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Bounded elements in certain topological partial *-algebras

by

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Abstract. We continue our study of topological partial *-algebras, focusing on the interplay between various partial multiplications. The special case of partial *-algebras of operators is examined first, in particular the link between strong and weak multiplications, on one hand, and invariant positive sesquilinear (ips) forms, on the other. Then the analysis is extended to abstract topological partial *-algebras, emphasizing the crucial role played by appropriate bounded elements, called $\mathcal{M}$-bounded. Finally, some remarks are made concerning representations in terms of so-called partial GC^*-algebras of operators.

1. Introduction. Studies on partial *-algebras have provided so far a considerable amount of information about their representation theory and their structure. In particular, many results have been obtained for *concrete* partial *-algebras, i.e., partial *-algebras of closable operators (called partial O^*-algebras). A full analysis of these aspects has been developed by Inoue and two of us and it can be found in the monograph [2], where earlier articles are cited.

In a recent paper [4], we have started the analysis of spectral properties of partial *-algebras and, in particular, partial O^*-algebras. We continue this study in the present work, focusing now on the interplay between different partial multiplications. Indeed, the main feature of partial O^*-algebras is that they carry two natural multiplications, the weak one and the strong one. Even though they are, in general, partial *-algebras only with respect to the first one, the interplay of the two multiplications allows a rather natural definition of inverse of an element and thus a good starting point for the spectral theory. These two ingredients (the possibility of defining a *strong* multiplication and the existence of bounded elements) are then introduced in the abstract context leading to the notion of *topologically regular* partial *-algebra. This, in turn, suggests characterizing a special class of topological partial *-algebras, called *partial GC^*-algebras*, both in

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an abstract version and in an operator version, i.e., as a special class of partial O^*-algebras.

In the case of a partial O^*-algebra $\mathfrak{A}$, the best situation for the spectral theory occurs when $\mathfrak{A}$ contains sufficiently many *bounded elements*, i.e., bounded operators. The same property will show up here. We will characterize an appropriate notion of bounded elements, which we call $\mathcal{M}$-bounded elements. The very name underlines that the construction derives from a (sufficiently large) family $\mathcal{M}$ of invariant positive sesquilinear (ips) forms. As a matter of fact, strong partial multiplication is also derived from this family, and so are the associated spectral results. For instance, an element $x \in \mathfrak{A}$ has a finite spectral radius if and only if it is $\mathcal{M}$-bounded. As a result, the whole picture becomes coherent.

The notion of bounded element of a topological *-algebra was first proposed by Allan in 1965 [1] with the goal of developing a spectral theory for these algebras. Allan's definition was applied to O^*-algebras by Schmüdgen [8], but he did not include this topic in his monograph [9]. Bounded elements in purely algebraic terms have been considered by Vidav [15] and Schmüdgen [11] with respect to some (positive) cone. This ingenious approach seems to be unfit for general partial *-algebras, since they may fail to possess a natural positive cone. Of course, if a locally convex partial *-algebra $\mathfrak{A}$ contains a dense *-algebra (like the $\mathfrak{A}_o$-regular partial *-algebras considered in Section 4), then it has a natural positive cone, namely, the closure of the positive cone of $\mathfrak{A}_o$. However, we will not pursue this direction here. Finally, Cimprič defines a notion of an element of a *-ring being bounded with respect to a given module. His construction, albeit in a totally different context, bears some analogy with the one we describe in Section 4, in particular with the C^*-seminorm used in Proposition 4.11.

The paper is organized as follows. After some preliminaries about partial *-algebras (Section 2), taken mostly from [2] and [4], we discuss in Section 3 the interplay between partial multiplications and sets of ips-forms. We show, in particular, how strong partial multiplication on the space $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ may be characterized in terms of ips-forms. Then, in Section 4, which is the core of the paper, we show how a sufficient family $\mathcal{M}$ of ips-forms leads one to an appropriate notion of $\mathcal{M}$-bounded elements and of strong partial multiplication induced by $\mathcal{M}$. The corresponding spectral elements are defined and they are shown to behave as expected. Finally, in Section 5, we make some remarks on representations. In particular, we examine under which conditions a partial GC^*-algebra may have a faithful representation by a partial GC^*-algebra of operators, that is, a representation in some space $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$. It is worth mentioning that the family $\mathcal{M}$ of ips-forms defines in a locally convex partial *-algebra a cone of positive elements, making possible a generalization of Schmüdgen's approach in [11] to the present framework. We leave this investigation to a future paper.

2. Preliminaries. For general aspects of the theory of partial *-algebras and of their representations, we refer to the monograph [2]. For the convenience of the reader, however, we repeat here the essential definitions, and the notation given there.

First we recall that a *partial *-algebra* $\mathfrak{A}$ is a complex vector space with conjugate linear involution * and a distributive partial multiplication $\cdot$, defined on a subset $\Gamma \subset \mathfrak{A} \times \mathfrak{A}$, satisfying the property that $(x, y) \in \Gamma$ if, and only if, $(y^*, x^*) \in \Gamma$, and $(x \cdot y)^* = y^* \cdot x^*$. From now on we will write simply xy instead of $x \cdot y$ whenever $(x, y) \in \Gamma$. For every $y \in \mathfrak{A}$, the set of *left* (resp. *right*) *multipliers* of y is denoted by $L(y)$ (resp. $R(y)$), i.e., $L(y) = \{x \in \mathfrak{A} : (x, y) \in \Gamma\}$ and $R(y) = \{x \in \mathfrak{A} : (y, x) \in \Gamma\}$. We denote by $L\mathfrak{A}$ (resp. $R\mathfrak{A}$) the space of universal left (resp. right) multipliers of $\mathfrak{A}$.

In general, a partial *-algebra is not associative, but in several situations a weaker form of associativity holds. More precisely, we say that $\mathfrak{A}$ is *semi-associative* if $y \in R(x)$ implies $yz \in R(x)$ for every $z \in R\mathfrak{A}$ and

$$(xy)z = x(yz).$$

Throughout this paper we will only consider partial *-algebras with unit: this means that there exists an element $e \in \mathfrak{A}$ such that $e = e^*$, $e \in R\mathfrak{A} \cap L\mathfrak{A}$ and $xe = ex = x$ for every $x \in \mathfrak{A}$.

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{D}$ a dense subspace of $\mathcal{H}$. We denote by $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ the set of all (closable) linear operators X such that $D(X) = \mathcal{D}$ and $D(X^*) \supseteq \mathcal{D}$. The set $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is a partial *-algebra with respect to the following operations: the usual sum $X_1 + X_2$, the scalar multiplication λX, the involution $X \mapsto X^\dagger := X^* \upharpoonright \mathcal{D}$ and the (*weak*) partial multiplication $X_1 \,\square\, X_2 = X_1^{\dagger *} X_2$, defined whenever X_2 is a weak right multiplier of X_1 (we shall write $X_2 \in R^{\mathrm{w}}(X_1)$ or $X_1 \in L^{\mathrm{w}}(X_2)$), that is, whenever $X_2 \mathcal{D} \subset \mathcal{D}(X_1^{\dagger *})$ and $X_1^* \mathcal{D} \subset \mathcal{D}(X_2^*)$.

It is easy to check that $X_1 \in L^{\mathrm{w}}(X_2)$ if, and only if, there exists $Z \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ such that

$$\langle X_2 \xi \mid X_1^\dagger \eta \rangle = \langle Z \xi \mid \eta \rangle, \quad \forall \xi, \eta \in \mathcal{D}. \tag{2.1}$$

In this case $Z = X_1 \,\square\, X_2$. The *-algebra $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is neither associative nor semi-associative. If I denotes the identity operator of $\mathcal{H}$, we put $I_{\mathcal{D}} = I \upharpoonright \mathcal{D}$. Then $I_{\mathcal{D}}$ is the unit of the partial *-algebra $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$.

If $\mathfrak{N} \subseteq \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ we denote by $R^{\mathrm{w}}\mathfrak{N}$ the set of right multipliers of all elements of $\mathfrak{N}$. We recall that

$$R^{\mathrm{w}}\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) = \{A \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) : A \text{ is bounded and } A : \mathcal{D} \to \mathcal{D}^*\},$$

where

$$\mathcal{D}^* = \bigcap_{X\in\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})} D(X^{\dagger *}).$$

We denote by $\mathcal{L}^\dagger_{\mathrm{b}}(\mathcal{D},\mathcal{H})$ the bounded part of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, i.e., $\mathcal{L}^\dagger_{\mathrm{b}}(\mathcal{D},\mathcal{H}) = \{X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}) : X \text{ is a bounded operator}\} = \{X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}) : \overline{X} \in \mathcal{B}(\mathcal{H})\}$.

A †-invariant subspace $\mathfrak{M}$ of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is called a *(weak) partial O^*-algebra* if $X \,\square\, Y \in \mathfrak{M}$ for every $X, Y \in \mathfrak{M}$ such that $X \in L^{\mathrm{w}}(Y)$. Thus $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is the maximal partial O^*-algebra on $\mathcal{D}$.

The set $\mathcal{L}^\dagger(\mathcal{D}) := \{X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}) : X, X^\dagger : \mathcal{D} \to \mathcal{D}\}$ is a *-algebra; more precisely, it is the maximal O^*-algebra on $\mathcal{D}$ (for the theory of O^*-algebras and their representations we refer to [9]).

Some interesting classes of partial O^*-algebras (such as partial GW^*-algebras) can be defined with the help of certain topologies on $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ and its commutants.

The *weak topology* t_{w} on $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is defined by the seminorms

$$r_{\xi,\eta}(X) = |\langle X\xi \,|\, \eta\rangle|, \qquad X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}),\ \xi,\eta \in \mathcal{D}.$$

The *strong topology* t_{s} on $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is defined by the seminorms

$$p_\xi(X) = \|X\xi\|, \qquad X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}),\ \xi \in \mathcal{D}.$$

The *strong* topology* $\mathsf{t}_{\mathrm{s}^*}$ on $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is usually defined by the seminorms

$$p^*_\xi(X) = \max\{\|X\xi\|, \|X^\dagger\xi\|\}, \qquad X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}),\ \xi \in \mathcal{D}.$$

If $\mathfrak{N}$ is a †-invariant subset of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, the *weak unbounded commutant* of $\mathfrak{N}$ is defined by

$$\mathfrak{N}'_\sigma = \{Y \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}) :\ \langle X\xi \,|\, Y^\dagger\eta\rangle = \langle Y\xi \,|\, X^\dagger\eta\rangle,\ \forall X \in \mathfrak{N},\ \xi,\eta \in \mathcal{D}\}.$$

The *weak bounded commutant* $\mathfrak{N}'_{\mathrm{w}}$ of $\mathfrak{N}$ is defined by $\mathfrak{N}'_{\mathrm{w}} = \{Y \in \mathfrak{N}'_\sigma : Y \text{ is bounded}\}$.

If $\mathfrak{N}$ is a partial O^*-algebra, the *quasi-weak bounded commutant* $\mathfrak{N}'_{\mathrm{qw}}$ of $\mathfrak{N}$ is defined as follows:

$$\mathfrak{N}'_{\mathrm{qw}} = \{C \in \mathfrak{N}'_{\mathrm{w}} : \langle CX^\dagger\xi \,|\, Y^\dagger\eta\rangle = \langle C\xi \,|\, (X \,\square\, Y)\eta\rangle,\quad \forall X \in L(Y),\ \xi,\eta \in \mathcal{D}\}.$$

If $\mathfrak{N}$ is an O^*-algebra of bounded operators on $\mathcal{D}$, then $\mathfrak{N}''_{\mathrm{w}\sigma} = \overline{\mathfrak{N}}^{\mathsf{t}_{\mathrm{s}^*}}$. This applies, in particular, to the set $\mathfrak{P} := \{X \in \mathcal{L}^\dagger_{\mathrm{b}}(\mathcal{D},\mathcal{H}) : X, X^\dagger : \mathcal{D} \to \mathcal{D}\}$, which is an O^*-algebra of bounded operators on $\mathcal{D}$ (it is in fact the bounded part of $\mathcal{L}^\dagger(\mathcal{D})$) and $\mathfrak{P} \subset R^{\mathrm{w}}\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$. Then $\mathfrak{P}''_{\mathrm{w}\sigma} = \overline{\mathfrak{P}}^{\mathsf{t}_{\mathrm{s}^*}}$. The fact that $\mathfrak{P}'_{\mathrm{w}} = \mathbb{C}I_{\mathcal{D}}$ implies that $\overline{\mathfrak{P}}^{\mathsf{t}_{\mathrm{s}^*}} = \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, and thus $R^{\mathrm{w}}\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is $\mathsf{t}_{\mathrm{s}^*}$-dense in $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$.

In $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ we can also consider the so-called *strong multiplication* $\circ$. It is defined in the following way:

$$\begin{cases} X\circ Y \text{ is well-defined if } Y:\mathcal{D}\to D(\overline{X}),\ X^\dagger:\mathcal{D}\to D(\overline{Y^\dagger}), \\ (X\circ Y)\xi = \overline{X}(Y\xi), \quad \forall\, \xi\in\mathcal{D}. \end{cases} \tag{2.2}$$

We shall write $Y\in R^{\mathrm{s}}(X)$ (or $X\in L^{\mathrm{s}}(Y)$). In general, this strong multiplication does not make $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ into a partial *-algebra, since the distributive law fails. However, a subspace $\mathfrak{M}$ of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ may happen to be a partial *-algebra with respect to strong multiplication. In this case we say, as in [2], that $\mathfrak{M}$ is a strong partial O^*-algebra.

A **-representation* of a partial *-algebra $\mathfrak{A}$ in the Hilbert space $\mathcal{H}$ is a linear map $\pi:\mathfrak{A}\to\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ such that: (i) $\pi(x^*)=\pi(x)^\dagger$ for every $x\in\mathfrak{A}$; (ii) $x\in L(y)$ in $\mathfrak{A}$ implies $\pi(x)\in L^{\mathrm{w}}(\pi(y))$ and $\pi(x)\,\square\,\pi(y)=\pi(xy)$. The *-representation π is said to be *bounded* if $\overline{\pi(x)}\in\mathcal{B}(\mathcal{H})$ for every $x\in\mathfrak{A}$.

Let φ be a positive sesquilinear form on $D(\varphi)\times D(\varphi)$, where $D(\varphi)$ is a subspace of $\mathfrak{A}$. Then we have

$$\varphi(x,y)=\overline{\varphi(y,x)}, \qquad \forall x,y\in D(\varphi), \tag{2.3}$$

$$|\varphi(x,y)|^2\le\varphi(x,x)\varphi(y,y), \quad \forall x,y\in D(\varphi). \tag{2.4}$$

We put

$$N_\varphi=\{x\in D(\varphi):\varphi(x,x)=0\}.$$

By (2.4), we have

$$N_\varphi=\{x\in D(\varphi):\varphi(x,y)=0,\ \forall y\in D(\varphi)\},$$

so N_φ is a subspace of $D(\varphi)$, and the quotient space $D(\varphi)/N_\varphi:=\{\lambda_\varphi(x)\equiv x+N_\varphi : x\in D(\varphi)\}$ is a pre-Hilbert space with respect to the inner product $\langle\lambda_\varphi(x)\,|\,\lambda_\varphi(y)\rangle=\varphi(x,y)$, $x,y\in D(\varphi)$. We denote by $\mathcal{H}_\varphi$ the Hilbert space obtained by completion of $D(\varphi)/N_\varphi$.

A positive sesquilinear form φ on $\mathfrak{A}\times\mathfrak{A}$ is said to be *invariant*, and called an *ips-form*, if there exists a subspace $B(\varphi)$ of $\mathfrak{A}$ (called a *core* for φ) with the properties

(ips_1) $B(\varphi)\subset R\mathfrak{A}$;
(ips_2) $\lambda_\varphi(B(\varphi))$ is dense in $\mathcal{H}_\varphi$;
(ips_3) $\varphi(ax,y)=\varphi(x,a^*y)$ for all $a\in\mathfrak{A}$ and $x,y\in B(\varphi)$;
(ips_4) $\varphi(a^*x,by)=\varphi(x,(ab)y)$ for all $a\in L(b)$ and $x,y\in B(\varphi)$.

In other words, an ips-form is an *everywhere defined* biweight, in the sense of [2].

To every ips-form φ on $\mathfrak{A}$ with core $B(\varphi)$, there corresponds a triple $(\pi_\varphi,\lambda_\varphi,\mathcal{H}_\varphi)$, where $\mathcal{H}_\varphi$ is a Hilbert space, λ_φ is a linear map from $B(\varphi)$ into $\mathcal{H}_\varphi$, and π_φ is a *-representation on $\mathfrak{A}$ in the Hilbert space $\mathcal{H}_\varphi$. We refer to [2] for more details on this celebrated *GNS construction*.

Let $\mathfrak{A}$ be a partial *-algebra with unit e. We assume that $\mathfrak{A}$ is a locally convex Hausdorff vector space under the topology τ defined by a (directed) set $\{p_\alpha\}_{\alpha\in\mathcal{I}}$ of seminorms. Assume that ([1])

(cl) for every $x \in \mathfrak{A}$, the linear map $\mathsf{L}_x : R(x) \to \mathfrak{A}$ with $\mathsf{L}_x(y) = xy$, $y \in R(x)$, is closed with respect to τ, in the sense that if $\{y_\alpha\} \subset R(x)$ is a net such that $y_\alpha \to y$ and $xy_\alpha \to z \in \mathfrak{A}$, then $y \in R(x)$ and $z = xy$.

Starting from the family of seminorms $\{p_\alpha\}_{\alpha\in\mathcal{I}}$, we can define a second topology τ^* on $\mathfrak{A}$ by introducing the set of seminorms $\{p^*_\alpha(x)\}$, where

$$p^*_\alpha(x) = \max\{p_\alpha(x), p_\alpha(x^*)\}, \quad x \in \mathfrak{A}.$$

The involution $x \mapsto x^*$ is automatically τ^*-continuous. By (cl) it follows that, for every $x \in \mathfrak{A}$, L_x is τ^*-closed. And it turns out that the map R_y : $x \in L(y) \mapsto xy \in \mathfrak{A}$ is also τ^*-closed.

If $\mathfrak{A}_o$ is a τ^*-dense subspace of $R\mathfrak{A}$, then the restriction $\mathsf{L}_x\lceil\mathfrak{A}_o$ is τ-closable. Let us denote by L°_x its τ-closure defined on the following subspace of $\mathfrak{A}$:

$$D(\mathsf{L}^\circ_x) = \{y \in \mathfrak{A} : \exists\{y_\alpha\} \subset \mathfrak{A}_o,\, y_\alpha \xrightarrow{\tau} y,\, xy_\alpha \xrightarrow{\tau} z \in \mathfrak{A}\}.$$

In terms of the latter, we may define a new multiplication $\bullet$ on $\mathfrak{A}$ by

$$\begin{cases} y \in R_{\mathfrak{A}_o}(x) \Leftrightarrow y \in D(\mathsf{L}^\circ_x) \text{ and } x^* \in D(\mathsf{L}^\circ_{y^*}), \\ x \bullet y := \mathsf{L}^\circ_x y = \lim_\alpha (\mathsf{L}_x\lceil\mathfrak{A}_o) y_\alpha. \end{cases}$$

We refer to the multiplication $\bullet$ as the strong multiplication *induced* by $\mathfrak{A}_o$. Clearly, $R_{\mathfrak{A}_o}(x) \subset R(x)$, i.e., if $x \bullet y$ is well-defined, then $y \in R(x)$ and $x \bullet y = xy$. On the other hand, if $y \in R(x)$, $x \bullet y$ need not be defined. The definition itself implies that $x \bullet y$ is well-defined if and only if $y^* \bullet x^*$ is well-defined, and one has

$$(x \bullet y)^* = y^* \bullet x^*.$$

We remark that in general $\bullet$ does not make $\mathfrak{A}$ into a partial *-algebra, since the distributive law may fail.

Let $\mathfrak{A}$ be a partial *-algebra with unit e and assume that $\mathfrak{A}$ is a locally convex space with respect to a given topology τ. Then $\mathfrak{A}$ is called *topologically regular* if it satisfies (cl) and $R\mathfrak{A} \cap L\mathfrak{A}$ contains a *distinguished* *-algebra $\mathfrak{A}_o$, i.e., $\mathfrak{A}_o$ is a τ^*-dense *-subalgebra of $\mathfrak{A}$ (containing the unit e) such that, for the multiplication $\bullet$ induced by $\mathfrak{A}_o$, the following associative law holds, for all $x, y, z \in \mathfrak{A}$:

$$\text{if} \quad z \in R(y),\, yz \in R(x) \text{ and } y \in R_{\mathfrak{A}_o}(x), \quad \text{then} \quad z \in R(x \bullet y),$$

([1]) Condition (cl) was called (t1) in [4].

and

$$x(yz) = (x \bullet y)z. \tag{2.5}$$

In particular the following semi-associativity with respect to $\mathfrak{A}_o$ holds: if $x \bullet y$ is well-defined, then $x \bullet (yb)$ is well defined for every $b \in \mathfrak{A}_o$ and

$$(x \bullet y)b = x(yb),$$

which follows easily from (2.5).

An element $a \in \mathfrak{A}$ of a topologically regular partial *-algebra $\mathfrak{A}$ is called *left τ-bounded* if there exists $\gamma_a > 0$ such that

$$p_\alpha(ax) \le \gamma_a p_\alpha(x), \qquad \forall x \in R\mathfrak{A},\ \alpha \in \mathcal{I}. \tag{2.6}$$

The set of all left τ-bounded elements of $\mathfrak{A}$ is denoted by $\mathfrak{A}_{\mathsf{lb}}$. In general, $x \in \mathfrak{A}_{\mathsf{lb}}$ does *not* imply that $x^* \in \mathfrak{A}_{\mathsf{lb}}$. For $a \in \mathfrak{A}_{\mathsf{lb}}$ we put

$$\|a\|_{\mathsf{lb}} = \sup\{p_\alpha(ax) : \alpha \in \mathcal{I},\ x \in R\mathfrak{A},\ p_\alpha(x) = 1\}.$$

It is easily seen that $\|\cdot\|_{\mathsf{lb}}$ is a norm on $\mathfrak{A}_{\mathsf{lb}}$ [4].

A topologically regular partial *-algebra $\mathfrak{A}$ with a distinguished *-subalgebra $\mathfrak{A}_o$ is called a *partial GC^*-algebra* if

(i) $\mathfrak{A}$ is τ^*-complete;
(ii) $\mathfrak{A}_o \subset \mathfrak{A}_{\mathsf{lb}}$ and $\mathfrak{A}_o$ is τ^*-dense in $\mathfrak{A}$;
(iii) $\mathfrak{A}_{\mathsf{lb}}$ is a C^*-algebra with respect to the norm $\|\cdot\|_{\mathsf{lb}}$.

3. Partial multiplication vs. ips-forms. We begin by examining in some detail the topological structure of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ (or, more generally, of a partial O^*-algebra $\mathfrak{M}$) when it is endowed with the topology t_{s} or $\mathsf{t}_{\mathrm{s}^*}$.

As already mentioned, $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ contains a distinguished *-algebra $\mathfrak{P}$, which is t_{s^*}-dense. It is easily seen that both left and right multiplication by any fixed element of $\mathfrak{P}$ are continuous for the two topologies t_{s} and $\mathsf{t}_{\mathrm{s}^*}$.

Remark 3.1. The semi-associativity with respect to $\mathfrak{P}$ can be easily checked as follows, without invoking the topological regularity. Let $A_1, A_2 \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ with $A_1 \,\square\, A_2$ well-defined and $B \in \mathfrak{P}$. Then $A_2 : \mathcal{D} \to D(A_1^{\dagger *})$ and $A_1 : \mathcal{D} \to D(A_2^*)$. Since $B : \mathcal{D} \to \mathcal{D}$, this implies that $A_2 \,\square\, B : \mathcal{D} \to D(A_1^{\dagger *})$. On the other hand, for $\xi, \eta \in \mathcal{D}$,

$$\langle A_2 \,\square\, B\xi \mid A_1^\dagger \eta\rangle = \langle B\xi \mid A_2^\dagger \,\square\, A_1^\dagger \eta\rangle = \langle \xi \mid B^*(A_2^\dagger \,\square\, A_1^\dagger)\eta\rangle;$$

this implies that $A_1^\dagger \eta \in D((A_2 \,\square\, B)^*) = D((A_2 B)^*)$. In conclusion, $A_1 \in L^{\mathrm{w}}(A_2 \,\square\, B)$.

Elements of $\mathfrak{P}$ are left t_{s}-bounded in the sense of (2.6) (see also [4]); but the set of all left t_{s}-bounded elements is larger, namely it is $\mathcal{L}^\dagger_{\mathrm{b}}(\mathcal{D},\mathcal{H})$, and it is a C^*-algebra.

Another relevant feature of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is the existence of sufficiently many ips-forms. Indeed, if $\xi \in \mathcal{D}$, then every positive sesquilinear form φ_ξ with

$$\varphi_\xi(X,Y) := \langle X\xi \,|\, Y\xi \rangle$$

is a t_{s}-continuous (and, *a fortiori*, $\mathsf{t}_{\mathrm{s}^*}$-continuous) ips-form. Here *sufficiently many* means that the unique element $X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ such that $\varphi_\xi(X,X) = 0$ for every $\xi \in \mathcal{D}$ is $X = 0$.

The family $\mathcal{M} = \{\varphi_\xi : \xi \in \mathcal{D}\}$ can also be used to describe the weak multiplication $\square$ of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$. Indeed, we have:

Proposition 3.2. *The weak product $X \square Y$ of $X, Y \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is well-defined if and only if there exists $Z \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ such that*

$$\varphi_\xi(YA, X^\dagger B) = \varphi_\xi(ZA, B), \qquad \forall \xi \in \mathcal{D},\ A, B \in \mathfrak{P}. \tag{3.1}$$

Proof. The necessity of the condition follows easily from (2.1). As for the sufficiency, one can put $A = B = I_{\mathcal{D}}$ in (3.1) and use the polarization identity to get (2.1). ∎

Another characterization of the existence of weak product can be given in terms of approximation by elements of $\mathfrak{P}$.

Proposition 3.3. *The weak product $X \square Y$ of $X, Y \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is well-defined if and only if there exists a net $\{B_\alpha\}$ in $\mathfrak{P}$ such that*

$$B_\alpha \xrightarrow{\mathsf{t}_{\mathrm{s}}} Y \quad \text{and} \quad X \square B_\alpha \text{ converges weakly to some } Z \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}). \tag{3.2}$$

Proof. Assume that Y satisfies (3.2). Then, for every $\xi, \eta \in \mathcal{D}$,

$$\langle Y\xi \,|\, X^\dagger \eta\rangle = \lim_\alpha \langle B_\alpha \xi \,|\, X^\dagger \eta\rangle = \lim_\alpha \langle X \square B_\alpha \xi \,|\, \eta\rangle = \langle Z\xi \,|\, \eta\rangle.$$

The statement then follows from (2.1).

On the other hand, assume that $X \square Y$ is well-defined and let $\{B_\alpha\}$ be a net in $\mathfrak{P}$ converging to Y. Then, for every $\xi, \eta \in \mathcal{D}$,

$$\lim_\alpha \langle X \square B_\alpha \xi \,|\, \eta\rangle = \lim_\alpha \langle B_\alpha \xi \,|\, X^\dagger \eta\rangle = \langle Y\xi \,|\, X^\dagger \eta\rangle = \langle X \square Y\xi \,|\, \eta\rangle. \ \blacksquare$$

The strong multiplication of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, given by (2.2), can be conveniently described also by means of vector forms defined by the inner product of $\mathcal{H}$. To prove this we need the following lemma.

Lemma 3.4. *Let $X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$. Then*

(i) *the operator $S(X) := (I + \overline{X}X^*)^{-1}\restriction\mathcal{D}$ is a weak left multiplier of X and a weak right multiplier of $X^\dagger$;*
(ii) *$S(X)\mathcal{D}$ is a core for X^*.*

Proof. (i) We need to prove that $X : \mathcal{D} \to D(S(X)^{\dagger *})$ and $S(X)^\dagger : \mathcal{D} \to D(X^*)$. The first condition is trivially satisfied since $D(S(X)^{\dagger *}) = \mathcal{H}$, the operator $S(X)$ being symmetric and bounded. For the second, we have

$$S(X)^\dagger \mathcal{D} = S(X)\mathcal{D} \subset \overline{S(X)}\mathcal{H} = D(\overline{X}X^*) \subset D(X^*).$$

(ii) First, we check that $S(X)\mathcal{D}$ is dense in $\mathcal{H}$. Let $\eta \in \mathcal{H}$ be such that $\langle S(X)\xi \,|\, \eta\rangle = 0$ for every $\xi \in \mathcal{D}$. Then

$$\langle \xi \,|\, S(X)\eta\rangle = \langle S(X)\xi \,|\, \eta\rangle = 0, \quad \forall \xi \in \mathcal{D}.$$

By the density of $\mathcal{D}$, we get $S(X)\eta = 0$. But $S(X)$ is one-to-one, thus $\eta = 0$. To prove that $S(X)\mathcal{D}$ is a core for X^*, it is enough to show that the unique vector $\{\phi, X^*\phi\}$ in the graph of X^* which is orthogonal to $\{\{\eta, X^*\eta\} : \eta \in S(X)\mathcal{D}\}$ is zero. Indeed, putting $\eta = S(X)\xi$ with $\xi \in \mathcal{D}$, we have

$$\begin{aligned}
&\langle \{\phi, X^*\phi\} \,|\, \{\eta, X^*\eta\}\rangle = \langle \{\phi, X^*\phi\} \,|\, \{S(X)\xi, X^*S(X)\xi\}\rangle \\
&\quad = \langle \phi \,|\, S(X)\xi\rangle + \langle X^*\phi \,|\, X^*S(X)\xi\rangle = \langle \phi \,|\, S(X)\xi\rangle + \langle \phi \,|\, \overline{X}X^*S(X)\xi\rangle \\
&\quad = \langle \phi \,|\, (I + \overline{X}X^*)S(X)\xi\rangle = \langle \phi \,|\, (I + \overline{X}X^*)(I + \overline{X}X^*)^{-1}\xi\rangle \\
&\quad = \langle \phi \,|\, \xi\rangle = 0, \quad \forall \xi \in \mathcal{D}.
\end{aligned}$$

Hence $\phi = 0$. In the previous computation we took into account the following facts: (a) the operator $X^*S(X)$ is bounded; (b) $\overline{X}X^*(I + \overline{X}X^*)^{-1}$ is everywhere defined and bounded; hence $X^*(I + \overline{X}X^*)^{-1}\xi \in D(\overline{X})$ for every $\xi \in \mathcal{D}$. ■

THEOREM 3.5. *Let $X, Y \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. The following statements are equivalent:*

(i) $X \in L^{\mathrm{s}}(Y)$.
(ii) $X \in L^{\mathrm{w}}(Y)$ *and*

(ii$_1$) $\langle (X \,\square\, Y)\xi \,|\, Z^\dagger\eta\rangle = \langle Y\xi \,|\, (X^\dagger \,\square\, Z^\dagger)\eta\rangle$, $\forall Z \in L^{\mathrm{w}}(X)$, $\xi, \eta \in \mathcal{D}$;
(ii$_2$) $\langle (Y^\dagger \,\square\, X^\dagger)\xi \,|\, V\eta\rangle = \langle X^\dagger\xi \,|\, (Y \,\square\, V)\eta\rangle$, $\forall V \in R^{\mathrm{w}}(Y)$, $\xi, \eta \in \mathcal{D}$.

Proof. Let $X, Y \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. The implication (i)⇒(ii) is easy. We prove that (ii)⇒(i). Let $X, Y \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ satisfy (ii). We begin by observing that conditions (ii$_1$) and (ii$_2$) are, respectively, equivalent to the following [2]:

$$Y : \mathcal{D} \to D((X^* \upharpoonright Z^\dagger\mathcal{D})^*), \quad \forall Z \in L^{\mathrm{w}}(X);$$
$$X^\dagger : \mathcal{D} \to D((Y^{\dagger *} \upharpoonright V\mathcal{D})^*), \quad \forall V \in R^{\mathrm{w}}(Y).$$

By Lemma 3.4, $S(X) \in L^{\mathrm{w}}(X)$ and since $S(X)\mathcal{D}$ is a core for X^*,

$$(X^* \upharpoonright S(X)\mathcal{D})^* = (X^*)^* = \overline{X}.$$

Thus, $Y : \mathcal{D} \to D(\overline{X})$. By applying again Lemma 3.4 to the operator $Y^\dagger$ we find that $S(Y^\dagger)$ is a right multiplier of Y and $S(Y^\dagger)\mathcal{D}$ is a core for $Y^{\dagger *}$. Then $X^\dagger : \mathcal{D} \to D(\overline{Y^\dagger})$. In conclusion, $Y \in L^{\mathrm{s}}(X)$. ■

[2] We remind the reader that if T is not densely defined, then $D(T^*) = \{\eta \in \mathcal{H} : \exists \eta^* \in \mathcal{H} : \langle T\xi \,|\, \eta\rangle = \langle \xi \,|\, \eta^*\rangle, \forall \xi \in D(T)\}$ is not necessarily the domain of a well-defined operator.

An interesting aspect of the interplay of weak and strong multiplication in $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is the following *mixed* associativity property [4, Prop. 3.5], which proves to be useful in many situations.

PROPOSITION 3.6. *Let $X, Y, Z \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$. Assume that $X \,\square\, Y$, $(X \,\square\, Y) \,\square\, Z$ and $Y \circ Z$ are all well-defined. Then $X \in L^{\mathrm{w}}(Y \circ Z)$ and*

$$X \,\square\, (Y \circ Z) = (X \,\square\, Y) \,\square\, Z. \tag{3.3}$$

In other words, (2.5) is valid in $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ with strong partial multiplication.

REMARK 3.7. The partial O^*-algebra $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is topologically regular when endowed with the strong topology t_{s}. Indeed, the multiplication induced by $\mathfrak{P}$ is a restriction of the strong multiplication of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, since if $Y \in D(\mathsf{L}^\circ_X)$, then there exists a net $\{Y_\alpha\} \subset \mathfrak{P}$ and $Z \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ such that $Y_\alpha\xi \to Y\xi$ and $X \,\square\, Y_\alpha\xi \to Z\xi$ for every $\xi \in \mathcal{D}$. This implies that $Y\xi \in D(\overline{X})$ and $Z\xi = \overline{X}Y\xi$ for every $\xi \in \mathcal{D}$. In a similar way, one proves that, if $X^\dagger \in D(\mathsf{L}^\circ_{Y^\dagger})$, then $X^\dagger\xi \in D(\overline{Y^\dagger})$. Hence, if the product of X and Y induced by $\mathfrak{P}$ is well-defined, then $X \circ Y$ is also well-defined and the two products coincide. The statement then follows from (3.3).

The next two statements are analogues of Propositions 3.2 and 3.3 and can be proved in a similar way.

PROPOSITION 3.8. *The strong product $X \circ Y$ of $X, Y \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is well-defined if and only if there exists $W \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ such that*

$$\begin{aligned} \varphi_\xi(WA, Z^\dagger B) &= \varphi_\xi(YA, (X^\dagger \,\square\, Z^\dagger)B) \quad \text{if } Z \in L^{\mathrm{w}}(X),\ \xi \in \mathcal{D},\ A, B \in \mathfrak{P},\\ \varphi_\xi(W^\dagger A, VB) &= \varphi_\xi(X^\dagger A, (Y \,\square\, V)B) \quad \text{if } V \in R^{\mathrm{w}}(Y),\ \xi \in \mathcal{D},\ A, B \in \mathfrak{P}. \end{aligned}$$

PROPOSITION 3.9. *The strong product $X \circ Y$ of $X, Y \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is well-defined if and only if there exist $W \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ and a net $\{C_\alpha\}$ in $\mathfrak{P}$ such that $C_\alpha \xrightarrow{\mathsf{t}_{\mathrm{s}}} Y$ and*

$$\begin{aligned} \varphi_\xi((X \,\square\, C_\alpha - W)A, Z^\dagger B) &\to 0 \quad \text{if } Z \in L^{\mathrm{w}}(X),\ \xi \in \mathcal{D},\ A, B \in \mathfrak{P},\\ \varphi_\xi((C_\alpha^\dagger \,\square\, X^\dagger - W^\dagger)A, VB) &\to 0 \quad \text{if } V \in R^{\mathrm{w}}(Y),\ \xi \in \mathcal{D},\ A, B \in \mathfrak{P}. \end{aligned}$$

The family $\mathcal{M} = \{\varphi_\xi : \xi \in \mathcal{D}\}$ plays an important role in the preceding discussion. Even though the elements of $\mathcal{M}$ do not exhaust the family of all strongly continuous ips-forms on $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, it is not restrictive to confine the analysis to them, since every t_{s}-continuous ips-form on $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is a linear combination of elements of $\mathcal{M}$. Indeed:

THEOREM 3.10. *Let $\mathfrak{M}$ be a partial O^*-algebra on $\mathcal{D}$, and $\mathfrak{M}_0$ a *-algebra of bounded operators contained in $\mathfrak{M}$ and strongly* dense in $\mathfrak{M}$.*

(i) *Every strongly continuous invariant positive sesquilinear form φ on $\mathcal{D}\times\mathcal{D}$, with core $\mathfrak{M}_0$, can be represented as*

$$\varphi(X,Y)=\sum_{i=1}^{n}\langle S_iX\xi_i\,|\,S_iY\xi_i\rangle,\quad X,Y\in\mathfrak{M}, \tag{3.4}$$

for some vectors $\xi_1,\dots,\xi_n$ in $\mathcal{D}$ and positive operators $S_1,\dots,S_n$ such that $S_1^2,\dots,S_n^2\in\mathfrak{M}'_{\mathrm{qw}}$.

(ii) *If $\mathfrak{M}=\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ and $\mathfrak{M}_0=\mathfrak{P}$, then*

$$\varphi(X,Y)=\sum_{i=1}^{n}\langle X\xi_i\,|\,Y\xi_i\rangle,\quad X,Y\in\mathfrak{M},$$

for some vectors $\xi_1,\dots,\xi_n$ in $\mathcal{D}$.

Proof. The strong continuity of φ implies that there exist $\xi_1,\dots,\xi_n\in\mathcal{D}$ such that

$$|\varphi(X,Y)|\le\sum_{i=1}^{n}p_{\xi_i}(X)\cdot\sum_{i=1}^{n}p_{\xi_i}(Y).$$

Let $\mathcal{H}_\oplus:=\bigoplus_{i=1}^n\mathcal{H}$, the direct sum of n copies of $\mathcal{H}$. We will write $\oplus\xi_i$ instead of $(\xi_1,\dots,\xi_n)$, $\xi_i\in\mathcal{H}$. Let $\mathcal{D}_\oplus=\bigoplus_{i=1}^n\mathcal{D}$.

We define a *-representation π of $\mathfrak{M}$ in $\mathcal{L}^\dagger(\mathcal{D}_\oplus,\mathcal{H}_\oplus)$ by

$$\pi(X)(\oplus\eta_i)=\oplus X\eta_i,\quad \eta_i\in\mathcal{D},\ i=1,\dots,n.$$

Let us consider the following subspaces of $\bigoplus_{i=1}^n\mathcal{H}$:

$$\mathcal{E}=\{\pi(X)(\oplus\xi_i):X\in\mathfrak{M}\},\quad \mathcal{E}_0=\{\pi(A)(\oplus\xi_i):A\in\mathfrak{M}_0\}.$$

The strong *-density of $\mathfrak{M}_0$ implies that $\overline{\mathcal{E}_0}=\overline{\mathcal{E}}$.

Define

$$\varTheta(\pi(X)(\oplus\xi_i),\pi(Y)(\oplus\xi_i)):=\varphi(X,Y).$$

The sesquilinear form $\varTheta$ is bounded on $\mathcal{E}\times\mathcal{E}$ and extends to $\overline{\mathcal{E}}\times\overline{\mathcal{E}}$. Then there exists a positive bounded operator T on the Hilbert space $\overline{\mathcal{E}}$ such that

$$\varTheta(\pi(X)(\oplus\xi_i),\pi(Y)(\oplus\xi_i))=\langle T(\oplus X\xi_i)\,|\,\oplus Y\xi_i\rangle.$$

The condition $\varphi(X\,\square\,A,B)=\varphi(A,X^\dagger\,\square\,B)$ implies the equality

$$\langle T\pi(X\,\square\,A)\oplus\xi_i\,|\,\pi(B)\oplus\xi_i\rangle=\langle T\pi(A)\oplus\xi_i\,|\,\pi(X^\dagger\,\square\,B)\oplus\xi_i\rangle,$$

or

$$\begin{aligned}&\langle T(\pi(X)\,\square\,\pi(A))\oplus\xi_i\,|\,\pi(B)\oplus\xi_i\rangle\\&\qquad=\langle T\pi(A)\oplus\xi_i\,|\,(\pi(X^\dagger)\,\square\,\pi(B))\oplus\xi_i\rangle.\end{aligned}\tag{3.5}$$

Now, for every $X\in\mathfrak{M}$, we define an operator $\pi_{\mathcal{E}}$ on $\mathcal{E}_0$ by

$$\pi_{\mathcal{E}}(X)(\pi(A)\oplus\xi_i):=(\pi(X)\,\square\,\pi(A))\oplus\xi_i,\quad A\in\mathfrak{M}_0.$$

It is easily seen that $\pi_{\mathcal{E}}(X) \in \mathcal{L}^\dagger(\mathcal{E}_0, \overline{\mathcal{E}})$. With this notation, (3.5) reads

$$\langle T\pi_{\mathcal{E}}(X)(\pi(A) \oplus \xi_i) \,|\, \pi(B) \oplus \xi_i\rangle = \langle T\pi(A) \oplus \xi_i \,|\, (\pi_{\mathcal{E}}(X^\dagger)(\pi(B) \oplus \xi_i)\rangle.$$

Hence $T \in \pi_{\mathcal{E}}(\mathfrak{M})'_{\mathrm{w}}$.

Now we extend T to a bounded operator $T_\oplus$ on $\mathcal{H}_\oplus$ by defining it to be 0 on the orthogonal complement of $\overline{\mathcal{E}}$.

Now we prove that $T_\oplus \in \pi(\mathfrak{M})'_{\mathrm{w}}$. Recalling that $\pi(\mathfrak{M}_0)$ is a *-algebra of bounded operators, we begin by showing that $T_\oplus \in \overline{\pi(\mathfrak{M}_0)}'$ (the ordinary commutant of bounded operators). Let $P_{\mathcal{E}}$ denote the projection of $\mathcal{H}_\oplus$ onto $\overline{\mathcal{E}}$. Since every $\overline{\pi(A)}$, $A \in \mathfrak{M}_0$, leaves $\overline{\mathcal{E}}$ invariant, it follows that $\overline{\pi(A)}P_{\mathcal{E}} = P_{\mathcal{E}}\overline{\pi(A)}$ for every $A \in \mathfrak{M}_0$. Moreover, if $\oplus\eta_i \in \mathcal{H}_\oplus$, there exists a sequence $\{B_n\}$ in $\mathfrak{M}_0$ such that $P_{\mathcal{E}} \oplus \eta_i = \lim_{n\to\infty} \pi(B_n) \oplus \xi_i$. From these facts, we get

$$\begin{aligned} T_\oplus\overline{\pi(A)}P_{\mathcal{E}} \oplus \eta_i &= T_\oplus\overline{\pi(A)}(\lim_{n\to\infty} \pi(B_n) \oplus \xi_i) = \lim_{n\to\infty} T_\oplus\overline{\pi(A)}\pi(B_n) \oplus \xi_i \\ &= \lim_{n\to\infty} \overline{\pi(A)}T_\oplus\pi(B_n) \oplus \xi_i = \overline{\pi(A)}T_\oplus P_{\mathcal{E}} \oplus \eta_i. \end{aligned}$$

Moreover, from the definition of $T_\oplus$ we find that $T_\oplus\overline{\pi(A)}(I - P_{\mathcal{E}}) \oplus \eta_i = T_\oplus(I - P_{\mathcal{E}})\overline{\pi(A)} \oplus \eta_i = 0$ and thus $T_\oplus \in \overline{\pi(\mathfrak{M}_0)}'$. Since $\overline{\mathfrak{M}_0}^{\mathrm{s}^*} \supseteq \mathfrak{M}$, it follows that $\pi(\mathfrak{M})'_{\mathrm{w}} = \pi(\mathfrak{M}_0)'_{\mathrm{w}} = \overline{\pi(\mathfrak{M}_0)}'$ and we finally conclude that $T_\oplus \in \pi(\mathfrak{M})'_{\mathrm{w}}$.

On the other hand, the condition $\varphi(X^\dagger \,\square\, A, YB) = \varphi(A, (X \,\square\, Y) \,\square\, B)$, whenever $X \,\square\, Y$ is defined, implies, in a similar way, that $T_\oplus \in \pi(\mathfrak{M})'_{\mathrm{qw}}$. Let now T_i denote the projection of T onto the subspace generated by $X\xi_i$, $X \in \mathfrak{M}$, and then extended to $\mathcal{H}$ by defining it to be 0 on the orthogonal complement. It is easily seen that $T_\oplus \in \pi(\mathfrak{M})'_{\mathrm{qw}}$ if and only if $T_i \in \mathfrak{M}'_{\mathrm{qw}}$ for each i. Hence,

$$\varphi(X, Y) = \sum_{i=1}^n \langle T_iX\xi_i \,|\, Y\xi_i\rangle, \qquad \xi_i \in \mathcal{D},\ T_i \in \mathfrak{M}'_{\mathrm{qw}}\,.$$

If we put $S_i = T_i^{1/2}$, then we get the representation (3.4). If $\mathfrak{M} = \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, then (ii) follows from the equality $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})'_{\mathrm{qw}} = \mathbb{C}I$. ■

With a similar proof, one also gets

THEOREM 3.11. *Let $\mathfrak{M}$ be a partial O^*-algebra on $\mathcal{D}$. Every strongly continuous linear functional Φ can be represented as*

$$\Phi(X) = \sum_{i=1}^n \langle X\xi_i \,|\, \eta_i\rangle, \quad X \in \mathfrak{M},$$

with $\xi_1, \ldots, \xi_n \in \mathcal{D}$ and $\eta_1, \ldots, \eta_n \in \mathcal{H}$.

In [4] we gave the following definition of a partial GC^*-algebra of operators.

Definition 3.12. A partial O^*-algebra $\mathfrak{M}$ on $\mathcal{D}$ is called a *partial GC^*-algebra of operators* over $\mathfrak{M}_0$ if

(i) $\mathfrak{M}$ is t_{s^*}-closed;
(ii) $\mathfrak{M}$ contains a t_{s^*}-dense *-algebra $\mathfrak{M}_0$ of bounded operators on $\mathcal{D}$;
(iii) $\mathfrak{M}_{\mathsf{lb}} = \mathfrak{M} \cap \mathcal{L}^\dagger_{\mathrm{b}}(\mathcal{D},\mathcal{H}) =: \mathfrak{M}_{\mathrm{b}}$ is a C^*-algebra.

Remark 3.13. Every partial GC^*-algebra of operators is topologically regular. Indeed, the argument used in Remark 3.7 can be easily adapted to the present situation. Hence, every partial GC^*-algebra of operators is a partial GC^*-algebra in the sense of Section 2.

Clearly $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ fulfills the conditions of this definition if $\mathfrak{M}_0 = \mathfrak{P}$. So it is natural to consider under which conditions a locally convex partial *-algebra $\mathfrak{A}[\tau]$ can be represented in a partial GC^*-algebra of operators. Some results in this direction were given in [4], but a deeper analysis shows that the conditions given there were sometimes unnecessarily strong. The crucial point for the existence of a nice *-representation of $\mathfrak{A}[\tau]$ is that it possesses a sufficient family of ips-forms as $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ itself does. This will be the starting point of the present discussion.

4. Sufficient families of ips-forms; $\mathcal{M}$-bounded elements

Definition 4.1. Let $\mathfrak{A}$ be a partial *-algebra endowed with a locally convex topology τ generated by a directed set $\{p_\alpha\}_{\alpha\in I}$ of seminorms. We say that $\mathfrak{A}[\tau]$ is *$\mathfrak{A}_o$-regular* if there exists a *-algebra $\mathfrak{A}_o \subset R\mathfrak{A}$ with the following properties:

(d_1) $\mathfrak{A}_o$ is τ-dense in $\mathfrak{A}$;
(d_2) for every $b\in\mathfrak{A}_o$, the maps $x\mapsto xb$ and $x\mapsto bx$, $x\in\mathfrak{A}$, are continuous.

Remark 4.2. We warn the reader that an $\mathfrak{A}_o$-regular partial *-algebra $\mathfrak{A}[\tau]$ is not necessarily a locally convex partial *-algebra in the sense of [2], since the definition of the latter requires stronger conditions (for instance, the continuity of the involution and of the multiplication $x \mapsto xb$ for every fixed $b \in R\mathfrak{A}$).

Let now $\mathcal{M}$ be a family of positive sesquilinear forms on $\mathfrak{A}\times\mathfrak{A}$ for which the conditions (ips_1), (ips_3) and (ips_4) are satisfied with respect to $\mathfrak{A}_o$ and such that every $\varphi \in \mathcal{M}$ is τ-continuous, i.e., there exist p_α and $\gamma > 0$ such that

$$|\varphi(x,y)| \le \gamma p_\alpha(x)p_\alpha(y).$$

Then (ips_2) is also satisfied, and therefore $\mathfrak{A}_o$ is a core for every $\varphi \in \mathcal{M}$, so that every $\varphi \in \mathcal{M}$ is an ips-form.

As announced above, the crucial condition is that $\mathfrak{A}$ possesses sufficiently many ips-forms. Hence, as in [4], we introduce

Definition 4.3. A family $\mathcal{M}$ of ips-forms on $\mathfrak{A} \times \mathfrak{A}$ with the above properties is *sufficient* if the conditions $x \in \mathfrak{A}$ and $\varphi(x,x) = 0$ for every $\varphi \in \mathcal{M}$ imply $x = 0$.

This definition is not empty, as the following examples show. Take $L^p[0,1]$ with its usual norm as a partial *-algebra and $\mathcal{M}$ the family of all continuous ips-forms on it, with core the algebra $C([0,1)])$ of continuous functions. Then for $1 \le p < 2$ the family $\mathcal{M}$ is trivial, so is not sufficient. For $p \ge 2$, the family $\mathcal{M}$ is sufficient. In the example $L^p[0,1] \oplus L^r[0,1]$ for $1 \le p < 2$ and $r \ge 2$, the corresponding family of continuous ips-forms is neither sufficient, nor trivial.

Of course, if the family $\mathcal{M}$ is sufficient, any larger family $\mathcal{M}' \supset \mathcal{M}$ is also sufficient. The maximal sufficient family is obviously the set $\mathcal{P}_{\mathfrak{A}_o}(\mathfrak{A})$ of *all* continuous ips-forms with core $\mathfrak{A}_o$, but we prefer to use the present notion, since it provides more flexibility.

When $\mathfrak{A}$ possesses a sufficient family $\mathcal{M}$ of ips-forms, we can define an *extension* of multiplication in the following way.

We say that the *weak product* $x \,\square\, y$ is well-defined if there exists $z \in \mathfrak{A}$ such that

$$\varphi(ya, x^*b) = \varphi(za, b), \qquad \forall a, b \in \mathfrak{A}_o,\ \varphi \in \mathcal{M}.$$

In this case, we put $x \,\square\, y := z$.

The following result is immediate.

Proposition 4.4. *If the partial *-algebra $\mathfrak{A}$ possesses a sufficient family $\mathcal{M}$ of ips-forms, then $\mathfrak{A}$ is also a partial *-algebra with respect to weak multiplication.*

From now on we will consider only the case where $\mathfrak{A}$ possesses a sufficient family $\mathcal{M}$ of ips-forms.

Remark 4.5. The sesquilinear forms of $\mathcal{M}$ define the topologies generated by the following families of seminorms:

$\tau_{\mathrm{w}}^{\mathcal{M}}$: $x \mapsto |\varphi(xa,b)|, \quad \varphi \in \mathcal{M},\ a, b \in \mathfrak{A}_o$;
$\tau_{\mathrm{s}}^{\mathcal{M}}$: $x \mapsto \varphi(x,x)^{1/2}, \quad \varphi \in \mathcal{M}$;
$\tau_{\mathrm{s}^*}^{\mathcal{M}}$: $x \mapsto \max\{\varphi(x,x)^{1/2}, \varphi(x^*,x^*)^{1/2}\}, \quad \varphi \in \mathcal{M}$.

From the continuity of $\varphi \in \mathcal{M}$ it follows that all the topologies $\tau_{\mathrm{w}}^{\mathcal{M}}$, $\tau_{\mathrm{s}}^{\mathcal{M}}$ (and also $\tau_{\mathrm{s}^*}^{\mathcal{M}}$ if the involution is τ-continuous) are coarser than the initial topology τ.

Proposition 4.6. *The weak product $x \,\square\, y$ is defined if and only if there exists a net $\{b_\alpha\}$ in $\mathfrak{A}_o$ such that $b_\alpha \xrightarrow{\tau} y$ and $xb_\alpha \xrightarrow{\tau_{\mathrm{w}}^{\mathcal{M}}} z \in \mathfrak{A}$.*

Proof. Assume that $x \,\square\, y$ is defined. From the τ-density of $\mathfrak{A}_o$, there exists a net $\{b_\alpha\}$ in $\mathfrak{A}_o$ such that $b_\alpha \xrightarrow{\tau} y$. Then for every $c, c' \in \mathfrak{A}_o$

one has $\varphi((xb_\alpha)c, c') = \varphi(b_\alpha c, x^* c') \to \varphi(yc, x^* c') = \varphi((x \,\square\, y)c, c')$, that is, $xb_\alpha \xrightarrow{\tau_{\mathrm{w}}^{\mathcal{M}}} x \,\square\, y$. Conversely, assume that there exists a net $\{b_\alpha\}$ in $\mathfrak{A}_o$ such that $b_\alpha \xrightarrow{\tau} y$ and $xb_\alpha \xrightarrow{\tau_{\mathrm{w}}^{\mathcal{M}}} z \in \mathfrak{A}$. Then, for every $a, a' \in \mathfrak{A}_o$, $\varphi(ya, x^* a') = \lim_\alpha \varphi(b_\alpha a, x^* a') = \lim_\alpha \varphi((xb_\alpha)a, a') = \varphi(za, a')$, that is, $x \,\square\, y$ is defined. ■

In the case of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, the weak multiplication $\square$ coincides with the weak multiplication defined here by means of ips-forms (Proposition 3.2). By analogy, from now on we will always assume the following:

(wp) xy exists if and only if $x \,\square\, y$ exists. In this case $xy = x \,\square\, y$.

Then, of course, $L(x) = L^{\mathrm{w}}(x)$ and $R(x) = R^{\mathrm{w}}(x)$.

The first result is that if $\mathfrak{A}$ is a partial *-algebra with a sufficient family $\mathcal{M}$ of ips-forms, and satisfying (wp), then it satisfies condition (cl) with respect to the topology $\tau_{\mathrm{s}}^{\mathcal{M}}$.

Proposition 4.7. *Let $\mathfrak{A}$ be a partial *-algebra with a sufficient family $\mathcal{M}$ of ips-forms, and satisfying* (wp)*. Then, for every $x \in \mathfrak{A}$, the linear map $\mathsf{L}_x : R(x) \to \mathfrak{A}$ with $\mathsf{L}_x(y) = xy$, $y \in R(x)$, is closed with respect to $\tau_{\mathrm{s}}^{\mathcal{M}}$, in the sense that if $y_\alpha \xrightarrow{\tau_{\mathrm{s}}^{\mathcal{M}}} y$ with $y_\alpha \in R(x)$ and $xy_\alpha \xrightarrow{\tau_{\mathrm{s}}^{\mathcal{M}}} z \in \mathfrak{A}$, then $y \in R(x)$ and $z = xy$.*

Proof. Let $y_\alpha \xrightarrow{\tau_{\mathrm{s}}^{\mathcal{M}}} y$ with $y_\alpha \in R(x)$ and $xy_\alpha \xrightarrow{\tau_{\mathrm{s}}^{\mathcal{M}}} z \in \mathfrak{A}$. Then, again by ($\mathsf{ips}_4$), for every $\varphi \in \mathcal{M}$,

$$\begin{aligned}\varphi((xy_\alpha - z)a, a') = \varphi((xy_\alpha)a, a') - \varphi(za, a') &= \varphi(y_\alpha a, x^* a') - \varphi(za, a') \\ &\to \varphi(ya, x^* a') - \varphi(za, a') = 0.\end{aligned}$$

Hence, since $\mathcal{M}$ is sufficient, $y \in R(x)$ and $z = xy$. ■

Remark 4.8. It is clear that the statement of Proposition 4.7 holds for any topology finer than $\tau_{\mathrm{s}}^{\mathcal{M}}$, and then, in particular, for the initial topology τ of $\mathfrak{A}$.

Now we are ready to introduce the appropriate notion of bounded elements.

Definition 4.9. Let $\mathfrak{A}$ be a partial *-algebra with a sufficient family $\mathcal{M}$ of ips-forms, and satisfying (wp). An element $x \in \mathfrak{A}$ is called *$\mathcal{M}$-bounded* if there exists $\gamma > 0$ such that

$$|\varphi(xa, b)| \le \gamma \varphi(a, a)^{1/2} \varphi(b, b)^{1/2}, \quad \forall \varphi \in \mathcal{M},\ a, b \in \mathfrak{A}_o.$$

Proposition 4.10. *Let $\mathfrak{A}[\tau]$ be an $\mathfrak{A}_o$-regular partial *-algebra satisfying condition* (wp)*. Then an element $x \in \mathfrak{A}$ is $\mathcal{M}$-bounded if and only if there exists $\gamma \in \mathbb{R}$ such that $\varphi(xa, xa) \le \gamma^2 \varphi(a, a)$ for all $\varphi \in \mathcal{M}$ and $a \in \mathfrak{A}_o$.*

Proof. Assume that $x \in \mathfrak{A}$ is $\mathcal{M}$-bounded. By the density of $\mathfrak{A}_o$, there exists a net $\{x_\alpha\} \subset \mathfrak{A}_o$ such that τ-$\lim_\alpha x_\alpha = x$. The continuity of φ then implies

$$\begin{aligned}|\varphi(xa, xb)| &= \lim_\alpha |\varphi(xa, x_\alpha b)| \le \gamma\varphi(a,a)^{1/2} \lim_\alpha \varphi(x_\alpha b, x_\alpha b)^{1/2} \\ &= \gamma\varphi(a,a)^{1/2}\varphi(xb,xb)^{1/2}.\end{aligned}$$

In particular, it follows that

$$\varphi(xa,xa) \le \gamma\varphi(a,a)^{1/2}\varphi(xa,xa)^{1/2},$$

that is, $\varphi(xa,xa) \le \gamma^2\varphi(a,a)$.

Conversely, we have

$$|\varphi(xa,b)| \le \varphi(xa,xa)^{1/2}\varphi(b,b)^{1/2} \le \gamma\varphi(a,a)^{1/2}\varphi(b,b)^{1/2}. \ \blacksquare$$

From the last proposition, it follows obviously that an element x of $\mathfrak{A}$ is $\mathcal{M}$-bounded if and only if x is left $\tau_{\mathrm{s}}^{\mathcal{M}}$-bounded in the sense of [4]. Define

$$\begin{aligned} q_{\mathcal{M}}(x) :=&\ \inf\{\gamma > 0 : \varphi(xa,xa) \le \gamma^2\varphi(a,a),\ \forall\varphi \in \mathcal{M},\ a \in \mathfrak{A}_o\} \\ =&\ \sup\{\varphi(xa,xa)^{1/2} : \varphi \in \mathcal{M},\ a \in \mathfrak{A}_o,\ \varphi(a,a)^{1/2} = 1\}\end{aligned}$$

Hence $q_{\mathcal{M}}$ coincides with the norm $\|\cdot\|_{\mathsf{lb}}$ obtained by giving $\mathfrak{A}$ the topology $\tau_{\mathrm{s}}^{\mathcal{M}}$ (see also [6] for a similar approach).

PROPOSITION 4.11. *Let x, y be $\mathcal{M}$-bounded elements of $\mathfrak{A}$. Then:*

(i) *x^* is $\mathcal{M}$-bounded and $q_{\mathcal{M}}(x) = q_{\mathcal{M}}(x^*)$;*
(ii) *if xy is well-defined, then xy is $\mathcal{M}$-bounded and*

$$q_{\mathcal{M}}(xy) \le q_{\mathcal{M}}(x)\, q_{\mathcal{M}}(y).$$

Proof. The first part of (i) is a direct consequence of the definition, and the second part follows from the fact that $|\varphi(xa,b)| = |\varphi(a,x^*b)| = |\varphi(x^*b,a)|$, by Proposition 4.10 and the definition of $q_{\mathcal{M}}(x)$. Moreover,

$$\begin{aligned}|\varphi((xy)a,b)| &= |\varphi(ya,x^*b)| \le \varphi(ya,ya)^{1/2}\varphi(x^*b,x^*b)^{1/2} \\ &\le q_{\mathcal{M}}(x)q_{\mathcal{M}}(y)\varphi(a,a)^{1/2}\gamma_2\varphi(b,b)^{1/2}.\end{aligned}$$

Taking the sup on the l.h.s., we get the inequality of (ii). $\blacksquare$

PROPOSITION 4.12. *$q_{\mathcal{M}}$ is an unbounded C^*-norm on $\mathfrak{A}$ with domain $\mathcal{D}(q_{\mathcal{M}}) := \{x \in \mathfrak{A} : x$ is $\mathcal{M}$-bounded$\}$.*

Proof. This can be deduced from [14], or computed directly. $\blacksquare$

The existence of a sufficient family $\mathcal{M}$ of ips-forms allows the definition of a stronger multiplication on $\mathfrak{A}$, which will play a role similar to strong partial multiplication on $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$.

Definition 4.13. If the family $\mathcal{M}$ of ips-forms is sufficient, we say that the *strong product* $x \bullet y$ is well-defined (and write $x \in L^{\mathrm{s}}(y)$ or $y \in R^{\mathrm{s}}(x)$) if $x \in L(y)$ and:

(sm_1) $\varphi((xy)a, z^*b) = \varphi(ya, (x^*z^*)b)$, $\forall z \in L(x)$, $\varphi \in \mathcal{M}$, $a, b \in \mathfrak{A}_0$;
(sm_2) $\varphi((y^*x^*)a, vb) = \varphi(x^*a, (yv)b)$, $\forall v \in R(y)$, $\varphi \in \mathcal{M}$, $a, b \in \mathfrak{A}_0$.

The following characterization is immediate.

Proposition 4.14. *If the family $\mathcal{M}$ of ips-forms is sufficient, the strong product $x \bullet y$ is well-defined (and $x \in L^s(y)$ or $y \in R^s(x)$) if and only if there exists $w \in \mathfrak{A}$ such that*

$$\varphi(wa, z^*b) = \varphi(ya, (x^*z^*)b) \quad \textit{whenever} \quad z \in L(x),\ \varphi \in \mathcal{M},\ a, b \in \mathfrak{A}_o,$$
$$\varphi(w^*a, vb) = \varphi(x^*a, (yv)b) \quad \textit{whenever} \quad v \in R(y),\ \varphi \in \mathcal{M},\ a, b \in \mathfrak{A}_o.$$

In this case, we put $x \bullet y := w$.

Remark 4.15. The uniqueness of w results from the sufficiency of the family $\mathcal{M}$. Clearly, if $\mathfrak{A}$ has a unit, then $x \bullet y = w$ implies that xy is defined and $x \bullet y = xy = w$.

Proposition 4.16. *The strong product $x \bullet y$ of $x, y \in \mathfrak{A}$ is well-defined if and only if there exist $w \in \mathfrak{A}$ and a net $\{c_\alpha\}$ in $\mathfrak{A}_o$ such that $c_\alpha \xrightarrow{\tau_{\mathrm{s}}^{\mathcal{M}}} y$ and*

$$\varphi((x \,\square\, c_\alpha - w)a, z^*b) \to 0 \qquad \textit{if } z \in L(x),\ \varphi \in \mathcal{M},\ a, b \in \mathfrak{A}_o,$$
$$\varphi((c_\alpha^* \,\square\, x^* - w^*)a, vb) \to 0 \qquad \textit{if } v \in R(y),\ \varphi \in \mathcal{M},\ a, b \in \mathfrak{A}_o.$$

Proof. If $x \bullet y$ is well-defined, then xy is well-defined. Then, by Proposition 4.6, there exists a net $\{c_\alpha\} \subset \mathfrak{A}_o$ such that $c_\alpha \xrightarrow{\tau_{\mathrm{s}}^{\mathcal{M}}} y$ and $xc_\alpha \xrightarrow{\tau_w^{\mathcal{M}}} xy$. Hence, by ($\mathsf{ips}_4$) and by the continuity of every $\varphi \in \mathcal{M}$,

$$\varphi((xc_\alpha - xy)a, a') = \varphi(x(c_\alpha - y)a, a') = \varphi((c_\alpha - y)a, x^*a') \to 0$$

and

$$\varphi((c_\alpha^* x^* - y^*x^*)a, a') = \varphi(x^*a, (c_\alpha - y)a') \to 0.$$

The converse is straightforward. ■

Proposition 4.17. *Let x, y be $\mathcal{M}$-bounded elements of $\mathfrak{A}$. Then $x \bullet y$ is well-defined if and only if xy is well-defined.*

Proof. If $x \bullet y$ is well-defined, then xy is obviously well-defined. Assume that xy is well-defined. Then by Proposition 4.11, xy is bounded. Let $z \in L(x)$. For $\varphi \in \mathcal{M}$ we denote by π_φ the corresponding GNS representation. Then, as is easily seen, for every $\mathcal{M}$-bounded element z, $\pi_\varphi(z)$ is a bounded operator. Hence, for every $a, b \in \mathfrak{A}_o$ and $\varphi \in \mathcal{M}$,

$$\begin{aligned} |\varphi((xy)a, z^*b)| &= |\langle \pi_\varphi(xy)\lambda_\varphi(a) \,|\, \pi_\varphi(z^*)\lambda_\varphi(b)\rangle| \\ &= |\langle \pi_\varphi(x) \,\square\, \pi_\varphi(y)\lambda_\varphi(a) \,|\, \pi_\varphi(z^*)\lambda_\varphi(b)\rangle| \\ &\le \|\overline{\pi_\varphi(x)}\| \, \|\pi_\varphi(y)\lambda_\varphi(a)\| \, \|\pi_\varphi(z^*)\lambda_\varphi(b)\|. \end{aligned}$$

This implies that $\pi_\varphi(z^*)\lambda_\varphi(b) \in D(\pi_\varphi(x)^*)$. Hence,

$$\begin{aligned} \varphi((xy)a, z^*b) &= \langle \pi_\varphi(y)\lambda_\varphi(a) \,|\, \pi_\varphi(x)^*\pi_\varphi(z^*)\lambda_\varphi(b)\rangle \\ &= \langle \pi_\varphi(y)\lambda_\varphi(a) \,|\, \pi_\varphi(x^*) \,\square\, \pi_\varphi(z^*)\lambda_\varphi(b)\rangle \\ &= \langle \pi_\varphi(y)\lambda_\varphi(a) \,|\, \pi_\varphi(x^*z^*)\lambda_\varphi(b)\rangle = \varphi(ya, (x^*z^*)b). \end{aligned}$$

Condition (sm_2) is proved in a similar way. ■

PROPOSITION 4.18. *Let x, y be $\mathcal{M}$-bounded elements of $\mathfrak{A}$. Then, for every $\varphi \in \mathcal{M}$, $\pi_\varphi(x) \,\square\, \pi_\varphi(y)$ is well-defined.*

Proof. Indeed, for every $a, b \in \mathfrak{A}_o$,

$$\begin{aligned} |\langle \pi_\varphi(y)\lambda_\varphi(a) \,|\, \pi_\varphi(x^*)\lambda_\varphi(b)\rangle| &= |\varphi(ya, x^*b) \le \varphi(ya, ya)^{1/2}\varphi(x^*b, x^*b)^{1/2} \\ &\le q_{\mathcal{M}}(x)q_{\mathcal{M}}(y)\varphi(a,a)^{1/2}\varphi(b,b)^{1/2}. \end{aligned}$$

Then, by the representation theorem for bounded sesquilinear forms in Hilbert space, there exists $Z_\varphi \in \mathcal{B}(\mathcal{H}_\varphi)$ such that

$$\langle \pi_\varphi(y)\lambda_\varphi(a) \,|\, \pi_\varphi(x^*)\lambda_\varphi(b)\rangle = \langle Z_\varphi\lambda_\varphi(a) \,|\, \lambda_\varphi(b)\rangle.$$

This implies that $\pi_\varphi(x) \,\square\, \pi_\varphi(y)$ is well-defined. ■

REMARK 4.19. We emphasize that this does *not* imply that there exists $z \in \mathfrak{A}$ such that $\pi_\varphi(x) \,\square\, \pi_\varphi(y) = \pi_\varphi(z)$. This fact will motivate a further restriction on the family $\mathcal{M}$: see Definition 4.26 below.

It is natural to ask under which assumptions $\mathfrak{A}_o$ itself consists of bounded elements.

PROPOSITION 4.20. *Let $\mathfrak{A}[\tau]$ be $\mathfrak{A}_o$-regular. Assume that the directed family $\{p_\alpha\}_{\alpha\in I}$ defining the topology τ has the property that, for every $\alpha \in I$,*

$$\liminf_{n\to\infty} (p_\alpha((a^*a)^{2^n}))^{2^{-n}} < \infty, \quad \forall a \in \mathfrak{A}_o. \tag{4.1}$$

Then every $a \in \mathfrak{A}_o$ is $\mathcal{M}$-bounded.

Proof. Let $a, b \in \mathfrak{A}_o$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \varphi(ab, ab) = \varphi(b, a^*ab) &\le \varphi(b,b)^{1/2}\varphi(a^*ab, a^*ab)^{1/2} \\ &= \varphi(b,b)^{1/2}\varphi(b, (a^*a)^2b)^{1/2}. \end{aligned}$$

Iterating, one obtains first

$$\varphi(ab, ab) \le \varphi(b,b)^{1/2+1/4}\varphi((a^*a)^2b, (a^*a)^2b)^{1/4},$$

and then the following Kaplansky-like inequality:

$$\varphi(ab,ab) \le \varphi(b,b)^{1-2^{-(n+1)}} \varphi((a^*a)^{2^n} b, (a^*a)^{2^n} b)^{2^{-(n+1)}}.$$

By the continuity of φ and of the right multiplication by $b \in \mathfrak{A}_o$, we can find a continuous seminorm p such that

$$\varphi(ab,ab) \le \varphi(b,b)^{1-2^{-(n+1)}} (p((a^*a)^{2^n}))^{2^{-n}} p(b)^{2^{-n}}.$$

On the other hand, there exist α and $\gamma > 0$ such that $p(x) \le \gamma p_\alpha(x)$ for every $x \in \mathfrak{A}$. Hence,

$$\varphi(ab,ab) \le \varphi(b,b)^{1-2^{-(n+1)}} \gamma^{2^{-n+1}} (p_\alpha((a^*a)^{2^n}))^{2^{-n}} p_\alpha(b)^{2^{-n}}.$$

Taking the lim inf of the r.h.s., we finally obtain

$$\varphi(ab,ab) \le \gamma_a \varphi(b,b), \quad \forall b \in \mathfrak{A}_o,$$

where $\gamma_a := \liminf_{n\to\infty} (p_\alpha((a^*a)^{2^n}))^{2^{-n}}$. ■

Example 4.21. As shown in Section 3, $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})[\mathsf{t}_\mathrm{s}]$ is a $\mathfrak{P}$-regular partial *-algebra. The seminorms defining t_s satisfy (4.1), since the elements of $\mathfrak{P}$ are bounded operators in Hilbert space. Indeed, if $A \in \mathfrak{P}$,

$$\|(A^*A)^{2^n}\xi\| \le \|A\|^{2^{n+1}} \|\xi\|, \quad \forall \xi \in \mathcal{D},$$

and so (4.1) holds in this case.

The following *mixed* associativity in $\mathfrak{A}$, similar to (2.5), can be easily proved by using Definition 4.14.

Proposition 4.22. *Let $x,y,z \in \mathfrak{A}$. Assume that $x \,\square\, y$, $(x \,\square\, y) \,\square\, z$ and $y \bullet z$ are all well-defined. Then $x \in L(y \bullet z)$ and*

$$x \,\square\, (y \bullet z) = (x \,\square\, y) \,\square\, z.$$

As we have seen in Section 2, the $\tau_{\mathrm{s}^*}^{\mathcal{M}}$-density of $\mathfrak{A}_o$ and Proposition 4.7 imply the existence of a strong multiplication *induced* by $\mathfrak{A}_o$. But this multiplication is, in general, only a restriction of the multiplication $\bullet$ defined above. However, let us assume that $\mathfrak{A}$ is *semi-associative with respect to* $\mathfrak{A}_o$, by which we mean that

$$(4.2) \qquad (xa)b = x(ab), \quad a(xb) = (ax)b, \quad \forall x \in \mathfrak{A},\ a,b \in \mathfrak{A}_o.$$

In other words, $(\mathfrak{A}, \mathfrak{A}_o)$ is a quasi *-algebra. In that case, Proposition 4.22 implies the topological regularity of $\mathfrak{A}[\tau]$.

Proposition 4.23. *Let $\mathfrak{A}$ be semi-associative with respect to $\mathfrak{A}_o$. Then $\mathfrak{A}[\tau_\mathrm{s}^{\mathcal{M}}]$ (and hence $\mathfrak{A}[\tau]$) is topologically regular.*

Proof. By Proposition 4.7, the operator of left multiplication L_x defined on $R(x)$ is τ-closed, for every $x \in \mathfrak{A}$. Let $y \in D(\mathsf{L}_x^\circ)$, where L_x° denotes the closure of the restriction of L_x to $\mathfrak{A}_o$. Then there exists a net $\{y_\alpha\} \subset \mathfrak{A}_o$ and $w \in \mathfrak{A}$ such that $y_\alpha \xrightarrow{\tau_\mathrm{s}^{\mathcal{M}}} y$ and $xy_\alpha \xrightarrow{\tau_\mathrm{s}^{\mathcal{M}}} w$. Thus, $x \in L(y)$, and by (ips_4),

$$\begin{aligned}\varphi((xy)a, z^*b) &= \lim_\alpha \varphi((xy_\alpha)a, z^*b) = \lim_\alpha \varphi(x(y_\alpha a), z^*b)\\ &= \lim_\alpha \varphi(y_\alpha a, (x^*z^*)b) = \varphi(ya, (x^*z^*)b), \quad \forall \varphi \in \mathcal{M},\ a, b \in \mathfrak{A}_o.\end{aligned}$$

Hence (sm_1) holds. The proof of (sm_2) is similar. ■

REMARK 4.24. If $\mathfrak{A}$ is semi-associative with respect to $\mathfrak{A}_o$, then $\mathfrak{A}_o \subset R^{\mathrm{s}}\,\mathfrak{A}$, the set of universal strong right multipliers of $\mathfrak{A}$.

An element x has a *strong inverse* if there exists $x^{-1} \in \mathfrak{A}$ such that $x \bullet x^{-1} = x^{-1} \bullet x = e$. The mixed associativity of Proposition 4.22 implies that if a strong inverse of x exists, then it is unique.

THEOREM 4.25. *Let $\mathfrak{A}[\tau]$ be an $\mathfrak{A}_o$-regular partial *-algebra satisfying condition (wp) and let $\mathcal{M}$ be the set of all continuous ips-forms with core $\mathfrak{A}_o$. Let π be a $(\tau, \mathsf{t}_{\mathrm{s}})$-continuous *-representation of $\mathfrak{A}$ (that is, $\pi : \mathfrak{A}[\tau] \to \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})[\mathsf{t}_{\mathrm{s}}]$ continuously). Then an element $x \in \mathfrak{A}$ is $\mathcal{M}$-bounded if and only if $\pi(x)$ is a bounded operator.*

Proof. Let us define the following positive sesquilinear form:

$$\varphi_\xi(x, y) := \langle \pi(x)\xi \,|\, \pi(y)\xi \rangle.$$

The conditions (ips_3) and (ips_4) are easily verified. By the continuity of π,

$$|\varphi_\xi(x, y)| = |\langle \pi(x)\xi \,|\, \pi(y)\xi \rangle| \le \|\pi(x)\xi\|\,\|\pi(y)\xi\| \le \gamma p_\alpha(x) p_\alpha(y)$$

for some $\gamma > 0$. Thus φ_ξ is an ips-form and $\varphi_\xi \in \mathcal{M}$.

If x is $\mathcal{M}$-bounded, by definition we have

$$\varphi_\xi(xa, xa) \le q_{\mathcal{M}}(x)^2 \varphi_\xi(a, a), \qquad \forall \xi \in \mathcal{D},\ a \in \mathfrak{A}_o.$$

For $a = e$, one has $\varphi_\xi(x, x) = \|\pi(x)\xi\|^2 \le q_{\mathcal{M}}(x)\varphi_\xi(e, e) = q_{\mathcal{M}}(x)\|\xi\|^2$.

Conversely, suppose that $\pi(x)$ is bounded for every $(\tau, \mathsf{t}_{\mathrm{s}})$-continuous *-representation π of $\mathfrak{A}$. In particular, the GNS representation π_φ defined by $\varphi \in \mathcal{M}$ is $(\tau, \mathsf{t}_{\mathrm{s}})$-continuous, so it is bounded on $\mathcal{D}_\varphi := \{\lambda_\varphi(a) : a \in \mathfrak{A}_o\}$. Then there exists $\gamma > 0$ such that $\|\pi_\varphi(x)\xi\|^2 \le \gamma^2\|\xi\|^2$ for all $\xi \in \mathcal{D}_\varphi$. Since $\xi = \lambda_\varphi(a)$ for all $a \in \mathfrak{A}_o$, we have $\|\pi_\varphi(x)\lambda_\varphi(a)\|^2 \le \gamma^2\|\lambda_\varphi(a)\|^2$ for some $a \in \mathfrak{A}_o$, i.e., $\varphi(xa, xa) \le \gamma^2\varphi(a, a)$ and x is $\mathcal{M}$-bounded. ■

We expect that $\mathcal{M}$-bounded elements can also be characterized in terms of their spectral behavior. For this, some additional assumptions on the family $\mathcal{M}$ of ips-forms are needed.

DEFINITION 4.26. Let $\mathcal{M}$ be a family of continuous ips-forms on $\mathfrak{A} \times \mathfrak{A}$. For every $\varphi \in \mathcal{M}$, let π_φ denote the corresponding GNS representation. We say that $\mathcal{M}$ is *well-behaved* if

- (wb_1) $\mathcal{M}$ is sufficient;
- (wb_2) for every $\varphi \in \mathcal{M}$ and every $a \in \mathfrak{A}$, also $\varphi_a \in \mathcal{M}$, where $\varphi_a(x, y) := \varphi(xa, ya)$;

(wb_3) if $x, y \in \mathfrak{A}$ and $\pi_\varphi(x) \,\Box\, \pi_\varphi(y)$ is well-defined for every $\varphi \in \mathcal{M}$, then there exists $z \in \mathfrak{A}$ such that $\pi_\varphi(x) \,\Box\, \pi_\varphi(y) = \pi_\varphi(z)$ for every $\varphi \in \mathcal{M}$;

(wb_4) $\mathfrak{A}$ is $\tau_{s^*}^{\mathcal{M}}$-complete.

To give an example, if $\mathfrak{M} = \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})[\mathsf{t}_{s^*}]$ or if $\mathfrak{M}$ is any partial GC^*-algebra of operators, the family

$$\mathcal{M} := \{\psi_\xi : \xi \in \mathcal{D},\ \psi_\xi(X, Y) = \langle X\xi \mid Y\xi\rangle,\ X, Y \in \mathfrak{M}\}$$

is well-behaved.

Proposition 4.27. *If $\mathcal{M}$ is well-behaved, then $\mathcal{D}(q_{\mathcal{M}})$ is a C^*-algebra with the strong multiplication $\bullet$ and the norm $q_{\mathcal{M}}$.*

Proof. By Proposition 4.18, if $x, y \in \mathcal{D}(q_{\mathcal{M}})$, then $\pi_\varphi(x) \,\Box\, \pi_\varphi(y)$ is well-defined. Thus, by (wb_3), there exists $z \in \mathfrak{A}$ such that $\pi_\varphi(x) \,\Box\, \pi_\varphi(y) = \pi_\varphi(z)$ for every $\varphi \in \mathcal{M}$. Then, for every $\varphi \in \mathcal{M}$ and $a, b \in \mathfrak{A}_o$,

$$\begin{aligned}\varphi(ya, x^*b) &= \langle \pi_\varphi(y)\lambda_\varphi(a) \mid \pi_\varphi(x^*)\lambda_\varphi(b)\rangle \\ &= \langle \pi_\varphi(x) \,\Box\, \pi_\varphi(y)\lambda_\varphi(a) \mid \lambda_\varphi(b)\rangle \\ &= \langle \pi_\varphi(z)\lambda_\varphi(a) \mid \lambda_\varphi(b)\rangle = \varphi(za, b).\end{aligned}$$

Hence xy is well-defined and, by Proposition 4.17, $x \bullet y$ is also well-defined. Since $q_{\mathcal{M}}$ is a C^*-norm on $\mathcal{D}(q_{\mathcal{M}})$, we only need to prove the completeness of $\mathcal{D}(q_{\mathcal{M}})$ to get the result.

Let $\{x_n\}$ be a Cauchy sequence with respect to the norm $q_{\mathcal{M}}$. Then $\{x_n^*\}$ is Cauchy too. Hence, by (wb_2), for every $\varphi \in \mathcal{M}$ and $a \in \mathfrak{A}_o$ we have

$$\begin{aligned}&\varphi((x_n - x_m)a, (x_n - x_m)a) \to 0 \quad \text{as } n, m \to \infty,\\ &\varphi((x_n^* - x_m^*)a, (x_n^* - x_m^*)a) \to 0 \quad \text{as } n, m \to \infty.\end{aligned}$$

Therefore, $\{x_n\}$ is also Cauchy with respect to $\tau_{s^*}^{\mathcal{M}}$. Then, by (wb_4), there exists $x \in \mathfrak{A}$ such that $x_n \xrightarrow{\tau_{s^*}^{\mathcal{M}}} x$. Since

$$\varphi(xa, xa) = \lim_{n\to\infty} \varphi(x_n a, x_n a) \le \limsup_{n\to\infty} q_{\mathcal{M}}(x_n)^2 \varphi(a, a)$$

and $\limsup_{n\to\infty} q_{\mathcal{M}}(x_n)^2 < \infty$ (by the boundedness of the sequence $\{q_{\mathcal{M}}(x_n)\}$), we conclude that x is $\mathcal{M}$-bounded. Finally, by the Cauchy condition, for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $q_{\mathcal{M}}(x_n - x_m) < \epsilon$ for all $n, m > n_\epsilon$. This implies that

$$\varphi((x_n - x_m)a, (x_n - x_m)a) < \epsilon\varphi(a, a), \quad \forall \varphi \in \mathcal{M},\ a \in \mathfrak{A}_o.$$

If we fix $n > n_\epsilon$ and let $m \to \infty$, we obtain

$$\varphi((x_n - x)a, (x_n - x)a) \le \epsilon\varphi(a, a), \quad \forall \varphi \in \mathcal{M},\ a \in \mathfrak{A}_o.$$

This, in turn, implies that $q_{\mathcal{M}}(x_n - x) \le \epsilon$, and completes the proof. ∎

Let us now introduce the usual spectral elements adapted to the present situation.

DEFINITION 4.28. Let $x \in \mathfrak{A}$. The *resolvent* $\rho^{\mathcal{M}}(x)$ of x is defined by

$$\rho^{\mathcal{M}}(x) := \left\{\lambda \in \mathbb{C} : (x - \lambda e)^{-1} \text{ exists in } \mathcal{D}(q_{\mathcal{M}})\right\}.$$

The corresponding *spectrum* of x is defined as $\sigma^{\mathcal{M}}(x) := \mathbb{C} \setminus \rho^{\mathcal{M}}(x)$.

As in [12], it can be proved that if $\mathcal{M}$ is well-behaved, then (a) $\rho^{\mathcal{M}}(x)$ is an open subset of the complex plane; (b) the map $\lambda \in \rho^{\mathcal{M}}(x) \mapsto (x - \lambda e)^{-1} \in \mathcal{D}(q_{\mathcal{M}})$ is analytic in each connected component of $\rho^{\mathcal{M}}(x)$; (c) $\sigma^{\mathcal{M}}(x)$ is nonempty.

As usual, we define the *spectral radius* of $x \in \mathfrak{A}$ by

$$r^{\mathcal{M}}(x) := \sup\{|\lambda| : \lambda \in \sigma^{\mathcal{M}}(x)\}.$$

THEOREM 4.29. *Assume that $\mathcal{M}$ is well-behaved and let $x \in \mathfrak{A}$. Then $r^{\mathcal{M}}(x) < \infty$ if and only if $x \in \mathcal{D}(q_{\mathcal{M}})$.*

Proof. If $x \in \mathcal{D}(q_{\mathcal{M}})$, then $\sigma^{\mathcal{M}}(x)$ coincides with the spectrum of x as an element of the C^*-algebra $\mathcal{D}(q_{\mathcal{M}})$ and so $\sigma^{\mathcal{M}}(x)$ is compact. Conversely, assume that $r^{\mathcal{M}}(x) < \infty$. Then the function $\lambda \mapsto (x - \lambda e)^{-1}$ is $q_{\mathcal{M}}$-analytic in the region $|\lambda| > r^{\mathcal{M}}(x)$. Therefore it has there a $q_{\mathcal{M}}$-convergent Laurent expansion

$$(x - \lambda e)^{-1} = \sum_{k=1}^{\infty} \frac{a_k}{\lambda^k}, \qquad |\lambda| > r^{\mathcal{M}}(x),$$

with $a_k \in \mathcal{D}(q_{\mathcal{M}})$ for each $k \in \mathbb{N}$. As usual

$$a_k = \frac{1}{2\pi i} \int_{\gamma} \frac{(x - \lambda e)^{-1}}{\lambda^{-k+1}}\, d\lambda, \qquad k \in \mathbb{N},$$

where $\gamma := \{\lambda \in \mathbb{C} : |\lambda| = R \text{ for some } R > r^{\mathcal{M}}(x)\}$ and the integral on the r.h.s. is meant to converge with respect to $q_{\mathcal{M}}$.

For every $\varphi \in \mathcal{M}$ and $b, b' \in \mathfrak{A}_o$, we have

$$\begin{aligned}
\varphi(a_k b, x^* b') &= \frac{1}{2\pi i} \int_{\gamma} \frac{\varphi((x - \lambda e)^{-1} b, x^* b')}{\lambda^{-k+1}}\, d\lambda \\
&= \frac{1}{2\pi i} \int_{\gamma} \frac{\varphi((x - \lambda e)^{-1} b, (x^* - \overline{\lambda} e) b')}{\lambda^{-k+1}}\, d\lambda \\
&\quad + \frac{1}{2\pi i} \int_{\gamma} \frac{\varphi((x - \lambda e)^{-1} b, \overline{\lambda} b')}{\lambda^{-k+1}}\, d\lambda \\
&= \frac{1}{2\pi i} \int_{\gamma} \frac{\varphi(b, b')}{\lambda^{-k+1}}\, d\lambda + \frac{1}{2\pi i} \int_{\gamma} \frac{\varphi((x - \lambda e)^{-1} b, b')}{\lambda^{-k}}\, d\lambda = \varphi(a_{k+1} b, b').
\end{aligned}$$

This implies that $x a_k$ is well-defined for every $k \in \mathbb{N}$, and $x a_k = a_{k+1}$.

In particular,

$$\varphi(a_1 b, x^* b') = \frac{1}{2\pi i} \int_\gamma \varphi((x-\lambda e)^{-1} b, x^* b')\, d\lambda$$
$$= \frac{1}{2\pi i} \varphi\Big(\Big(\int_\gamma (x-\lambda e)^{-1} d\lambda\Big) b, x^* b'\Big) = \frac{1}{2\pi i} \varphi(-b, x^* b').$$

Hence $xa_1 = -x$. Thus finally $x = -a_2 \in \mathcal{D}(q_{\mathcal{M}})$. ∎

In our previous paper [4], we have introduced a notion of strong inverse based on the multiplication obtained by closure, and this has allowed us to derive a number of spectral properties. Now the notion of strong multiplication $\bullet$ defined here (Definition 4.13) allows us to obtain similar results. In particular, Proposition 4.13 of [4] may be generalized as follows.

Proposition 4.30. *Assume that $\mathfrak{A}$ is topologically regular over $\mathfrak{A}_o$ and let $x \in \mathfrak{A}$. Then every $\lambda \in \mathbb{C}$ such that $|\lambda| > q_{\mathcal{M}}(x)$ belongs to $\rho^{\mathcal{M}}(x)$.*

Proof. Let x^{-1} be the strong inverse by closure of $x \in \mathfrak{A}$ so that $x^{-1} \in L(x) \cap R(x)$. Of course, we may assume that $x \in \mathcal{D}(q_{\mathcal{M}})$. Then the following analogue of (sm_1) holds true:

$$\varphi((xx^{-1})a, z^* b) = \varphi(a, z^* b) = \varphi(x^{-1} a, (x^* z^*) b), \qquad \forall z \in L(x),\ \varphi \in \mathcal{M},\ a, b \in \mathfrak{A}_o.$$

Let indeed $x^{-1} \in D(\mathsf{L}_x^\circ)$. Then there exists a net $\{w_\alpha\} \subset \mathfrak{A}_o$ such that $w_\alpha \xrightarrow{\tau_s^{\mathcal{M}}} x^{-1}$ and $xw_\alpha \xrightarrow{\tau_s^{\mathcal{M}}} e$. Then, using the continuity of $\varphi \in \mathcal{M}$ and of multiplication by $\mathfrak{A}_o$, and (ips_4), we have

$$\varphi(x^{-1} a, (x^* z^*) b) = \lim_\alpha \varphi(w_\alpha a, (x^* z^*) b) = \lim_\alpha \varphi((xw_\alpha) a, z^* b) = \varphi(a, z^* b).$$

In the same way, one proves the following analogue of (sm_2):

$$\varphi((x^{-1*} x^*) a, vb) = \varphi(x^* a, (x^{-1} v) b), \qquad \forall v \in R(y),\ \varphi \in \mathcal{M},\ a, b \in \mathfrak{A}_o.$$

Since $x \bullet x^{-1} = x^{-1} \bullet x = e$, one shows in the same way, for $x \in D(\mathsf{L}^\circ_{x^{-1}})$, that

$$\varphi((x^{-1} x) a, z^* b) = \varphi(a, z^* b) = \varphi(x^{-1} a, (x^{-1*} z^*) b), \qquad \forall z \in L(x),\ \varphi \in \mathcal{M},\ a, b \in \mathfrak{A}_o,$$

and

$$\varphi((x^* x^{-1*}) a, vb) = \varphi(x^{-1*} a, xvb), \qquad \forall v \in R(y),\ \varphi \in \mathcal{M},\ a, b \in \mathfrak{A}_o.$$

Thus we have proved that if x^{-1} is the strong inverse by closure of $x \in \mathfrak{A}$, as defined in [4], then x^{-1} is also the strong inverse with respect to the strong multiplication $\bullet$ (the converse is not true in general).

Combining this fact with Proposition 4.13 of [4], we can conclude that $(x - \lambda e)^{-1}$ exists as a strong inverse, which proves the statement. ∎

REMARK 4.31. The previous proposition implies that, for every $x \in \mathfrak{A}$, $r^{\mathcal{M}}(x) \leq q_{\mathcal{M}}(x)$ for every choice of the sufficient family $\mathcal{M}$. Clearly, if $x \notin \mathcal{D}(q_{\mathcal{M}})$, then both $r^{\mathcal{M}}(x)$ and $q_{\mathcal{M}}(x)$ are infinite.

5. Existence of faithful representations. The lesson of Theorem 4.25 is essentially that the notion of $\mathcal{M}$-bounded element given above is reasonable: as for the case of locally convex *-algebras, a good notion of boundedness of an element is equivalent to the boundedness of the operators representing it. This definition will be even more significant if the locally convex partial *-algebra under consideration possesses *sufficiently many* *-representations. This fact is expressed, in the case of locally convex *-algebras, through the notion of **-semisimplicity* which we will extend to locally convex partial *-algebras in a natural way.

A *-representation of a partial *-algebra $\mathfrak{A}$ is a *-homomorphism $\pi : \mathfrak{A} \to \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. If $\mathfrak{A}[\tau]$ is $\mathfrak{A}_o$-regular, then, by definition, it has a τ^*-dense distinguished *-subalgebra $\mathfrak{A}_o$. Clearly, $\pi(\mathfrak{A}_o)$ is a *-algebra of operators, but in general $\pi(\mathfrak{A}_o) \not\subset \mathcal{L}^\dagger(\mathcal{D})$. However, we can always guarantee this property by changing the domain. Indeed:

PROPOSITION 5.1. *Let $\mathfrak{A}$ be an $\mathfrak{A}_o$-regular partial *-algebra and let π be a *-representation of $\mathfrak{A}$ with domain $\mathcal{D}$ in $\mathcal{H}$. Put*

$$\mathcal{D}_1 := \Big\{\xi_0 + \sum_{i=1}^{n} \pi(b_i)\xi_i : b_i \in \mathfrak{A}_o,\ \xi_0, \ldots, \xi_n \in \mathcal{D};\ n \in \mathbb{N}\Big\}$$

and define

$$\pi_1(a)\Big(\xi_0 + \sum_{i=1}^{n} \pi(b_i)\xi_i\Big) := \pi(a)\xi_0 + \sum_{i=1}^{n} \pi(a) \,\square\, \pi(b_i)\xi_i.$$

Then π_1 is a *-representation of $\mathfrak{A}$ with domain $\mathcal{D}_1 \supset \mathcal{D}$ and $\pi(\mathfrak{A}_o) \subset \mathcal{L}^\dagger(\mathcal{D}_1)$.

The proof of this proposition is given in Appendix A. Thus we can conclude that it is not restrictive to suppose that $\pi(\mathfrak{A}_o) \subset \mathcal{L}^\dagger(\mathcal{D})$.

Now we can state the result announced at the end of Section 3.

THEOREM 5.2. *Let $\mathfrak{A}$ be an $\mathfrak{A}_o$-regular partial *-algebra, with a sufficient family $\mathcal{M}$ of ips-forms, in particular, a partial GC*-algebra. Then:*

- (i) *$\mathfrak{A}$ has a faithful, (τ, t_s)-continuous representation into a partial GC*-algebra of operators.*
- (ii) *Assume, in addition, that the family $\mathcal{M}$ is well-behaved. Then $\mathfrak{A}$ has a faithful, (τ, t_s)-continuous representation* onto *a partial GC*-algebra of operators.*

Proof. (i) For every $\varphi \in \mathcal{M}$, let $(\pi_\varphi, \lambda_\varphi, \mathcal{H}_\varphi)$ be the corresponding GNS construction. Define, as usual, $\mathcal{H} := \bigoplus_{\varphi \in \mathcal{M}} \mathcal{H}_\varphi$ and

$$\mathcal{D}(\pi) := \Big\{\xi = (\lambda_\varphi(a)),\, a \in \mathfrak{A}_o : \sum_{\varphi\in\mathcal{M}} \|\pi_\varphi(x)\lambda_\varphi(a)\|^2 < \infty,\, \forall x \in \mathfrak{A}\Big\}.$$

Then, putting

$$\pi(x)\xi := (\pi_\varphi(x)\lambda_\varphi(a)), \qquad a \in \mathfrak{A}_o,$$

one defines a faithful representation of $\mathfrak{A}$.

Taking into account the continuity of $\varphi \in \mathcal{M}$ and of multiplication by $\mathfrak{A}_o$, we have

$$\|\pi_\varphi(x)\lambda_\varphi(a)\|^2 = \varphi(xa, xa) \le p(xa)^2 \le p'(xa)$$

for some τ-continuous seminorms p, p'. This implies that π is $(\tau, \mathsf{t}_\mathrm{s})$-continuous. So, by Theorem 4.25, if $x \in \mathcal{D}(q_\mathcal{M})$, then $\pi(x)$ is bounded and one checks directly that

$$\|\overline{\pi(x)}\| \le q_\mathcal{M}(x).$$

(ii) Now suppose $\mathcal{M}$ is well-behaved. Then $\mathcal{D}(q_\mathcal{M})$ is a C^*-algebra, and hence

$$\|\overline{\pi(x)}\| = q_\mathcal{M}(x), \qquad \forall x \in \mathcal{D}(q_\mathcal{M}),$$

and $\pi(\mathcal{D}(q_\mathcal{M}))$ is a C^*-algebra.

Moreover, if $\mathfrak{A}_o \subset \mathcal{D}(q_\mathcal{M})$, then $\mathcal{D}(q_\mathcal{M})$ is τ^*-dense in $\mathfrak{A}$. Hence, if $x \in \mathfrak{A}$, there exists a net $\{x_\alpha\} \subset \mathfrak{A}_o$ such that $x_\alpha \xrightarrow{\tau^*} x$. This implies that $x_\alpha \xrightarrow{\tau} x$ and $x_\alpha^* \xrightarrow{\tau} x^*$.

Then, since π is $(\tau, \mathsf{t}_\mathrm{s})$-continuous, we deduce that $\pi(x_\alpha)\xi \to \pi(x)\xi$ and $\pi(x_\alpha^*)\xi \to \pi(x^*)\xi$ for all $\xi \in \mathcal{D}(\pi)$. This implies that $\pi(x_\alpha)\xi \xrightarrow{\mathsf{t}_{\mathrm{s}^*}} \pi(x)\xi$. Hence, $\pi(\mathcal{D}(q_\mathcal{M}))$ is $\mathsf{t}_{\mathrm{s}^*}$-dense in $\pi(\mathfrak{A})$.

The construction of π implies that $\pi(\mathfrak{A})$ is a partial *-algebra. Assume indeed that $\pi(x) \,\square\, \pi(y)$ is well-defined. Then $\pi_\varphi(x) \,\square\, \pi_\varphi(y)$ is well-defined for every $\varphi \in \mathcal{M}$. Hence there exists a $z \in \mathfrak{A}$ such that $\pi_\varphi(x) \,\square\, \pi_\varphi(y) = \pi_\varphi(z)$. This in turn implies that $\pi(x) \,\square\, \pi(y) = \pi(z)$.

In general, however, $\pi(\mathfrak{A})$ need not be complete with respect to $\mathsf{t}_{\mathrm{s}^*}$. Assume that $\{\pi(x_\alpha)\}$ is a net in $\pi(\mathfrak{A})$ with

$$\pi(x_\alpha) \xrightarrow{\mathsf{t}_{\mathrm{s}^*}} Z \in \mathcal{L}^\dagger(\mathcal{D}(\pi), \mathcal{H}).$$

Then, by the definition of π,

$$\pi_\varphi(x_\alpha) \xrightarrow{\mathsf{t}_{\mathrm{s}^*}} Z_\varphi, \quad \text{where} \quad Z_\varphi \xi_\varphi = (Z(\xi_\varphi))_\varphi.$$

This implies that, for every $a \in \mathfrak{A}_o$,

$$\pi_\varphi(x_\alpha)\lambda_\varphi(a) \to Z_\varphi\lambda_\varphi(a), \qquad \pi_\varphi(x_\alpha^*)\lambda_\varphi(a) \to Z_\varphi^*\lambda_\varphi(a).$$

Hence

$$\varphi((x_\alpha - x_\beta)a, (x_\alpha - x_\beta)a) = \|\pi_\varphi(x_\alpha - x_\beta)\lambda_\varphi(a)\|^2 \to 0$$

for α, β "large" enough and every $a \in \mathfrak{A}_o$. Similarly,

$$\varphi((x_\alpha^* - x_\beta^*)a, (x_\alpha^* - x_\beta^*)a) = \|\pi_\varphi(x_\alpha^* - x_\beta^*)\lambda_\varphi(a)\|^2 \to 0.$$

Since $\mathcal{M}$ is well-behaved, $\{x_\alpha\}$ is a $\tau_{s^*}^{\mathcal{M}}$-Cauchy net. Thus there exists $x \in \mathfrak{A}$ such that

$$\varphi(x_\alpha - x, x_\alpha - x) \to 0, \quad \forall \varphi \in \mathcal{M}.$$

By (wb_2), it follows that

$$\varphi((x_\alpha - x)a, (x_\alpha - x)a) \to 0, \quad \forall \varphi \in \mathcal{M},\ a \in \mathfrak{A}_o.$$

Consequently, $\pi_\varphi(x_\alpha) \xrightarrow{\mathsf{t}_{s^*}} \pi_\varphi(x)$ for all $\varphi \in \mathcal{M}$, and hence $\pi(x_\alpha) \xrightarrow{\mathsf{t}_s} \pi(x)$. Thus $\pi(\mathfrak{A})$ is t_{s^*}-closed and hence t_{s^*}-complete, if one remembers that $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})[\mathsf{t}_{s^*}]$ is complete. This concludes the proof. ∎

For topological *-algebras the set of elements which belong to the intersection of the kernels of all continuous *-representations is called the *-*radical* of $\mathfrak{A}$ (see e.g. [5, 7]).

In a previous paper [3], we have introduced the notions of the algebraic *-radical and of an algebraically *-semisimple partial *-algebra. In the present context, the presence of a sufficient family of continuous ips-forms allows one to introduce similar concepts at the topological level as well. Thus the notion of *-radical has a natural extension to our case.

Let in fact $\mathfrak{A}[\tau]$ be an $\mathfrak{A}_o$-regular partial *-algebra. We define the *-*radical* of $\mathfrak{A}$ by

$$\mathcal{R}^*(\mathfrak{A}) := \{x \in \mathfrak{A} : \pi(x) = 0 \text{ for all } (\tau, \mathsf{t}_{s^*})\text{-continuous } ^*\text{-representations } \pi\}.$$

We put $\mathcal{R}^*(\mathfrak{A}) := \mathfrak{A}$ if $\mathfrak{A}[\tau]$ has no (τ, t_{s^*})-continuous *-representations.

PROPOSITION 5.3. *Let $\mathfrak{A}[\tau]$ be an $\mathfrak{A}_o$-regular partial *-algebra and $\mathcal{P}_{\mathfrak{A}_o}(\mathfrak{A})$ the set of* all *τ-continuous ips-forms with core $\mathfrak{A}_o$. For an element $x \in \mathfrak{A}$ the following statements are equivalent:*

(i) *$x \in \mathcal{R}^*(\mathfrak{A})$.*
(ii) *$\varphi(x, x) = 0$ for every $\varphi \in \mathcal{P}_{\mathfrak{A}_o}(\mathfrak{A})$.*
(iii) *x^*x is well-defined and $x^*x = 0$.*

Proof. (i)⇒(ii). Assume that, for all $x \in \mathfrak{A}$, $x \neq 0$, there exists a continuous ips-form with core $\mathfrak{A}_o$ such that $\varphi(x, x) > 0$. Let $(\pi_\varphi, \mathcal{H}_\varphi, \lambda_\varphi)$ be the corresponding GNS construction. The GNS *-representation is $(\tau^*, \mathsf{t}_{s^*})$-continuous. Indeed, if $a \in \mathfrak{A}_o$, we have

$$\|\pi_\varphi(x)\lambda_\varphi(a)\|^2 = \varphi(xa, xa) \le \gamma^2 p_\alpha^*(xa) \le \gamma' p_\beta^*(x).$$

On the other hand,

$$\|\pi_\varphi(x^*)\lambda_\varphi(a)\|^2 = \varphi(x^*a, x^*a) \le \gamma^2 p_\alpha^*(x^*a) \le \gamma'' p_\beta^*(x^*) = \gamma''' p_\beta^*(x).$$

Finally, $\|\pi_\varphi(x)\lambda_\varphi(e)\|^2 = \varphi(x, x) > 0$, and this implies $\pi_\varphi(x) \neq 0$.

(ii)⇒(iii). Assume that $\varphi(x,x) = 0$ for all $\varphi \in \mathcal{P}_{\mathfrak{A}_o}(\mathfrak{A})$. For $a \in \mathfrak{A}_o$, we have $\varphi_a(x,x) = 0$, since clearly $\varphi_a \in \mathcal{P}_{\mathfrak{A}_o}(\mathfrak{A})$. By the Cauchy–Schwarz inequality, it follows that $\varphi(xa, xb) = 0$ for all $a, b \in \mathfrak{A}_o$. By (wp), this means that $x^* \,\square\, x = x^*x$ is well-defined and $x^*x = 0$.

(iii)⇒(i). Assume now that x^*x is well-defined and $x^*x = 0$. If π is a (τ, t_{s^*})-continuous *-representation of $\mathfrak{A}$, then $\pi(x^*) \,\square\, \pi(x) = \pi(x)^\dagger \,\square\, \pi(x)$ is well-defined and equals 0. Hence, for every $\xi \in \mathcal{D}(\pi)$,

$$\begin{aligned}\|\pi(x)\xi\|^2 &= \langle \pi(x)\xi \,|\, \pi(x)\xi\rangle = \langle \pi(x)\xi \,|\, \pi(x)\xi\rangle = \langle \pi(x)^\dagger \,\square\, \pi(x)\xi \,|\, \xi\rangle \\ &= \langle \pi(x^*) \,\square\, \pi(x)\xi \,|\, \xi\rangle = \langle \pi(x^*x)\xi \,|\, \xi\rangle = 0.\end{aligned}$$

Hence $\pi(x) = 0$. ∎

Clearly if $\mathfrak{A}$ possesses a sufficient family $\mathcal{M}$ of τ-continuous ips-forms, then $\mathcal{P}_{\mathfrak{A}_o}(\mathfrak{A})$ itself is sufficient, and Proposition 5.3 implies that $\mathcal{R}^*(\mathfrak{A}) = \{0\}$. Conversely, if $\mathcal{R}^*(\mathfrak{A}) = \{0\}$, then $\mathcal{P}_{\mathfrak{A}_o}(\mathfrak{A})$ is sufficient. Our choice of considering a sufficient family $\mathcal{M}$ instead of the whole $\mathcal{P}_{\mathfrak{A}_o}(\mathfrak{A})$ is motivated by the fact that characterizing the space $\mathcal{P}_{\mathfrak{A}_o}(\mathfrak{A})$ in concrete examples is much more difficult than choosing a sufficient subfamily.

As for the case of topological algebras, it is natural, in the light of the previous discussion, to call an $\mathfrak{A}_o$-regular partial *-algebra $\mathfrak{A}[\tau]$ **-semisimple* if $\mathcal{R}^*(\mathfrak{A}) = \{0\}$. We hope to carry out a more detailed analysis of this situation in another paper.

Appendix A. Proof of Proposition 5.1. The argument is very similar to that given in [10, Proposition 1] in a different context. To lighten notation, we assume that $\pi(e) = I_{\mathcal{D}}$. The general case can be proved by a slight modification of the argument below. Note that all the sums considered are finite.

We have to check that $\pi_1(a)$ is well-defined for every $a \in \mathfrak{A}$ and that π_1 is a *-representation of $\mathfrak{A}$. We have

$$\begin{aligned}&\Big\langle \sum_i (\pi(a) \,\square\, \pi(b_i))\xi_i \,\Big|\, \sum_j \pi(c_j)\eta_j \Big\rangle \\ &\qquad = \sum_{i,j} \langle \pi(ab_i)\xi_i \,|\, \pi(c_j)\eta_j\rangle = \sum_{i,j} \langle \xi_i \,|\, \pi(ab_i)^\dagger \pi(c_j)\eta_j\rangle \\ &\qquad = \sum_{i,j} \langle \xi_i \,|\, \pi(b_i^* a^*) \,\square\, \pi(c_j)\eta_j\rangle = \sum_{i,j} \langle \xi_i \,|\, \pi((b_i^* a^*)c_j)\eta_j\rangle \\ &\qquad = \sum_{i,j} \langle \xi_i \,|\, (\pi(b_i^*) \,\square\, \pi(a^* c_j))\eta_j\rangle = \sum_i \Big\langle \pi(b_i)\xi_i \,\Big|\, \sum_j (\pi(a^* c_j)\eta_j \Big\rangle.\end{aligned}$$

Hence, if $\sum_i \pi(b_i)\xi_i = 0$, then $\xi := \sum_i (\pi(a) \,\square\, \pi(b_i))\xi_i$ is orthogonal to every element of $\mathcal{D}_1$, which is dense in $\mathcal{H}$. Thus $\xi = 0$. This proves that

$\pi_1(a)$ is, for every $a \in \mathfrak{A}$, a well-defined linear map of $\mathcal{D}_1$ into $\mathcal{H}$. Clearly, $\pi_1(\mathfrak{A}_o) \subset \mathcal{L}^\dagger(\mathcal{D})$. Moreover, the above equalities also imply that

$$\Big\langle \pi_1(a)\Big(\sum_i \pi(b_i)\xi_i\Big) \,\Big|\, \sum_j \pi(c_j)\eta_j \Big\rangle = \Big\langle \sum_i \pi(b_i)\xi_i \,\Big|\, \pi_1(a^*) \sum_j \pi(c_j)\eta_j \Big\rangle.$$

Hence, $\pi_1(a)^\dagger = \pi_1(a^*)$.

Let now $a_1, a_2 \in \mathfrak{A}$ with $a_1 a_2$ well-defined. We have to prove that the product $\pi_1(a_1) \,\Box\, \pi_1(a_2)$ is well-defined and $\pi_1(a_1) \,\Box\, \pi_1(a_2) = \pi_1(a_1 a_2)$:

$$\Big\langle \pi_1(a_1 a_2)\Big(\sum \pi(b_i)\xi_i\Big) \,\Big|\, \sum \pi(c_j)\eta_j \Big\rangle = \sum_{i,j} \langle \pi(a_1 a_2) \,\Box\, \pi(b_i)\xi_i \,|\, \pi(c_j)\eta_j \rangle.$$

On the other hand,

$$\begin{aligned} &\Big\langle \pi_1(a_2)\Big(\sum \pi(b_i)\xi_i\Big) \,\Big|\, \pi_1(a_1)^\dagger \sum \pi(c_j)\eta_j \Big\rangle \\ &\qquad = \sum_{i,j} \langle \pi(a_2) \,\Box\, \pi(b_i)\xi_i \,|\, \pi(a_1^*) \,\Box\, \pi(c_j)\eta_j \rangle = \sum_{i,j} \langle \pi(a_2 b_i)\xi_i \,|\, \pi(a_1^* c_j)\eta_j \rangle. \end{aligned}$$

Now, since $c_j^*((a_1 a_2) b_i) = (c_j^*(a_1 a_2)) b_i$, we have

$$\begin{aligned} \sum_{i,j} \langle \pi(a_2 b_i)\xi_i \,|\, \pi(a_1^* c_j)\eta_j \rangle &= \sum_{i,j} \langle \pi(c_j^* a_1) \,\Box\, \pi(a_2 b_i)\xi_i \,|\, \eta_j \rangle \\ &= \sum_{i,j} \langle (\pi^\dagger(c_j) \,\Box\, \pi(a_1)) \,\Box\, \pi(a_2 b_i)\xi_i \,|\, \eta_j \rangle \\ &= \sum_{i,j} \langle \pi^\dagger(c_j) \,\Box\, (\pi(a_1) \,\Box\, \pi(a_2 b_i))\xi_i \,|\, \eta_j \rangle \\ &= \sum_{i,j} \langle (\pi(a_1) \,\Box\, \pi(a_2 b_i))\xi_i \,|\, \pi(c_j)\eta_j \rangle \\ &= \sum_{i,j} \langle ((\pi(a_1) \,\Box\, \pi(a_2)) \,\Box\, \pi(b_i)\xi_i \,|\, \pi(c_j)\eta_j \rangle \\ &= \Big\langle \Big(\sum \pi(a_1 a_2)\Big) \,\Box\, \pi(b_i)\xi_i \,\Big|\, \sum \pi(c_j)\eta_j \Big\rangle \\ &= \Big\langle \pi_1(a_1 a_2) \sum \pi(b_i)\xi_i \,\Big|\, \sum \pi(c_j)\eta_j \Big\rangle. \end{aligned}$$

This proves the statement.

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