On Lindenstrauss–Pełczyński spaces

by

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Abstract. We consider some stability aspects of the classical problem of extension of $C(K)$-valued operators. We introduce the class $\mathcal{LP}$ of Banach spaces of Lindenstrauss–Pełczyński type as those such that every operator from a subspace of c_0 into them can be extended to c_0. We show that all $\mathcal{LP}$-spaces are of type $\mathcal{L}_\infty$ but not conversely. Moreover, $\mathcal{L}_\infty$-spaces will be characterized as those spaces E such that E-valued operators from $w^*(l_1, c_0)$-closed subspaces of l_1 extend to l_1. Regarding examples we will show that every separable $\mathcal{L}_\infty$-space is a quotient of two $\mathcal{LP}$-spaces; also, $\mathcal{L}_\infty$-spaces not containing c_0 are $\mathcal{LP}$-spaces; the complemented subspaces of $C(K)$ and the separably injective spaces are subclasses of the $\mathcal{LP}$-spaces and we show that the former does not contain the latter. Regarding stability properties, we prove that quotients of an $\mathcal{LP}$-space by a separably injective space and twisted sums of $\mathcal{LP}$-spaces are $\mathcal{LP}$-spaces.

1. Introduction and preliminaries. In this work we shall be concerned with some stability aspects of the classical problem of extension of $C(K)$-valued and $\mathcal{L}_\infty$-valued operators. Let us describe and motivate them. In a 1971 paper [26] Lindenstrauss and Pełczyński proved:

THEOREM 1. *Let K be a compact Hausdorff space. Every $C(K)$-valued operator defined on a subspace of c_0 admits an extension to the whole space.*

The result remained isolated until 1989 when Johnson and Zippin obtained in [17] an extension to subspaces of $c_0(\Gamma)$, and later in 1995, in [18], the analogous result for $w(l_1, c_0)$-closed subspaces of l_1. Further proofs of the Lindenstrauss–Pełczyński theorem have been provided by Zippin [34, 35]. The paper [8] contains a homological approach to both results showing that they are in a sense dual to each other.

The general problem of extension of operators admits a natural formulation in homological terms. We shall assume from the reader some familiarity

with the basic notions and constructions of the theory of exact sequences of Banach spaces; the necessary background can be found in [9] and, operatively defined, below. We shall write $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0 \equiv F$ to represent an exact sequence of Banach spaces and operators, which is a diagram where the kernel of each operator coincides with the image of the preceding one. The open mapping theorem makes Y a subspace of X through the embedding j and Z the corresponding quotient space through q. The reader can view F just as the name of the sequence; however, those familiar with the theory of quasi-linear maps created in [19, 22] can in fact consider F as a quasi-linear map associated to the exact sequence.

We shall consider exact sequences of Banach spaces modulo the natural equivalence relation: two sequences F and G are said to be equivalent if there is a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\ j\ } & X & \longrightarrow & Z & \longrightarrow & 0 \equiv F \\
& & \| & & T\downarrow & & \| & & \\
0 & \longrightarrow & Y & \longrightarrow & X' & \longrightarrow & Z & \longrightarrow & 0 \equiv G
\end{array}
$$

In this case we write $F \equiv G$. The space of equivalence classes of exact sequences with Y as subspace and Z as quotient will be denoted $\mathrm{Ext}(Z, Y)$. It is a vector space under some natural operations (see [13, III.2]) and the 0 element is the sequence $0 \to Y \to Y \oplus Z \to Z \to 0$ with inclusion $y \mapsto (y, 0)$ and quotient map $(y, z) \mapsto z$. We shall say that F is *trivial* or *splits* when $F \equiv 0$. This means, in classical terms, that $j(Y)$ is complemented in X. Recall that a property $\mathcal{P}$ is said to be a 3-*space property* if whenever one has an exact sequence $0 \to Y \to X \to Z \to 0$ in which both Y and Z have $\mathcal{P}$ then also X has $\mathcal{P}$; see [9] for most of the available information about 3-space problems.

The lower sequence in a diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\ j\ } & X & \xrightarrow{\ q\ } & Z & \longrightarrow & 0 \equiv F \\
& & \| & & \uparrow & & \uparrow T & & \\
0 & \longrightarrow & Y & \longrightarrow & X' & \longrightarrow & E & \longrightarrow & 0
\end{array}
$$

is a called a *pull-back sequence* and is naturally denoted FT. The middle space X' is called the pull-back of q and T. The sequence FT splits if and only if T can be lifted to X through q. The lower sequence in a diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\ j\ } & X & \longrightarrow & Z & \longrightarrow & 0 \equiv F \\
& & T\downarrow & & \downarrow t & & \| & & \\
0 & \longrightarrow & E & \xrightarrow{\ i\ } & X' & \longrightarrow & Z & \longrightarrow & 0
\end{array}
$$

is called the *push-out sequence* and is naturally denoted TF. The middle space X' is called the push-out of j and T. This space has the universal property that given operators $a : E \to B$ and $b : X \to B$ such that $aT = bj$ there exists a unique operator $p : X' \to B$ such that $pi = a$ and $pt = b$. Extending an operator $T : Y \to E$ through j is the same as saying that TF is trivial. The lifting property of $l_1(\Gamma)$ and the fact that every Banach space Z admits an exact sequence $0 \to K(Z) \to l_1(\Gamma) \to Z \to 0$, called a *projective presentation* of Z, imply that every exact sequence $0 \to Y \to X \to Z \to 0$ is a push-out of a projective presentation of Z. Hence $\mathrm{Ext}(Z, Y) = 0$ is equivalent to the statement "every operator $K(Z) \to Y$ can be extended to $l_1(\Gamma)$".

That all operators $Y \to E$ can be extended to X through j admits an even simpler formulation: the restriction operator $j^* : \mathcal{L}(X, E) \to \mathcal{L}(Y, E)$ is surjective. The following terminology is quite natural and will prove to be very useful:

DEFINITION 1. Let $\mathcal{A}$ be a class of Banach spaces. We say that an exact sequence $0 \to Y \xrightarrow{j} X \to Z \to 0 \equiv F$ is $\mathcal{A}$-*trivial* (or that F $\mathcal{A}$-*splits*) if for every $A \in \mathcal{A}$ the restriction operator $j^* : \mathcal{L}(X, A) \to \mathcal{L}(Y, A)$ is surjective. We will also say that Y is $\mathcal{A}$-*complemented* in X.

Sometimes the quantitative version of the previous notion will be used: given $\lambda \geq 1$, the exact sequence F will be said to be $(\lambda, \mathcal{A})$-*trivial* if for every $A \in \mathcal{A}$ every operator $T : Y \to A$ admits an extension $\widehat{T} : X \to A$ such that $\|\widehat{T}\| \leq \lambda \|T\|$.

This notion of $\mathcal{A}$-triviality unifies different notions appearing in the literature: (i) trivial sequences, which correspond to $\mathcal{A} = $ all Banach spaces; (ii) Kalton's *locally trivial*, or locally split, sequences (see [20]), corresponding to $\mathcal{A} = l_\infty(G_n)$, where G_n is a dense (in the Banach–Mazur distance) sequence of finite-dimensional Banach spaces (see also [14]); (iii) Zippin's *almost trivial* sequences (see [33–36]), which correspond to the choice $\mathcal{A} = C(K)$-spaces.

In this work we are concerned with $C(K)$-trivial and $\mathcal{L}_\infty$-trivial sequences. In Section 2 we study the stability of $C(K)$-trivial sequences by amalgams and duality. We first show that l_p- and c_0-amalgams of $C(K)$-trivial sequences are $C(K)$-trivial. Concerning the stability of $C(K)$-trivial sequences by duality, the Lindenstrauss–Pełczyński and Johnson–Zippin theorems suggest that it could be that the dual of a $C(K)$-trivial sequence is $C(K)$-trivial since the former implies that every exact sequence $0 \to H \to X \to S \to 0 \equiv F$ with H a subspace of c_0 and S separable is $C(K)$-trivial; and the latter yields (see [10]) that its dual sequence F^* is $C(K)$-trivial. However, the situation covered by those two theorems proves to be quite peculiar; we give examples at the end of Section 2 to show that the dual and bidual sequences of a $C(K)$-trivial sequence need not be $C(K)$-trivial.

In Section 3 our attention turns to those Banach spaces which can play the role of $C(K)$-spaces in the Lindenstrauss–Pełczyński theorem. We call such spaces Lindenstrauss–Pełczyński ($\mathcal{LP}$, for short) spaces. Our motivation to introduce those spaces comes from [26, p. 234, remark 2] where Lindenstrauss and Pełczyński assert that isometric L_1-preduals can play the role of $C(K)$-spaces regarding extension of operators from subspaces of c_0. After showing that every $\mathcal{LP}$-space is an $\mathcal{L}_\infty$-space we then face the unavoidable question: Must every $\mathcal{L}_\infty$-space be an $\mathcal{LP}$-space? The answer is no, which solves Problem 6.15 of Zippin in [36]. We shall show that the same approach to the Johnson–Zippin theorem just provides a new characterization of $\mathcal{L}_\infty$-spaces (see Proposition 3.1).

As regards the problem of the identification of $\mathcal{LP}$-spaces, it is clear that complemented subspaces of $C(K)$-spaces and separably injective spaces are $\mathcal{LP}$-spaces. On the way, we will give perhaps the first example of a separably injective space that is not complemented in any $C(K)$-space. The previous examples do not exhaust the class of $\mathcal{LP}$-spaces: we shall show that $\mathcal{L}_\infty$-spaces not containing c_0, the new exotic $\mathcal{L}_\infty$-spaces constructed in [7], quotients of $\mathcal{LP}$-spaces by separably injective subspaces and c_0-vector sums of *uniformly* $\mathcal{LP}$-spaces are $\mathcal{LP}$-spaces.

In Section 4 we tackle the 3-space problem for the class of $\mathcal{LP}$-spaces, which needs the development of a new method of proof and new characterizations of $\mathcal{LP}$-spaces. Section 5 contains further remarks, examples and open problems.

2. On the stability of $C(K)$-trivial sequences by amalgams and duality. To study the stability of $C(K)$-trivial sequences we need to know their behavior with respect to the basic homological pull-back and push-out constructions.

PROPOSITION 2.1. *Let $\mathcal{A}$ be a class of Banach spaces.*

(1) *A pull-back sequence of an $\mathcal{A}$-trivial sequence is $\mathcal{A}$-trivial.*
(2) *A push-out sequence of an $\mathcal{A}$-trivial sequence is $\mathcal{A}$-trivial.*

Proof. The first assertion is obvious. The second is a consequence of the universal property of the push-out construction. ∎

When, moreover, the push-out is obtained from a surjective operator one has:

LEMMA 1. *Let $\mathcal{A}$ be a class of Banach spaces. Consider the completed push-out diagram of Banach spaces*

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
B & = & B \\
\downarrow a & & \downarrow b
\end{array}
$$

$$
0 \longrightarrow Y \xrightarrow{\ j\ } X \longrightarrow Z \longrightarrow 0 \equiv F
$$

$$
\downarrow c \qquad \downarrow d \qquad \parallel
$$

$$
0 \longrightarrow C \xrightarrow{\ i\ } D \longrightarrow Z \longrightarrow 0 \equiv G
$$

$$
\begin{array}{ccc}
\downarrow & & \downarrow \\
0 & & 0 \\
\parallel\!\parallel\!\parallel & & \parallel\!\parallel\!\parallel \\
H & & V
\end{array}
$$

(i) V and G are $\mathcal{A}$-trivial if and only if F and H are $\mathcal{A}$-trivial.

(ii) If F is $\mathcal{A}$-trivial then G is $\mathcal{A}$-trivial; if V is $\mathcal{A}$-trivial, H is $\mathcal{A}$-trivial.

Proof. The second part of (ii) follows from Proposition 2.1(1). Let $A \in \mathcal{A}$ and notice that from the diagram in the hypothesis we can construct the commutative diagram

$$
\begin{array}{ccc}
\uparrow & & \uparrow \\
\mathcal{L}(B, A) & = & \mathcal{L}(B, A) \\
\uparrow b^* & & \uparrow a^*
\end{array}
$$

$$
0 \longrightarrow \mathcal{L}(Z, A) \longrightarrow \mathcal{L}(X, A) \xrightarrow{\ j^*\ } \mathcal{L}(Y, A) \longrightarrow
$$

$$
\parallel \qquad \uparrow d^* \qquad \uparrow c^*
$$

$$
0 \longrightarrow \mathcal{L}(Z, A) \longrightarrow \mathcal{L}(D, A) \xrightarrow{\ i^*\ } \mathcal{L}(C, A) \longrightarrow
$$

$$
\begin{array}{ccc}
\uparrow & & \uparrow \\
0 & & 0
\end{array}
$$

Now, the first part of (ii) can be easily obtained by diagram chasing while (i) follows by simply observing that the restriction operators b^* and i^* are surjective if and only if j^* and a^* are surjective. ∎

REMARK. Pełczyński's Proposition 2.6 of [30] can be considered a rudimentary version of this principle.

The characterization of $C(K)$-trivial extensions that Zippin formulates and proves in [33] is especially interesting for us.

LEMMA 2. *A sequence* $0 \to Y \xrightarrow{j} X \to Z \to 0$ *is* $(\lambda, C(K))$-*trivial if and only if there is a* w^*-*continuous map* $\omega : B_{Y^*} \to \lambda B_{X^*}$ *such that* $j^*\omega = \mathrm{id}$.

The map ω will be called a λ-w^*-*selector* for j^*. Zippin [34, 35] uses this criterion to obtain different proofs of the Lindenstrauss–Pełczyński theorem. It is inspired by the most natural possible situation: the sequence $0 \to Y \xrightarrow{\delta_Y} C(B_{Y^*}) \to C(B_{Y^*})/Y \to 0 \equiv \complement_Y$, in which $\delta_Y : Y \to C(B_{Y^*})$ is the canonical embedding, is $(1, C(K))$-trivial; indeed, the map $\omega : B_{Y^*} \to B_{C(B_{Y^*})}$ defined as $\omega(x^*)(f) = f(x^*)$ is a 1-w^*-selector for δ_Y^*. Let us remark that every $C(K)$-trivial sequence is a pull-back of $\complement_Y$ and conversely.

It will be useful to notice that some properties of the w^*-topology in l_p, $1 \le p \le \infty$, pass to l_p-vector sums of Banach spaces. Given an l_p-sum $l_p(X_n)$ we denote by $\pi_j : l_p(X_n) \to X_j$ the natural projections. It is straightforward that given a sequence $(E_n^*)_n$ of dual spaces, a bounded net $(x_\alpha)_\alpha$ in $l_p(E_n^*)$, $1 \le p \le \infty$, is w^*-null if and only if for each j the net $(\pi_j(x_\alpha))_\alpha$ is w^*-null.

Given a family of exact sequences $0 \to A_n \to B_n \to C_n \to 0 \equiv F_n$, we call $0 \to l_p(A_n) \to l_p(B_n) \to l_p(C_n) \to 0 \equiv l_p(F_n)$ the l_p-*amalgam* of $(F_n)_n$, for $1 \le p \le \infty$. Analogously, the c_0-*amalgam* of (F_n) will be denoted $c_0(F_n)$. One has:

PROPOSITION 2.2. *The* c_0- *and* l_p-*amalgams of* $(\lambda, C(K))$-*trivial exact sequences,* $1 \le p < \infty$, *are* $(\lambda, C(K))$-*trivial*.

Proof. For each n, let $\omega_n : B_{A_n^*} \to \lambda B_{B_n^*}$ be a λ-w^*-selector for j_n^*. If we have the l_p-amalgam

$$0 \to l_p(A_n) \xrightarrow{\chi} l_p(B_n) \to l_p(C_n) \to 0 \equiv l_p(F_n),$$

it follows from the observation above that the map $\Omega : B_{l_{p^*}(A_n^*)} \to \lambda B_{l_{p^*}(B_n^*)}$ defined by $\Omega[(a_n^*)] = [\omega_n(a_n^*)]$ is a λ-w^*-selector for χ^*. ∎

The situation for l_∞-amalgams is entirely different because a subspace X of l_∞ can only be $C(K)$-complemented if X enjoys both the Dunford–Pettis property (weakly compact operators transform weakly compact sets into relatively compact sets) and the Grothendieck property (all operators with separable range are weakly compact): Indeed, if E is separable and $\tau : X \to E$ is an operator, then $\delta_E \tau : X \xrightarrow{\tau} E \xrightarrow{\delta_E} C(B_{E^*})$ should extend to an operator $l_\infty \to C(B_{E^*})$; by the Grothendieck property of l_∞ (see [12, Cor. 12, p. 156]) this operator is weakly compact, hence $\delta_E \tau$ is weakly compact, as also is τ. We have shown that X has the Grothendieck property. Since l_∞ has the Dunford–Pettis property, every weakly compact operator $X \to c_0$ must do the same, which means that X also enjoys the Dunford–

Pettis property. In particular, subspaces of l_∞ such as $l_\infty(l_2^n)$, which does not have the Dunford–Pettis property because it contains a complemented copy of l_2, provide sequences $0 \to l_\infty(l_2^n) \to l_\infty \to Q \to 0$ that are not $C(K)$-trivial. Therefore, the l_∞-amalgam of the sequences

$$0 \to l_2^n \to l_\infty^{m(n)} \to l_\infty^{m(n)}/l_2^n \to 0$$

in which the embeddings are $(1+\varepsilon)$-isometries cannot be $C(K)$-trivial. Nevertheless, those sequences are all $((1+\varepsilon)^2, C(K))$-trivial by the existence of the Bartle–Graves continuous selection.

Later on we shall also show that the c_0-amalgam of $\mathcal{L}_\infty$-trivial sequences is not necessarily $\mathcal{L}_\infty$-trivial. Our concern now is to study the stability of $C(K)$-trivial sequences by duality. Let us start by observing that, for every subspace H of c_0 and every separable Banach space S, the sequence $0 \to H \to X \to S \to 0 \equiv F$ is $C(K)$-trivial as is its dual F^*. That F is $C(K)$-trivial was observed by Lindenstrauss and Pełczyński in [26, Cor. 2]. The assertion about F^* (actually, that every exact sequence $0 \to Y \to X \to H^* \to 0$ is $C(K)$-trivial for every subspace H of c_0) directly follows from the equality $\mathrm{Ext}(H^*, C(K)) = 0$, which is a consequence of the Johnson–Zippin theorem. It is not true, in general, that the dual or bidual of a $C(K)$-trivial sequence is $C(K)$-trivial:

EXAMPLES. Consider the $C(K)$-trivial sequence $0 \to l_2 \xrightarrow{\delta} C(B_{l_2}) \to Q \to 0 \equiv F$. We claim that the dual sequence $0 \to Q^* \to L_1 \xrightarrow{\delta^*} l_2 \to 0 \equiv F^*$ is not $C(K)$-trivial. Assume otherwise. Consider a projective presentation of l_2,

$$0 \to K(l_2) \to l_1 \to l_2 \to 0 \equiv P.$$

The space $K(l_2)$ is complemented in its bidual (see [23]). Hence $\mathrm{Ext}(L_1, K(l_2)) = 0$ by the Lindenstrauss lifting principle (see [25, 23], but also [5]). Thus, the lower pull-back sequence in the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K(l_2) & \longrightarrow & l_1 & \longrightarrow & l_2 & \longrightarrow & 0 \equiv P \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle \delta^*} & & \\
0 & \longrightarrow & K(l_2) & \longrightarrow & PB & \longrightarrow & L_1 & \longrightarrow & 0 \equiv P\delta^*
\end{array}
$$

splits, and therefore the quotient map $\delta^* : L_1 \to l_2$ can be lifted to l_1. In this way, there is an operator ϕ such that $P \equiv \phi F^*$ (see [10]) and P is a push-out of F^*; by Lemma 2.1, it must also be $C(K)$-trivial. But if a projective presentation of l_2 is $C(K)$-trivial then $\mathrm{Ext}(l_2, C(K)) = 0$, and it was proved by Kalton in [21] (see also [7]) that $\mathrm{Ext}(l_2, C[0,1]) \neq 0$.

The bidual sequence

$$0 \to l_2 \xrightarrow{\delta^{**}} C(B_{l_2})^{**} \to Q^{**} \to 0 \equiv F^{**}$$

is also not $C(K)$-trivial: since $I = C(B_{l_2})^{**}$ is injective, it has the Grothen-
dieck property (see [12, Cor. 12, p. 156]) and thus every operator $I \to S$ into
a separable space is weakly compact. By the Dunford–Pettis property of I,
weakly compact operators on I transform weakly compact sets onto norm
compact sets; thus, the canonical inclusion $l_2 \to C(B_{l_2})$ cannot be extended
to I. Another example is provided by sequences having the form

$$0 \to c_0(A_n) \to c_0(l_\infty^{m(n)}) \to c_0(C_n) \to 0 \equiv F;$$

they have the property that F and F^* do $C(K)$-split although F^{**} does not
necessarily $C(K)$-split.

3. On Banach spaces of Lindenstrauss–Pełczyński type. There
is an obvious difference between the Lindenstrauss–Pełczyński [26] and the
Johnson–Zippin [18] theorems. While the former asserts that every sequence
$0 \to H \to c_0 \to c_0/H \to 0 \equiv F$ is $C(K)$-trivial the latter establishes that
the dual sequence F^* is $\mathcal{L}_\infty$-trivial. Let us see that the class $\mathcal{L}_\infty$ cannot be
enlarged, obtaining in this way a new characterization of $\mathcal{L}_\infty$-spaces.

PROPOSITION 3.1. *For a Banach space E the following are equivalent*:

(1) *E is an $\mathcal{L}_\infty$-space.*
(2) *Every E-valued operator defined on a $w(l_1, c_0)$-closed subspace of l_1
can be extended to l_1.*

Proof. Recall that $w(l_1, c_0)$-closed subspaces of l_1 are precisely the or-
thogonal complements $H^\perp$ to subspaces H of c_0. So, as we said in Section 1,
(2) can be written as: $\mathrm{Ext}(H^*, E) = 0$ for every subspace H of c_0.

General structure results of Johnson–Rosenthal and Zippin (see [28, 1.g.2
and 2.d.1]) imply that given a subspace H of c_0 there exist sequences (A_n)
and (B_n) of finite-dimensional spaces such that there is an exact sequence
$0 \to c_0(A_n) \to H \to c_0(B_n) \to 0$. Therefore, there is an exact sequence
$0 \to l_1(B_n^*) \to H^* \to l_1(A_n^*) \to 0$. A simple 3-space argument (see [6] or [7,
Cor. 1.2]) then shows that $\mathrm{Ext}(H^*, E) = 0$ is equivalent to $\mathrm{Ext}(l_1(A_n^*), E) =
0 = \mathrm{Ext}(l_1(B_n^*), E)$. Therefore, (2) is equivalent to $\mathrm{Ext}(l_1(G_n), E) = 0$ for
every sequence (G_n) of finite-dimensional spaces. It was already observed by
Johnson [14] that a sequence $0 \to Y \to X \to Z \to 0 \equiv F$ locally splits if and
only if $FT \equiv 0$ for every operator $T : l_1(G_n) \to Z$ with G_n finite-dimensional.
Thus, every sequence $0 \to E \to X \to Z \to 0$ locally splits and in particular
so does any sequence $0 \to E \to l_\infty(I) \to Q \to 0 \equiv G$. Recall from [20] that
an exact sequence F locally splits if and only if F^{**} splits. Hence, the bidual
sequence G^{**} splits, E^{**} must be complemented in an $\mathcal{L}_\infty$-space, so it is an
$\mathcal{L}_\infty$-space and E must itself be an $\mathcal{L}_\infty$-space. ∎

Not entirely trivial is the observation that condition (2) can be replaced
by $(2')$ for every set Γ, every E-valued operator defined on a $w(l_1(\Gamma), c_0(\Gamma))$-

closed subspace of $l_1(\Gamma)$ can be extended to $l_1(\Gamma)$. The proof only requires to take into consideration the decomposition lemma of [17].

The situation outlined for $w^*(l_1, c_0)$-closed subspaces of l_1, together with Lindenstrauss–Pełczyński's remark in [26] asserting that operators with ranges being isometric preduals of L_1 extend from subspaces of c_0 to the whole space, suggest investigating how much the class of $C(K)$-spaces can be enlarged in the Lindenstrauss–Pełczyński theorem.

DEFINITION 2. We shall say that a Banach space E is a *Lindenstrauss–Pełczyński space*, for short an $\mathcal{LP}$-space, if all operators from subspaces of c_0 into E can be extended to c_0.

We shall also need the quantitative version: when every operator $T : H \to E$ admits an extension $\widehat{T} : c_0 \to E$ such that $\|\widehat{T}\| \leq \lambda\|T\|$ we shall say that E is an $\mathcal{LP}_\lambda$-space. It is not hard to see that every $\mathcal{LP}$-space is an $\mathcal{LP}_\lambda$-space for some λ.

LEMMA 3. *An $\mathcal{LP}_\lambda$-space is, for every $\varepsilon > 0$, an $\mathcal{L}_{\infty, 2\lambda+\varepsilon}$-space.*

Proof. Let E be an $\mathcal{LP}_\lambda$-space. Let $T : Y \to E$ be a compact operator from a subspace Y of a separable space X. Schauder's theorem (T is compact if and only if T^* is compact) plus the characterization of compact sets in Banach spaces as subsets of the closed convex hull of a norm null sequence immediately show that T factorizes through some subspace $i : H \to c_0$ as $T = BA$ with $A : Y \to H$ and $B : H \to E$. By definition, there is an extension $B_1 : c_0 \to E$ of B with $\|B_1\| \leq \lambda\|B\|$; moreover Sobczyk's theorem gives an extension $A_1 : X \to c_0$ of iA with $\|A_1\| \leq 2\|iA\|$. The composition $B_1A_1 : X \to E$ extends T and satisfies $\|B_1A_1\| \leq 2\lambda\|T\|$.

Let us now show that "if E-valued compact norm one operators can be extended with norm at most c then for every $\varepsilon > 0$, E is an $\mathcal{L}_{\infty,c+\varepsilon}$-space". To this end, let $\tau : Y \to E$ be any (compact or not) norm one operator. Each restriction $\tau_F : F \to E$ of τ to a finite-dimensional subspace F of Y is compact and can therefore be extended to an operator $T_F : X \to E$ with norm at most c. Define the operator $T : X \to E^{**}$ by

$$T(x) = w^*\text{-}\lim_{\mathcal{U}(F)} T_F(x)$$

where $\mathcal{U}$ is a free ultrafilter on the set of finite-dimensional subspaces of Y compatible with the natural ordering. This operator T has norm at most c and extends $\tau : Y \to E^{**}$. In particular, given an exact sequence $0 \to E \to V \to Q \to 0$, the canonical embedding $\delta : E \to E^{**}$ can be extended to V with norm at most c. So E^{**} is complemented in V^{**} with norm $c + \varepsilon$ (because of the principle of local reflexivity). Applying this to any embedding $0 \to E \to l_\infty(I) \to Q \to 0$ one finds that E^{**} is an $\mathcal{L}_{\infty,c+\varepsilon}$-space, as also is E. ∎

The above argument is modeled over Lindenstrauss' construction in [24]. A direct application of Lindenstrauss' result is awkward: as the referee of the paper remarked, in his memoir, Lindenstrauss considered what he calls $\mathcal{N}_\lambda$-spaces, which are spaces that are the closure of a directed union of finite-dimensional λ-injective spaces; today we know that an $\mathcal{N}_\lambda$-space is an $\mathcal{L}_{\infty,\beta}$-space, but we do not know how β depends on λ.

The converse fails: we show that not every $\mathcal{L}_\infty$-space is an $\mathcal{LP}$-space. This solves Zippin's Problem 6.15 in [36]. The example (which was sketched in [8]) is based on the Bourgain–Pisier construction [3] which shows that for every separable Banach space X there is an exact sequence

$$0 \to X \to \mathcal{L}_\infty(X) \to \mathcal{L}_\infty(X)/X \to 0 \equiv \mathrm{BP}_X,$$

in which $\mathcal{L}_\infty(X)$ is a separable $\mathcal{L}_\infty$-space and $\mathcal{L}_\infty(X)/X$ has the Schur property (weakly convergent sequences are norm convergent).

PROPOSITION 3.2. *Let H be a subspace of c_0 such that c_0/H is not isomorphic to c_0. Then $\mathcal{L}_\infty(H)$ is not an $\mathcal{LP}$-space. In fact, there exists an operator $H \to \mathcal{L}_\infty(H)$ which does not extend to c_0.*

Proof. Consider the sequence $0 \to H \xrightarrow{j} c_0 \to c_0/H \to 0 \equiv F$. Let $0 \to H \xrightarrow{i} \mathcal{L}_\infty(H) \to S \to 0 \equiv \mathrm{BP}_H$ be a Bourgain–Pisier sequence associated with H. Assume that i extends to c_0 through j, so that F is a pull-back of BP_H. By Sobczyk's theorem [28, 2.f.5], BP_H is a pull-back of F. Applying the diagonal principle [27, Thm. 2] one gets an isomorphism

$$\mathcal{L}_\infty(H) \oplus c_0/H \simeq c_0 \oplus S.$$

In particular, c_0/H is a complemented subspace of $c_0 \oplus S$. Since S and c_0 are totally incomparable by the Schur property of S the decomposition theorem of Edelstein–Wojtaszczyk (see [28, Thm. 2.c.13]) ensures that c_0/H is isomorphic to some $A \oplus B$ with A complemented in c_0 and B complemented in S. Since c_0/H is a subspace of c_0 [28, 2.f.6], B can only be finite-dimensional, hence $c_0/H \simeq c_0$, contrary to hypothesis. ∎

This example immediately implies

EXAMPLE. *The c_0-amalgam of $(\mu, \mathcal{L}_\infty)$-trivial sequences is not necessarily $\mathcal{L}_\infty$-trivial.*

Proof. As already mentioned, given a subspace H of c_0 there exist sequences (A_n) and (B_n) of finite-dimensional spaces such that there is an exact sequence $0 \to c_0(A_n) \to H \to c_0(B_n) \to 0$. Clearly, the exact sequences $0 \to A_n \to l_\infty^{m(n)} \to C_n \to 0$ are $(\lambda, \mathcal{L}_{\infty,\lambda})$-trivial. If all the amalgams $0 \to c_0(A_n) \to c_0(l_\infty^{m(n)}) \to c_0(C_n) \to 0$ were $\mathcal{L}_\infty$-trivial then every sequence $0 \to H \to c_0 \to c_0/H \to 0$ would also be $\mathcal{L}_\infty$-trivial: indeed, there

would be a complete push-out diagram

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
c_0(A_n) & = & c_0(A_n) \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle i} \\
\end{array}
$$

$$
0 \longrightarrow \quad H \quad \overset{j}{\longrightarrow} \quad c_0 \quad \longrightarrow \quad c_0/H \quad \longrightarrow 0 \equiv F
$$

$$
\begin{array}{ccc}
\downarrow{\scriptstyle b} & \downarrow{\scriptstyle c} & \| \\
\end{array}
$$

$$
0 \longrightarrow c_0(B_n) \overset{}{\underset{d}{\longrightarrow}} Z \longrightarrow Z/c_0(B_n) \longrightarrow 0 \equiv bF
$$

$$
\begin{array}{ccc}
\downarrow & & \downarrow \\
0 & & 0 \\
||| & & ||| \\
Vd & & V
\end{array}
$$

to which we apply Lemma 1(i): if V and bF are $\mathcal{L}_\infty$-trivial then F is $\mathcal{L}_\infty$-trivial (which we know it is not). It remains to check that V and bF are $\mathcal{L}_\infty$-trivial. The sequence V is $\mathcal{L}_\infty$-trivial by our assumption and the Lindenstrauss–Rosenthal theorem that asserts that c_0 is automorphic (see [27, 10]); i.e., there exists an isomorphism $\tau : c_0 \to c_0$ making the diagram

$$
\begin{array}{ccccccc}
0 \longrightarrow & c_0(A_n) & \longrightarrow & c_0 & \longrightarrow & Z & \longrightarrow 0 \\
 & \| & & \downarrow{\scriptstyle \tau} & & \downarrow & \\
0 \longrightarrow & c_0(B_n) & \longrightarrow & c_0(l_\infty^{m(n)}) & \longrightarrow & c_0(l_\infty^{m(n)}/A_n) & \longrightarrow 0
\end{array}
$$

commutative and therefore the two sequences $\mathcal{L}_\infty$-split simultaneously. The sequence bF is $\mathcal{L}_\infty$-trivial by essentially the same arguments taking into account that Z must be a subspace of c_0. ∎

The problem of identifying $\mathcal{LP}$-spaces is still far from being solved, and it actually gives rise to interesting questions. Observe that, in addition to $C(K)$-spaces, it is clear that complemented subspaces of $C(K)$-spaces and separably injective spaces are also $\mathcal{LP}$-spaces. The reader might be surprised by the distinction between the two, especially regarding the fact that every injective space is complemented in some $C(K)$-space. Let us show that the two classes are indeed distinct.

PROPOSITION 3.3. *There exists a separably injective space that is not complemented in any* $C(K)$-*space.*

Proof. Consider the pull-back diagram

$$0 \longrightarrow c_0 \longrightarrow l_\infty \longrightarrow l_\infty/c_0 \longrightarrow 0 \equiv \mathcal{I}$$

$$\| \qquad\qquad \uparrow \qquad\qquad \uparrow \lambda\cdot 1$$

$$0 \longrightarrow c_0 \longrightarrow P(\lambda) \longrightarrow l_\infty/c_0 \longrightarrow 0 \equiv \mathcal{I}_\lambda$$

Benyamini shows in [1] that $P(\lambda)$ is no less than λ^{-1} complemented in any $C(K)$-space. Thus, the c_0-amalgam of the family $(\mathcal{I}_{n^{-1}})$

$$0 \to c_0(c_0) \to c_0(P(n^{-1})) \to c_0(l_\infty/c_0) \to 0 \equiv c_0(\mathcal{I}_{n^{-1}})$$

provides an exact sequence in which both $c_0(c_0)$ as well as $c_0(l_\infty/c_0)$ are $C(K)$-spaces. However, the space $c_0(P(n^{-1}))$ cannot be complemented in any $C(K)$-space. That c_0 is separably injective is precisely Sobczyk's theorem. That l_∞/c_0 is separably injective is well known and follows from Proposition 4.3 below. It was shown in [15, 32] that when X is separably injective then $c_0(X)$ is separably injective as well. Finally, separable injectivity is a 3-space property [6]. ∎

There are other $\mathcal{LP}$-spaces. As mentioned before, according to [26, p. 234, Remark 2], isometric preduals of L_1 are $\mathcal{LP}$-spaces. So, it is quite natural to ask whether the previous classes (namely: complemented subspaces of a $C(K)$-space, separably injective spaces and isometric preduals of L_1) exhaust the $\mathcal{LP}$-spaces. The answer is no.

PROPOSITION 3.4. *Every $\mathcal{L}_\infty$-space not containing c_0 is an $\mathcal{LP}$-space.*

Proof. Let X be a Banach space. If T is a noncompact operator whose domain is a subspace of c_0 then T is an isomorphism onto a copy of c_0; hence, when Y contains no copy of c_0 every operator $H \to Y$ must be compact. Now Lindenstrauss' extension theorem for compact operators [24] yields the result. ∎

By a result of Johnson and Zippin [16] separable isometric L_1-preduals are quotients of $C[0, 1]$. Observe that $\mathcal{L}_\infty$-spaces not containing c_0 cannot be quotients of $C(K)$-spaces (since, by [29], every operator on a $C(K)$-space is either weakly compact or an isomorphism onto a copy of c_0). Concrete examples of $\mathcal{L}_\infty$-spaces not containing c_0 can be obtained by applying the Bourgain–Pisier construction $0 \to X \to \mathcal{L}_\infty(X) \to S \to 0$ to spaces X without copies of c_0 by a simple 3-space argument (see [9, Thm. 3.2.e]).

New examples of $\mathcal{LP}$-spaces can be obtained by showing that this class has the 3-space property.

4. The 3-space problem for $\mathcal{LP}$-spaces. The purpose of this section is to show:

THEOREM 2. *The class of $\mathcal{LP}$-spaces has the 3-space property.*

The proof is not simple and requires both a different characterization of $\mathcal{LP}$-spaces and a new method to obtain 3-space properties. We assume from the reader some acquaintance with the theory of operator ideals as developed by Pietsch in [31]. Recall that an operator ideal $\mathfrak{A}$ is said to be *surjective* (see [31, 4.7.9]) if whenever Q is a quotient map and $TQ \in \mathfrak{A}$ then $T \in \mathfrak{A}$; dually, the ideal $\mathfrak{A}$ is *injective* (see [31, 4.6.9]) if whenever J is an into isomorphism and $JT \in \mathfrak{A}$ then $T \in \mathfrak{A}$. Consider the operator ideal $\mathfrak{J}_0$ of those operators that factorize through a subspace of c_0. Recall that a functor is called *exact* when it transforms exact sequences into exact sequences.

PROPOSITION 4.1. *A Banach space E is an $\mathcal{LP}$-space if and only if the functor $\mathfrak{J}_0(\cdot, E)$ is exact when acting on the category of separable Banach spaces.*

Proof. Let $j : Y \to X$ be an into isomorphism, and let $T \in \mathfrak{J}_0(Y, E)$. Write $T = RS$ with $S \in \mathfrak{L}(Y, H)$, $R \in \mathfrak{L}(H, E)$ and let $i : H \to c_0$ be an into isomorphism. The operator R can be extended to an operator $R_1 : c_0 \to E$ through i since E is an $\mathcal{LP}$-space; moreover Sobczyk's theorem allows one to extend $iS : Y \to c_0$ to an operator $S_1 : X \to c_0$ through j. The operator $R_1 S_1$ is an extension of T through j. The other implication is immediate. ∎

The new method to obtain 3-space properties is the following.

PROPOSITION 4.2. *Let $\mathfrak{A}$ be a surjective and injective operator ideal. The class of all Banach spaces E such that the functor $\mathfrak{A}(\cdot, E)$ is exact has the 3-space property.*

Proof. The surjectivity of $\mathfrak{A}$ implies that given an exact sequence $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0$ and a space E the induced sequence

$$0 \to \mathfrak{A}(Z, E) \xrightarrow{q^*} \mathfrak{A}(X, E) \xrightarrow{j^*} \mathfrak{A}(Y, E)$$

is exact. Now, let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be an exact sequence. By assumption, both $\mathfrak{A}(\cdot, A)$ and $\mathfrak{A}(\cdot, C)$ are exact functors and we need to prove that also $\mathfrak{A}(\cdot, B)$ is exact. To this end we construct the commutative diagram

The rows are exact by the surjectivity of $\mathfrak{U}$, while the columns are also exact by injectivity of $\mathfrak{U}$. By hypothesis,

$$\mathfrak{A}(Y, A)/j^*(\mathfrak{A}(X, A)) = \mathfrak{A}(Y, C)/j^*(\mathfrak{A}(X, C)) = 0$$

and the exactness of the fourth column implies that

$$\mathfrak{A}(Y, B)/j^*(\mathfrak{A}(X, B)) = 0,$$

hence $\mathfrak{A}(\cdot, B)$ is exact. ∎

Since the separability assumption of the previous characterization of $\mathcal{LP}$-spaces does not affect the method of Proposition 4.2, the proof of Theorem 2 will be complete after showing:

LEMMA 1. *The ideal $\mathfrak{J}_0$ is injective and surjective.*

Proof. The injectivity is a direct consequence of the definition. To show the surjectivity, let $\tau : X \to E$ be an operator which factorizes as $\tau = \varphi_0 \varphi_1$ through a subspace H of c_0 in a diagram

$$0 \longrightarrow Y \xrightarrow{\ j\ } X \xrightarrow{\ p\ } Z \longrightarrow 0$$

$$\varphi_1 \downarrow$$

$$H$$

$$\varphi_0 \downarrow$$

$$E$$

Assume that $\tau j = 0$. One then has the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\ j\ } & X & \xrightarrow{\ p\ } & Z & \longrightarrow & 0 \\
& & \downarrow & & \varphi_1 \downarrow & & \widetilde{\varphi}_1 \downarrow & & \\
0 & \longrightarrow & \ker \varphi_0 & \longrightarrow & H & \xrightarrow[P]{} & H/\ker \varphi_0 & \longrightarrow & 0 \\
& & & & \varphi_0 \downarrow & & & & \\
& & & & E & & & &
\end{array}
$$

It is clear that there exists an operator $\widetilde{\varphi}_0 : H/\ker \varphi_0 \to E$ such that $\widetilde{\varphi}_0 P = \varphi_0$. It is then obvious that $\widetilde{\varphi}_0 \widetilde{\varphi}_1 p = \varphi_0 \varphi_1$. Moreover the operator $\widetilde{\varphi}_0 \widetilde{\varphi}_1$ is in $\mathfrak{J}_0(Z, E)$ since $H/\ker \varphi_0$, as a quotient of a subspace of c_0, is itself a subspace of a quotient of c_0, hence [28, 2.f.6] a subspace of c_0. ∎

Theorem 2, in particular, yields:

COROLLARY 1. *Every twisted sum of $C(K)$-spaces is an $\mathcal{LP}$-space.*

The paper [7] contains most of the available information about how to construct twisted sums of $C(K)$-spaces. For instance, it is shown that for

every separable Banach space X not containing l_1 there exists an exact sequence

$$0 \to C[0,1] \xrightarrow{i} \Omega(X) \xrightarrow{q} X \to 0$$

with strictly singular quotient map. Of course, the space $\Omega(X)$ is not a quotient of a $C(K)$-space. Using Theorem 4.7 of [7] one can obtain examples of $\mathcal{LP}$-spaces not containing l_1 which are not $C(K)$-spaces. We do not know if there exist $\mathcal{L}_\infty$-spaces not containing l_1 which are not $\mathcal{LP}$-spaces.

New nonseparable $\mathcal{LP}$-spaces can be obtained through the following stability result.

PROPOSITION 4.3. *Every quotient of an $\mathcal{LP}$-space by a separably injective space is an $\mathcal{LP}$-space.*

Proof. Consider an exact sequence $0 \to \mathrm{SI} \to \mathcal{LP} \xrightarrow{q} Q \to 0$ in which the middle term is a Lindenstrauss–Pełczyński space and the subspace is separably injective. Let $\phi : H \to Q$ be an operator from a subspace H of c_0. Since SI is separably injective, $\mathrm{Ext}(H, \mathrm{SI}) = 0$. Hence $F\phi$ splits and ϕ can be lifted through q to an operator $\psi : H \to \mathcal{LP}$. This operator can be extended to an operator $\Psi : c_0 \to \mathcal{LP}$. The operator $q\Psi : c_0 \to Q$ is the desired extension of ϕ. ∎

The same proof shows that the quotient of two separably injective spaces is separably injective. In particular, l_∞/c_0 is separably injective. However, the following example shows that the quotient of two $\mathcal{LP}$-spaces is not necessarily an $\mathcal{LP}$-space.

PROPOSITION 4.4. *Every separable $\mathcal{L}_\infty$-space is a quotient of two $\mathcal{LP}$-spaces.*

Proof. Let X be a separable $\mathcal{L}_\infty$-space. Consider the following push-out diagram:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & l_1 & \longrightarrow & X & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \parallel & & \\
0 & \longrightarrow & \mathcal{L}_\infty(K) & \longrightarrow & \mathrm{PO} & \longrightarrow & X & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & & & \\
& & S & = \!\!=\!\!= & S & & & & \\
& & \downarrow & & \downarrow & & & & \\
& & 0 & & 0 & & & &
\end{array}
$$

The push-out space PO is an $\mathcal{L}_\infty$-space by a simple 3-space argument (see [9, Thm. 3.3.b]). Since the Schur property is also a 3-space property (see [9, 6.1]), both $\mathcal{L}_\infty(K)$ and PO are therefore $\mathcal{L}_\infty$-spaces with the Schur property. Thus, they are both $\mathcal{LP}$-spaces, while their quotient is X. ∎

COROLLARY 2. *Every separable $\mathcal{L}_\infty$-space is a quotient of two $\mathcal{L}_\infty$-spaces with the Schur property.*

Therefore, the choice $X = \mathcal{L}_\infty(H)$ of a separable $\mathcal{L}_\infty$-space that is not an $\mathcal{LP}$-space, as the one constructed in Proposition 3.2, shows that the quotient of two $\mathcal{LP}$-spaces need not be an $\mathcal{LP}$-space, and similarly for the quotient of two $\mathcal{L}_\infty$-spaces with the Schur property.

5. Some open questions. We have already shown that the quotient of two $\mathcal{LP}$-spaces is not necessarily an $\mathcal{LP}$-space. An especially interesting case is:

QUESTION 1. Is $l_\infty/C[0,1]$ an $\mathcal{LP}$-space?

A variation of this question is:

QUESTION 2. Must $\mathcal{L}_\infty$-spaces which are quotients of $C[0,1]$ be $\mathcal{LP}$-spaces?

As mentioned in the introduction, Johnson and Zippin proved in [17] that every extension $0 \to H \to c_0(\Gamma) \to Z \to 0$ is $C(K)$-trivial. It would be nice to know if $\mathcal{LP}$-spaces can play the role of $C(K)$-spaces in this result.

QUESTION 3. Given a subspace H of $c_0(\Gamma)$, does every operator $H \to \mathcal{LP}$ have an extension to $c_0(\Gamma)$?

Needless to say, the extension property one would like to get from $\mathcal{LP}$-spaces is: every $C(K)$-trivial sequence is also $\mathcal{LP}$-trivial. Unfortunately, this does not hold.

EXAMPLE. We already know that the Bourgain–Pisier space $\mathcal{L}_\infty(l_2)$ is an $\mathcal{LP}$-space. Consider then the sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & l_2 & \xrightarrow{\ \delta_2\ } & C(B_{l_2}) & \longrightarrow & Q & \longrightarrow & 0 \equiv \complement_{l_2} \\
 & & \Big\| & & & & & & \\
0 & \longrightarrow & l_2 & \xrightarrow{\ j\ } & \mathcal{L}_\infty(l_2) & \longrightarrow & S & \longrightarrow & 0 \equiv \mathrm{BP}_{l_2}
\end{array}
$$

If j could be extended to an operator $J : C(B_{l_2}) \to \mathcal{L}_\infty(l_2)$ through δ_2 this would be a weakly compact operator since $\mathcal{L}_\infty(l_2)$ does not contain c_0. Hence J would be completely continuous by the Dunford–Pettis property of $C(K)$-spaces. It is therefore impossible that $J\delta_2 = j$.

The method of proof developed in Proposition 4.2 is new. It is moreover clear that it can be applied to other injective and surjective operator ideals appearing in the literature. It is easy to check that the following ideals are injective and surjective (see also [31]): $\mathfrak{L}$ = all operators; $\mathfrak{F}$ = finite rank operators; $\mathfrak{K}$ = compact operators; $\mathfrak{W}$ = weakly compact operators; $\mathfrak{U}$ = unconditionally summing operators; $\mathfrak{J}_2$ = operators factorable through a Hilbert space. Let us introduce some notation: given an injective and surjective operator ideal $\mathfrak{U}$ let $E(\mathfrak{U})$ be the class of all Banach spaces E such that the functor $\mathfrak{U}(\cdot, E)$ is exact. From Proposition 4.2 we have obtained easy proofs that the following classes have the 3-space property: $E(\mathfrak{L})$ = injective spaces; applying the method only to separable spaces one obtains the class of separably injective spaces; $E(\mathfrak{K}) = \mathcal{L}_\infty$-spaces (by [24]); $E(\mathfrak{W})$ = $\mathcal{L}_\infty$-spaces with the Schur property (shown in [2]). The classes $E(\mathfrak{U})$ and $E(\mathfrak{J}_2)$ seem not to have been characterized yet.

A simple homological duality argument yields:

PROPOSITION 5.1. *Let $\mathfrak{A}$ be an injective and surjective operator ideal. The class of all Banach spaces E such that the functor $\mathfrak{A}(E, \cdot)$ is exact has the 3-space property.*

If we write $\exists(\mathfrak{A})$ for the previous class determined by the ideal $\mathfrak{A}$ then the only nontrivial case identified is $\exists(\mathfrak{K}) = \mathcal{L}_1$-spaces.

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