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A remark on separate holomorphy

by

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Abstract. Let X be a Riemann domain over $\mathbb{C}^k \times \mathbb{C}^l$. If X is a domain of holomorphy with respect to a family $\mathcal{F} \subset \mathcal{O}(X)$, then there exists a pluripolar set $P \subset \mathbb{C}^k$ such that every slice X_a of X with $a \notin P$ is a region of holomorphy with respect to the family $\{f|_{X_a} : f \in \mathcal{F}\}$.

1. Introduction: Riemann regions of holomorphy. Let (X,p) be a *Riemann region over* $\mathbb{C}^n$, i.e. X is an n-dimensional complex manifold and $p : X \to \mathbb{C}^n$ is a locally biholomorphic mapping (see [Jar-Pfl 2000] for details). If X is connected, then (X,p) is said to be a *Riemann domain*. We say that two Riemann regions (X,p) and (Y,q) over $\mathbb{C}^n$ are *isomorphic* (written $(X,p) \simeq (Y,q)$) if there exists a biholomorphic mapping $\varphi : X \to Y$ such that $q \circ \varphi = p$. Throughout, isomorphic Riemann regions will be identified.

We say that an open set $U \subset X$ is *univalent* (*schlicht*) if $p|_U$ is injective. Note that X is univalent iff $(X,p) \simeq (\Omega, \mathrm{id}_\Omega)$, where Ω is an open set in $\mathbb{C}^n$.

Let $f \in \mathcal{O}(X)$. For any $\alpha \in \mathbb{Z}_+^n$ ($\mathbb{Z}_+$ stands for the set of non-negative integers) and $x_0 \in X$, let $D^\alpha f(x_0)$ denote the *α-partial derivative of f at x_0*,

$$D^\alpha f(x_0) := D^\alpha (f \circ (p|_U)^{-1})(p(x_0)),$$

where U is an open univalent neighborhood of x_0 and D^α on the right hand side means the standard α-partial derivative operator in $\mathbb{C}^n$. Let $T_{x_0} f$ denote the *Taylor series of f at x_0*, i.e. the formal power series

$$\sum_{\alpha \in \mathbb{Z}_+^n} \frac{1}{\alpha!} D^\alpha f(x_0)(z - p(x_0))^\alpha, \qquad z \in \mathbb{C}^n.$$

For $x_0 \in X$ and $0 < r \le \infty$ let $\mathbb{P}_X(x_0, r)$ denote an open univalent neighborhood of x_0 such that $p(\mathbb{P}_X(x_0,r)) = \mathbb{P}(p(x_0), r) =$ the polydisc

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with center at $p(x_0)$ and radius r. Let $d_X(x_0)$ denote the maximal r such that $\mathbb{P}_X(x_0, r)$ exists. Put $\mathbb{P}_X(x_0) := \mathbb{P}_X(x_0, d_X(x_0))$.

For $f \in \mathcal{O}(X)$ and $x_0 \in X$, let $d(T_{x_0}f)$ denote the *radius of convergence of* $T_{x_0}f$, i.e.

$$d(T_{x_0}f) := \sup\{r > 0 : \text{the series } T_{x_0}f \text{ is convergent in } \mathbb{P}(p(x_0), r)\}.$$

Obviously, $d(T_{x_0}f) \geq d_X(x_0)$ and $f(x) = T_{x_0}f(p(x))$, $x \in \mathbb{P}_X(x_0)$. Notice that

$$\frac{1}{d(T_{x_0}f)} = \limsup_{\nu\to\infty} \Big(\max_{\alpha \in \mathbb{Z}_+^n : |\alpha| = \nu} \frac{1}{\alpha!} |D^\alpha f(x_0)| \Big)^{1/\nu}.$$

From now on we assume that all Riemann regions considered are countable at infinity.

Let $\emptyset \neq \mathcal{F} \subset \mathcal{O}(X)$. We say that (X, p) is an $\mathcal{F}$*-region of existence* if

$$d_X(x) = \inf\{d(T_x f) : f \in \mathcal{F}\}, \quad x \in X.$$

We say that an $\mathcal{F}$-region of existence (X, p) is an $\mathcal{F}$*-region of holomorphy* if $\mathcal{F}$ *weakly separates points in* X, i.e. for any $x', x'' \in X$ with $x' \neq x''$ and $p(x') = p(x'')$, there exists an $f \in \mathcal{F}$ such that $T_{x'}f \neq T_{x''}f$ (as formal power series).

Remark 1.1 (Properties of regions of holomorphy). (a) Let (X, p) be an $\mathcal{F}$-region of holomorphy and let $U \subset X$ be a univalent domain for which there exists a domain $V \supset p(U)$ such that for every $f \in \mathcal{F}$ there exists a function $F_f \in \mathcal{O}(V)$ such that $F_f = f \circ (p|_U)^{-1}$ on $p(U)$. Then there exists a univalent domain $W \supset U$ with $p(W) = V$.

Indeed, we only need to observe that we may always assume that (X, p) is realized as an open subset of the sheaf of $\mathcal{F}$-germs of holomorphic functions (cf. [Jar-Pfl 2000, proof of Theorem 1.8.4]) and, consequently, we may put

$$W := \{[(D, (F_f)_{f\in\mathcal{F}})]_{\underset{\sim}{z}} : z \in V\}$$

(cf. [Jar-Pfl 2000, Example 1.6.6]).

(b) ([Jar-Pfl 2000, Proposition 1.8.10]) Let $A \subset X$ be a dense subset such that $A = p^{-1}(p(A))$. Then the following conditions are equivalent:

(i) (X, p) is an $\mathcal{F}$-region of holomorphy;
(ii) $d_X(x) = \inf\{d(T_x f) : f \in \mathcal{F}\}$, $x \in A$, and for any $x', x'' \in A$ with $x' \neq x''$ and $p(x') = p(x'')$, there exists an $f \in \mathcal{F}$ such that $T_{x'}f \neq T_{x''}f$.

(c) If (X, p) is an $\mathcal{F}$-region of holomorphy, then there exists a finite or countable subfamily $\mathcal{F}_0 \subset \mathcal{F}$ such that (X, p) is an $\mathcal{F}_0$-region of holomorphy.

Indeed, we may assume that X is connected. The case where $(X, p) \simeq (\mathbb{C}^n, \mathrm{id}_{\mathbb{C}^n})$ is trivial. Thus assume that $d_X(x) < \infty$, $x \in X$. Let $A \subset X$ be a countable dense subset such that $A = p^{-1}(p(A))$. By (b), for any $x \in A$ and $r > d_X(x)$ there exists an $f_{x,r} \in \mathcal{F}$ such that $d(T_x f_{x,r}) < r$, and for $x', x'' \in A$ with $x' \neq x''$ and $p(x') = p(x'')$, there exists an $f_{x',x''} \in \mathcal{F}$ such that $T_{x'} f_{x',x''} \neq T_{x''} f_{x',x''}$. Now, we may take

$$\begin{aligned}\mathcal{F}_0 := \{f_{x,r} : x \in A,\ \mathbb{Q} \ni r > d_X(x)\} \\ \cup \{f_{x',x''} : x', x'' \in A,\ x' \neq x'',\ p(x') = p(x'')\}.\end{aligned}$$

2. Main results: separate holomorphy. Let (X, p) be a Riemann domain over $\mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^l$,

$$p = (u, v) : X \to \mathbb{C}^k \times \mathbb{C}^l.$$

Put $D := p(X)$, $D_k := u(X)$, $D^l := v(X)$. For $a \in D_k$ define $X_a := u^{-1}(a)$, $p_a := v|_{X_a}$. Similarly, for $b \in D^l$, put $X^b := v^{-1}(b)$, $p^b := u|_{X^b}$.

REMARK 2.1. For every $a \in D_k$, (X_a, p_a) is a Riemann region over $\mathbb{C}^l$, countable at infinity.

Let $\emptyset \neq \mathcal{F} \subset \mathcal{O}(X)$. For $a \in D_k$ define $f_a := f|_{X_a}$, $\mathcal{F}_a := \{f_a : f \in \mathcal{F}\} \subset \mathcal{O}(X_a)$, and analogously, $f^b := f|_{X^b}$, $\mathcal{F}^b := \{f^b : f \in \mathcal{F}\} \subset \mathcal{O}(X^b)$, $b \in D^l$.

The main result of the paper is the following

THEOREM 2.2.

(a) *Let $\emptyset \neq \mathcal{F} \subset \mathcal{O}(X)$ and assume that (X, p) is an $\mathcal{F}$-domain of holomorphy. Then there exists a pluripolar set $S_k \subset D_k$ such that for every $a \in D_k \setminus S_k$, (X_a, p_a) is an $\mathcal{F}_a$-region of holomorphy.*
(b) *Assume that $(X, p) \simeq (D, \mathrm{id}_D)$, where $D \subset \mathbb{C}^k \times \mathbb{C}^l$ is a fat domain (i.e. $D = \operatorname{int} \overline{D}$) and there exist sets $S_k \subset D_k$, $S^l \subset D^l$ such that*:
   - $\operatorname{int} S_k = \emptyset$, $\operatorname{int} S^l = \emptyset$,
   - *for any $a \in D_k \setminus S_k$, D_a is an $\mathcal{F}_a$-region of holomorphy*,
   - *for any $b \in D^l \setminus S^l$, D^b is an $\mathcal{F}^b$-region of holomorphy.*

   Then D is an $\mathcal{F}$-domain of holomorphy.

Proof. (a) By Remark 1.1(c), we may assume that $\mathcal{F}$ is finite or countable.

STEP 1. *There exists a pluripolar set $P \subset D_k$ such that for any $a \in D_k \setminus P$, (X_a, p_a) is an $\mathcal{F}_a$-region of existence.*

Define $R_{f,b}(x) := d(T_x f_{u(x)})$, $f \in \mathcal{F}$, $b \in D^l$, $x \in X^b$. Recall that

$$\frac{1}{R_{f,b}(x)} = \limsup_{\nu \to \infty} \Big(\max_{\beta \in \mathbb{Z}_+^l : |\beta| = \nu} \frac{1}{\beta!} |D^{(0,\beta)} f(x)| \Big)^{1/\nu}, \qquad x \in X^b.$$

Obviously, $R_{f,b}(x) \geq d_X(x)$, $x \in X^b$. By the Cauchy inequalities, we get

$$\frac{1}{\beta!}\,|D^{(0,\beta)}f(x)| \leq \frac{\sup_{\mathbb{P}_X(x_0,r)}|f|}{r^{|\beta|}},\qquad 0 < r < d_X(x_0),\ x \in \mathbb{P}_X(x_0, r/2),\ \beta \in \mathbb{Z}_+^l.$$

Consequently, the function $-\log(R_{f,b})_*$ (where $_*$ denotes the lower semicontinuous regularization on X^b) is plurisubharmonic on X^b. Put

$$P_{f,b} := u(\{x \in X^b : (R_{f,b})_*(x) < R_{f,b}(x)\}) \subset D_k.$$

It is known that $P_{f,b}$ is pluripolar (cf. [Jar-Pfl 2000, Theorem 2.1.41(b)]). Put

$$R_b := \inf_{f\in\mathcal{F}} R_{f,b}, \qquad \widehat{R}_b := \inf_{f\in\mathcal{F}} (R_{f,b})_*.$$

Observe that $-\log(\widehat{R}_b)_*$ is plurisubharmonic on X^b. Put

$$P_b := u(\{x \in X^b : (\widehat{R}_b)_*(x) < \widehat{R}_b(x)\}) \subset D_k.$$

The set P_b is also pluripolar (cf. [Jar-Pfl 2000, Theorem 2.1.41(a)]). Now let $B \subset D^l$ be a dense countable set. Define

$$P := \Big(\bigcup_{f\in\mathcal{F},\, b\in B} P_{f,b}\Big) \cup \Big(\bigcup_{b\in B} P_b\Big) \subset D_k.$$

Then P is pluripolar.

Take an $a \in D_k \setminus P$ and suppose that X_a is not an $\mathcal{F}_a$-region of existence. Then there exist a point $x_0 \in X_a$ and a number $r > d_{X_a}(x_0)$ such that $b := v(x_0) \in B$ and $R_b(x_0) > r$. Since $a \notin P$, we have

$$(\widehat{R}_b)_*(x_0) = \widehat{R}_b(x_0) = \inf_{f\in\mathcal{F}} (R_{f,b})_* = \inf_{f\in\mathcal{F}} R_{f,b} = R_b(x_0) > r.$$

In particular, there exists $0 < \varepsilon < d_X(x_0)$ such that $(\widehat{R}_b)_*(x) > r$ for $x \in \mathbb{P}_{X^b}(x_0)$. Since

$$R_b(x) = \inf_{f\in\mathcal{F}} R_{f,b}(x) \geq \inf_{f\in\mathcal{F}} (R_{f,b})_*(x) = \widehat{R}_b(x) \geq (\widehat{R}_b)_*(x),$$

we conclude that $R_b(x) > r$ for $x \in \mathbb{P}_{X^b}(x_0)$. Put $U := \mathbb{P}_X(x_0, \varepsilon)$. Hence, by the classical Hartogs lemma, for every $f \in \mathcal{F}$, the function $f \circ (p|_U)^{-1}$ extends holomorphically to $V := \mathbb{P}(a, \varepsilon) \times \mathbb{P}(b, r)$. Since (X, p) is an $\mathcal{F}$-domain of holomorphy, by Remark 1.1(a), there exists a univalent domain $W \subset X$, $U \subset W$, such that $p(W) = V$. In particular, $d_{X_a}(x_0) \geq r$, a contradiction.

STEP 2. *There exists a pluripolar set $P \subset D_k$ such that for any $a \in D_k \setminus P$ the family $\mathcal{F}_a$ weakly separates points in X_a.*

Take $a \in D_k$, $x', x'' \in X_a$ with $x' \neq x''$ and $p_a(x') = p_a(x'') =: b$. Since $\mathcal{F}$ weakly separates points in X, there exists an $f \in \mathcal{F}$ such that $T_{x'} f \neq T_{x''} f$.

Put $r := \min\{d(T_{x'}f),\ d(T_{x''}f)\}$ and let

$$P_{a,x',x''} := \bigcap_{w\in\mathbb{P}(b,r)} \{z \in \mathbb{P}(a,r) : T_{x'}f(z,w) = T_{x''}f(z,w)\}.$$

Then $P_{a,x',x''} \subsetneq \mathbb{P}(a,r)$ is an analytic subset. For any $z \in \mathbb{P}(a,r) \setminus P_{a,x',x''}$ we have $T_{x'}f(z,\cdot) \not\equiv T_{x''}f(z,\cdot)$ on $\mathbb{P}(b,r)$.

Take a countable dense set $A \subset D_k$. For any $a \in A$ let $B_a \subset X_a$ be a countable dense subset such that $p_a^{-1}(p_a(B_a)) = B_a$. Then

$$P := \bigcup_{\substack{a\in A,\, x',x''\in B_a\\ x'\neq x'',\, p_a(x')=p_a(x'')}} P_{a,x',x''}$$

is a pluripolar set.

Fix $a_0 \in D_k \setminus P$, $x_0', x_0'' \in X_{a_0}$ with $x_0' \neq x_0''$ and $p_{a_0}(x_0') = p_{a_0}(x_0'') =: b_0$. Put $r := \min\{d_X(x_0'),\ d_X(x_0'')\}$. Let $a \in A \cap \mathbb{P}(a_0, r/2)$ and $x', x'' \in B_a$ be such that $x' \in \mathbb{P}_X(x_0', r/2)$, $x'' \in \mathbb{P}_X(x_0'', r/2)$ and $p_a(x') = p_a(x'')$. Since $a_0 \notin P$, we conclude that $T_{x'}f(a_0,\cdot) \not\equiv T_{x''}f(a_0,\cdot)$ on $\mathbb{P}(b_0, r/2)$. Consequently, $T_{x_0'}f(a_0,\cdot) \not\equiv T_{x_0''}f(a_0,\cdot)$ on $\mathbb{P}(b_0, r/2)$, which implies that $T_{x_0'}f_{a_0} \neq T_{x_0''}f_{a_0}$.

(b) Suppose there exist $x_0 = (a_0, b_0) \in G$ and $r > d_D(x_0) =: r_0$ such that $d(T_{x_0}f) \geq r$ for $f \in \mathcal{F}$. Then $d(T_{b_0}f_a) \geq r$ for any $f \in \mathcal{F}$ and $a \in \mathbb{P}(a_0, r_0)$. Consequently, $(\mathbb{P}(a_0, r_0) \setminus S_k) \times \mathbb{P}(b_0, r) \subset D$. Since $\operatorname{int} S_k = \emptyset$ and D is fat, we conclude that $\mathbb{P}(a_0, r_0) \times \mathbb{P}(b_0, r) \subset D$. Now, we see that $d(T_{a_0}f^b) \geq r$ for any $f \in \mathcal{F}$ and $b \in \mathbb{P}(b_0, r)$. Consequently, $\mathbb{P}(a_0, r) \times (\mathbb{P}(b_0, r) \setminus S^l) \subset D$. Hence $\mathbb{P}(a_0, r) \times \mathbb{P}(b_0, r) \subset D$, a contradiction. ∎

REMARK 2.3. The following natural question arises from the discussion above: is it possible to sharpen Theorem 2.2(a) so that the exceptional set there is even a countable union of locally analytic sets? The following example will show that the answer is, in general, negative.

Let $C_1 \subset \mathbb{D}$ (the unit disc) be a compact polar set which is uncountable (take, for example, an appropriate Cantor set). Define $C := C_1 \cup C_2$, where $C_2 := \mathbb{D} \cap \mathbb{Q}^2$. Then C is polar and a countable union of compact sets.

Using Example 2 from [Ter 1972], we find a function $f : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ with the following properties:

- $f(\cdot, w) \in \mathcal{O}(\mathbb{D})$ for all $w \in \mathbb{D}$,
- $f(z, \cdot) \in \mathcal{O}(\mathbb{D})$ for all $z \in C$,
- f is unbounded near some point $(z_0, 0) \in \mathbb{D} \times \mathbb{D}$.

Using the corollary to Lemma 8 of [Ter 1972], we conclude that there is a non-empty domain $V \subset\subset \mathbb{D}$ such that $f|_{\mathbb{D}\times V} \in \mathcal{O}(\mathbb{D} \times V)$. Set $\mathcal{F} := \{f|_{\mathbb{D}\times V}, g|_{\mathbb{D}\times V}\}$, where $g \in \mathcal{O}(\mathbb{D} \times \mathbb{D})$ is chosen in such a way that $\mathbb{D} \times \mathbb{D}$ is the existence domain of g. Denote by (D', p) the $\mathcal{F}$-envelope of holomorphy of $\mathbb{D}\times V$. Then $p(D')\subset\mathbb{D}\times\mathbb{D}$. Moreover, using the fact that C is dense in $\mathbb{D}$ one sees that D' is univalent. Indeed, let us take a sequence $G_j=G'_j\times G''_j\subset\mathbb{D}\times\mathbb{D}$, $j=1,\dots,N$, of bidiscs, $G_j\cap G_{j+1}\neq\emptyset$, and functions $f_j\in\mathcal{O}(G_j)$, $j=1,\dots,N$, such that $G_1\subset\mathbb{D}\times V$, $f_1=f|_{G_1}$, and $f_j|_{G_j\cap G_{j+1}}=f_{j+1}|_{G_j\cap G_{j+1}}$, $j=1,\dots,N-1$. We claim that then $f_N=f|_{G_N}$, which implies that D' is univalent. By induction we may assume that $f_j=f|_{G_j}$ for a $j<N$. Then for any point $a\in C\cap G'_j\cap G'_{j+1}$ we have two holomorphic functions $f(a,\cdot)$ and $f_{j+1}(a,\cdot)$ on G''_{j+1}. They coincide on $G''_j\cap G''_{j+1}$, and so on G''_{j+1}. Now fix a $b\in G''_{j+1}$. Then $f(\cdot,b)$ and $f_{j+1}(\cdot,b)$ are holomorphic in G'_{j+1} and they coincide on $C\cap G'_j\cap G'_{j+1}$; hence they are equal on G''_{j+1}, i.e. $f|_{G_{j+1}}=f_{j+1}$.

Set $D:=p(D')$. Then D is an $\widehat{\mathcal{F}}$-domain of holomorphy, where

$$\widehat{\mathcal{F}}:=\{\widehat{g}:=g|_D,\widehat{f}:=f|_D\}.$$

Observe that for any $a\in C$, the functions $\widehat{f}(a,\cdot),\widehat{g}(a,\cdot)$ extend to the whole of $\mathbb{D}$.

Fix $R'<R\in(0,1)$ such that $V\subset\subset\mathbb{P}(0,R')$. Suppose that there is an $a_0\in C$ with $\{a_0\}\times\mathbb{P}(0,R)\subset D$. Then there is a small open neighborhood $U\subset\mathbb{D}$ of a_0 such that $U\times\mathbb{P}(0,R')\subset D$. In view of the Hartogs lemma we conclude that f is holomorphic on $\mathbb{D}\times\mathbb{P}(0,R')$, in particular a holomorphic extension of $f|_{\mathbb{D}\times V}$, and therefore bounded near $(z_0,0)$, a contradiction. Thus, the singular set S_1 for D must contain C.

Remark 2.4. Observe that Theorem 2.2(b) need not be true if D is not fat. For example, let $D:=\mathbb{D}^2\setminus\{(0,0)\}\subset\mathbb{C}^2$, $\mathcal{F}:=\mathcal{O}(D)$. By the Hartogs extension theorem, any function from $\mathcal{F}$ extends holomorphically to $\mathbb{D}^2$. Thus D is not an $\mathcal{F}$-domain of holomorphy. Observe that for any $a\in\mathbb{D}\setminus\{0\}$, $D_a=\mathbb{D}$ and $\mathcal{F}_a=\mathcal{O}(\mathbb{D})$. Similarly, for any $b\in\mathbb{D}\setminus\{0\}$, $D^b=\mathbb{D}$ and $\mathcal{F}^b=\mathcal{O}(\mathbb{D})$.

3. Applications: separately holomorphic functions. Directly from Theorem 2.2 we get the following useful corollary.

Corollary 3.1. *Let $D\subset\mathbb{C}^k\times\mathbb{C}^l$ be a domain, let $\emptyset\neq\mathcal{F}\subset\mathcal{O}(D)$ and let $A\subset\operatorname{pr}_{\mathbb{C}^k}(D)$. Assume that for any $a\in A$ we are given a domain $G(a)\supset D_a$ in $\mathbb{C}^l$ such that*:

- *for any $f\in\mathcal{F}$, the function $f(a,\cdot)$ extends to an $\widehat{f}_a\in\mathcal{O}(G(a))$,*
- *the domain $G(a)$ is an $\{\widehat{f}_a:f\in\mathcal{F}\}$-domain of holomorphy.*

Let (X,p) be the $\mathcal{F}$-envelope of holomorphy of D. Then there exists a pluripolar set $P\subset A$ such that for every $a\in A\setminus P$ we have $(X_a,p_a)\simeq(G(a),\mathrm{id}_{G(a)})$.

Recall a version of the cross theorem for separately holomorphic functions with pluripolar singularities (cf. [Jar-Pfl 2003, Main Theorem]).

THEOREM 3.2. *Let $D \subset \mathbb{C}^k$, $G \subset \mathbb{C}^l$ be domains of holomorphy and let $A \subset D$, $B \subset G$ be locally pluriregular sets* (1). *Consider the* cross

$$X = \mathbb{X}(A, B; D, G) := (A \times G) \cap (D \times B)$$

and let

$$\widehat{X} = \widehat{\mathbb{X}}(A, B; D, G) := \{(z, w) \in D \times G : \omega_{A,D}(z) + \omega_{B,G}(w) < 1\},$$

where $\omega_{A,D}$ and $\omega_{B,G}$ are generalized relative extremal functions. Let $M \subset X$ be a relatively closed set (2) *such that*:

- *for every $a \in A$ the fiber $M_a := \{w \in G : (a, w) \in M\}$ is pluripolar*,
- *for every $b \in B$ the fiber $M^b := \{z \in D : (z, b) \in M\}$ is pluripolar.*

Let $\mathcal{F} = \mathcal{O}_{\mathrm{s}}(X \setminus M)$ denote the set of all functions separately holomorphic on $X \setminus M$, *i.e. of those functions $f : X \setminus M \to \mathbb{C}$ for which*:

- *for every $a \in A$, $f(a, \cdot) \in \mathcal{O}(G \setminus M_a)$*,
- *for every $b \in A$, $f(\cdot, b) \in \mathcal{O}(D \setminus M^b)$.*

Then there exists a relatively closed pluripolar set $S \subset \widehat{X}$ such that:

- $S \cap X \subset M$,
- *for every $f \in \mathcal{F}$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{X} \setminus S)$ with $\widehat{f} = f$ on $X \setminus M$*,
- *$\widehat{X} \setminus S$ is an $\{\widehat{f} : f \in \mathcal{F}\}$-domain of holomorphy.*

In the proof presented in [Jar-Pfl 2003] the assumption that M is relatively closed in X played an important role. Observe that from the point of view of the formulation of the above theorem, we only need to assume that all the fibers M_a and M^b are relatively closed. Corollary 3.1 permits us to clarify this problem in certain cases.

LEMMA 3.3. *Let $D \subset \mathbb{C}^k$, $G_0 \subset G \subset \mathbb{C}^l$ be domains of holomorphy and let $A \subset D$. Assume that for every $a \in A$ we are given a relatively closed pluripolar set $M(a) \subset G$. Let $\mathcal{F}$ denote the set of all functions $f \in \mathcal{O}(D \times G_0)$ such that for every $a \in A$, the function $f(a, \cdot)$ extends to an $\widehat{f}_a \in \mathcal{O}(G \setminus M(a))$. Assume that for every $a \in A$ the set $M(a)$ is singular with respect to the family $\{\widehat{f}_a : f \in \mathcal{F}\}$* (3). *Then there exists a pluripolar*

(1) A non-empty set A is said to be *locally pluriregular* if for any $a \in A$, the set A is *locally pluriregular at a*, i.e. for any open neighborhood U of a we have $h^*_{A\cap U,U}(a) = 0$, where $h_{A\cap U,U}$ denotes the relative extremal function of $A \cap U$ in U. For an arbitrary set A define $A^* := \{a \in \overline{A} : A \text{ is locally pluriregular at } a\}$. It is known that the set $Z := A \setminus A^*$ is pluripolar. In particular, if A is non-pluripolar, then $A \setminus Z$ is locally pluriregular.

(2) That is, M is closed in X.

(3) Recall that a relatively closed pluripolar set $M \subset G$ is said to be *singular* with respect to a family $\emptyset \neq \mathcal{G} \subset \mathcal{O}(G \setminus M)$ if there is no point $a \in M$ which has an open neighborhood $U \subset G$ such that every function from $\mathcal{G}$ extends holomorphically to U (cf. [Jar-Pfl 2000, §3.4]).

set $P \subset A$ such that if we put $A_0 := A \setminus P$, then the set

$$M(A_0) := \bigcup_{a \in A_0} \{a\} \times M(a)$$

is relatively closed in $A_0 \times G$.

Proof. First observe that every function from $\mathcal{O}(G)$ may be regarded as an element of $\mathcal{F}$, which implies that for every $a \in A$ the domain $G(a) := G \setminus M(a)$ is a $\{\widehat{f}_a : f \in \mathcal{F}\}$-domain of holomorphy.

Let (X, p) be the $\mathcal{F}$-envelope of holomorphy of $D \times G_0$. Since D and G are domains of holomorphy, we may assume that $p(X) \subset D \times G$.

By Corollary 3.1, there exists a pluripolar set $P \subset A$ such that for every $a \in A_0 := A \setminus P$ we have $(X_a, p_a) \simeq (G(a), \mathrm{id}_{G(a)})$. Thus p is injective on the set $B := p^{-1}(A_0 \times G)$ and $p(B) = \bigcup_{a \in A_0} \{a\} \times G(a) = (A_0 \times G) \setminus M(A_0)$. Hence $p(B) = p(X) \cap (A_0 \times G)$ and, consequently, $p(B)$ is relatively open in $A_0 \times G$. ∎

Consequently, we get the following generalization of Theorem 3.2.

Theorem 3.4. *Let $D_0 \subset D \subset \mathbb{C}^k$, $G_0 \subset G \subset \mathbb{C}^l$ be domains of holomorphy and let $A \subset D_0$, $B \subset G_0$ be non-pluripolar sets. Let $M \subset X := \mathbb{X}(A, B; D, G)$ be such that*:

- *for every $a \in A$ the fiber M_a is a closed pluripolar subset of G,*
- *for every $b \in B$ the fiber M^b is a closed pluripolar subset of D.*

Let $\mathcal{F}$ denote the set of all functions $f \in \mathcal{O}(D_0 \times G_0)$ such that:

- *for every $a \in A$ the function $f(a, \cdot)$ extends holomorphically to $G \setminus M_a$,*
- *for every $b \in A$ the function $f(\cdot, b)$ extends holomorphically to $D \setminus M^b$.*

Then there exist:

- *pluripolar sets $P \subset A$, $Q \subset B$ such that the sets $A_0 := A \setminus P$ and $B_0 := B \setminus Q$ are locally pluriregular,*
- *a relatively closed pluripolar set $S \subset \widehat{X}_0 := \widehat{\mathbb{X}}(A_0, B_0; D, G)$*

such that:

- $S \cap \mathbb{X}(A_0, B_0; D, G) \subset M$,
- *for every $f \in \mathcal{F}$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{X}_0 \setminus S)$ with $\widehat{f} = f$ on $D_0 \times G_0$,*
- *$\widehat{X}_0 \setminus S$ is an $\{\widehat{f} : f \in \mathcal{F}\}$-domain of holomorphy.*

Remark 3.5. More general versions of the cross theorem (also for N-fold crosses) will be discussed in our forthcoming paper.

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