STUDIA MATHEMATICA 209 (2) (2012)

Boundedness of Riesz transforms on weighted Carleson measure spaces

by

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Abstract. Let w be in the Muckenhoupt A_∞ weight class. We show that the Riesz transforms are bounded on the weighted Carleson measure space CMO^p_w, the dual of the weighted Hardy space H^p_w, $0 < p \le 1$.

1. Introduction. One of the principal interests of $H^p(\mathbb{R}^n)$ theory is to give a natural extension of the boundedness on L^p, $1 < p < \infty$, for maximal functions and singular integrals to the Hardy space $H^p(\mathbb{R}^n)$ for $p \le 1$. It is well known that the Riesz transforms are bounded on $H^p(\mathbb{R}^n)$, $0 < p \le 1$, and $\mathrm{BMO}(\mathbb{R}^n)$, the dual of H^1. For $p < 1$, the dual of $H^p(\mathbb{R}^n)$ can be identified with a Campanato space (see [CW], [FS], and [GR]). Moreover, it was proved that Campanato spaces are equivalent to Lipschitz spaces (see [FS, Theorem 5.39]). Lemarié [L, Theorem A] proved that Calderón–Zygmund singular integral operators satisfying certain conditions are bounded on Lipschitz spaces (cf. [MC, Chapter 10, §4]). Therefore, these results imply that the Riesz transforms are bounded on the dual of $H^p(\mathbb{R}^n)$.

For the weighted case, Lee et al. [LLY] showed that the Riesz transforms are bounded on weighted Hardy spaces H^p_w, $0 < p \le 1$ for $w \in A_1$. Recently, Ding et al. [DHLW] extend the H^p_w-boundedness of the Riesz transforms to $w \in A_\infty$. A natural question arises: Are the Riesz transforms bounded on the dual of the weighted Hardy space H^p_w for $0 < p \le 1$ and $w \in A_\infty$? The purpose of this paper is to give an affirmative answer. In 2001, García-Cuerva and Martell [GM] gave a wavelet characterization of weighted Hardy spaces $H^p_w(\mathbb{R}^n)$. In [LLL], Lee et al. introduced the weighted Carleson measure space $\mathrm{CMO}^p_w(\mathbb{R})$ and showed that $\mathrm{CMO}^p_w(\mathbb{R})$ is the dual of the weighted Hardy space $H^p_w(\mathbb{R})$. To state the duality result of [LLL], we first recall the definition of the weighted Carleson measure spaces CMO^p_w. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy

2010 *Mathematics Subject Classification*: Primary 42B20.
Key words and phrases: weighted Carleson measure spaces, duality, weighted Hardy spaces, Riesz transforms.

DOI: 10.4064/sm209-2-5

$$\operatorname{supp}\widehat{\psi} \subset \{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\} \tag{1.1}$$

and

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^{-j}\xi)|^2 = 1 \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{1.2}$$

Set $\psi_j(x) = 2^{jn}\psi(2^j x)$. Denote by $\mathcal{S}_\infty(\mathbb{R}^n)$ the functions $f \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} f(x)x^\alpha\,dx = 0$ for $|\alpha| \ge 0$. We define $\mathrm{CMO}^p_w(\mathbb{R}^n)$ as follows.

Definition 1.1. Let $0 < p \le 1$ and $w \in A_\infty$. We say that $f \in \mathrm{CMO}^p_w(\mathbb{R}^n)$ if $f \in (\mathcal{S}_\infty)'(\mathbb{R}^n)$ with the finite norm defined by

$$\|f\|_{\mathrm{CMO}^p_w(\mathbb{R}^n)} := \sup_J \biggl\{\frac{1}{w(J)^{2/p-1}} \sum_{j\in\mathbb{Z}} \sum_{I\subset J} |(\psi_j * f)(x_I)|^2 \frac{|I|^2}{w(I)}\biggr\}^{1/2},$$

where J is a dyadic cube in $\mathbb{R}^n$ and I is a dyadic cube in $\mathbb{R}^n$ with edge-length 2^{-j} and lower-left corner x_I. Note that $x_I = 2^{-j}\mathbf{k}$, where $j \in \mathbb{Z}, \mathbf{k} = (k_1,\dots,k_n) \in \mathbb{Z}^n$ and $I = \{(x_1,\dots,x_n) \in \mathbb{R}^n : k_i \le 2^j x_i < k_i + 1,\ i = 1,\dots,n\}$. This convention will be used throughout the paper.

By the same argument of [LLL], the dual space of $H^p_w(\mathbb{R}^n)$, $0 < p \le 1$, can be identified with $\mathrm{CMO}^p_w(\mathbb{R}^n)$ as follows.

Theorem A. *Let $0 < p \le 1$ and $w \in A_\infty$. The dual of H^p_w is CMO^p_w in the following sense:*

(a) *For each $g \in \mathrm{CMO}^p_w$, there is a linear functional ℓ_g, initially defined on $H^p_w \cap L^2$, which has a continuous extension onto H^p_w and $\|\ell_g\| \le C\|g\|_{\mathrm{CMO}^p_w}$.*
(b) *Conversely, every continuous linear functional ℓ on H^p_w can be realized as $\ell = \ell_g$ with $g \in \mathrm{CMO}^p_w$ and $\|g\|_{\mathrm{CMO}^p_w} \le C\|\ell\|$.*

In particular for $p = 1$, $\mathrm{CMO}^1_w(\mathbb{R}^n) = \mathrm{BMO}_w(\mathbb{R}^n)$.

Since CMO^p_w is the dual of H^p_w, the definition of CMO^p_w is independent of the choice of the function ψ. However, we would like to show this independence by using the following inequality for CMO^p_w, which will also be used for the proof of the main result in this paper.

Theorem 1.2. *Let $0 < p \le 1$, $w \in A_\infty$ and ψ, ϕ satisfy* (1.1)–(1.2). *Then, for all $f \in (\mathcal{S}_\infty)'$,*

$$\sup_J \biggl\{\frac{1}{w(J)^{2/p-1}} \sum_{j\in\mathbb{Z}} \sum_{I\subset J} |(\psi_j * f)(x_I)|^2 \frac{|I|^2}{w(I)}\biggr\}^{1/2} \approx \sup_J \biggl\{\frac{1}{w(J)^{2/p-1}} \sum_{j\in\mathbb{Z}} \sum_{I\subset J} |(\phi_j * f)(x_I)|^2 \frac{|I|^2}{w(I)}\biggr\}^{1/2}.$$

Let R_j, $j = 1, \dots, n$, denote the *Riesz transforms* in $\mathbb{R}^n$ defined by

$$R_j f(x) = \text{p.v. } (K_j * f)(x), \quad \text{where} \quad K_j(x) = \pi^{-(n+1)/2}\Gamma\left(\frac{n+1}{2}\right)\frac{x_j}{|x|^{n+1}}.$$

For $n = 1$, the Riesz transform reduces to the Hilbert transform

$$Hf(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int\limits_{|y| \ge \varepsilon} \frac{f(x-y)}{y}\, dy.$$

Note that by Definition 1.1, $\mathrm{CMO}^p_w \subset (\mathcal{S}_\infty)'$. In general, R_j may not be well defined on CMO^p_w. Accordingly, to obtain the boundedness of an operator R_j on CMO^p_w, we need first to define $R_j f$ for $f \in \mathrm{CMO}^p_w$. Indeed, the same problem appeared even in the study of the boundedness of singular integral operators on the classical Hardy spaces H^p. The key method used in the classical case was to consider the dense subspace $L^2 \cap H^p$ of H^p. Thus, to show the H^p boundedness of singular integral operators, by the density argument, it suffices to prove the boundedness of operators on $L^2 \cap H^p$. However, this method does not work in our case because $L^2 \cap \mathrm{CMO}^p_w$ is not dense in CMO^p_w. But we will prove in Proposition 4.1 below that $L^2 \cap \mathrm{CMO}^p_w$ is dense in CMO^p_w in the weak topology $(H^p_w, \mathrm{CMO}^p_w)$. Hence, for $f \in \mathrm{CMO}^p_w$, $\langle R_j f, g\rangle$ is well defined for $g \in \mathcal{S}_\infty$. This means that for $f \in \mathrm{CMO}^p_w$, $R_j f$ is well defined as a distribution in $(\mathcal{S}_\infty)'$. The main result of this paper is the following

Theorem 1.3. *Let $w \in A_\infty$. Then there exists a constant C such that*

$$\|R_j f\|_{\mathrm{CMO}^p_w} \le C\|f\|_{\mathrm{CMO}^p_w} \quad \text{for } 0 < p \le 1 \text{ and } j = 1, \dots, n.$$

Remark. Theorem 1.3 cannot be directly obtained by duality from the H^p_w-boundedness of Riesz transforms since we do not have $\|f\|_{\mathrm{CMO}^p_w} \approx \sup_{\|g\|_{H^p_w} \le 1} |\langle f, g\rangle|$.

Throughout the article the letter C will denote a positive constant that may vary from line to line but remains independent of the main variables. We use $j \wedge k$ to denote the minimum of j and k and use $a \approx b$ to denote the equivalence of a and b, that is, there exist two positive constants C_1, C_2 independent of a, b such that $C_1 a \le b \le C_2 a$.

2. Preliminaries. The class A_p was used by Muckenhoupt [M], Hunt–Muckenhoupt–Wheeden [HMW], and Coifman–Fefferman [CF] to investigate the weighted L^p boundedness of Hardy–Littlewood maximal functions, the Hilbert transform and Calderón–Zygmund singular integral operators, respectively. In this article a weight means an A_p weight. More precisely, let w be a nonnegative function defined on $\mathbb{R}^n$. We say that $w \in A_p, 1 < p < \infty$, if

$$\Big(\int_I w(x)\,dx\Big)\Big(\int_I w(x)^{-1/(p-1)}\,dx\Big)^{p-1} \le C|I|^p \quad \text{for every cube } I \subseteq \mathbb{R}^n,$$

where C is a positive constant independent of I and $0\cdot\infty$ is taken to be 0. A function w satisfies the condition A_∞ if given $\varepsilon > 0$ there exists $\delta > 0$ such that if I is a cube and $E \subseteq I$ with $|E| < \delta|I|$, then

$$\int_E w(x)\,dx < \varepsilon \int_I w(x)\,dx.$$

For the case $p = 1, w \in A_1$ if

$$\frac{1}{|I|}\int_I w(x)\,dx \le C \operatorname*{ess\,inf}_{x\in I} w(x) \quad \text{for every cube } I \subseteq \mathbb{R}^n.$$

It is well known that a locally integrable function satisfies the condition A_∞ if and only if it satisfies the condition A_p for some $p > 1$. Also, if $w \in A_p$ with $1 < p < \infty$, then $w \in A_r$ for all $r > p$ and $w \in A_q$ for some $1 < q < p$. We thus use $q_w \equiv \inf\{q > 1 : w \in A_q\}$ to denote the *critical index* of w and define the weighted measure of a set $E \subseteq I$ by $w(E) = \int_E w(x)\,dx$.

For any cube I and $\lambda > 0$, we shall denote by λI the cube concentric with I each of whose edges is λ times as long as the edges of I. It is known that for $w \in A_p$, $p \ge 1, w$ satisfies the *doubling condition*, that is, there exists an absolute constant C such that $w(2I) \le Cw(I)$.

Closely related to A_p is the reverse Hölder condition. If there exist $r > 1$ and a fixed constant $C > 0$ such that

$$\Big(\frac{1}{|I|}\int_I w(x)^r\,dx\Big)^{1/r} \le C\Big(\frac{1}{|I|}\int_I w(x)\,dx\Big) \quad \text{for every cube } I \subseteq \mathbb{R}^n,$$

we say that w satisfies the *reverse Hölder condition of order* r and write $w \in RH_r$. It follows from Hölder's inequality that $w \in RH_r$ implies $w \in RH_s$ for all $s < r$. It is known that $w \in A_\infty$ if and only if $w \in RH_r$ for some $r > 1$. Moreover, if $w \in RH_r$, $r > 1$, then $w \in RH_{r+\varepsilon}$ for some $\varepsilon > 0$. We thus write $r_w \equiv \sup\{r > 1 : w \in RH_r\}$ to denote the *critical index of* w *for the reverse Hölder condition.*

For the comparison between the Lebesgue measure of a set E and its weighted measure $w(E)$, we have the following

Theorem B ([GR, GW]). *Let $w \in A_p \cap RH_r$ with $p \ge 1$ and $r > 1$. Then there exist constants $C_1, C_2 > 0$ such that*

$$C_1\Big(\frac{|E|}{|I|}\Big)^p \le \frac{w(E)}{w(I)} \le C_2\Big(\frac{|E|}{|I|}\Big)^{(r-1)/r}$$

for any measurable subset E of a cube I.

For the integral with respect to the measure $w(x)dx$, we have the following estimate which can be found in [GR, p. 412].

LEMMA C. *Let $w \in A_q$, $q > 1$. Then, for all $r > 0$, there exists a constant C independent of r such that*

$$\int_{|x|\geq r} \frac{w(x)}{|x|^{nq}}\,dx \leq Cr^{-nq}w(I_r),$$

where I_r is the cube centered at 0 with edge-length $2r$.

For $f \in (\mathcal{S}_\infty)'$, we define the *discrete Littlewood–Paley square function* $\mathcal{G}(f)$ by

$$\mathcal{G}(f)(x) = \Big(\sum_{j\in\mathbb{Z}}\sum_{I} |(\psi_j * f)(x_I)|^2\chi_I(x)\Big)^{1/2}.$$

It is known that $\mathcal{G}$ is bounded on L^q_w, $1 < q < \infty$, provided $w \in A_q$. The following discrete Calderón identity on $\mathbb{R}^n$ was proved in [FJ]:

THEOREM D. *Suppose that ψ satisfies* (1.1) *and* (1.2). *Then, for $f \in L^2(\mathbb{R}^n)$, $\mathcal{S}_\infty(\mathbb{R}^n)$, or $(\mathcal{S}_\infty)'(\mathbb{R}^n)$,*

$$f(x) = \sum_{j\in\mathbb{Z}}\sum_{I} 2^{-jn}(\psi_j * f)(x_I)\psi_j(x - x_I),$$

where the series converges in $L^2(\mathbb{R}^n)$, $\mathcal{S}_\infty(\mathbb{R}^n)$, or $(\mathcal{S}_\infty)'(\mathbb{R}^n)$, respectively.

3. The proof of Theorem 1.2. For $f \in (\mathcal{S}_\infty)'$, we use Theorem D to get

$$(\psi_j * f)(z) = \sum_{j'\in\mathbb{Z}}\sum_{I'} 2^{-j'n}(\phi_{I'} * f)(x_{I'})(\psi_j * \phi_{I'})(z - x_{I'}),$$

where $\phi_{I'} := \phi_{j'}$ if $\ell(I') = 2^{-j'}$. Note that ϕ_{I_1} and ϕ_{I_2} represent the same operator if I_1 and I_2 have the same edge-length. For $L, M > 0$, the almost orthogonality (cf. [HS, Lemma 4.3]) gives

$$|(\psi_j * \phi_{j'})(z - x_{I'})| \leq C2^{-|j-j'|L}\frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')} + |z - x_{I'}|)^{n+M}}. \tag{3.1}$$

Hence,

$$\begin{aligned}
&|(\psi_j * f)(z)| \\
&\quad\leq C\sum_{j'}\sum_{I'} |I'|\,2^{-|j-j'|L}\frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')} + |z - x_{I'}|)^{n+M}}|(\phi_{I'} * f)(x_{I'})| \\
&\quad\leq C\sum_{j'}\sum_{I'} |I'|\,2^{-|j-j'|L}\frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')} + |x_I^c - x_{I'}^c|)^{n+M}}|(\phi_{I'} * f)(x_{I'})|, \quad z \in I,
\end{aligned}$$

where x_I^c, $x_{I'}^c$ denote the centers of I, I', respectively. Taking the supremum over $z \in I$, we get

$$\begin{aligned}
&\sup_{z\in I} |(\psi_j * f)(z)| \\
&\qquad \le C \sum_{j'} \sum_{I'} |I'|\, 2^{-|j-j'|L} \frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')} + |x_I^c - x_{I'}^c|)^{n+M}} |(\phi_{I'} * f)(x_{I'})|.
\end{aligned}$$

Schwarz's inequality gives

$$\begin{aligned}
&\Big(\sup_{z\in I} |(\psi_j * f)(z)|\Big)^2 \\
&\qquad \le C\bigg(\sum_{j'} 2^{-|j-j'|L} \bigg\{ \sum_{I'} |I'| \frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')} + |x_I^c - x_{I'}^c|)^{n+M}} \bigg\}^{1/2} \\
&\qquad\qquad \times \bigg\{ \sum_{I'} |I'| \frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')} + |x_I^c - x_{I'}^c|)^{n+M}} |(\phi_{I'} * f)(x_{I'})|^2 \bigg\}^{1/2}\bigg)^2.
\end{aligned}$$

A direct computation yields

$$\sum_{I'} |I'| \frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')} + |x_I^c - x_{I'}^c|)^{n+M}} \le C. \tag{3.2}$$

By Schwarz's inequality again,

$$\begin{aligned}
&\Big(\sup_{z\in I} |(\psi_j * f)(z)|\Big)^2 \\
&\qquad \le C\bigg(\sum_{j'} 2^{-|j-j'|L}\bigg)\bigg(\sum_{j'} 2^{-|j-j'|L} \\
&\qquad\qquad \times \sum_{I'} |I'| \frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')} + |x_I^c - x_{I'}^c|)^{n+M}} |(\phi_{I'} * f)(x_{I'})|^2\bigg) \\
&\qquad \le C \sum_{j'} \sum_{I'} 2^{-|j-j'|L} |I'| \frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')} + |x_I^c - x_{I'}^c|)^{n+M}} |(\phi_{I'} * f)(x_{I'})|^2.
\end{aligned}$$

Given a dyadic cube P, say $\ell(P) = 2^{-j_0}$, we have

$$\begin{aligned}
&\frac{1}{w(P)^{2/p-1}} \sum_{I\subset P} \Big(\sup_{z\in I} |(\psi_j * f)(z)|\Big)^2 \frac{|I|^2}{w(I)} \\
&\qquad \le \frac{C}{w(P)^{2/p-1}} \sum_{j=j_0}^{\infty} \sum_{\substack{I\subset P \\ \ell(I)=2^{-j}}} \sum_{j'=j_0}^{\infty} \sum_{I'} 2^{-|j-j'|L} |I'| \\
&\qquad\qquad \times \frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')} + |x_I^c - x_{I'}^c|)^{n+M}} |(\phi_{I'} * f)(x_{I'})|^2 \frac{|I|^2}{w(I)}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{C}{w(P)^{2/p-1}}\sum_{j=j_0}^{\infty}\sum_{\substack{I\subset P\\ \ell(I)=2^{-j}}}\sum_{j'=-\infty}^{j_0-1}\sum_{I'}2^{-|j-j'|L}|I'|\\
&\qquad\times\frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')}+|x_I^c-x_{I'}^c|)^{n+M}}|(\phi_{I'}*f)(x_{I'})|^2\frac{|I|^2}{w(I)}\\
&:=A_1+A_2.
\end{aligned}
$$

A_1 can be further decomposed as

$$
\begin{aligned}
A_1=\frac{C}{w(P)^{2/p-1}}&\Bigg(\sum_{j=j_0}^{\infty}\sum_{\substack{I\subset P\\ \ell(I)=2^{-j}}}\sum_{j'=j_0}^{\infty}\sum_{\substack{I'\subset 3P\\ \ell(I')=2^{-j'}}}+\sum_{j=j_0}^{\infty}\sum_{\substack{I\subset P\\ \ell(I)=2^{-j}}}\sum_{j'=j_0}^{\infty}\sum_{\substack{I'\cap 3P=\emptyset\\ \ell(I')=2^{-j'}}}\Bigg)\\
&2^{-|j-j'|L}|I'|\frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')}+|x_I^c-x_{I'}^c|)^{n+M}}|(\phi_{I'}*f)(x_{I'})|^2\frac{|I|^2}{w(I)}\\
:=A_{11}+A_{12}.&
\end{aligned}
$$

Let $w\in A_\infty$. There exist $q,r>1$ such that $w\in A_q\cap RH_r$. The definition of A_q and Hölder's inequality show that

$$|I|^q\approx w(I)(w(I)^{1-q'})^{q-1}. \tag{3.3}$$

Hence,

$$
\begin{aligned}
&\sum_{\substack{I\subset P\\ \ell(I)=2^{-j}}}\frac{|I|^2}{w(I)}\frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')}+|x_I^c-x_{I'}^c|)^{n+M}}\\
&\le\sum_{\substack{I\subset P\\ \ell(I)=2^{-j}}}|I|^{2-q}(w(I)^{1-q'})^{q-1}\frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')}+|x_I^c-x_{I'}^c|)^{n+M}}\\
&\le 2^{-jn(2-q)}\bigg(\int_P\frac{2^{-(j\wedge j')\frac{M}{q-1}}}{(2^{-(j\wedge j')}+|x_I^c-x_{I'}^c|)^{\frac{n+M}{q-1}}}w(x)^{1-q'}\,dx\bigg)^{q-1}\\
&\le C2^{-jn(2-q)}\bigg(\bigg(\int_{|x-x_{I'}^c|\le 2^{-j'}}+\int_{|x-x_{I'}^c|>2^{-j'}}\bigg)\\
&\qquad\qquad\frac{2^{-(j\wedge j')\frac{M}{q-1}}}{(2^{-(j\wedge j')}+|x-x_{I'}^c|)^{\frac{n+M}{q-1}}}w(x)^{1-q'}\,dx\bigg)^{q-1}.
\end{aligned}\tag{3.4}
$$

Since $w\in A_q$ it follows that $w^{1-q'}\in A_{q'}$. If we take $M>nq'(q-1)-n$, Lemma C yields

$$\int\limits_{|x-x_{I'}^c|>2^{-j'}} \frac{2^{-(j\wedge j')\frac{M}{q-1}}}{(2^{-(j\wedge j')}+|x_I^c-x_{I'}^c|)^{\frac{n+M}{q-1}}} w(x)^{1-q'}\,dx$$
$$\leq C2^{-(j\wedge j')\frac{M}{q-1}+j'\frac{n+M}{q-1}} w(I')^{1-q'}.$$

Inserting the above estimate into the last term in (3.4) implies

$$\sum_{\substack{I\subset P\\ \ell(I)=2^{-j}}} \frac{|I|^2}{w(I)} \frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')}+|x_I^c-x_{I'}^c|)^{n+M}}$$
$$\leq 2^{-jn(2-q)}(2^{(j\wedge j'-j')n}+2^{(j'-j\wedge j')M})|I'|^{-1}(w(I')^{1-q'})^{q-1}.$$

By (3.3) again,

$$\sum_{\substack{I\subset P\\ \ell(I)=2^{-j}}} \frac{|I|^2}{w(I)} \frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')}+|x_I^c-x_{I'}^c|)^{n+M}} \tag{3.5}$$
$$\leq C2^{-jn(2-q)}(2^{(j\wedge j'-j')n}+2^{(j'-j\wedge j')M})|I'|^{q-1}w(I')^{-1}.$$

Thus,

$$A_{11} \leq \frac{C}{w(P)^{2/p-1}} \sum_{j=j_0}^{\infty}\sum_{j'=j_0}^{\infty} \sum_{\substack{I'\subset 3P\\ \ell(I')=2^{-j'}}}$$
$$2^{-|j-j'|L+(j-j')n(q-2)}(2^{(j\wedge j'-j')n}+2^{(j'-j\wedge j')M})\frac{|I'|^2}{w(I')}|(\phi_{I'}*f)(x_{I'})|^2.$$

Since there are 3^n dyadic cubes in $3P$ with the same edge-length as P,

$$\sum_{\substack{I'\subset 3P\\ \ell(I')\leq\ell(P)}} |(\phi_{I'}*f)(x_{I'})|^2 \frac{|I'|^2}{w(I')} \leq 3^n \sup_{\substack{P'\subset 3P\\ \ell(P')=\ell(P)}} \sum_{I'\subset P'} |(\phi_{I'}*f)(x_{I'})|^2\,\frac{|I'|^2}{w(I')}.$$

Choosing $L>\max\{M-n(q-2)-n, n(q-2)\}$, we have

$$A_{11} \leq \frac{C}{w(P)^{2/p-1}} \sum_{j'=j_0}^{\infty} \sum_{\substack{I'\subset 3P\\ \ell(I')=2^{-j'}}} |(\phi_{I'}*f)(x_{I'})|^2\,\frac{|I'|^2}{w(I')}$$
$$\times \sum_{j=j_0}^{\infty} 2^{-|j-j'|L+(j-j')n(q-2)}(2^{(j\wedge j'-j')n}+2^{(j'-j\wedge j')M})$$
$$\leq \frac{C}{w(P)^{2/p-1}} \sup_{\substack{P'\subset 3P\\ \ell(P')=\ell(P)}} \sum_{I'\subset P'} |(\phi_{I'}*f)(x_{I'})|^2\,\frac{|I'|^2}{w(I')}.$$

Next we decompose the set of dyadic cubes $\{I : I \cap 3P = \emptyset \text{ and } \ell(I) = \ell(P)\}$ into the disjoint union of $\{H_i\}_{i\in\mathbb{N}}$ according to the distance between each I and P. Namely, for each $i \in \mathbb{N}$,

$$H_i := \{P' : P' \cap 3P = \emptyset,\ \ell(P') = \ell(P),\ 2^{i-j_0} \le |x^c_{P'} - x^c_P| < 2^{i+1-j_0}\},$$

where x^c_P and $x^c_{P'}$ denote the centers of P and P', respectively. Thus,

$$A_{12} \le C \sum_{i=1}^{\infty} \sum_{P'\in H_i} \frac{1}{w(P)^{2/p-1}} \sum_{j=j_0}^{\infty} \sum_{\substack{I\subset P\\ \ell(I)=2^{-j}}} \frac{|I|^2}{w(I)} \sum_{j'=j_0}^{\infty} \sum_{\substack{I'\subset P'\\ \ell(I')=2^{-k'}}} 2^{-|j-j'|L}|I'|$$
$$\times \frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')} + |x^c_P - x^c_{P'}|)^{n+M}} |(\phi_{I'} * f)(x_{I'})|^2.$$

Let Q_i be the cube with center x^c_P and $\ell(Q_i) = 2^{i+2-j_0}$. Then $P \subset Q_i$ and $P' \subset Q_i$ for any $P' \in H_i$. Theorem B shows that, for any $P' \in H_i$,

$$\frac{w(P')}{w(P)} \le C2^{-i\frac{r-1}{r}+iq}.$$

Note that $|x^c_{P'} - x^c_P| \approx 2^{i-j_0}$ for $P' \in H_i$. By (3.5) for $M > 2(nq'(q-1)-n)$,

$$A_{12} \le C \sum_{i=1}^{\infty} \sum_{P'\in H_i} \frac{1}{w(P')^{2/p-1}} 2^{(-i+j_0)M/2+i(q-\frac{r-1}{r})(2/p-1)}$$
$$\times \sum_{j'=j_0}^{\infty} \sum_{j=j_0}^{\infty} 2^{-|j-j'|L-(j\wedge j')M/2} 2^{(j-j')n(q-2)} (2^{(j\wedge j'-j')n} + 2^{(j'-j\wedge j')M/2})$$
$$\times \sum_{\substack{I'\subset P'\\ \ell(I')=2^{-j'}}} |(\phi_{I'} * f)(x_{I'})|^2 \frac{|I'|^2}{w(I')}.$$

Since there are at most $2^{(i+2)n}$ cubes P' in H_i and for $j' \ge j_0$,

$$\sum_{j=j_0}^{\infty} 2^{-|j-j'|L-(j\wedge j')M/2} 2^{(j-j')n(q-2)} (2^{(j\wedge j'-j')n} + 2^{(j'-j\wedge j')M/2}) \le C2^{-j_0M/2},$$

we choose $M > 2q(2/p-1)$ to get

$$A_{12} \le C \sup_{P'} \frac{1}{w(P')^{2/p-1}} \sum_{j'=j_0}^{\infty} \sum_{\substack{I'\subset P'\\ \ell(I')=2^{-j'}}} |(\phi_{I'} * f)(x_{I'})|^2 \frac{|I'|^2}{w(I')}$$
$$\times \sum_{i=1}^{\infty} 2^{-iM/2+i(q-\frac{r-1}{r})(2/p-1)}$$
$$\le C \sup_{P'} \frac{1}{w(P')^{2/p-1}} \sum_{I'\subset P'} |(\phi_{I'} * f)(x_{I'})|^2 \frac{|I'|^2}{w(I')}.$$

To estimate A_2, we use (3.3) to obtain

$$\begin{aligned}
A_2 &\le \frac{C}{w(P)^{2/p-1}} \sum_{j=j_0}^{\infty} \sum_{\substack{I\subset P\\ \ell(I)=2^{-j}}} \sum_{j'=-\infty}^{j_0-1} \sum_{I'} \frac{|I|^2}{w(I)} 2^{-(j-j')L} \\
&\qquad \times |I'| \frac{2^{-j'M}}{(2^{-j'}+|x_P^c - x_{I'}^c|)^{n+M}} |(\phi_{I'} * f)(x_{I'})|^2 \\
&\le \frac{C}{w(P)^{2/p-1}} \sum_{j=j_0}^{\infty} \sum_{\substack{I\subset P\\ \ell(I)=2^{-j}}} \sum_{j'=-\infty}^{j_0-1} \sum_{I'} 2^{-(j-j')L} 2^{-jn(2-q)} (w(I)^{1-q'})^{q-1} \\
&\qquad \times |I'| \frac{2^{-j'M}}{(2^{-j'}+|x_P^c - x_{I'}^c|)^{n+M}} |(\phi_{I'} * f)(x_{I'})|^2 \\
&= \frac{C}{w(P)^{2/p-1}} \sum_{j=j_0}^{\infty} \sum_{j'=-\infty}^{j_0-1} \sum_{I'} 2^{-(j-j')L} 2^{-jn(2-q)} (w(P)^{1-q'})^{q-1} \\
&\qquad \times |I'| \frac{2^{-j'M}}{(2^{-j'}+|x_P^c - x_{I'}^c|)^{n+M}} |(\phi_{I'} * f)(x_{I'})|^2
\end{aligned}$$

Let $E_k^0 = \{I' : \ell(I') = 2^k \ell(P) \text{ and } |x_P^c - x_{I'}^c| \le \ell(I')\}$ and $E_k^i = \{I' : \ell(I') = 2^k \ell(P) \text{ and } 2^{i-1}\ell(I') < |x_P^c - x_{I'}^c| \le 2^i \ell(I')\}$ for $i \in \mathbb{N}$. Then the cube Q_k^i with center x_P^c and $\ell(Q_k^i) = 2^{i+k-j_0+2}$ contains P and I' for any $I' \in E_k^i$. By Theorem B,

$$\frac{w(I')}{w(P)} \le C 2^{(i+k)(q-\frac{r-1}{r})} \quad \text{for any } I' \in E_k^i.$$

Since $w^{1-q'} \in A_{q'}$, there exists $\bar{r} > 1$ such that $w^{1-q'} \in RH_{\bar{r}}$. Using Theorem B again, we have

$$\frac{w(P)^{1-q'}}{w(I')^{1-q'}} \le C 2^{-(i+k)(q'-\frac{\bar{r}-1}{\bar{r}})} \quad \text{for any } I' \in E_k^i.$$

By the above two inequalities and (3.3),

$$\begin{aligned}
A_2 \le C \sum_{j=j_0}^{\infty} \sum_{k=1}^{\infty} \sum_{\{I' : \ell(I')=2^k\ell(P)\}} & \frac{2^{(i+k)(q-\frac{r-1}{r})(2/p-1)}}{w(I')^{2/p-1}} 2^{-(j-j')L} 2^{(j-j')n(q-2)} \\
& \times 2^{-(i+k)(q'-\frac{\bar{r}-1}{\bar{r}})(q-1)} \frac{2^{-j'(M+n)}}{(2^{-j'}+|x_P^c - x_{I'}^c|)^{n+M}} \frac{|I'|^2}{w(I')} |(\phi_{I'} * f)(x_{I'})|^2.
\end{aligned}$$

Choosing $L = n(q-2) + M + n$, we then have

$$A_2 = C2^{-j_0(M+n)} \sum_{k=1}^{\infty} \sum_{\{I' : \ell(I')=2^k\ell(P)\}} \frac{2^{-(i+k)[(q'-\frac{\bar{r}-1}{\bar{r}})(q-1)-(q-\frac{r-1}{r})(2/p-1)]}}{w(I')^{2/p-1}(\ell(I') + |x_P^c - x_{I'}^c|)^{n+M}} \times |(\phi_{I'} * f)(x_{I'})|^2 \frac{|I'|^2}{w(I')}$$

$$\leq \sum_{k=1}^{\infty}\sum_{i=0}^{\infty}\sum_{I'\in E_j^i} \frac{2^{-(i+k)[M+(q'-\frac{\bar{r}-1}{\bar{r}})(q-1)-(q-\frac{r-1}{r})(2/p-1)]}}{w(I')^{2/p-1}} 2^{-(i+k)(n+M)} \times |(\phi_{I'} * f)(x_{I'})|^2 \frac{|I'|^2}{w(I')}.$$

There are at most 2^{in} dyadic cubes $I' \in E_k^i$ for $i \in \mathbb{N}$, and at most 3^n dyadic cubes $I' \in E_k^0$. Thus,

$$A_2 \leq C\bigg(\sup_P \frac{1}{|P|^{2/p-1}} \sum_{I'\subset P} |(\phi_{I'} * f)(x_{I'})|^2 \frac{|I'|^2}{w(I')}\bigg) \times \sum_{k=1}^{\infty} 2^{-k[2M+n+(q'-\frac{\bar{r}-1}{\bar{r}})(q-1)-(q-\frac{r-1}{r})(2/p-1)]}$$

$$\leq C \sup_P \frac{1}{|P|^{2/p-1}} \sum_{I'\subset P} |(\phi_{I'} * f)(x_{I'})|^2 \frac{|I'|^2}{w(I')}$$

since $M > 2q(2/p-1)$. The proof of Theorem 1.2 is complete.

4. Some results on CMO_w^p**.** We now use Theorem D to obtain the following density statement.

PROPOSITION 4.1. *Let* $0<p\leq 1$ *and* $w\in A_\infty$*. Then* $L^2(\mathbb{R}^n)\cap \mathrm{CMO}_w^p(\mathbb{R}^n)$ *is dense in* $\mathrm{CMO}_w^p(\mathbb{R}^n)$ *in the weak topology* $(H_w^p, \mathrm{CMO}_w^p)$*. More precisely, for any* $f\in \mathrm{CMO}_w^p(\mathbb{R}^n)$*, there exists a sequence* $\{f_N\}\subset L^2(\mathbb{R}^n)\cap \mathrm{CMO}_w^p(\mathbb{R}^n)$ *satisfying* $\|f_N\|_{\mathrm{CMO}_w^p} \leq C\|f\|_{\mathrm{CMO}_w^p}$ *such that, for each* $g \in H_w^p(\mathbb{R}^n)$*,* $\lim_{N\to\infty}\langle f_N, g\rangle = \langle f, g\rangle$*, where the constant* C *is independent of* N *and* f*.*

Suppose that $f \in \mathrm{CMO}_w^p(\mathbb{R}^n)$. Denote

$$E_N = \{(j,\mathbf{j}) \in \mathbb{Z}\times\mathbb{Z}^n : |j| \leq N,\ |\mathbf{j}| \leq N\}.$$

Set

$$f_N(x) = \sum_{(j,\mathbf{j})\in E_N} 2^{-jn}(\psi_j * f)(x_I)\psi_j(x-x_I), \tag{4.1}$$

where ψ satisfies (1.1)–(1.2). It is easy to see that $f_N \in L^2(\mathbb{R}^n)$.

To show Proposition 4.1, we need the following lemma.

LEMMA 4.2. *Let $w \in A_\infty$. Suppose that $f \in \mathrm{CMO}^p_w(\mathbb{R}^n)$ and f_N is given by* (4.1). *Then $f_N \in \mathrm{CMO}^p_w(\mathbb{R}^n)$ and $\|f_N\|_{\mathrm{CMO}^p_w} \le C\|f\|_{\mathrm{CMO}^p_w}$, where the constant C is independent of N.*

Proof. It suffices to show

$$\sup_P \bigg\{\frac{1}{w(P)^{2/p-1}} \sum_{j\in\mathbb{Z}} \sum_{I\subset P} |(\psi_j * f_N)(x_I)|^2 \frac{|I|^2}{w(I)}\bigg\}^{1/2} \le C \sup_P \bigg\{\frac{1}{w(P)^{2/p-1}} \sum_{j\in\mathbb{Z}} \sum_{I\subset P} |(\psi_j * f)(x_I)|^2 \frac{|I|^2}{w(I)}\bigg\}^{1/2}.$$

The proof of the above inequality is similar to the proof of Theorem 1.2. We omit the details. ∎

We use Lemma 4.2 to show Proposition 4.1.

Proof of Proposition 4.1. Without loss of generality, we may choose ψ to satisfy (1.1)–(1.2) with $\psi(x) = \psi(-x)$. For each $h \in \mathcal{S}_\infty$, by Theorem D and (4.1),

$$\begin{aligned}\langle f - f_N, h\rangle &= \Big\langle \sum_{(j,\mathbf{j})\in(E_N)^c} 2^{-nj}(\psi_j * f)(x_I)\psi_j(\cdot - x_I), h\Big\rangle \\ &= \Big\langle f, \sum_{(j,\mathbf{j})\in(E_N)^c} 2^{-nj}(\psi_j * h)(x_I)\psi_j(\cdot - x_I)\Big\rangle.\end{aligned}$$

By Theorem D,

$$\sum_{(j,\mathbf{j})\in(E_N)^c} 2^{-nj}(\psi_j * h)(x_I)\psi_j(x - x_I)$$

tends to zero in $\mathcal{S}_\infty(\mathbb{R}^n)$ as $N \to \infty$ and hence, for each $h \in \mathcal{S}_\infty(\mathbb{R}^n)$, $\langle f - f_N, h\rangle$ tends to zero as $N \to \infty$. Since $\mathcal{S}_\infty$ is dense in H^p_w, it follows that for each $g \in H^p_w$, $\langle f - f_N, g\rangle$ tends to 0 as $N \to \infty$. Indeed, for any given $\varepsilon > 0$, there exists $h \in \mathcal{S}_\infty$ such that $\|g - h\|_{H^p_w} \le \varepsilon$. It follows from Lemma 4.2, $\|f_N\|_{\mathrm{CMO}^p_w} \le C\|f\|_{\mathrm{CMO}^p_w}$, and Theorem A that

$$\begin{aligned}|\langle f - f_N, g\rangle| &\le |\langle f - f_N, g - h\rangle| + |\langle f - f_N, h\rangle| \\ &\le C\|f - f_N\|_{\mathrm{CMO}^p_w}\|g - h\|_{H^p_w} + |\langle f - f_N, h\rangle| \\ &\le C\varepsilon\|f\|_{\mathrm{CMO}^p_w} + |\langle f - f_N, h\rangle|.\end{aligned}$$

This implies $\langle f - f_N, g\rangle \to 0$ as $N \to \infty$. ∎

5. The proof of Theorem 1.3. We define R_j on $\mathrm{CMO}^p_w(\mathbb{R}^n)$ as follows. Given $f \in \mathrm{CMO}^p_w(\mathbb{R}^n)$, by Proposition 4.1, there is a sequence $\{f_N\} \subset L^2 \cap \mathrm{CMO}^p_w$ such that $\|f_N\|_{\mathrm{CMO}^p_w} \le C\|f\|_{\mathrm{CMO}^p_w}$ and, for each $g \in L^2 \cap H^p_w$,

$\langle f_N, g\rangle \to \langle f, g\rangle$ as $N \to \infty$. Thus, for $f \in \mathrm{CMO}^p_w$, define

$$\langle R_j f, g\rangle = \lim_{N\to\infty} \langle R_j f_N, g\rangle \quad \text{for } g \in L^2 \cap H^p_w.$$

To see the existence of this limit, we write $\langle (R_j(f_i - f_k), g\rangle = \langle f_i - f_k, R_j^*(g)\rangle$ since both $f_i - f_k$ and g belong to L^2, and R_j is bounded on L^2. It is known that R_j is bounded on H^p_w and hence $R_j^* g \in L^2 \cap H^p_w$. Consequently, by Proposition 4.1 again, $\langle f_i - f_k, R_j^* g\rangle$ tends to zero as $i, k \to \infty$. It is also easy to see that the above definition of $R_j f$ is independent of the choice of the sequence $\{f_N\}$ which satisfies the conditions in Proposition 4.1. We now show the boundedness of R_j on $L^2 \cap \mathrm{CMO}^p_w$.

THEOREM 5.1. *Suppose that $w \in A_\infty$. For $f \in L^2(\mathbb{R}^n) \cap \mathrm{CMO}^p_w(\mathbb{R}^n)$,*

$$\|R_j f\|_{\mathrm{CMO}^p_w} \le C\|f\|_{\mathrm{CMO}^p_w},$$

where the constant C is independent of f.

To show Theorem 5.1, we need a discrete Calderón-type identity on $L^2 \cap \mathrm{CMO}^p_w$. For this purpose, let $\phi \in \mathcal{S}$ with $\operatorname{supp} \phi \subset B(0,1)$,

$$\sum_{j\in\mathbb{Z}} |\widehat{\phi}(2^{-j}\xi)|^2 = 1 \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}, \tag{5.1}$$

and

$$\int_{\mathbb{R}^n} \phi(x) x^\alpha\, dx = 0 \quad \text{for all } |\alpha| \le 10M, \tag{5.2}$$

where M is any fixed large positive integer.

The discrete Calderón-type identity on $L^2 \cap \mathrm{CMO}^p_w$ is given by the following

LEMMA 5.2. *Let $0 < p \le 1$, $w \in A_\infty$ and ϕ satisfy conditions* (5.1)–(5.2) *with a large M depending on p. Then for any $f \in L^2 \cap \mathrm{CMO}^p_w$, there exists $h \in L^2 \cap \mathrm{CMO}^p_w$ such that, for sufficiently large $N \in \mathbb{N}$,*

$$f(x) = \sum_{j\in\mathbb{Z}} \sum_{\widetilde{I}}^{(N)} |\widetilde{I}| \phi_j(x - x_{\widetilde{I}})(\phi_j * h)(x_{\widetilde{I}}),$$

where the series converges in L^2 and, hereafter, $\sum_{\widetilde{I}}^{(N)}$ denotes summation over $\widetilde{I}$ running over dyadic cubes in $\mathbb{R}^n$ with edge-lengths 2^{-j-N} and lower-left corners $x_{\widetilde{I}}$. Moreover,

$$\|f\|_{L^2} \approx \|h\|_{L^2} \quad \textit{and} \quad \|f\|_{\mathrm{CMO}^p_w} \approx \|h\|_{\mathrm{CMO}^p_w}.$$

Proof. By taking the Fourier transform, it is easy to see that

$$f(x) = \sum_{j\in\mathbb{Z}} (\phi_j * \phi_j * f)(x) \quad \text{for } f \in L^2.$$

Applying Coifman's decomposition of the identity operator, we obtain

$$\begin{aligned} f(x) &= \sum_{j\in\mathbb{Z}} \sum_{\widetilde{I}}^{(N)} |\widetilde{I}|\phi_j(x-x_{\widetilde{I}})(\phi_j * f)(x_{\widetilde{I}}) + \mathcal{R}_N f(x) \\ &:= T_N f(x) + \mathcal{R}_N f(x), \end{aligned}$$

where

$$\begin{aligned} \mathcal{R}_N f(x) &= \sum_{j\in\mathbb{Z}} \sum_{\widetilde{I}}^{(N)} \int_{\widetilde{I}} [\phi_j(x-u)(\phi_j * f)(u) - \phi_j(x-x_{\widetilde{I}})(\phi_j * f)(x_{\widetilde{I}})]\, du \\ &= \sum_{j\in\mathbb{Z}} \sum_{\widetilde{I}}^{(N)} \int_{\widetilde{I}} [\phi_j(x-u) - \phi_j(x-x_{\widetilde{I}})](\phi_j * f)(u)\, du \\ &\quad + \sum_{j\in\mathbb{Z}} \sum_{\widetilde{I}}^{(N)} \int_{\widetilde{I}} \phi_j(x-x_{\widetilde{I}})[(\phi_j * f)(u) - (\phi_j * f)(x_{\widetilde{I}})]\, du \\ &:= \mathcal{R}_N^1 f(x) + \mathcal{R}_N^2 f(x). \end{aligned}$$

We claim that, for $f \in L^2 \cap \mathrm{CMO}_w^p$,

$$\|\mathcal{R}_N^i f\|_2 \le C2^{-N}\|f\|_2, \qquad i = 1,2, \tag{5.3}$$

$$\|\mathcal{R}_N^i f\|_{\mathrm{CMO}_w^p} \le C2^{-N}\|f\|_{\mathrm{CMO}_w^p}, \quad i = 1,2, \tag{5.4}$$

where C is a constant independent of f and N.

Assume the claim for the moment. Then, by choosing N sufficiently large, $T_N^{-1} = \sum_{n=0}^{\infty}(\mathcal{R}_N)^n$ is bounded on both L^2 and CMO_w^p, which implies

$$\|T_N^{-1} f\|_2 \approx \|f\|_2 \quad \text{and} \quad \|T_N^{-1} f\|_{\mathrm{CMO}_w^p} \approx \|f\|_{\mathrm{CMO}_w^p}.$$

Moreover, for any $f \in L^2 \cap \mathrm{CMO}_w^p$, set $h = T_N^{-1} f$. We obtain

$$f(x) = T_N(T_N^{-1} f)(x) = \sum_{j\in\mathbb{Z}} \sum_{\widetilde{I}}^{(N)} |\widetilde{I}|\phi_j(x-x_{\widetilde{I}})(\phi_j * h)(x_{\widetilde{I}}),$$

where the series converges in L^2.

Now we prove (5.3) and (5.4). Since the proofs for $\mathcal{R}_N^1$ and $\mathcal{R}_N^2$ are similar, we give the proof for $\mathcal{R}_N^1$ only. Let $f \in L^2 \cap \mathrm{CMO}_w^p$. By Theorem D,

$$\begin{aligned} &(\psi_{j'} * \mathcal{R}_N^1 f)(x) \\ &= \sum_{j\in\mathbb{Z}} \sum_{\widetilde{I}}^{(N)} \int_{\widetilde{I}} (\psi_{j'} * [\phi_j(\cdot-u) - \phi_j(\cdot - x_{\widetilde{I}})])(x)(\phi_j * f)(u)\, du \\ &= \sum_{j\in\mathbb{Z}} \sum_{\widetilde{I}}^{(N)} \int_{\widetilde{I}} (\psi_{j'} * [\phi_j(\cdot-u) - \phi_j(\cdot - x_{\widetilde{I}})])(x) \\ &\qquad \times \Big(\phi_j * \Big\{ \sum_{j''\in\mathbb{Z}} \sum_{I''} |I''|\psi_{j''}(\cdot - x_{I''})(\psi_{j''} * f)(x_{I''}) \Big\}\Big)(u)\, du, \end{aligned} \tag{5.5}$$

where I'' are dyadic cubes in $\mathbb{R}^n$ with edge-lengths $2^{-j''}$ and lower-left corners $x_{I''}$.

Set $\widetilde{\phi}_j(z) = \phi_j(z-u) - \phi_j(z - x_{\widetilde{I}})$, where $u \in \widetilde{I}$. Note that $\widetilde{\phi}_j \in \mathcal{S}$ and $|\widetilde{\phi}_j(x)| \le C2^{-N}2^{jn}(1+2^j|x-u|)^{-M}$ for any $M \in \mathbb{N}$ since, if $u \in \widetilde{I}$, then $|u - x_{\widetilde{I}}| \le C2^{-j-N}$. Thus, by an almost orthogonality argument, for large positive integers M we obtain

$$\begin{aligned} |(\psi_{j'} * \widetilde{\phi}_j)(x)| &\le C2^{-N}2^{-10M|j-j'|}\frac{2^{n(j\wedge j')}}{(1+2^{j\wedge j'}|x-u|)^{n+M}} \\ &\le C2^{-N}2^{-5M|j-j'|}\frac{2^{nj'}}{(1+2^{j'}|x-u|)^{n+M}}. \end{aligned}$$

Similarly, for $u \in \widetilde{I}$,

$$|(\phi_j * \psi_{j''})(u - x_{I''})| \le C2^{-5M|j-j''|}\frac{2^{nj''}}{(1+2^{j''}|u-x_{I''}|)^{n+M}}.$$

Substituting these estimates into the last term in (5.5) yields

$$\begin{aligned} &|(\psi_{j'} * \mathcal{R}_N^1 f)(x)| \\ &\quad\le C2^{-N}\sum_{j''\in\mathbb{Z}}\sum_{I''}|I''|\,|(\psi_{j''} * f)(x_{I''})|\sum_{j\in\mathbb{Z}}\sum_{\widetilde{I}}^{(N)}\int_{\widetilde{I}} 2^{-5M|j-j'|} \\ &\qquad\times \frac{2^{nj'}}{(1+2^{j'}|x-u|)^{n+M}}2^{-5M|j-j''|}\frac{2^{nj''}}{(1+2^{j''}|u-x_{I''}|)^{n+M}}\,du \\ &\quad\le C2^{-N}\sum_{j''\in\mathbb{Z}}\sum_{I''}2^{-5M|j'-j''|}|I''|\frac{2^{n(j'\wedge j'')}}{(1+2^{j'\wedge j''}|x-x_{I''}|)^{n+M}}|(\psi_{j''} * f)(x_{I''})|. \end{aligned}$$

By the equivalence $\|\mathcal{G}(f)\|_2 \approx \|f\|_2$ and Hölder's inequality,

$$\begin{aligned} \|\mathcal{R}_N^1 f\|_2 &\le C\|\mathcal{G}(\mathcal{R}_N^1 f)\|_2 \\ &\le C2^{-N}\Big\|\Big\{\sum_{j''\in\mathbb{Z}}\sum_{I''}|(\psi_{j''} * f)(x_{I''})|^2\chi_{I''}\Big\}^{1/2}\Big\|_2 \le C2^{-N}\|f\|_2. \end{aligned}$$

Similarly, repeating the same proof of Theorem 1.2 yields

$$\|\mathcal{R}_N^1 f\|_{\mathrm{CMO}_w^p} \le C2^{-N}\|f\|_{\mathrm{CMO}_w^p}.$$

Thus both (5.3) and (5.4) are proved and Lemma 5.2 follows. ∎

As a consequence of Lemma 5.2, we give an equivalent norm for functions in $L^2 \cap \mathrm{CMO}_w^p$.

Corollary 5.3. *Let $w \in A_\infty$ and $0 < p \le 1$. Suppose ϕ_j's satisfy the same conditions as in Lemma* 5.2. *Then for a fixed large N as in Lemma* 5.2

and $f \in L^2 \cap \mathrm{CMO}^p_w$,

$$\|f\|_{\mathrm{CMO}^p_w} \approx \sup_P \left\{ \frac{1}{w(P)^{2/p-1}} \sum_{j\in\mathbb{Z}} \sum_{\widetilde{I}\subset P}^{(N)} |(\phi_j * f)(x_{\widetilde{I}})|^2 \frac{|\widetilde{I}|^2}{w(\widetilde{I})} \right\}^{1/2}.$$

Proof. Suppose $f \in L^2 \cap \mathrm{CMO}^p_w$. Let $T_N f$ be as in Lemma 5.2. The boundedness of T_N^{-1} on $L^2 \cap \mathrm{CMO}^p_w$ gives

$$\|f\|_{\mathrm{CMO}^p_w} = \|T_N^{-1} T_N f\|_{\mathrm{CMO}^p_w} \le C \|T_N f\|_{\mathrm{CMO}^p_w}.$$

For any dyadic cube $P \subset \mathbb{R}^n$, by the definition of T_N,

$$\begin{aligned} &\sum_{j\in\mathbb{Z}} \sum_{I\subset P} |(\psi_j * T_N f)(x_I)|^2 \frac{|I|^2}{w(I)} \\ &\quad = \sum_{j\in\mathbb{Z}} \sum_{I\subset P} \Big| \sum_{j'\in\mathbb{Z}} \sum_{\widetilde{I}'}^{(N)} (\psi_j * \phi_{j'})(x_I - x_{\widetilde{I}'})(\phi_{j'} * f)(x_{\widetilde{I}'}) |\widetilde{I}'| \Big|^2 \frac{|I|^2}{w(I)}, \end{aligned} \tag{5.6}$$

where ψ_j and $\phi_{j'}$ are as in Theorem 1.2 and Lemma 5.2, respectively.

Applying the classical almost orthogonality estimates, we have

$$|\psi_j * \phi_{j'}(x)| \le C 2^{-|j-j'|L} \frac{2^{n(j\wedge j')}}{(1 + 2^{j\wedge j'}|x|)^{n+M}}. \tag{5.7}$$

This, together with Hölder's inequality, shows that the right hand side in (5.6) is dominated by

$$\begin{aligned} &C \sum_{j\in\mathbb{Z}} \sum_{I\subset P} \sum_{j'\in\mathbb{Z}} \sum_{\widetilde{I}'}^{(N)} 2^{-|j-j'|L} \\ &\qquad \times \frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')} + |x_I - x_{\widetilde{I}'}|)^{n+M}} |\widetilde{I}'| (\phi_{j'} * f)(x_{\widetilde{I}'})^2 \frac{|I|^2}{w(I)}. \end{aligned}$$

Applying a similar argument to the proof of Theorem 1.2, we obtain

$$\begin{aligned} &\|f\|_{\mathrm{CMO}^p_w} \le C \|T_N f\|_{\mathrm{CMO}^p_w} \\ &\qquad \le C \sup_P \left(\frac{1}{w(P)^{2/p-1}} \sum_{j'\in\mathbb{Z}} \sum_{\widetilde{I}'\subset P}^{(N)} |(\phi_{j'} * f)(x_{\widetilde{I}'})|^2 \frac{|\widetilde{I}'|^2}{w(\widetilde{I}')} \right)^{1/2}. \end{aligned}$$

On the other hand, applying first the discrete Calderón identity (Lemma 5.2) and then the orthogonality estimates (5.7), we also find that, for any dyadic cube $P \subset \mathbb{R}^n$,

$$\begin{aligned}
&\sum_{j'\in\mathbb{Z}} \sum_{\widetilde{I'}\subset P}^{(N)} |(\phi_{j'} * f)(x_{\widetilde{I'}})|^2 \frac{|\widetilde{I'}|^2}{w(\widetilde{I'})} \\
&\quad = \sum_{j'\in\mathbb{Z}} \sum_{\widetilde{I'}\subset P}^{(N)} \Big| \sum_j \sum_I (\phi_{j'} * \psi_j)(x_{\widetilde{I'}} - x_I)(\psi_j * f)(x_I)|I| \Big|^2 \frac{|\widetilde{I'}|^2}{w(\widetilde{I'})} \\
&\quad \le C \sum_{j'\in\mathbb{Z}} \sum_{\widetilde{I'}\subset P}^{(N)} \sum_j \sum_I 2^{-|j-j'|L} \\
&\qquad\qquad \times \frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')} + |x_{\widetilde{I'}} - x_I|)^{n+M}} |I| (\psi_j * f)(x_I)^2 \frac{|\widetilde{I'}|^2}{w(\widetilde{I'})},
\end{aligned}$$

where I and I' are as in (5.6).

Using again a similar argument to the proof of Theorem 1.2, we have

$$\sup_P \bigg\{ \frac{1}{w(P)^{2/p-1}} \sum_{\widetilde{I}\subset P}^{(N)} |(\phi_j * f)(x_{\widetilde{I}})|^2 \frac{|\widetilde{I}|^2}{w(\widetilde{I})} \bigg\}^{1/2} \le C\|f\|_{\mathrm{CMO}_w^p},$$

completing the proof. ∎

We are ready to show Theorem 5.1.

Proof of Theorem 5.1. By Corollary 5.3, it suffices to show that for any dyadic cube P,

$$\bigg(\frac{1}{w(P)^{2/p-1}} \sum_{i\in\mathbb{Z}} \sum_{\widetilde{I}\subset P}^{(N)} |(\phi_i * R_j f)(x_{\widetilde{I}})|^2 \frac{|\widetilde{I}|^2}{w(\widetilde{I})} \bigg)^{1/2} \le C\|f\|_{\mathrm{CMO}_w^p},$$

where ϕ_i and I satisfy the conditions as in Lemma 5.2 and the constant C is independent of P and f.

Using the L^2 boundedness of R_j and the discrete Carderón-type identity given in Lemma 5.2, we write

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}} \sum_{\widetilde{I}\subset P}^{(N)} |(\phi_i * R_j f)(x_{\widetilde{I}})|^2 \frac{|\widetilde{I}|^2}{w(\widetilde{I})} \\
&\quad = \sum_{i\in\mathbb{Z}} \sum_{\widetilde{I}\subset P}^{(N)} \Big| \sum_{i'\in\mathbb{Z}} \sum_{\widetilde{I'}}^{(N)} (\phi_{i'} * h)(x_{\widetilde{I'}})|\widetilde{I'}|(K_j * \phi_i * \phi_{i'})(x_{\widetilde{I}} - x_{\widetilde{I'}}) \Big|^2 \frac{|\widetilde{I}|^2}{w(\widetilde{I})},
\end{aligned}$$

where $\|h\|_{\mathrm{CMO}_w^p} \le C\|f\|_{\mathrm{CMO}_w^p}$.

We claim that

$$|(K_j * \phi_i)(x)| \le C \frac{2^{in}}{(1 + 2^i|x|)^{n+M}}. \tag{5.8}$$

To show (5.8), we consider the following two cases. For $|x| \le 2^{1-i}$, by the support condition on ϕ_i,

$$\begin{aligned}|(K_j * \phi_i)(x)| &= \Big|\lim_{\varepsilon_1\to 0} \int\limits_{\varepsilon_1 \le |x-u| \le 3\cdot 2^{-i}} K_j(x-u)\phi_i(u)\,du\Big| \\ &= \Big|\lim_{\varepsilon_1\to 0} \int\limits_{\varepsilon_1 \le |x-u| \le 3\cdot 2^{-i}} K_j(x-u)[\phi_i(u)-\phi_i(x)]\,du\Big| \\ &\le C2^{i(n+1)} \int\limits_{|x-u|\le 3\cdot 2^{-i}} |x-u|^{-n+1}\,du \\ &\le C2^{in} \le C\frac{2^{in}}{(1+2^i|x|)^{n+M}}.\end{aligned}$$

For $|x| > 2^{1-i}$, by the cancellation condition on ϕ_i with order M,

$$\begin{aligned}|(K_j * \phi_i)(x)| &= \Big|\int\limits_{|u|\le 2^{-i}} \Big[K_j(x-u) - \sum_{|\alpha|\le M} \frac{1}{\alpha!}\partial_x^\alpha K_j(x)u^\alpha\Big]\phi_i(u)\,du\Big| \\ &\le C\int\limits_{|u|\le 2^{-i}} \frac{|u|^{M+1}}{|x|^{n+M+1}}|\phi_i(u)|\,du \le C\frac{2^{in}}{(1+2^i|x|)^{n+M}}.\end{aligned}$$

Estimate (5.8) and the classical orthogonality estimate

$$|(\phi_i * \phi_{i'})(x)| \le C2^{-|i-i'|L}\frac{2^{n(i\wedge i')}}{(1+2^{i\wedge i'}|x|)^{n+M}}$$

imply

$$|(K_j * \phi_i * \phi_{i'})(x)| \le C2^{-|i-i'|L}\frac{2^{n(i\wedge i')}}{(1+2^{i\wedge i'}|x|)^{n+M}}.$$

Therefore, the same argument as in Theorem 1.2 yields

$$\|R_j f\|_{\mathrm{CMO}_w^p} \le C\|h\|_{\mathrm{CMO}_w^p} \le C\|f\|_{\mathrm{CMO}_w^p}$$

for $f \in L^2 \cap \mathrm{CMO}_w^p$. ■

We now prove the main result of this article.

Proof of Theorem 1.3. By the definition of $R_j f$ for $f \in \mathrm{CMO}_w^p$ and the boundedness of R_j on $L^2 \cap \mathrm{CMO}_w^p$, we choose a sequence $\{f_N\} \subset L^2 \cap \mathrm{CMO}_w^p$ such that $\|f_N\|_{\mathrm{CMO}_w^p} \le C\|f\|_{\mathrm{CMO}_w^p}$ and

$$\begin{aligned}\|R_j f\|_{\mathrm{CMO}_w^p} &\le \liminf_{N\to\infty} \|R_j f_N\|_{\mathrm{CMO}_w^p} \\ &\le C\liminf_{N\to\infty} \|f_N\|_{\mathrm{CMO}_w^p} \le C\|f\|_{\mathrm{CMO}_w^p}.\end{aligned}$$

This completes the proof. ■

Acknowledgements. This research was supported by NSC of Taiwan under Grant #NSC 99-2115-M-008-002-MY3.

References

[CF] R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. 51 (1974), 241–250.

[CW] R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. 83 (1977), 569–645.

[DHLW] Y. Ding, Y. Han, G. Lu and X. Wu, *Boundedness of singular integrals on multiparameter weighted Hardy spaces* $H_w^p(\mathbb{R}^n \times \mathbb{R}^m)$, Potential Anal.; DOI 10.1007/s11118-011-9244-y.

[FS] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, Math. Notes 28, Princeton Univ. Press, Princeton, NJ, 1982.

[FJ] M. Frazier and B. Jawerth, *A discrete transform and decompositions of distribution spaces*, J. Funct. Anal. 93 (1990), 34–170.

[GM] J. García-Cuerva and J. M. Martell, *Wavelet characterization of weighted spaces*, J. Geom. Anal. 11 (2001), 241–264.

[GR] J. García-Cuerva and J. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, Amsterdam, 1985.

[GW] R. F. Gundy and R. L. Wheeden, *Weighted integral inequalities for the nontangential maximal function, Lusin area integral, and Walsh–Paley series*, Studia Math. 49 (1974), 107–124.

[HS] Y. Han and E. T. Sawyer, *Para-accretive functions, the weak boundedness property and the Tb theorem*, Rev. Mat. Iberoamer. 6 (1990), 17–41.

[HMW] R. Hunt, B. Muckenhoupt, and R. Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc. 176 (1973), 227–251.

[L] P. Lemarié, *Continuité sur les espaces de Besov des opérations définies par des intégrales singulières*, Ann. Inst. Fourier (Grenoble) 35 (1985), no. 4, 175–187.

[LLL] M.-Y. Lee, C.-C. Lin and Y.-C. Lin, *A wavelet characterization for the dual of weighted Hardy spaces*, Proc. Amer. Math. Soc. 137 (2009), 4219–4225.

[LLY] M.-Y. Lee, C.-C. Lin and W.-C. Yang, H_w^p *boundedness of Riesz transforms*, J. Math. Anal. Appl. 301 (2005), 394–400.

[MC] Y. Meyer and R. R. Coifman, *Wavelets. Calderón–Zygmund and Multilinear Operators*, Cambridge Stud. Adv. Math. 48, Cambridge Univ. Press, Cambridge, 1997.

[M] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207–226.

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Received November 8, 2011
Revised version April 4, 2012 (7350)