{(k)})_{n=-k+2}^{\infty}$ is called the *orthonormal spline system of order k corresponding to the sequence $(t_n)_{n=0}^{\infty}$*.

We will frequently omit the parameter k and write f_n and $\mathcal{S}_n$ instead of $f_n^{(k)}$ and $\mathcal{S}_n^{(k)}$, respectively.

Note that the case $k=2$ corresponds to orthonormal systems of piecewise linear functions, i.e. general Franklin systems.

We are interested in characterizing sequences $\mathcal{T}$ of knots such that the system $(f_n^{(k)})_{n=-k+2}^{\infty}$ is an unconditional basis in $H^1 = H^1[0,1]$. By $H^1 = H^1[0,1]$ we mean the atomic Hardy space on $[0,1]$ (see [6]). A function $a : [0,1] \to \mathbb{R}$ is called an *atom* if either $a \equiv 1$ or there exists an interval Γ such that:

(i) $\operatorname{supp} a \subset \Gamma$,
(ii) $\|a\|_\infty \leq |\Gamma|^{-1}$,
(iii) $\int_0^1 a(x)\,dx = \int_\Gamma a(x)\,dx = 0$.

Then, by definition, H^1 consists of all functions f with a representation

$$f = \sum_{n=1}^{\infty} c_n a_n$$

for some atoms $(a_n)_{n=1}^{\infty}$ and real scalars $(c_n)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} |c_n| < \infty$. The space H^1 becomes a Banach space under the norm

$$\|f\|_{H^1} := \inf \sum_{n=1}^{\infty} |c_n|,$$

where the inf is taken over all atomic representations $\sum c_n a_n$ of f.

To formulate our result, we need to introduce some regularity conditions for a sequence $\mathcal{T}$.

For $n \geq 2$, $\ell \leq k$ and $k - \ell + 1 \leq i \leq n + k - 1$, we define $D_{n,i}^{(\ell)}$ to be the interval $[\tau_{n,i}, \tau_{n,i+\ell}]$.

Definition 2.2. Let $\ell \leq k$ and $(t_n)_{n=0}^{\infty}$ be an ℓ-admissible (and therefore k-admissible) point sequence. This sequence is called *ℓ-regular with parameter $\gamma \geq 1$* if

$$\frac{|D_{n,i}^{(\ell)}|}{\gamma} \leq |D_{n,i+1}^{(\ell)}| \leq \gamma |D_{n,i}^{(\ell)}|, \qquad n \geq 2,\ k - \ell + 1 \leq i \leq n + k - 2.$$

In other words, (t_n) is ℓ-regular if there is a uniform finite bound $\gamma \geq 1$ such that for all n, the ratios of the lengths of neighboring supports of B-spline functions (cf. Section 3.2) of order ℓ in the grid $\mathcal{T}_n$ are bounded by γ.

The following characterization for $(f_n^{(k)})$ to be a basis in H^1 is the main result of [16]:

Theorem 2.3 ([16]). *Let $k \geq 1$ and let (t_n) be a k-admissible sequence of knots in $[0, 1]$ with the corresponding orthonormal spline system $(f_n^{(k)})$ of order k. Then $(f_n^{(k)})$ is a basis in H^1 if and only if (t_n) is k-regular with some parameter $\gamma \geq 1$,*

In this paper, we prove a characterization for $(f_n^{(k)})$ to be an unconditional basis in H^1. The main result of our paper is the following:

Theorem 2.4. *Let (t_n) be a k-admissible sequence of points. Then the corresponding orthonormal spline system $(f_n^{(k)})$ is an unconditional basis in H^1 if and only if (t_n) is $(k-1)$-regular with some parameter $\gamma \geq 1$.*

Let us note that in case $k = 2$, i.e. for general Franklin systems, both Theorems 2.3 and 2.4 were obtained by G. G. Gevorkyan and A. Kamont [15]. (In the terminology of the current paper, strong regularity from [15] is 1-regularity, and strong regularity for pairs from [15] is 2-regularity.)

The proof of Theorem 2.4 follows the same general scheme as the proof of Theorem 2.2 in [15]. In Section 5 we introduce four conditions (A)–(D) for series with respect to orthonormal spline systems of order k corresponding to a k-admissible sequence of points. Then we study relations between these conditions under various regularity assumptions on the underlying sequence of points. Finally, we prove Theorem 2.4 in Section 6.

3. Preliminaries. The parameter $k \geq 2$ will always be used for the order of the underlying polynomials or splines. We use the notation $A(t) \sim B(t)$ to indicate the existence of two constants $c_1, c_2 > 0$ such that $c_1 B(t) \leq A(t) \leq c_2 B(t)$ for all t, where t denotes all implicit and explicit dependencies that the expressions A and B might have. If the constants c_1, c_2 depend on an additional parameter p, we write $A(t) \sim_p B(t)$. Correspondingly, we use the symbols $\lesssim, \gtrsim, \lesssim_p, \gtrsim_p$. For a subset E of the real line, we denote by $|E|$ its Lebesgue measure and by $\mathbb{1}_E$ the characteristic function of E. If $f : \Omega \to \mathbb{R}$ is a real valued function and λ is a real parameter, we write $[f > \lambda] := \{\omega \in \Omega : f(\omega) > \lambda\}$.

3.1. Properties of regular sequences of points. The following lemma describes geometric decay of intervals in regular sequences (recall the notation $D_{n,i}^{(\ell)} = [\tau_{n,i}, \tau_{n,i+\ell}]$):

LEMMA 3.1. *Let (t_n) be a k-admissible sequence of points that is ℓ-regular for some $1 \leq \ell \leq k$ with parameter γ and let $D_{n_1,i_1}^{(\ell)} \supset \cdots \supset D_{n_{2\ell},i_{2\ell}}^{(\ell)}$ be a strictly decreasing sequence of sets defined above. Then*

$$|D_{n_{2\ell},i_{2\ell}}^{(\ell)}| \leq \frac{\gamma^\ell}{1+\gamma^\ell} |D_{n_1,i_1}^{(\ell)}|.$$

Proof. We set $V_j := D_{n_j,i_j}^{(\ell)}$ for $1 \leq j \leq 2\ell$. Then, by definition, V_1 contains $\ell+1$ grid points from $\mathcal{T}_{n_1}$ and at least 3ℓ grid points of $\mathcal{T}_{n_{2\ell}}$. As a consequence, there exists an interval $D_{n_{2\ell},m}^{(\ell)}$ for some m that satisfies

$$\operatorname{int}(D_{n_{2\ell},m}^{(\ell)} \cap V_{2\ell}) = \emptyset, \qquad D_{n_{2\ell},m}^{(\ell)} \subset V_1, \qquad \operatorname{dist}(D_{n_{2\ell},m}^{(\ell)}, V_{2\ell}) = 0.$$

The ℓ-regularity of (t_n) now implies

$$|V_{2\ell}| \leq \gamma^\ell |D_{n_{2\ell},m}^{(\ell)}| \leq \gamma^\ell (|V_1| - |V_{2\ell}|),$$

i.e., $|V_{2\ell}| \leq \frac{\gamma^\ell}{1+\gamma^\ell} |V_1|$, which proves the assertion of the lemma. ∎

3.2. Properties of B-spline functions. We define $(N_{n,i}^{(k)})_{i=1}^{n+k-1}$ to be the collection of B-spline functions of order k corresponding to the partition $\mathcal{T}_n$. Those functions are normalized so that they form a partition of unity, i.e., $\sum_{i=1}^{n+k-1} N_{n,i}^{(k)}(x) = 1$ for all $x \in [0,1]$. Associated to this basis, there exists a biorthogonal basis of $\mathcal{S}_n$, denoted by $(N_{n,i}^{(k)*})_{i=1}^{n+k-1}$. If the parameters k and n are clear from the context, we also denote those functions by $(N_i)_{i=1}^{n+k-1}$ and $(N_i^*)_{i=1}^{n+k-1}$, respectively.

We will need the following well known formula for the derivative of a linear combination of B-spline functions: if $g = \sum_{j=1}^{n+k-1} a_j N_{n,j}^{(k)}$, then

$$g' = (k-1) \sum_{j=2}^{n+k-1} (a_j - a_{j-1}) \frac{N_{n,j}^{(k-1)}}{|D_{n,j}^{(k-1)}|}. \tag{3.1}$$

We now recall an elementary property of polynomials.

Proposition 3.2. *Let $0 < \rho < 1$. Let I be an interval and $A \subset I$ be a subset of I with $|A| \geq \rho |I|$. Then, for every polynomial Q of order k on I,*

$$\max_{t\in I} |Q(t)| \lesssim_{\rho,k} \sup_{t\in A} |Q(t)| \quad \textit{and} \quad \int_I |Q(t)|\,dt \lesssim_{\rho,k} \int_A |Q(t)|\,dt.$$

We recall a few important results on B-splines (N_i) and their dual functions (N_i^*).

Proposition 3.3. *Let $1 \leq p \leq \infty$ and $g = \sum_{j=1}^{n+k-1} a_j N_j$, where $(N_i)_{i=1}^{n+k-1}$ are the B-splines of order k corresponding to the partition $\mathcal{T}_n$. Then*

$$|a_j| \lesssim_k |J_j|^{-1/p} \|g\|_{L^p(J_j)}, \qquad 1 \leq j \leq n+k-1, \tag{3.2}$$

where J_j is a subinterval $[\tau_{n,i}, \tau_{n,i+1}]$ of $[\tau_{n,j}, \tau_{n,j+k}]$ of maximal length. Furthermore,

$$\|g\|_p \sim_k \Big(\sum_{j=1}^{n+k-1} |a_j|^p |D_{n,j}^{(k)}| \Big)^{1/p} = \|(a_j |D_{n,j}^{(k)}|^{1/p})_{j=1}^{n+k-1}\|_{\ell^p}. \tag{3.3}$$

Moreover, if $h = \sum_{j=1}^{n+k-1} b_j N_j^*$, then

$$\|h\|_p \lesssim_k \Big(\sum_{j=1}^{n+k-1} |b_j|^p |D_{n,j}^{(k)}|^{1-p} \Big)^{1/p} = \|(b_j |D_{n,j}^{(k)}|^{1/p-1})_{j=1}^{n+k-1}\|_{\ell^p}. \tag{3.4}$$

The inequalites (3.2) and (3.3) are Lemmas 4.1 and 4.2 in [10, Chapter 5], respectively. Inequality (3.4) is a consequence of Shadrin's theorem [22] that the orthogonal projection onto $\mathcal{S}_n^{(k)}$ is bounded on L^∞ independently of n and $\mathcal{T}_n$. For a deduction of (3.4) from this result, see [4, Property P.7].

We next consider estimates for the inverse $(b_{ij})_{i,j=1}^{n+k-1}$ of the Gram matrix $(\langle N_i, N_j\rangle)_{i,j=1}^{n+k-1}$. Later, we will need a special property of this matrix, of being *checkerboard*, i.e.,

$$(-1)^{i+j} b_{ij} \geq 0 \quad \text{for all } i, j. \tag{3.5}$$

This is a simple consequence of the total positivity of the Gram matrix (cf. [7, 18]). Moreover, we need the lower estimate for $b_{i,i}$,

$$|D_{n,i}^{(k)}|^{-1} \lesssim_k b_{i,i}. \tag{3.6}$$

This is a consequence of the total positivity of the B-spline Gram matrix, the L^2-stability of B-splines and the following lemma:

Lemma 3.4 ([20]). *Let $C = (c_{ij})_{i,j=1}^n$ be a symmetric positive definite matrix. Then for $(d_{ij})_{i,j=1}^n = C^{-1}$ we have*

$$c_{ii}^{-1} \le d_{ii}, \qquad 1 \le i \le n.$$

3.3. Some results for orthonormal spline systems. We now recall two results concerning orthonormal spline series.

Theorem 3.5 ([21]). *Let $(f_n)_{n=-k+2}^\infty$ be the orthonormal spline system of order k corresponding to an arbitrary k-admissible point sequence $(t_n)_{n=0}^\infty$. Then, for every $f \in L^1 \equiv L^1[0,1]$, the series $\sum_{n=-k+2}^\infty \langle f, f_n\rangle f_n$ converges to f almost everywhere.*

Let $f \in L^p \equiv L^p[0,1]$ for some $1 \le p < \infty$. Since the orthonormal spline system $(f_n)_{n\ge -k+2}$ is a basis in L^p, we can write $f = \sum_{n=-k+2}^\infty a_n f_n$. Based on this expansion, we define the *square function* $Pf := (\sum_{n=-k+2}^\infty |a_n f_n|^2)^{1/2}$ and the *maximal function* $Sf := \sup_m |\sum_{n\le m} a_n f_n|$. Moreover, given a measurable function g, we denote by $\mathcal{M}g$ the *Hardy–Littlewood maximal function* of g defined as

$$\mathcal{M}g(x) := \sup_{I \ni x} |I|^{-1} \int_I |g(t)|\, dt,$$

where the supremum is taken over all intervals I containing x. The connection between the maximal function Sf and the Hardy–Littlewood maximal function is given by the following result:

Theorem 3.6 ([21]). *If $f \in L^1$, then*

$$Sf(t) \lesssim_k \mathcal{M}f(t), \qquad t \in [0,1].$$

4. Properties of orthogonal spline functions and characteristic intervals

4.1. Estimates for f_n. This section concerns the calculation and estimation of one explicit orthonormal spline function $f_n^{(k)}$ for fixed $k \in \mathbb{N}$ and $n \ge 2$ induced by a k-admissible sequence $(t_n)_{n=0}^\infty$. Most of the results are taken from [20].

Here, we change our notation slightly. We fix n and let i_0 with $k+1 \le i_0 \le n+k-1$ be such that $\mathcal{T}_{n-1}$ equals $\mathcal{T}_n$ with the point τ_{i_0} removed. In the points of the partition $\mathcal{T}_n$, we omit the parameter n, and thus $\mathcal{T}_n$ is given by

$$\mathcal{T}_n = (0 = \tau_1 = \cdots = \tau_k < \tau_{k+1} \le \cdots \le \tau_{i_0} \\ \le \cdots \le \tau_{n+k-1} < \tau_{n+k} = \cdots = \tau_{n+2k-1} = 1).$$

We denote by $(N_i : 1 \le i \le n+k-1)$ the B-spline functions corresponding to $\mathcal{T}_n$.

An (unnormalized) orthogonal spline function $g \in \mathcal{S}_n^{(k)}$ that is orthogonal to $\mathcal{S}_{n-1}^{(k)}$, as calculated in [20], is given by

$$g = \sum_{j=i_0-k}^{i_0} \alpha_j N_j^* = \sum_{j=i_0-k}^{i_0} \sum_{\ell=1}^{n+k-1} \alpha_j b_{j\ell} N_\ell, \tag{4.1}$$

where $(b_{j\ell})_{j,\ell=1}^{n+k-1}$ is the inverse of the Gram matrix $(\langle N_j, N_\ell \rangle)_{j,\ell=1}^{n+k-1}$ and

$$\alpha_j = (-1)^{j-i_0+k} \Bigg(\prod_{\ell=i_0-k+1}^{j-1} \frac{\tau_{i_0} - \tau_\ell}{\tau_{\ell+k} - \tau_\ell} \Bigg) \Bigg(\prod_{\ell=j+1}^{i_0-1} \frac{\tau_{\ell+k} - \tau_{i_0}}{\tau_{\ell+k} - \tau_\ell} \Bigg), \quad i_0-k \le j \le i_0. \tag{4.2}$$

We remark that the sequence (α_j) alternates in sign, and since the matrix $(b_{j\ell})_{j,\ell=1}^{n+k-1}$ is checkerboard, the B-spline coefficients of g, that is,

$$w_\ell := \sum_{j=i_0-k}^{i_0} \alpha_j b_{j\ell}, \qquad 1 \le \ell \le n+k-1, \tag{4.3}$$

satisfy

$$\Big| \sum_{j=i_0-k}^{i_0} \alpha_j b_{j\ell} \Big| = \sum_{j=i_0-k}^{i_0} |\alpha_j b_{j\ell}|, \qquad 1 \le j \le n+k-1. \tag{4.4}$$

In the definition below, we assign to each orthonormal spline function a characteristic interval that is a grid point interval $[\tau_i, \tau_{i+1}]$ and lies close to the newly inserted point τ_{i_0}. The choice of this interval is crucial for proving important properties of the system $(f_n^{(k)})_{n=-k+2}^\infty$. This approach has its origins in [14], where it is proved that general Franklin systems are unconditional bases in L^p, $1 < p < \infty$.

Definition 4.1. Let $\mathcal{T}_n, \mathcal{T}_{n-1}$ be as above and τ_{i_0} be the new point in $\mathcal{T}_n$ that is not present in $\mathcal{T}_{n-1}$. We define the *characteristic interval J_n corresponding to the pair* $(\mathcal{T}_n, \mathcal{T}_{n-1})$ as follows.

(1) Let

$$\Lambda^{(0)} := \Big\{ i_0 - k \le j \le i_0 : |[\tau_j, \tau_{j+k}]| \le 2 \min_{i_0-k\le\ell\le i_0} |[\tau_\ell, \tau_{\ell+k}]| \Big\}$$

be the set of all j for which the support of the B-spline function N_j is approximately minimal. Observe that $\Lambda^{(0)}$ is nonempty.

(2) Define

$$\Lambda^{(1)} := \Big\{ j \in \Lambda^{(0)} : |\alpha_j| = \max_{\ell \in \Lambda^{(0)}} |\alpha_\ell| \Big\}.$$

For any fixed index $j^{(0)} \in \Lambda^{(1)}$, set $J^{(0)} := [\tau_{j^{(0)}}, \tau_{j^{(0)}+k}]$.

(3) The interval $J^{(0)}$ can now be written as the union of k grid intervals

$$J^{(0)} = \bigcup_{\ell=0}^{k-1} [\tau_{j^{(0)}+\ell}, \tau_{j^{(0)}+\ell+1}] \quad \text{with } j^{(0)} \text{ as above.}$$

We define the *characteristic interval* J_n to be one of the above k intervals that has maximal length.

A few clarifying comments are in order. Roughly speaking, we first take the B-spline support $[\tau_j, \tau_{j+k}]$ including the new point τ_{i_0} with minimal length and then we choose as J_n the largest grid point interval in $[\tau_j, \tau_{j+k}]$. This definition guarantees the concentration of f_n on J_n in terms of the L^p-norm (cf. Lemma 4.3) and the exponential decay of f_n away from J_n (cf. Lemma 4.4), which are crucial for further investigations. An important ingredient in the proof of Lemma 4.3 is Proposition 3.3, which justifies why we choose the largest grid point interval as J_n. Further important properties of the collection (J_n) of characteristic intervals are that they form a nested family of sets and for a subsequence of decreasing characteristic intervals, their lengths decay geometrically (cf. Lemma 4.5).

Next we remark that the constant 2 in step (1) of Definition 4.1 could also be an arbitrary number $C > 1$, but $C = 1$ is not allowed. This is in contrast to the definition of characteristic intervals in [14] for piecewise linear orthogonal functions ($k = 2$), where precisely $C = 1$ is chosen, step (2) is omitted and $j^{(0)}$ is an arbitrary index in $\Lambda^{(0)}$.

At first glance, it might seem natural to carry over the same definition to arbitrary spline orders k, but at a certain point in the proof of Theorem 4.2 below, we estimate $\alpha_{j^{(0)}}$ by the constant $C-1$ from below, which has to be strictly greater than zero in order to establish (4.5). Since Theorem 4.2 is also used in the proofs of both Lemmas 4.3 and 4.4, this is the reason for a different definition of characteristic intervals, in particular for step (2) of Definition 4.1.

THEOREM 4.2 ([20]). *With the above definition* (4.3) *of w_ℓ for $1 \le \ell \le n+k-1$ and with $j^{(0)}$ given in Definition* 4.1,

$$|w_{j^{(0)}}| \gtrsim_k b_{j^{(0)},j^{(0)}}. \tag{4.5}$$

LEMMA 4.3 ([20]). *Let $\mathcal{T}_n$, $\mathcal{T}_{n-1}$ be as above and g be the function given in* (4.1). *Then $f_n = g/\|g\|_2$ is the L^2-normalized orthogonal spline function* *corresponding to* $(\mathcal{T}_n, \mathcal{T}_{n-1})$ *and*

$$\|f_n\|_{L^p(J_n)} \sim_k \|f_n\|_p \sim_k |J_n|^{1/p-1/2} \sim_k |J_n|^{1/2}\|g\|_p, \qquad 1 \le p \le \infty,$$

where J_n *is the characteristic interval associated to* $(\mathcal{T}_n, \mathcal{T}_{n-1})$.

We denote by $d_n(x)$ the number of points in $\mathcal{T}_n$ between x and J_n counting endpoints of J_n. Correspondingly, for an interval $V \subset [0,1]$, we denote by $d_n(V)$ the number of points in $\mathcal{T}_n$ between V and J_n counting endpoints of both J_n and V.

LEMMA 4.4 ([20]). *Let* $\mathcal{T}_n, \mathcal{T}_{n-1}$ *be as above,* $g = \sum_{j=1}^{n+k-1} w_j N_j$ *be the function in* (4.1) *with* $(w_j)_{j=1}^{n+k-1}$ *as in* (4.3), *and* $f_n = g/\|g\|_2$. *Then there exists a constant* $0 < q < 1$ *that depends only on* k *such that*

(4.6)

$$|w_j| \lesssim_k \frac{q^{d_n(\tau_j)}}{|J_n| + \operatorname{dist}(\operatorname{supp} N_j, J_n) + |D^k_{n,j}|} \quad \text{for all } 1 \le j \le n+k-1.$$

Moreover, if $x < \inf J_n$, *we have*

$$\|f_n\|_{L^p(0,x)} \lesssim_k \frac{q^{d_n(x)}|J_n|^{1/2}}{(|J_n| + \operatorname{dist}(x, J_n))^{1-1/p}}, \qquad 1 \le p \le \infty. \tag{4.7}$$

Similarly, for $x > \sup J_n$,

$$\|f_n\|_{L^p(x,1)} \lesssim_k \frac{q^{d_n(x)}|J_n|^{1/2}}{(|J_n| + \operatorname{dist}(x, J_n))^{1-1/p}}, \qquad 1 \le p \le \infty. \tag{4.8}$$

4.2. Combinatorics of characteristic intervals. Next, we recall a combinatorial result about the relative positions of different characteristic intervals:

LEMMA 4.5 ([20]). *Let* $x, y \in (t_n)_{n=0}^{\infty}$ *with* $x < y$. *Then there exists a constant* F_k *only depending on* k *such that*

$$N_0 := \operatorname{card}\{n : J_n \subseteq [x,y],\ |J_n| \ge |[x,y]|/2\} \le F_k,$$

where $\operatorname{card} E$ *denotes the cardinality of the set* E.

Similarly to [14] and [15], we need the following estimate involving characteristic intervals and orthonormal spline functions:

LEMMA 4.6. *Let* (t_n) *be a* k*-admissible point sequence in* $[0,1]$ *and let* $(f_n)_{n\ge -k+2}$ *be the corresponding orthonormal spline system of order* k. *Then, for each interval* $V = [\alpha, \beta] \subset [0,1]$,

$$\sum_{n:\, J_n \subset V} |J_n|^{1/2} \int_{V^c} |f_n(t)|\, dt \lesssim_k |V|.$$

Once we know the estimates for orthonormal spline functions as in Lemma 4.4 and the basic combinatorial result for their characteristic intervals, i.e. Lemma 4.5, this result follows by the same argument that was used in the proof of Lemma 4.6 in [14], so we skip its proof.

Instead of Lemma 3.4 of [15], we will use the following:

LEMMA 4.7. *Let $(t_n)_{n=0}^{\infty}$ be a k-admissible knot sequence that is $(k-1)$-regular, and let $\Delta = D_{m,i}^{(k-1)}$ for some m and i. For $\ell \geq 0$, let*

$$N(\Delta) := \{n : \operatorname{card}(\Delta \cap \mathcal{T}_n) = k,\ J_n \subset \Delta\},$$
$$M(\Delta, \ell) := \{n : d_n(\Delta) = \ell,\ \operatorname{card}(\Delta \cap \mathcal{T}_n) \geq k,\ |J_n \cap \Delta| = 0\},$$

where in both definitions we count the points in $\Delta \cap \mathcal{T}_n$ including multiplicities. Then

$$\frac{1}{|\Delta|} \sum_{n \in N(\Delta)} |J_n| \lesssim_k 1, \qquad \sum_{n \in M(\Delta,\ell)} \frac{|J_n|}{\operatorname{dist}(J_n, \Delta) + |\Delta|} \lesssim_{k,\gamma} (\ell+1)^2. \tag{4.9}$$

Proof. For every $n \in N(\Delta)$, there are only the $k-1$ possibilities $D_{m,i}^{(1)}, \ldots, D_{m,i+k-2}^{(1)}$ for J_n and by Lemma 4.5, each interval $D_{m,j}^{(1)}$, $j = i, \ldots, i+k-2$, occurs at most F_k times as a characteristic interval. This implies the first inequality in (4.9).

To prove the second, assume that each J_n, $n \in M(\Delta, \ell)$, lies to the right of Δ, since the other case is handled similarly. The argument is split into two parts depending on the value of ℓ, beginning with $\ell \leq k$. In that case, for $n \in M(\Delta, \ell)$, let $J_n^{1/2}$ be the unique interval determined by the conditions

$$\sup J_n^{1/2} = \sup J_n, \qquad |J_n^{1/2}| = |J_n|/2.$$

Since $d_n(\Delta) = \ell$ is constant, we group the intervals J_n into packets, where all intervals in one packet have the same left endpoint and maximal intervals from different packets are disjoint (up to possibly one point). By Lemma 4.5, each $t \in [0,1]$ belongs to at most F_k intervals $J_n^{1/2}$. The $(k-1)$-regularity and $\ell \leq k$ now imply $|J_n| \lesssim_{k,\gamma} |\Delta|$ and $\operatorname{dist}(\Delta, J_n) \lesssim_{k,\gamma} |\Delta|$ for $n \in M(\Delta, \ell)$, and thus every interval J_n with $n \in M(\Delta, \ell)$ is a subset of a fixed interval whose length is comparable to $|\Delta|$. Putting these things together, we obtain

$$\sum_{n \in M(\Delta,\ell)} \frac{|J_n|}{\operatorname{dist}(J_n, \Delta) + |\Delta|} \leq \frac{1}{|\Delta|} \sum_{n \in M(\Delta,\ell)} |J_n| = \frac{2}{|\Delta|} \sum_{n \in M(\Delta,\ell)} \int_{J_n^{1/2}} dx \lesssim_{k,\gamma} 1,$$

which completes the case of $\ell \leq k$.

Next, assume $\ell \geq k+1$ and define $(L_j)_{j=1}^{\infty}$ as the strictly decreasing sequence of all sets L that satisfy

$$L = D_{n,i}^{(k-1)} \quad \text{and} \quad \sup L = \sup \Delta$$

for some n and i. Moreover, set

$$M_j(\Delta,\ell) := \{n \in M(\Delta,\ell) : \operatorname{card}(L_j \cap \mathcal{T}_n) = k\},$$

i.e., L_j is a union of $k-1$ grid point intervals in the grid $\mathcal{T}_n$. Then, since $|\Delta| + \operatorname{dist}(J_n,\Delta) \gtrsim_\gamma |\Delta| + \operatorname{dist}(t,\Delta)$ for $t \in J_n^{1/2}$ by $(k-1)$-regularity,

$$\sum_{n\in M_j(\Delta,\ell)} \frac{|J_n|}{\operatorname{dist}(J_n,\Delta)+|\Delta|} \lesssim_{k,\gamma} \sum_{n\in M_j(\Delta,\ell)} \int_{J_n^{1/2}} \frac{1}{\operatorname{dist}(t,\Delta)+|\Delta|}\,dt.$$

If $n \in M_j(\Delta,\ell)$ we get, again due to $(k-1)$-regularity,

$$\inf J_n^{1/2} \ge \inf J_n \ge \gamma^{-k}|L_j| + \sup \Delta,$$

and

$$\sup J_n^{1/2} \le \inf J_n + |J_n| \le C_k\gamma^\ell |L_j| + \sup \Delta$$

for some constant C_k only depending on k. Combining this with Lemma 4.5, which implies that each point t belongs to at most F_k intervals $J_n^{1/2}$, we get

$$\sum_{n\in M_j(\Delta,\ell)} \int_{J_n^{1/2}} \frac{1}{\operatorname{dist}(t,\Delta)+|\Delta|}\,dt \lesssim \int_{\gamma^{-k}|L_j|+|\Delta|}^{C_k\gamma^\ell|L_j|+|\Delta|} \frac{1}{s}\,ds. \tag{4.10}$$

Next we will show that the above integration intervals can intersect for $\lesssim \ell$ indices j. Let $j_2 \ge j_1$, so that $L_{j_1} \supset L_{j_2}$, and write $j_2 = j_1 + 2kr + t$ with $t \le 2k-1$. Then, by Lemma 3.1,

$$C_k\gamma^\ell|L_{j_2}| \le C_k\gamma^\ell|L_{j_1+2kr}| \le C_k\gamma^\ell\eta^r|L_{j_1}|,$$

where $\eta = \gamma^{k-1}/(1+\gamma^{k-1}) < 1$. If now $r \ge C_{k,\gamma}\ell$ for a suitable constant $C_{k,\gamma}$ depending only on k and γ, we have

$$C_k\gamma^\ell|L_{j_2}| \le \gamma^{-k}|L_{j_1}|.$$

Thus, each point s in the integral in (4.10) for some j belongs to at most $C_{k,\gamma}\ell$ intervals $[\gamma^{-k}|L_j| + |\Delta|, C_k\gamma^\ell|L_j| + |\Delta|]$ where j is varying. So by summing over j we conclude

$$\sum_{n\in M(\Delta,\ell)} \frac{|J_n|}{\operatorname{dist}(J_n,\Delta)+|\Delta|} \le C_{k,\gamma}\ell \int_{|\Delta|}^{(1+C_k\gamma^\ell)|\Delta|} \frac{1}{s}\,ds \le C_{k,\gamma}\ell^2.$$

This completes the analysis of the case $\ell \ge k+1$, and the proof of the lemma. ∎

5. Four conditions on spline series and their relations. Let (t_n) be a k-admissible sequence of knots with the corresponding orthonormal spline system $(f_n)_{n\geq -k+2}$. For a sequence $(a_n)_{n\geq -k+2}$ of coefficients, let

$$P := \Big(\sum_{n=-k+2}^{\infty} a_n^2 f_n^2\Big)^{1/2} \quad \text{and} \quad S := \max_{m\geq -k+2}\Big|\sum_{n=-k+2}^{m} a_n f_n\Big|.$$

If $f \in L^1$, we denote by Pf and Sf the respective functions P and S corresponding to the coefficient sequence $a_n = \langle f, f_n\rangle$. Consider the following conditions:

(A) $P \in L^1$.
(B) The series $\sum_{n=-k+2}^{\infty} a_n f_n$ converges unconditionally in L^1.
(C) $S \in L^1$.
(D) There exists a function $f \in H^1$ such that $a_n = \langle f, f_n\rangle$.

We will discuss relations between those four conditions and prove the implications indicated in the diagram below; some results need regularity conditions on (t_n), which we also indicate.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
(A) \arrow[r, Rightarrow, shift left, "{\text{Proposition 5.2, } \sup_\varepsilon \|\sum \varepsilon_n a_n f_n\|_1 \lesssim_k \|P\|_1}"] \arrow[dr, Rightarrow, "{\text{Proposition 5.2, } \|S\|_1 \lesssim_k \|P\|_1}"] & (B) \arrow[l, Rightarrow, shift left, "{\|P\|_1 \lesssim \sup_\varepsilon \|\sum \varepsilon_n a_n f_n\|_1, \text{ Proposition 5.1}}"] \\
(D) \arrow[u, Rightarrow, "{\text{Proposition 5.3, } (k-1)\text{-reg.} \Rightarrow \|Pf\|_1 \lesssim_{k,\gamma} \|f\|_{H^1}}"] & (C) \arrow[l, Rightarrow, "{k\text{-reg.} \Rightarrow \|f\|_{H^1} \lesssim_{k,\gamma} \|Sf\|_1, \text{ Proposition 5.4}}"]
\end{tikzcd}$$

For orthonormal spline systems with dyadic knots, relations (and equivalences) of these conditions have been studied by several authors, also in the case $p < 1$ (see e.g. [23, 1, 12]). For general Franklin systems corresponding to arbitrary sequences of knots, relations of these conditions were discussed in [15] (and earlier in [13], also for $p < 1$, but for a restricted class of point sequences). Below, we follow the approach of [15], adapted to the case of spline orthonormal systems of order k.

We begin with the implication (B)⇒(A), which is a consequence of Khinchin's inequality:

Proposition 5.1 ((B)⇒(A)). *Let (t_n) be a k-admissible sequence of knots with the corresponding general orthonormal spline system (f_n), and* *let (a_n) be a sequence of coefficients. If the series $\sum_{n=-k+2}^{\infty} a_n f_n$ converges unconditionally in L^1, then $P \in L^1$. Moreover,*

$$\|P\|_1 \lesssim \sup_{\varepsilon\in\{-1,1\}^{\mathbb{Z}}} \Big\| \sum_{n=-k+2}^{\infty} \varepsilon_n a_n f_n \Big\|_1.$$

Next, we investigate the implications (A)⇒(B) and (A)⇒(C). Once we know the estimates and combinatorial results of Sections 3 and 4, the proof is the same as in [15, proof of Proposition 4.3], so we just state the result.

PROPOSITION 5.2 ((A)⇒(B) and (A)⇒(C)). *Let (t_n) be a k-admissible sequence of knots and let (a_n) be a sequence of coefficients such that $P \in L^1$. Then $S \in L^1$ and $\sum a_n f_n$ converges unconditionally in L^1; moreover,*

$$\sup_{\varepsilon\in\{-1,1\}^{\mathbb{Z}}} \Big\| \sum_{n=-k+2}^{\infty} \varepsilon_n a_n f_n \Big\| \lesssim_k \|P\|_1 \quad \textit{and} \quad \|S\|_1 \lesssim_k \|P\|_1.$$

Next we discuss (D)⇒(A).

PROPOSITION 5.3 ((D)⇒(A)). *Let (t_n) be a k-admissible point sequence that is $(k-1)$-regular with parameter γ. Then there exists a constant $C_{k,\gamma}$, depending only on k and γ, such that for each atom ϕ,*

$$\|P\phi\|_1 \le C_{k,\gamma}.$$

Consequently, if $f \in H^1$, then

$$\|Pf\|_1 \le C_{k,\gamma} \|f\|_{H^1}.$$

Before we proceed to the proof, let us remark that essentially the same arguments give a direct proof of (D)⇒(C), under the same assumption of $(k-1)$-regularity of (t_n), and moreover

$$\|Sf\|_1 \le C_{k,\gamma} \|f\|_{H^1}.$$

We do not present it here, since we have the implications (D)⇒(A) under the assumption of $(k-1)$-regularity and (A)⇒(C) under the assumption of k-admissibility only. Note that Proposition 6.1 below shows that without the assumption of $(k-1)$-regularity of the point sequence, the implications (D)⇒(A) and (D)⇒(C) need not be true.

Proof of Proposition 5.3. Let ϕ be an atom with $\int_0^1 \phi(u)\,du = 0$ and let $\Gamma = [\alpha,\beta]$ be an interval such that $\operatorname{supp}\phi \subset \Gamma$ and $\sup|\phi| \le |\Gamma|^{-1}$. Define $n_\Gamma := \max\{n : \operatorname{card}(\mathcal{T}_n \cap \Gamma) \le k-1\}$, where in the maximum, we also count multiplicities of knots. It will be shown that

$$\|P_1\phi\|_1, \|P_2\phi\|_1 \lesssim_{\gamma,k} 1,$$

where

$$P_1\phi = \Big(\sum_{n\le n_\Gamma} a_n^2 f_n^2\Big)^{1/2} \quad \text{and} \quad P_2\phi = \Big(\sum_{n> n_\Gamma} a_n^2 f_n^2\Big)^{1/2}.$$

First, we consider P_1 and prove the stronger inequality

$$\sum_{n\le n_\Gamma} |a_n|\,\|f_n\|_1 \lesssim_{k,\gamma} 1,$$

where $a_n = \langle \phi, f_n\rangle$. For each $n \le n_\Gamma$, we define $\Gamma_{n,\alpha}$ as the unique closed interval $D_{n,j}^{(k-1)}$ with minimal j such that

$$\alpha \le \min D_{n,j+1}^{(k-1)}.$$

We note that

$$\Gamma_{n_1,\alpha} \supseteq \Gamma_{n_2,\alpha} \quad \text{ for } n_1 \le n_2,$$

and, by $(k-1)$-regularity,

$$|\Gamma_{n,\alpha}| \gtrsim_{\gamma,k} |\Gamma|.$$

Let $g_n = \sum_{j=1}^{n+k-1} w_j N_{n,j}^{(k)}$ be the unnormalized orthogonal spline function as in (4.1) and with the coefficients (w_j) as in (4.3). For $\xi \in \Gamma$, we have (cf. (3.1))

$$|g_n'(\xi)| \lesssim_k \sum_j \frac{|w_j| + |w_{j-1}|}{|D_{n,j}^{(k-1)}|}, \tag{5.1}$$

where we sum over those j such that $\Gamma \cap \operatorname{supp} N_{n,j}^{(k-1)} = \Gamma \cap D_{n,j}^{(k-1)} \ne \emptyset$. By $(k-1)$-regularity, all lengths $|D_{n,j}^{(k-1)}|$ in this summation are comparable to $|\Gamma_{n,\alpha}|$. Moreover, by (4.6),

$$|w_j| \lesssim_k \frac{q^{d_n(\tau_{n,j})}}{|J_n| + \operatorname{dist}(D_{n,j}^{(k)}, J_n) + |D_{n,j}^{(k)}|}.$$

Again by $(k-1)$-regularity, for j in (5.1),

$$|D_{n,j}^{(k-1)}| \gtrsim_{k,\gamma} |\Gamma_{n,\alpha}|,$$
$$\operatorname{dist}(D_{n,j}^{(k)}, J_n) + |D_{n,j}^{(k)}| \gtrsim_{k,\gamma} \operatorname{dist}(J_n, \Gamma_{n,\alpha}) + |\Gamma_{n,\alpha}|.$$

Combining the above inequalities, we estimate the right hand side in (5.1) further and get, with the notation $\Gamma_n := \Gamma_{n,\alpha}$,

$$|g_n'(\xi)| \lesssim_{k,\gamma} \frac{1}{|\Gamma_n|}\,\frac{q^{d_n(\Gamma_n)}}{|J_n| + \operatorname{dist}(J_n,\Gamma_n) + |\Gamma_n|}. \tag{5.2}$$

As a consequence, for every $\tau \in \Gamma$,

$$|a_n| = \Big|\int_\Gamma \phi(t)[f_n(t) - f_n(\tau)]\,dt\Big| \le \int_\Gamma \frac{1}{|\Gamma|}\sup_{\xi\in\Gamma}|f_n'(\xi)|\,|t-\tau|\,dt$$
$$\lesssim_k |\Gamma|\,|J_n|^{1/2}\sup_{\xi\in\Gamma}|g_n'(\xi)| \lesssim_{k,\gamma} \frac{|\Gamma|}{|\Gamma_n|}\,\frac{|J_n|^{1/2}q^{d_n(\Gamma_n)}}{|J_n| + \operatorname{dist}(J_n,\Gamma_n) + |\Gamma_n|}.$$

Let $\Delta_1 \supset \cdots \supset \Delta_s$ be the collection of all different intervals appearing as Γ_n for $n \le n_\Gamma$. By Lemma 3.1, we have some geometric decay in the measure of $\varDelta_i$. Now fix $\varDelta_i$ and $\ell \geq 0$ and consider indices $n \leq n_\varGamma$ such that $\varGamma_n = \varDelta_i$ and $d_n(\varGamma_n) = \ell$. By the last display and Lemma 4.3,

$$|a_n|\,\|f_n\|_1 \lesssim_{k,\gamma} \frac{|\varGamma|}{|\varDelta_i|}\,\frac{|J_n|q^\ell}{|J_n| + \operatorname{dist}(J_n, \varDelta_i) + |\varDelta_i|},$$

and thus Lemma 4.7 implies

$$\sum_{n:\,\varGamma_n=\varDelta_i,\,d_n(\varGamma_n)=\ell} |a_n|\,\|f_n\|_1 \lesssim_{k,\gamma} (\ell+1)^2 q^\ell \frac{|\varGamma|}{|\varDelta_i|}.$$

Now, summing over ℓ and then over i (recall that $|\varDelta_i|$ decays like a geometric progression by Lemma 3.1 and $|\varDelta_i| \gtrsim_{k,\gamma} |\varGamma|$ since $n \leq n_\varGamma$) yields

$$\sum_{n \leq n_\varGamma} |a_n|\,\|f_n\|_1 \lesssim_{k,\gamma} 1.$$

This implies the desired inequality $\|P_1\phi\|_1 \lesssim_{k,\gamma} 1$ for the first part of $P\phi$.

Next, we look at $P_2\phi$ and define V to be the smallest interval whose endpoints in $\mathcal{T}_{n_\varGamma+1}$ and which contains $\varGamma$. Moreover, $\widetilde{V}$ is defined to be the smallest interval with endpoints in $\mathcal{T}_{n_\varGamma+1}$ and such that $\widetilde{V}$ contains k points in $\mathcal{T}_{n_\varGamma+1}$ to the left of $\varGamma$ and as well k points in $\mathcal{T}_{n_\varGamma+1}$ to the right of $\varGamma$. We observe that due to $(k-1)$-regularity and the fact that $\varGamma$ contains at least k points from $\mathcal{T}_{n_\varGamma+1}$,

$$\begin{aligned}&|V| \sim_{k,\gamma} |\widetilde{V}| \sim_{k,\gamma} |\varGamma|,\\ &|(\widetilde{V}\setminus V)\cap[0,\inf\varGamma]| \sim_{k,\gamma} |(\widetilde{V}\setminus V)\cap[\sup\varGamma,1]| \sim_{k,\gamma} |\widetilde{V}|.\end{aligned} \tag{5.3}$$

First, we consider the integral of $P_2\phi$ over $\widetilde{V}$ and obtain by the Cauchy–Schwarz inequality

$$\int_{\widetilde{V}} P_2\phi(t)\,dt \leq \|\mathbb{1}_{\widetilde{V}}\|_2\|\phi\|_2 \leq \frac{|\widetilde{V}|^{1/2}}{|\varGamma|^{1/2}} \lesssim_{k,\gamma} 1.$$

It remains to estimate $\int_{\widetilde{V}^c} P_2\phi(t)\,dt$. Since for $n > n_\varGamma$, the endpoints of $\widetilde{V}$ are in $\mathcal{T}_n$, either we have $J_n \subset \widetilde{V}$, or J_n is to the right of $\tilde{V}$, or J_n is to the left of $\widetilde{V}$. If $J_n \subset \widetilde{V}$, then

$$|a_n| = \Big|\int_\varGamma \phi(t) f_n(t)\,dt\Big| \leq \frac{\|f_n\|_1}{|\varGamma|} \lesssim_k \frac{|J_n|^{1/2}}{|\varGamma|},$$

and therefore, by Lemma 4.6 and (5.3),

$$\begin{aligned}\sum_{n:\,J_n\subset\widetilde{V},\,n>n_\varGamma} |a_n| \int_{\widetilde{V}^c} |f_n(t)|\,dt &\lesssim_k \frac{1}{|\varGamma|}\sum_{n:J_n\subset\widetilde{V}} |J_n|^{1/2}\int_{\widetilde{V}^c}|f_n(t)|\,dt\\ &\lesssim_k \frac{|\widetilde{V}|}{|\varGamma|} \lesssim_{k,\gamma} 1.\end{aligned}$$

Now, let J_n be to the right of $\widetilde{V}$; the case of J_n to the left of $\widetilde{V}$ is considered similarly. By (4.7) for $p = \infty$,

$$|a_n| \le \frac{1}{|\Gamma|} \int_\Gamma |f_n(t)|\,dt \le \frac{1}{|\Gamma|} \int_V |f_n(t)|\,dt \lesssim_{k,\gamma} \frac{q^{d_n(V)}|J_n|^{1/2}}{\operatorname{dist}(V, J_n) + |J_n|}.$$

This inequality, Lemma 4.3 and the fact that $\operatorname{dist}(V, J_n) \gtrsim_{k,\gamma} \operatorname{dist}(V, J_n) + |V|$ (cf. (5.3)) allow us to deduce

$$\sum_{\substack{n>n_\Gamma \\ J_n \text{ to the right of } \widetilde{V}}} |a_n|\,\|f_n\|_1 \lesssim_{k,\gamma} \sum_{\substack{n>n_\Gamma \\ J_n \text{ to the right of } \widetilde{V}}} \frac{q^{d_n(V)}|J_n|}{\operatorname{dist}(V, J_n) + |V|}.$$

Note that V can be a union of $k-1$, k or $k+1$ intervals from $\mathcal{T}_{n_\Gamma+1}$; therefore, let V^+ be a union of $k-1$ grid intervals from $\mathcal{T}_{n_\Gamma+1}$, with right endpoint of V^+ coinciding with the right endpoint of V. As J_n is to the right of V, we have $d_n(V) = d_n(V^+)$, $\operatorname{dist}(V, J_n) = \operatorname{dist}(V^+, J_n)$ and—by $(k-1)$-regularity—$|V| \sim_{k,\gamma} |V^+|$, which implies

$$\sum_{\substack{n>n_\Gamma \\ J_n \text{ to the right of } \widetilde{V}}} \frac{q^{d_n(V)}|J_n|}{\operatorname{dist}(V, J_n) + |V|} \lesssim_{k,\gamma} \sum_{\substack{n>n_\Gamma \\ J_n \text{ to the right of } \widetilde{V}}} \frac{q^{d_n(V^+)}|J_n|}{\operatorname{dist}(V^+, J_n) + |V^+|}.$$

Finally, we employ Lemma 4.7 to conclude

$$\begin{aligned}\sum_{\substack{n>n_\Gamma \\ J_n \text{ to the right of } \widetilde{V}}} |a_n|\,\|f_n\|_1 &\lesssim_{k,\gamma} \sum_{\ell=0}^\infty q^\ell \sum_{\substack{n>n_\Gamma \\ d_n(V^+)=\ell \\ J_n \text{ to the right of } \widetilde{V}}} \frac{|J_n|}{\operatorname{dist}(V^+, J_n) + |V^+|} \\ &\lesssim_{k,\gamma} \sum_{\ell=0}^\infty (\ell+1)^2 q^\ell \lesssim_k 1.\end{aligned}$$

To conclude the proof, note that if $f \in H^1$ and $f = \sum_{m=1}^\infty c_m\phi_m$ is an atomic decomposition of f, then $\langle f, f_n\rangle = \sum_{m=1}^\infty c_m\langle \phi_m, f_n\rangle$, and $Pf(t) \le \sum_{m=1}^\infty |c_m|P\phi_m(t)$. ∎

Finally, we discuss (C)⇒(D).

PROPOSITION 5.4 ((C)⇒(D)). *Let (t_n) be a k-admissible sequence of knots in $[0,1]$ which is k-regular with parameter γ and let (a_n) be a sequence of coefficients such that $S = \sup_m \big|\sum_{n\le m} a_n f_n\big| \in L^1$. Then there exists a function $f \in H^1$ with $a_n = \langle f, f_n\rangle$ for each n. Moreover,*

$$\|f\|_{H^1} \lesssim_{k,\gamma} \|Sf\|_1.$$

Proof. As $S \in L^1$, there is $f \in L^1$ such that $f = \sum_{n\ge -k+2} a_n f_n$ with convergence in L^1. Indeed, this is a consequence of the relative weak compactness of uniformly integrable subsets in L^1 and the basis property of (f_n)

in L^1. Thus, we need only show that $f \in H^1$, and this is done by finding a suitable atomic decomposition of f.

We define $E_0 = B_0 = [0,1]$ and, for $r \geq 1$,

$$E_r = [S > 2^r], \qquad B_r = [\mathcal{M}\mathbb{1}_{E_r} > c_{k,\gamma}],$$

where $\mathcal{M}$ denotes the Hardy–Littlewood maximal function and $0 < c_{k,\gamma} \leq 1/2$ is a small constant only depending on k and γ which is chosen according to a few restrictions that will be given during the proof. We note that

$$\mathcal{M}\mathbb{1}_{E_r}(t) = \sup_{I \ni t} \frac{|I \cap E_r|}{|I|}, \qquad t \in [0,1],$$

where the supremum is taken over all intervals containing t. Since $\mathcal{M}$ is of weak type $(1,1)$, we have $|B_r| \lesssim_{k,\gamma} |E_r|$. As $S \in L^1$, it follows that $|E_r| \to 0$ and hence $|B_r| \to 0$ as $r \to \infty$. Now, decompose the open set B_r into a countable union of disjoint open intervals,

$$B_r = \bigcup_{\kappa} \Gamma_{r,\kappa},$$

where for fixed r, no two intervals $\Gamma_{r,\kappa}$ have a common endpoint and the above equality is up to a measure zero set (each open set of real numbers can be decomposed into a countable union of open intervals, but it can happen that two intervals have the same endpoint; in that case, we collect those two intervals into one $\Gamma_{r,\kappa}$). This can be achieved by taking as $\Gamma_{r,\kappa}$ the collection of level sets of positive measure of the function $t \mapsto |[0,t] \cap B_r^c|$.

Moreover, observe that if $\Gamma_{r+1,\xi}$ is one of the intervals in the decomposition of B_{r+1}, then there is an interval $\Gamma_{r,\kappa}$ in the decomposition of B_r such that $\Gamma_{r+1,\xi} \subset \Gamma_{r,\kappa}$.

Based on this decomposition, we define the following functions for $r \geq 0$:

$$g_r(t) := \begin{cases} f(t), & t \in B_r^c, \\ \dfrac{1}{|\Gamma_{r,\kappa}|} \displaystyle\int_{\Gamma_{r,\kappa}} f(t)\,dt, & t \in \Gamma_{r,\kappa}. \end{cases}$$

Observe that $f = g_0 + \sum_{r=0}^{\infty}(g_{r+1} - g_r)$ in L^1 and $g_{r+1} - g_r = 0$ on B_r^c. As a consequence,

$$\begin{aligned} \int_{\Gamma_{r,\kappa}} g_{r+1}(t)\,dt &= \int_{\Gamma_{r,\kappa} \cap B_{r+1}^c} g_{r+1}(t)\,dt + \int_{\Gamma_{r,\kappa} \cap B_{r+1}} g_{r+1}(t)\,dt \\ &= \int_{\Gamma_{r,\kappa} \cap B_{r+1}^c} f(t)\,dt + \sum_{\xi:\, \Gamma_{r+1,\xi} \subset \Gamma_{r,\kappa}} \int_{\Gamma_{r+1,\xi}} f(t)\,dt \\ &= \int_{\Gamma_{r,\kappa}} f(t)\,dt = \int_{\Gamma_{r,\kappa}} g_r(t)\,dt. \end{aligned}$$

The main step of the proof is to show that

$$|g_r(t)| \le C_{k,\gamma} 2^r, \qquad \text{a.e. } t \in [0,1], \tag{5.4}$$

for some constant $C_{k,\gamma}$ only depending on k and γ. Once this inequality is proved, we take $\phi_0 \equiv 1$, $\eta_0 = \int_0^1 f(u)\,du$ and

$$\phi_{r,\kappa} := \frac{(g_{r+1} - g_r)\mathbb{1}_{\Gamma_{r,\kappa}}}{C_{k,\gamma} 2^r |\Gamma_{r,\kappa}|}, \qquad \eta_{r,\kappa} = C_{k,\gamma} 2^r |\Gamma_{r,\kappa}|$$

and observe that $f = \eta_0\phi_0 + \sum_{r,\kappa} \eta_{r,\kappa}\phi_{r,\kappa}$ is the desired atomic decomposition of f since

$$\begin{aligned} \sum_{r,\kappa} \eta_{r,\kappa} \le C_{k,\gamma} \sum_{r,\kappa} 2^r |\Gamma_{r,\kappa}| = C_{k,\gamma} \sum_r 2^r |B_r| \\ \lesssim_{k,\gamma} \sum_r 2^r |E_r| \lesssim \|S\|_1. \end{aligned}$$

Thus it remains to prove inequality (5.4).

To do so, we first assume $t \in B_r^c$. Additionally, assume that t is such that the series $\sum_n a_n f_n(t)$ converges to $f(t)$ and t is not in (t_n). By Theorem 3.5, this holds for a.e. $\in [0,1]$. We fix m and let V_m be the maximal interval where the function $S_m := \sum_{n\le m} a_n f_n$ is a polynomial of order k and that contains t. Then $V_m \not\subset B_r$ and since V_m is an interval containing t,

$$|V_m \cap E_r^c| \ge (1 - c_{k,\gamma})|V_m| \ge |V_m|/2.$$

Since $|S_m| \le 2^r$ on E_r^c, the above display and Proposition 3.2 imply that $|S_m| \lesssim_k 2^r$ on V_m and in particular $|S_m(t)| \lesssim_k 2^r$. Now, $S_m(t) \to f(t)$ as $m \to \infty$ by the assumptions on t, and thus

$$|g_r(t)| = |f(t)| \lesssim_k 2^r.$$

This concludes the proof of (5.4) in the case of $t \in B_r^c$.

Next, we fix κ and consider g_r on $\Gamma := [\alpha, \beta] := \Gamma_{r,\kappa}$. Let n_Γ be the first index such that there are $k+1$ points from $\mathcal{T}_{n_\Gamma}$ contained in Γ, i.e., there exists a support $D^{(k)}_{n_\Gamma,i}$ of a B-spline function of order k in the grid $\mathcal{T}_{n_\Gamma}$ that is contained in Γ. Additionally, we define

$$U_0 := [\tau_{n_\Gamma,i-k}, \tau_{n_\Gamma,i}], \qquad W_0 := [\tau_{n_\Gamma,i+k}, \tau_{n_\Gamma,i+2k}].$$

Note that if $\alpha \in \mathcal{T}_{n_\Gamma}$, then α is a common endpoint of U_0 and Γ, otherwise α is an interior point of U_0. Similarly, if $\beta \in \mathcal{T}_{n_\Gamma}$, then β is a common endpoint of W_0 and Γ, otherwise β is an interior point of W_0. By k-regularity of (t_n), we have $\max(|U_0|, |W_0|) \lesssim_{k,\gamma} |\Gamma|$. We first estimate the part $S_\Gamma := \sum_{n\le n_\Gamma} a_n f_n$ and show that $|S_\Gamma| \lesssim_{k,\gamma} 2^r$ on Γ. Observe that on $\Delta := U_0 \cup \Gamma \cup W_0$, S_Γ can be represented as a linear combination of B-splines (N_j) on the grid $\mathcal{T}_{n_\Gamma}$ of the form

$$S_\Gamma(t) = h(t) := \sum_{j=i-2k+1}^{i+2k-1} b_j N_j(t),$$

for some coefficients (b_j). For $j = i-2k+1, \dots, i+2k-1$, let J_j be a maximal interval of $\operatorname{supp} N_j$ and observe that due to k-regularity, $|J_j| \sim_{k,\gamma} |\Gamma| \sim_{k,\gamma} |\operatorname{supp} h|$.

If we assume that $\max_{J_j} |S_\Gamma| > C_k 2^r$, where C_k is the constant of Proposition 3.2 for $\rho = 1/2$, then Proposition 3.2 implies that $|S_\Gamma| > 2^r$ on a subset I_j of J_j with measure $\geq |J_j|/2$. Hence

$$|\operatorname{supp} h \cap E_r| \geq |J_j \cap E_r| \geq |J_j|/2 \gtrsim_{k,\gamma} |\operatorname{supp} h|.$$

We choose the constant $c_{k,\gamma}$ in the definition of B_r sufficiently small to guarantee that this last inequality implies $\operatorname{supp} h \subset B_r$. This contradicts the choice of Γ, which implies that our assumption $\max_{J_j} |S_\Gamma| > C_k 2^r$ is not true and thus

$$\max_{J_j} |S_\Gamma| \leq C_k 2^r, \qquad j = i-2k+1, \dots, i+2k-1.$$

By local stability of B-splines, i.e., inequality (3.2) in Proposition 3.3, this implies

$$|b_j| \lesssim_k 2^r, \qquad j = i-2k+1, \dots, i+2k-1,$$

and so $|S_\Gamma| \lesssim_k 2^r$ on Δ. This means

$$\int_\Gamma |S_\Gamma| \lesssim_k 2^r |\Gamma|, \tag{5.5}$$

which is inequality (5.4) for the part S_Γ.

In order to estimate the remaining part, we inductively define two sequences $(u_s, U_s)_{i\geq 0}$ and $(w_s, W_s)_{s\geq 0}$ consisting of integers and intervals. Set $u_0 = w_0 = n_\Gamma$ and inductively define u_{s+1} to be the first $n > u_s$ such that $t_n \in U_s$. Moreover, define U_{s+1} to be the B-spline support $D_{u_{s+1},i^{(k)}}$ in the grid $\mathcal{T}_{u_{s+1}}$, where i is minimal such that $D^{(k)}_{u_{s+1},i} \cap \Gamma \neq \emptyset$. Similarly, we define w_{s+1} to be the first $n > w_s$ such that $t_n \in W_s$ and W_{s+1} as the B-spline support $D^{(k)}_{w_{s+1},i}$ in the grid $\mathcal{T}_{w_{s+1}}$, where i is maximal such that $D^{(k)}_{w_{s+1},i} \cap \Gamma \neq \emptyset$. It can easily be seen that this construction implies $U_{s+1} \subset U_s$, $W_{s+1} \subset W_s$ and $\alpha \in U_s$, $\beta \in W_s$ for all $s \geq 0$, or more precisely: if $\alpha \in \mathcal{T}_{u_s}$, then α is either a common endpoint of U_s and Γ, or an inner point of U_s, and similarly if $\beta \in \mathcal{T}_{u_s}$, then β is either a common endpoint of W_s and Γ, or an inner point of W_s.

For a pair of indices ℓ, m, let

$$x_\ell := \sum_{\nu=0}^{k-1} N_{u_\ell,i+\nu} \mathbb{1}_{U_\ell}, \qquad y_m := \sum_{\nu=0}^{k-1} N_{w_m,j-\nu} \mathbb{1}_{W_m},$$

where $N_{u_\ell,i}$ is the B-spline function on the grid $\mathcal{T}_{u_\ell}$ with support U_ℓ, and $N_{w_m,j}$ is the B-spline function on $\mathcal{T}_{w_m}$ with support W_m. The function

$$\phi_{\ell,m} := x_\ell + \mathbb{1}_{\Gamma\setminus(U_\ell\cup W_m)} + y_m$$

is zero on $(U_\ell \cup \Gamma \cup W_m)^c$, one on $\Gamma \setminus (U_\ell \cup W_m)$ and a piecewise polynomial function of order k in between. For $\ell, m \geq 0$, consider the following subsets of $\{n : n > n_\Gamma\}$:

$$L(\ell) := \{n : u_\ell < n \leq u_{\ell+1}\}, \qquad R(m) := \{n : w_m < n \leq w_{m+1}\}.$$

If $n \in L(\ell) \cap R(m)$, we clearly have $\langle f_n, \phi_{\ell,m}\rangle = 0$ and thus

$$\int_\Gamma f_n(t)\,dt = \int_\Gamma f_n(t)\,dt - \int_0^1 f_n(t)\phi_{\ell,m}(t)\,dt = A_\ell(f_n) + B_m(f_n), \tag{5.6}$$

where

$$A_\ell(f_n) := \int_{\Gamma\cap U_\ell} f_n(t)\,dt - \int_{U_\ell} f_n(t) x_\ell(t)\,dt,$$
$$B_m(f_n) := \int_{\Gamma\cap W_m} f_n(t)\,dt - \int_{W_m} f_n(t) y_m(t)\,dt.$$

This implies

$$\begin{aligned} \Big|\int_\Gamma \sum_{n=n_\Gamma+1}^{\infty} a_n f_n(t)\,dt\Big| &= \Big|\sum_{\ell,m=0}^{\infty} \sum_{n\in L(\ell)\cap R(m)} a_n\big(A_\ell(f_n) + B_m(f_n)\big)\Big| \\ &\leq 2\sum_{\ell=0}^{\infty}\int_{U_\ell}\Big|\sum_{n\in L(\ell)} a_n f_n(t)\Big|\,dt + 2\sum_{m=0}^{\infty}\int_{W_m}\Big|\sum_{n\in R(m)} a_n f_n(t)\Big|\,dt. \end{aligned} \tag{5.7}$$

Consider the first sum on the right hand side. On $U_\ell = D^{(k)}_{u_\ell,i}$, the function $\sum_{n\in L(\ell)} a_n f_n$ can be represented as a linear combination of B-splines (N_j) on the grid $\mathcal{T}_{u_\ell}$ of the form

$$\sum_{n\in L(\ell)} a_n f_n = h_\ell := \sum_{j=i-k+1}^{i+k-1} b_j N_j,$$

for some coefficients (b_j). For $j = i-k+1, \ldots, i+k-1$, let J_j be a maximal grid interval of $\operatorname{supp} N_j$ and observe that due to k-regularity, $|J_j| \sim_{k,\gamma} |U_\ell| \sim_{k,\gamma} |\operatorname{supp} h_\ell|$. On J_j, the function $\sum_{n\in L(\ell)} a_n f_n$ is a polynomial of order k. If we assume $\max_{J_j} \big|\sum_{n\in L(\ell)} a_n f_n\big| > C_k 2^{r+1}$, where C_k is the constant of Proposition 3.2 for $\rho = 1/2$, then Proposition 3.2 implies that $\big|\sum_{n\in L(\ell)} a_nf_n\big| > 2^{r+1}$ on a set $J_j^* \subset J_j$ with $|J_j^*| = |J_j|/2$; but this means $\max(|\sum_{n\le u_\ell} a_nf_n|, |\sum_{n\le u_{\ell+1}} a_nf_n|) > 2^r$ on J_j^*. Hence

$$|E_r \cap \operatorname{supp} h_\ell| \ge |E_r \cap J_j| \ge |J_j^*| \ge |J_j|/2 \gtrsim_k |\operatorname{supp} h_\ell|.$$

We choose the constant $c_{k,\gamma}$ in the definition of B_r sufficiently small to guarantee that this last inequality implies $\operatorname{supp} h_\ell \subset B_r$. This contradicts the choice of Γ, which implies that our assumption $\max_{J_j} \big|\sum_{n\in L(\ell)} a_nf_n\big| > C_k2^r$ is not true and thus

$$\max_{J_j}\Big|\sum_{n\in L(\ell)} a_nf_n\Big| \le C_k2^r, \qquad j = i-k+1,\dots,i+k-1.$$

By local stability of B-splines, i.e., inequality (3.2), this implies

$$|b_j| \lesssim_k 2^r, \qquad j = i-k+1,\dots,i+k-1,$$

and so $\big|\sum_{n\in L(\ell)} a_nf_n\big| \lesssim_k 2^r$ on U_ℓ, which gives

$$\int_{U_\ell}\Big|\sum_{n\in L(\ell)} a_nf_n\Big| \lesssim_k 2^r|U_\ell|.$$

Combining Lemma 3.1, the inclusions $U_{\ell+1} \subset U_\ell$ and the inequality $|U_0| \lesssim_{k,\gamma} |\Gamma|$, we see that $\sum_{\ell=0}^\infty |U_\ell| \lesssim_{k,\gamma} |\Gamma|$. Thus we get

$$\sum_{\ell=0}^\infty \int_{U_\ell}\Big|\sum_{n\in L(\ell)} a_nf_n\Big| \lesssim_{k,\gamma} 2^r|\Gamma|.$$

The second sum on the right hand side of (5.7) is estimated similarly, which gives

$$\sum_{m=0}^\infty \int_{W_m}\Big|\sum_{n\in R(m)} a_nf_n\Big| \lesssim_{k,\gamma} 2^r|\Gamma|.$$

Combining these estimates with (5.7) and (5.5), we find

$$\Big|\int_\Gamma f(t)\,dt\Big| = \Big|\int_\Gamma \sum_n a_nf_n(t)\,dt\Big| \lesssim_{k,\gamma} 2^r|\Gamma|,$$

which implies (5.4) on Γ, and thus the proof is complete. ∎

6. Proof of the main theorem. For the proof of the necessity part of Theorem 2.4, we will use the following:

PROPOSITION 6.1. *Let (t_n) be a k-admissible sequence of knots that is k-regular with parameter γ, but not $(k-1)$-regular. Then*

$$\sup\Big\|\sup_n |a_n(\phi)f_n|\Big\|_1 = \infty,$$

where the first sup *is taken over all atoms ϕ, and $a_n(\phi) := \langle\phi, f_n\rangle$.*

Proposition 6.1 implies in particular that Proposition 5.3 cannot be extended to arbitrary partitions. For the proof of Proposition 6.1 we need the following technical lemma.

LEMMA 6.2. *Let (t_n) be a k-admissible sequence of knots that is k-regular with parameter $\gamma \geq 1$, but not $(k-1)$-regular. Let ℓ be an arbitrary positive integer. Then, for all $A \geq 2$, there exists a finite increasing sequence $(n_j)_{j=0}^{\ell-1}$ such that if τ_{n_j,i_j} is the new point in $\mathcal{T}_{n_j}$ not present in $\mathcal{T}_{n_j-1}$ and*

$$\Lambda_j := [\tau_{n_j,i_j-k}, \tau_{n_j,i_j-1}), \qquad L_j := [\tau_{n_j,i_j-1}, \tau_{n_j,i_j}), \qquad R_j := [\tau_{n_j,i_j}, \tau_{n_j,i_j+1}),$$

then for all i, j with $0 \leq i < j \leq \ell - 1$ we have:

(1) $R_i \cap R_j = \emptyset$,
(2) $\Lambda_i = \Lambda_j$,
(3) $(2\gamma - 1)|L_j| \geq |[\tau_{n_j,i_j-k-1}, \tau_{n_j,i_j-k}]| \geq |L_j|/(2\gamma)$,
(4) $|R_j| \leq (2\gamma - 1)|L_j|$,
(5) $|L_j| \leq 2(\gamma + 1)k|R_j|$,
(6) $\min(|L_j|, |R_j|) \geq A|\Lambda_j|$.

Proof. First, we choose a sequence $(n_j)_{j=0}^{lk}$ so that (1)–(4) hold. Next, we choose a subsequence $(n_{m_j})_{j=0}^{l-1}$ so that (5) and (6) hold as well.

Since (t_n) is not $(k-1)$-regular, for all C_0 there exist n_0 and i_0 such that

$$\text{(6.1)} \qquad \text{either } C_0|D_{n_0,i_0-k}^{(k-1)}| \leq |D_{n_0,i_0-k+1}^{(k-1)}| \text{ or } |D_{n_0,i_0-k}^{(k-1)}| \geq C_0|D_{n_0,i_0-k+1}^{(k-1)}|.$$

We choose C_0 sufficiently large such that with $C_j := C_{j-1}/\gamma - 1$ for $j \geq 1$ we have $C_{k\ell} \geq 2\gamma$. We will make an additional restriction on C_0 at the end of the proof. Without loss of generality, we can assume that the first inequality in (6.1) holds. Taking $\Lambda_0 = [\tau_{n_0,i_0-k}, \tau_{n_0,i_0-1})$ and $L_0 = [\tau_{n_0,i_0-1}, \tau_{n_0,i_0})$, $R_0 = [\tau_{n_0,i_0}, \tau_{n_0,i_0+1})$, we have

$$\text{(6.2)} \qquad |[\tau_{n_0,i_0-k+1}, \tau_{n_0,i_0}]| \geq C_0|\Lambda_0|.$$

A direct consequence of (6.2) is

$$\text{(6.3)} \qquad |L_0| \geq (C_0 - 1)|\Lambda_0|.$$

By k-regularity we have

$$|D_{n_0,i_0-k-1}^{(k)}| \geq \frac{|D_{n_0,i_0-k}^{(k)}|}{\gamma} = \frac{|\Lambda_0| + |L_0|}{\gamma},$$

which implies

(6.4) $$|[\tau_{n_0,i_0-k-1},\tau_{n_0,i_0-k}]| = |D^{(k)}_{n_0,i_0-k-1}| - |\Lambda_0| \ge \frac{|\Lambda_0|+|L_0|}{\gamma} - |\Lambda_0| \ge \frac{|L_0|}{2\gamma} + \frac{|\Lambda_0|}{\gamma} + \frac{C_0-1}{2\gamma}|\Lambda_0| - |\Lambda_0| = \frac{|L_0|}{2\gamma} + \left(\frac{C_0+1}{2\gamma} - 1\right)|\Lambda_0| \ge \frac{|L_0|}{2\gamma},$$

i.e., the right hand inequality of (3) for $j = 0$. To get the upper estimate, note that by k-regularity,

$$|\Lambda_0| + |[\tau_{n_0,i_0-k-1},\tau_{n_0,i_0-k}]| \le \gamma(|\Lambda_0| + |L_0|),$$

hence by (6.3),

(6.5) $$|[\tau_{n_0,i_0-k-1},\tau_{n_0,i_0-k}]| \le \gamma|L_0| + (\gamma-1)|\Lambda_0| \le (2\gamma-1)|L_0|.$$

This and the previous calculation give (3) for $j = 0$. Therefore, the construction can be continued either to the right or to the left of Λ_0.

We continue the construction to the right of Λ_0 by induction. Having defined n_j, Λ_j, L_j and R_j, we take

$$n_{j+1} := \min\{n > n_j : t_n \in \Lambda_j \cup L_j\}, \qquad j \ge 0.$$

By definition of R_j and n_{j+1}, property (1) is satisfied for all $j \ge 0$. We identify $t_{n_{j+1}} = \tau_{n_{j+1},i_{j+1}}$. Thus, by construction, $t_{n_j} = \tau_{n_j,i_j}$ is a common endpoint of L_j and R_j for $j \ge 1$.

In order to prove (2), we will show by induction that

(6.6) $$|[\tau_{n_j,i_j-k+1},\tau_{n_j,i_j}]| \ge C_j|\Lambda_j| \quad \text{and} \quad \Lambda_{j+1} = \Lambda_j$$

for all $j = 0,\dots,k\ell$. We remark that the equality $\Lambda_{j+1} = \Lambda_j$ is equivalent to the condition $\tau_{n_{j+1},i_{j+1}} \in L_j$.

The inequality of (6.6) for $j = 0$ is exactly (6.2). If the identity in (6.6) were not satisfied for $j = 0$, i.e., $\tau_{n_1,i_1} \in \Lambda_0$, by k-regularity of (t_n), applied to the partition $\mathcal{T}_{n_1}$, we would have

$$|\Lambda_0| \ge \frac{1}{\gamma}|L_0|,$$

which contradicts (6.3) for our choice of C_0. This means $\Lambda_1 = \Lambda_0$, and so (6.6) is true for $j = 0$. Next, assume that (6.6) is satisfied for $j-1$, where $1 \le j \le k\ell - 1$. By k-regularity, applied to $\mathcal{T}_{n_j}$, and employing (6.6) for $j-1$ repeatedly, we obtain

$$|\Lambda_j| + |L_j| = |\Lambda_j \cup L_j| \ge \frac{1}{\gamma}(\tau_{n_j,i_j+1} - \tau_{n_j,i_j-k+1}) = \frac{1}{\gamma}(\tau_{n_{j-1},i_{j-1}} - \tau_{n_{j-1},i_{j-1}-k+1}) \ge \frac{C_{j-1}}{\gamma}|\Lambda_{j-1}| = \frac{C_{j-1}}{\gamma}|\Lambda_j|.$$

This means, by the recursive definition of C_j, that

$$(6.7)\qquad |L_j| \geq C_j |\Lambda_j|,$$

and in particular the first identity in (6.6) is true for j. If the identity in (6.6) were not satisfied for j, i.e., $\tau_{n_{j+1},i_{j+1}} \in \Lambda_j$, by k-regularity of (t_n), applied to $\mathcal{T}_{n_{j+1}}$, we would have

$$|\Lambda_j| \geq \frac{1}{\gamma}|L_j|,$$

which contradicts (6.7) and our choice of C_0. This proves (6.6) for j, and thus property (2) is true for all $j = 0, \ldots, k\ell$.

Moreover, choosing C_0 sufficiently large, namely such that $C_{kl} \geq 2(\gamma+1)kA$, (6.7) implies

$$(6.8)\qquad |L_j| \geq 2(\gamma+1)kA|\Lambda_j|,$$

which is a part of (6).

The lower estimate in (3) is proved by repeating the argument giving (6.4) and using (6.7) instead of (6.4). The upper estimate uses the same arguments as the proof of (6.5), but now we have to use (6.7) as well.

Next, we look at (4). By k-regularity and (6.7), as $C_j > 1$, we have

$$|R_j| + |L_j| \leq \gamma(|L_j| + |\Lambda_j|) \leq 2\gamma|L_j|,$$

which is exactly (4).

We prove (5) by choosing a suitable subsequence of $(n_j)_{j=0}^{k\ell}$. First, assume that (5) fails for k consecutive indices, i.e., for some s,

$$(6.9)\qquad |R_{s+r}| < \alpha|L_{s+r}| \leq \alpha|L_s|, \qquad r = 1, \ldots, k,$$

where $\alpha := (2(\gamma+1)k)^{-1}$. We have $L_j = L_{j+1} \cup R_{j+1}$ for $0 \leq j \leq k\ell - 1$. Thus, on the one hand,

$$(6.10)\qquad |L_s \setminus L_{s+k}| = \sum_{r=1}^{k} |R_{s+r}| \leq \alpha k |L_s|$$

by (6.9); on the other hand, by k-regularity of $\mathcal{T}_{n_{s+k}}$,

$$(6.11)\qquad |L_s \setminus L_{s+k}| \geq \frac{1}{\gamma}|L_{s+k}| = \frac{1}{\gamma}\Big(|L_s| - \sum_{r=1}^{k}|R_{s+r}|\Big) \geq \frac{1-\alpha k}{\gamma}|L_s|.$$

Now, (6.10) contradicts (6.11) for our choice of α. We have thus proved that there is at least one index $s+r$, $1 \leq r \leq k$, such that (5) is satisfied for $s+r$. Hence we can extract a sequence of length ℓ from $(n_j)_{j=1}^{k\ell}$ satisfying (5). For simplicity, this subsequence is called $(n_j)_{j=0}^{\ell-1}$ again.

Property (6) for R_j is now a simple consequence of (6.8), property (5) and the choice of $(n_j)_{j=0}^{\ell-1}$. Thus, the proof of the lemma is complete. ■

Now, we are ready to proceed to the proof of Proposition 6.1.

Proof of Proposition 6.1. Let ℓ be an arbitrary positive integer and $A \geq 2$ a number to be chosen later. Lemma 6.2 gives a sequence $(n_j)_{j=0}^{\ell-1}$ such that conditions (1)–(6) in Lemma 6.2 are satisfied. We assume that $|\Lambda_0| > 0$. Let $\tau := \tau_{n_0,i_0-1}$, $x := \tau - 2|\Lambda_0|$ and $y := \tau + 2|\Lambda_0|$. Then we define an atom ϕ by

$$\phi \equiv \frac{1}{4|\Lambda_0|}(\mathbb{1}_{[x,\tau]} - \mathbb{1}_{[\tau,y]})$$

and let j be an arbitrary integer with $0 \leq j \leq \ell - 1$. By partial integration, the expression $a_{n_j}(\phi) = \langle \phi, f_{n_j} \rangle$ can be written as

$$\begin{aligned}
4|\Lambda_0|a_{n_j}(\phi) &= \int_x^\tau f_{n_j}(t)\,dt - \int_\tau^y f_{n_j}(t)\,dt \\
&= \int_x^\tau f_{n_j}(t) - f_{n_j}(\tau)\,dt - \int_\tau^y f_{n_j}(t) - f_{n_j}(\tau)\,dt \\
&= \int_x^\tau (x-t) f'_{n_j}(t)\,dt - \int_\tau^y (y-t) f'_{n_j}(t)\,dt.
\end{aligned}$$

In order to estimate $|a_{n_j}(\phi)|$ from below, we estimate the absolute values of $I_1 := \int_x^\tau (x-t)f'_{n_j}(t)\,dt$ from below and of $I_2 := \int_\tau^y (y-t)f'_{n_j}(t)\,dt$ from above. We begin with I_2.

Consider the function g_{n_j}, connected with f_{n_j} via $f_{n_j} = g_{n_j}/\|g_{n_j}\|_2$ and $\|g_{n_j}\|_2 \sim_k |J_{n_j}|^{-1/2}$ (cf. (4.1) and Lemma 4.3). In the notation of Lemma 6.2, g_{n_j} is obtained by inserting the point $t_{n_j} = \tau_{n_j,i_j}$ in $\mathcal{T}_{n_j-1}$, and it is a common endpoint of intervals L_i and R_i. By construction of the characteristic interval J_{n_j}, properties (4)–(6) of Lemma 6.2, and the k-regularity of (t_n), we have

$$|J_{n_j}| \sim_{k,\gamma} |L_j| \sim_{k,\gamma} |R_j|. \tag{6.12}$$

By property (6), we have $[\tau, y] \subset L_j$, and therefore on $[\tau, y]$, the derivative of g_{n_j} has the representation (cf. (3.1))

$$g'_{n_j}(u) = (k-1)\sum_{i=i_j-k+1}^{i_j-1} \xi_i N_{n_j,i}^{(k-1)}(u), \qquad u \in [\tau, y],$$

where $\xi_i = (w_i - w_{i-1})/|D_{n_j,i}^{(k-1)}|$ and the coefficients w_i are given by (4.3) associated to the partition $\mathcal{T}_{n_j}$. For $i = i_j - k + 1, \ldots i_j - 1$ we have $L_j \subset D_{n_j,i}^{(k-1)}$, which together with the k-regularity of (t_n) and property (6) implies

$$|J_{n_j}| \sim_k |L_j| \sim_{k,\gamma} |D_{n_j,i}^{(k-1)}|, \qquad i = i_j - k + 1, \ldots, i_j - 1. \tag{6.13}$$

Moreover, by Lemma 4.4,

$$|w_i| \lesssim_k \frac{1}{|J_{n_j}|}, \qquad 1 \le i \le n_j + k - 1.$$

Therefore

$$|f'_{n_j}(t)| \sim_k |J_{n_j}|^{1/2} |g'_{n_j}(t)| \lesssim_{k,\gamma} |L_j|^{-3/2} \qquad \text{for } t \in [\tau, y].$$

Consequently, putting the above facts together,

$$|I_2| \lesssim_{k,\gamma} |\Lambda_0|^2 \cdot |L_j|^{-3/2}. \tag{6.14}$$

We now estimate I_1. By properties (3) and (6) of Lemma 6.2 (with $A \ge 2\gamma$), we have $[x, \tau] \subset [\tau_{n_j, i_j - k - 1}, \tau_{n_j, i_j - 1}]$, and therefore on $[x, \tau]$, g'_{n_j} has the representation (cf. (3.1))

$$g'_{n_j}(u) = (k-1) \sum_{i = i_j - 2k + 1}^{i_j - 2} \xi_i N^{(k-1)}_{n_j, i}(u), \qquad u \in [x, \tau].$$

We split $I_1 = I_{1,1} + I_{1,2}$ according to whether $i \ne i_j - k$ or $i = i_j - k$ in the above representation of g'_{n_j} on $[x, \tau]$.

Note that $[\tau_{n_j, i_j - k - 1}, \tau_{n_j, i_j - k}] \subset D^{(k-1)}_{n_j, i}$ for $i_j - 2k + 1 \le i < i_j - k$ and $L_j \subset D^{(k-1)}_{n_j, i}$ for $i_j - k < i \le i_j - 2$. Therefore, by properties (3) and (6) of Lemma 6.2 and the k-regularity of the sequence of knots we have

$$|D^{(k-1)}_{n_j, i}| \sim_{k,\gamma} |L_j| \quad \text{for } i_j - 2k + 1 \le i \le i_j - 2,\ i \ne i_j - k.$$

So, by arguments analogous to the proof of (6.14) we get

$$|I_{1,1}| \sim_k |J_{n_j}|^{1/2} \Big| \int_x^\tau (t - x) \sum_{\substack{i = i_j - 2k + 1 \\ i \ne i_j - k}}^{i_j - 2} \xi_i N^{(k-1)}_{n_j, i}(t)\, dt \Big| \lesssim_{k,\gamma} |\Lambda_0|^2 \cdot |L_j|^{-3/2}. \tag{6.15}$$

Moreover, for $i = i_j - k$, we have $D^{(k-1)}_{n_j, i_j - k} = \Lambda_0$, so

$$\begin{aligned} |I_{1,2}| &\sim_k |J_{n_j}|^{1/2} \Big| \int_x^\tau (t - x) \xi_{i_j - k} N^{(k-1)}_{n_j, i_j - k}(t)\, dt \Big| \\ &\ge |\xi_{i_j - k}|\, |J_{n_j}|^{1/2} |\Lambda_0| \int_x^\tau N^{(k-1)}_{n_j, i_j - k}(t)\, dt \\ &= |\xi_{i_j - k}|\, |\Lambda_0|\, |J_{n_j}|^{1/2} \frac{|D^{(k-1)}_{n_j, i_j - k}|}{k - 1} = |\xi_{i_j - k}|\, |J_{n_j}|^{1/2} \frac{|\Lambda_0|^2}{k - 1}, \end{aligned} \tag{6.16}$$

because $t - x \ge |\Lambda_0|$ for $t \in \operatorname{supp} N^{(k-1)}_{n_j, i_j - k}$. Since the sequence w_j is checkerboard (cf. (4.4)),

$$|\xi_{i_j-k}| = \frac{|w_{i_j-k}| + |w_{i_j-k-1}|}{|D^{(k-1)}_{n_j,i_j-k}|} \ge \frac{|w_{i_j-k}|}{|D^{(k-1)}_{n_j,i_j-k}|}.$$

By definition of w_{i_j-k},

$$|w_{i_j-k}| \ge |\alpha_{i_j-k}|\,|b_{i_j-k,i_j-k}|,$$

where α_{i_j-k} is the factor from formula (4.2) and b_{i_j-k,i_j-k} is an entry of the inverse of the B-spline Gram matrix, both corresponding to the partition $\mathcal{T}_{n_j}$. Formulas (4.2) and (6.12) imply that α_{i_j-k} is bounded from below by a positive constant that only depends on k and γ ([1]). Moreover, $|b_{i_j-k,i_j-k}| \ge \|N^{(k)}_{n_j,i_j-k}\|_2^{-2} \gtrsim_k |D^{(k)}_{n_j,i_j-k}|^{-1}$ (cf. (3.6)). Note that $D^{(k)}_{n_j,i_j-k} = \Lambda_0 \cup L_j$, so $|D^{(k)}_{n_j,i_j-k}| \sim_{k,\gamma} |L_j|$. Thus, $|\xi_{i_j-k}| \gtrsim_{k,\gamma} |\Lambda_0|^{-1}|L_j|^{-1}$. Inserting the above calculations in (6.16), we find

$$|I_{1,2}| \gtrsim_{k,\gamma} |J_{n_j}|^{1/2}\frac{|\Lambda_0|}{|L_j|} \sim_{k,\gamma} |\Lambda_0|\,|L_j|^{-1/2}. \tag{6.17}$$

We now impose conditions on the constant $A \ge 2\gamma$ from the beginning of the proof and property (6) in Lemma 6.2. It follows from (6.17), (6.15) and (6.14) that there are $C_{k,\gamma} > 0$ and $c_{k,\gamma} > 0$, depending only on k and γ, such that

$$\begin{aligned} 4|\Lambda_0|\,|a_{n_j}(\phi)| &\ge |I_{1,2}| - |I_{1,1}| - |I_2| \ge C_{k,\gamma}|\Lambda_0|\,|L_j|^{-1/2} - c_{k,\gamma}|\Lambda_0|^2|L_j|^{-3/2} \\ &= |\Lambda_0|\,|L_j|^{-1/2}(C_{k,\gamma} - c_{k,\gamma}|\Lambda_0|\,|L_j|^{-1}). \end{aligned}$$

By property (6) in Lemma 6.2 we have $|\Lambda_0|\,|L_j|^{-1} \le 1/A$. Choosing A sufficiently large to guarantee

$$C_{k,\gamma} - \frac{c_{k,\gamma}}{A} \ge \frac{C_{k,\gamma}}{2},$$

we get a constant $m_{k,\gamma}$, depending only on k and γ, such that

$$m_{k,\gamma}|L_j|^{-1/2} \le |a_{n_j}(\phi)|, \qquad j = 0,\dots,\ell-1. \tag{6.18}$$

Next, we estimate $\int_{R_j} |g_{n_j}(t)|\,dt$ from below. First, Proposition 3.3, property (6) of Lemma 6.2 and the k-regularity of (t_n) yield

$$\int_{R_j} |g_{n_j}(t)|\,dt \gtrsim_{k,\gamma} |R_j|\,|w_{i_j}|,$$

([1]) Formula (4.2) is applied with $\mathcal{T}_n = \mathcal{T}_{n_j}$ and corresponding to $\tau_{i_0} = \tau_{n_j,i_j}$. Then $[\tau_{i_0-1},\tau_{i_0}] = L_j$ and $[\tau_{i_0},\tau_{i_0+1}] = R_j$. By k-regularity and $|\Lambda_0 \cup L_j| \sim_{k,\gamma} |L_j|$, each denominator in (4.2) is $\sim_{k,\gamma} |L_j|$. Each numerator in (4.2) is greater than either L_j or R_j, so by (6.12) and k-regularity it is $\sim_{k,\gamma} |L_j|$ as well.

where w_{i_j} corresponds to the partition $\mathcal{T}_{n_j}$. By definition of w_{i_j},

$$\int_{R_j} |g_{n_j}(t)|\,dt \gtrsim_{k,\gamma} |R_j|\,|\alpha_{i_j}|\,|b_{i_j,i_j}|.$$

By arguments similar to those above, $|\alpha_{i_j}|$ is bounded from below by a constant only depending on k and γ, and $|b_{i_j,i_j}| \gtrsim_k |D^{(k)}_{n_j,i_j}|^{-1}$. Since by k-regularity, $|R_j| \sim_{k,\gamma} |D^{(k)}_{n_j,i_j}|$, we finally get

$$\int_{R_j} |g_{n_j}(t)|\,dt \gtrsim_{k,\gamma} 1,$$

which means for f_{n_j} that

$$\int_{R_j} |f_{n_j}(t)|\,dt \gtrsim_{k,\gamma} |J_{n_j}|^{1/2} \gtrsim_{k,\gamma} |L_j|^{1/2}.$$

Combining this last estimate with (6.18) and (1) of Lemma 6.2 gives

$$\int_0^1 \sup_n |a_n(\phi) f_n(t)|\,dt \geq \sum_{j=1}^{\ell} \int_{R_j} |a_{n_j}(\phi) f_{n_j}(t)|\,dt \gtrsim_{k,\gamma} \ell.$$

This construction applies to every positive integer ℓ, proving the assertion of the proposition for $|\Lambda_0| > 0$.

The case $|\Lambda_0| = 0$ is handled similarly, with the difference that the atom ϕ is defined to be centered at τ_{n_0,i_0-1} and the length of the support is sufficiently small, depending on ℓ and $|L_0|$. ∎

With Proposition 6.1 and the results of Section 5 at hand, the proof of Theorem 2.4 follows the proof of Theorem 2.2 in [15], but we present it here for the sake of completeness.

Proof of Theorem 2.4. We start by proving the unconditional basis property of $(f_n) = (f_n^{(k)})$ assuming the $(k-1)$-regularity of (t_n). If (t_n) is $(k-1)$-regular, it is not difficult to check that it is also k-regular. As a consequence, Theorem 2.3 implies that (f_n) is a basis in H^1. Let $f \in H^1$ with $f = \sum a_n f_n$ and $\varepsilon \in \{-1,1\}^{\mathbb{Z}}$. We need to prove the convergence of $\sum \varepsilon_n a_n f_n$ in H^1. Let $m_1 \leq m_2$. Then

$$\begin{aligned}\Big\| \sum_{n=m_1}^{m_2} \varepsilon_n a_n f_n \Big\|_{H^1} &\lesssim_{k,\gamma} \Big\| S\Big(\sum_{n=m_1}^{m_2} \varepsilon_n a_n f_n\Big)\Big\|_1 \lesssim_k \Big\| P\Big(\sum_{n=m_1}^{m_2} \varepsilon_n a_n f_n\Big)\Big\|_1 \\ &= \Big\| P\Big(\sum_{n=m_1}^{m_2} a_n f_n\Big)\Big\|_1 \lesssim_{k,\gamma} \Big\| \sum_{n=m_1}^{m_2} a_n f_n \Big\|_{H^1}\end{aligned}$$

where we have used Propositions 5.4, 5.2 and 5.3 (cf. also the diagram on page 135). So, since $\sum a_n f_n$ converges in H^1, so does $f_\varepsilon := \sum \varepsilon_n a_n f_n$, and the same calculation as above shows

$$\|f_\varepsilon\|_{H^1} \lesssim_{k,\gamma} \|f\|_{H^1}.$$

This implies that (f_n) is an unconditional basis in H^1.

We now prove the converse: (f_n) being an unconditional basis in H^1 implies $(k-1)$-regularity. First, if (t_n) is not k-regular, (f_n) is not a basis in H^1 by Theorem 2.3. Thus, it remains to consider the case when (t_n) is k-regular, but not $(k-1)$-regular. By Theorem 2.3 again, (f_n) is then a basis in H^1. Suppose that (f_n) is an unconditional basis in H^1. Then, for $f = \sum a_n f_n$ and $\varepsilon \in \{-1,1\}^{\mathbb{Z}}$, the function $f_\varepsilon := \sum \varepsilon_n a_n f_n$ is also in H^1. Since $\|\cdot\|_1 \le \|\cdot\|_{H^1}$, the series $\sum a_n f_n$ also converges unconditionally in L^1, and thus Proposition 5.1 (i.e., Khinchin's inequality) implies

$$\|Pf\|_1 \lesssim \sup_\varepsilon \|f_\varepsilon\|_1 \le \sup_\varepsilon \|f_\varepsilon\|_{H^1} \lesssim \|f\|_{H^1},$$

which is impossible due to Proposition 6.1, even for atoms. This concludes the proof of Theorem 2.4. ∎

As an immediate consequence of Theorem 2.4, a fifth condition equivalent to (A)–(D) is the unconditional convergence of $\sum_n a_n f_n$ in H^1:

Corollary 6.3. *Let (t_n) be a k-admissible and $(k-1)$-regular sequence of points, with (f_n) the corresponding orthonormal spline system of order k. Let (a_n) be a sequence of coefficients. Then conditions* (A)–(D) *from Section* 5 *are equivalent. Moreover, they are equivalent to*

(E) *The series $\sum_n a_n f_n$ converges unconditionally in H^1.*

In addition, for $f \in H^1$, $f = \sum_n a_n f_n$, we have

$$\|f\|_{H^1} \sim \|Sf\|_1 \sim \|Pf\|_1 \sim \sup_{\varepsilon \in \{-1,1\}^{\mathbb{Z}}} \Big\| \sum_n \varepsilon_n a_n f_n \Big\|_1,$$

with the implied constants depending only on k and the parameter of $(k-1)$-regularity of (t_n).

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References

[1] S.-Y. A. Chang and Z. Ciesielski, *Spline characterizations of H^1*, Studia Math. 75 (1983), 183–192.

[2] Z. Ciesielski, *Properties of the orthonormal Franklin system*, Studia Math. 23 (1963), 141–157.

[3] Z. Ciesielski, *Properties of the orthonormal Franklin system. II*, Studia Math. 27 (1966), 289–323.

[4] Z. Ciesielski, *Orthogonal projections onto spline spaces with arbitrary knots*, in: Function Spaces (Poznań, 1998), Lecture Notes in Pure Appl. Math. 213, Dekker, New York, 2000, 133–140.

[5] Z. Ciesielski and A. Kamont, *Projections onto piecewise linear functions*, Funct. Approx. Comment. Math. 25 (1997), 129–143.

[6] R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. 83 (1977), 569–645.

[7] C. de Boor, *On the convergence of odd-degree spline interpolation*, J. Approx. Theory 1 (1968), 452–463.

[8] C. de Boor, *The quasi-interpolant as a tool in elementary polynomial spline theory*, in: Approximation Theory (Austin, TX, 1973), Academic Press, New York, 1973, 269–276.

[9] C. de Boor, *On a max-norm bound for the least-squares spline approximant*, in: Approximation and Function Spaces (Gdańsk, 1979), North-Holland, Amsterdam, 1981, 163–175.

[10] R. A. DeVore and G. G. Lorentz, *Constructive Approximation*, Grundlehren Math. Wiss. 303, Springer, Berlin, 1993.

[11] J. Domsta, *A theorem on B-splines*, Studia Math. 41 (1972), 291–314.

[12] G. G. Gevorkyan, *Some theorems on unconditional convergence and the majorant of Franklin series and their application to* $\operatorname{Re} H^p$, Trudy Mat. Inst. Steklova 190 (1989), 49–74 (in Russian).

[13] G. G. Gevorkyan and A. Kamont, *On general Franklin systems*, Dissertationes Math. 374 (1998), 59 pp.

[14] G. G. Gevorkyan and A. Kamont, *Unconditionality of general Franklin systems in* $L^p[0,1], 1 < p < \infty$, Studia Math. 164 (2004), 161–204.

[15] G. G. Gevorkyan and A. Kamont, *General Franklin systems as bases in* $H^1[0,1]$, Studia Math. 167 (2005), 259–292.

[16] G. G. Gevorkyan and A. Kamont, *Orthonormal spline systems with arbitrary knots as bases in* $H_1[0,1]$, East J. Approx. 14 (2008), 161–182.

[17] G. G. Gevorkyan and A. A. Sahakian, *Unconditional basis property of general Franklin systems*, Izv. Nats. Akad. Nauk Armenii Mat. 35 (2000), no. 4, 7–25 (in Russian).

[18] S. Karlin, *Total Positivity. Vol. I*, Stanford Univ. Press, Stanford, CA, 1968.

[19] B. S. Kashin and A. A. Saakyan, *Orthogonal Series*, Transl. Math. Monogr. 75, Amer. Math. Soc., Providence, RI, 1989.

[20] M. Passenbrunner, *Unconditionality of orthogonal spline systems in* L^p, Studia Math. 222 (2014), 51–86.

[21] M. Passenbrunner and A. Shadrin, *On almost everywhere convergence of orthogonal spline projections with arbitrary knots*, J. Approx. Theory 180 (2014), 77–89.

[22] A. Shadrin, *The* L_∞*-norm of the* L_2*-spline projector is bounded independently of the knot sequence: a proof of de Boor's conjecture*, Acta Math. 187 (2001), 59–137.

[23] P. Sjölin and J. O. Strömberg, *Basis properties of Hardy spaces*, Ark. Mat. 21 (1983), 111–125.
[24] M. v. Golitschek, *On the L^∞-norm of the orthogonal projector onto splines. A short proof of A. Shadrin's theorem*, J. Approx. Theory 181 (2014), 30–42.

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