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On the number of minimal pairs of compact convex sets that are not translates of one another

by

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Abstract. Let $[A,B]$ be the family of pairs of compact convex sets equivalent to (A,B). We prove that the cardinality of the set of minimal pairs in $[A,B]$ that are not translates of one another is either 1 or greater than $\aleph_0$.

Let $X=(X,\tau)$ be a topological vector space over the field $\mathbb{R}$. Let $\mathcal{K}(X)$ be the family of all nonempty compact convex subsets of X. For any $A,B\subset X$ the Minkowski sum is defined by $A+B=\{a+b \mid a\in A \,\text{and}\, b\in B\}$. For $(A,B),(C,D)\in\mathcal{K}^2(X)$, let $(A,B)\sim(C,D)$ if and only if $A+D=B+C$. Let $[A,B]$ be the equivalence class of (A,B) in $\mathcal{K}^2(X)/{\sim}$. For $(A,B),(C,D)\in\mathcal{K}^2(X)$ let $(A,B)\le(C,D)$ if and only if $(A,B)\sim(C,D)$, $A\subset C$ and $B\subset D$. Let $m[A,B]$ be the family of all elements of $[A,B]$ that are minimal with respect to the ordering $\le$. Let $A\vee B$ be the convex hull of $A\cup B$. For $A,B,C\in\mathcal{K}^2(X)$, we have the Pinsker formula $A\vee B+C=(A+C)\vee(B+C)$.

Minimal pairs of compact convex sets play an important role in quasidifferential calculus [5]–[7]. Minimal pairs were studied in numerous papers ([1]–[4], [8]–[14], [17], and others).

Let $(A,B)\in\mathcal{K}^2(X)$ and $n_{A,B}$ be the number of minimal pairs in $m[A,B]$ that are not translates of one another. If $X=\mathbb{R}^1$ or $\mathbb{R}^2$ then $n_{A,B}$ is always 1 ([8], [15]). In [13], there is an example of $A,B\in\mathcal{K}(\mathbb{R}^3)$ such that $n_{A,B}$ is the continuum.

In December 2000, Professor S. Rolewicz posed the problem whether $n_{A,B}$ can be finite and greater than 1. The following theorem implies a negative answer to this problem.

Theorem. *Let (A_1,B_1), (A_2,B_2) be two equivalent minimal pairs of compact convex sets such that (A_2,B_2) is not a translate of (A_1,B_1). Then there exists an uncountable family (A_λ,B_λ), $\lambda\in\Lambda$, of minimal pairs that are equivalent to (A_1,B_1) and no (A_λ,B_λ) is a translate of (A_μ,B_μ), $\lambda\ne\mu$.*

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Proof. Assume that $\{(A_n + x, B_n + x) \mid n \in \mathbb{N},\, x \in X\}$ is the family of all minimal pairs equivalent to (A_1, B_1) and (A_2, B_2). Let

$$k_3 = \min\{k \in \mathbb{N} \mid \exists \alpha \in (0, 1/2],\ \exists x \in X \text{ such that } (A_k + x, B_k + x) \leq (\alpha A_1 + (1-\alpha)A_2, \alpha B_1 + (1-\alpha)B_2)\}.$$

If $k_3 = 1$ then

$$\alpha A_1 + (1-\alpha)A_1 + x = A_1 + x \subset \alpha A_1 + (1-\alpha)A_2.$$

By the order law of cancellation [16]

$$(1-\alpha)A_1 + x \subset (1-\alpha)A_2 \quad \text{and so} \quad A_1 + \frac{x}{1-\alpha} \subset A_2.$$

In a similar way we prove that

$$B_1 + \frac{x}{1-\alpha} \subset B_2.$$

Since (A_2, B_2) is minimal,

$$A_1 + \frac{x}{1-\alpha} = A_2, \qquad B_1 + \frac{x}{1-\alpha} = B_2.$$

This contradicts the assumption of our theorem. Therefore, $k_3 \neq 1$. In a similar way we prove that $k_3 \neq 2$. Thus $k_3 > 2$. We can assume that $\alpha_1 \in (0, 1/2]$ and

$$(A_{k_3}, B_{k_3}) \leq (\alpha_1 A_1 + (1-\alpha_1)A_2, \alpha_1 B_1 + (1-\alpha_1)B_2).$$

Set $k_1 = 1$, $k_2 = 2$. Assume that $(A_{k_1}, B_{k_1}), \ldots, (A_{k_n}, B_{k_n})$ are minimal pairs such that $k_1 < \ldots < k_n$,

$$k_i = \min\{k \in \mathbb{N} \mid \exists \alpha \in (0, 1/2],\ \exists x \in X \text{ such that } (A_k + x, B_k + x) \leq (\alpha A_{k_{i-2}} + (1-\alpha)A_{k_{i-1}}, \alpha B_{k_{i-2}} + (1-\alpha)B_{k_{i-1}})\}$$

and

$$(A_{k_i}, B_{k_i}) \leq (\alpha_{i-2} A_{k_{i-2}} + (1-\alpha_{i-2})A_{k_{i-1}}, \alpha_{i-2} B_{k_{i-2}} + (1-\alpha_{i-2})B_{k_{i-1}})$$

for $i = 3, \ldots, n$. Let

$$k_{n+1} = \min\{k \in \mathbb{N} \mid \exists \alpha \in (0, 1/2],\ \exists x \in X \text{ such that } (A_k + x, B_k + x) \leq (\alpha A_{k_{n-1}} + (1-\alpha)A_{k_n}, \alpha B_{k_{n-1}} + (1-\alpha)B_{k_n})\}.$$

Of course, $k_{n+1} \neq k_{n-1}, k_n$. Define

$$\gamma_i^{n+1} = \alpha_i(1 - \alpha_{i+1}(\ldots(1 - \alpha_{n-2}(1-\alpha))\ldots)), \qquad i = 1, \ldots, n-2.$$

Notice that $\gamma_i^{n+1} \in (0, 1/2]$ and

$$(A_{k_{n+1}} + x, B_{k_{n+1}} + x) \leq (\gamma_i^{n+1} A_{k_i} + (1-\gamma_i^{n+1})A_{k_{i+1}}, \gamma_i^{n+1} B_{k_i} + (1-\gamma_i^{n+1})B_{k_{i+1}}).$$

Therefore,

$$k_{n+1} > k_n.$$

We can assume that $\alpha_{n-1} \in (0,1/2]$ and

$$(A_{k_{n+1}}, B_{k_{n+1}}) \le (\alpha_{n-1}A_{k_{n-1}} + (1-\alpha_{n-1})A_{k_n}, \alpha_{n-1}B_{k_{n-1}} + (1-\alpha_{n-1})B_{k_n}).$$

In this way we can choose infinite sequences $(\alpha_n)_n \subset (0,1/2]$ and $(k_n)_n$ such that $k_1 < k_2 < \dots$,

$$k_n = \min\{k \in \mathbb{N} \mid \exists\, \alpha \in (0,1/2],\ \exists x \in X \text{ such that } (A_k + x, B_k + x) \le (\alpha A_{k_{n-2}} + (1-\alpha)A_{k_{n-1}}, \alpha B_{k_{n-2}} + (1-\alpha)B_{k_{n-1}})\},$$

and

$$(A_{k_n}, B_{k_n}) \le (\alpha_{n-2}A_{k_{n-2}} + (1-\alpha_{n-2})A_{k_{n-1}}, \alpha_{n-2}B_{k_{n-2}} + (1-\alpha_{n-2})B_{k_{n-1}})$$

for all $n \ge 3$. Notice that $(A_{k_n} \vee A_{k_{n+1}}, B_{k_n} \vee B_{k_{n+1}})$ is equivalent to (A_1, B_1) for $n \in \mathbb{N}$ (see [12]). The sequences $(A_{k_n} \vee A_{k_{n+1}})_n$ and $(B_{k_n} \vee B_{k_{n+1}})_n$ are decreasing. Thus the pair (C, D) with

$$C = \bigcap_{n=1}^{\infty} (A_{k_n} \vee A_{k_{n+1}}), \qquad D = \bigcap_{n=1}^{\infty} (B_{k_n} \vee B_{k_{n+1}})$$

is equivalent to (A_1, B_1) (see [12]). There exists a minimal pair $(A, B) \le (C, D)$ (see [11]). Let

$$\gamma_i^{i+2} = \alpha_i, \quad i \in \mathbb{N}, \qquad \gamma_i^n = \alpha_i(1-\gamma_{i+1}^n), \quad i, n \in \mathbb{N},\ n \ge i+3.$$

Then

$$(A_{k_n}, B_{k_n}) \le (\gamma_i^n A_{k_i} + (1-\gamma_i^n)A_{k_{i+1}}, \gamma_i^n B_{k_i} + (1-\gamma_i^n)B_{k_{i+1}}).$$

Notice that

$$\begin{aligned} \gamma_{n-2}^{n+1} - \gamma_{n-2}^n &= -\alpha_{n-2} \cdot \alpha_{n-1}, && n \ge 3, \\ \gamma_i^{n+1} - \gamma_i^n &= -\alpha_i(\gamma_{i+1}^{n+1} - \gamma_{i+1}^n), && n \ge i+3. \end{aligned}$$

Then

$$\gamma_i^{n+1} - \gamma_i^n = (-1)^{n-1-i}\alpha_i \dots \alpha_{n-1}.$$

Since $\alpha_n \in (0,1/2]$ for all n, the sequence $(\gamma_i^n)_n$ converges to some $\gamma_i \in (0,1/2]$, $(\gamma_i^{i+2n})_n$ is decreasing and $(\gamma_i^{i+2n+1})_n$ is increasing. Therefore,

$$\begin{aligned} A_{k_n} \vee A_{k_{n+1}} &\subset (\gamma_i^n A_{k_i} + (1-\gamma_i^n)A_{k_{i+1}}) \vee (\gamma_i^{n+1} A_{k_i} + (1-\gamma_i^{n+1})A_{k_{i+1}}) \\ &= (\gamma' A_{k_i} + (1-\gamma'')A_{k_{i+1}} + (\gamma''-\gamma')A_{k_i}) \\ &\quad \vee (\gamma' A_{k_i} + (1-\gamma'')A_{k_{i+1}} + (\gamma''-\gamma')A_{k_{i+1}}) \\ &= \gamma' A_{k_i} + (1-\gamma'')A_{k_{i+1}} + (\gamma''-\gamma')(A_{k_i} \vee A_{k_{i+1}}) \end{aligned}$$

for all $i, n \in \mathbb{N}$, where $n \ge i+2$, $\gamma' = \min(\gamma_i^n, \gamma_i^{n+1})$, $\gamma'' = \max(\gamma_i^n, \gamma_i^{n+1})$. In the last equality we have applied the Pinsker formula (see [14]). We can assume that $0 \in A_{k_i} \cap A_{k_{i+1}}$. Then

$$A_{k_n} \vee A_{k_{n+1}} \subset \gamma_i A_{k_i} + (1-\gamma_i)A_{k_{i+1}} + |\gamma_i^{n+1} - \gamma_i^n|(A_{k_i} \vee A_{k_{i+1}}).$$

Hence

$$\begin{aligned}C &\subset \bigcap_{n=i+2}^{\infty} (\gamma_i A_{k_i} + (1-\gamma_i)A_{k_{i+1}} + |\gamma_i^{n+1} - \gamma_i^n|(A_{k_i} \vee A_{k_{i+1}})) \\ &= \gamma_i A_{k_i} + (1-\gamma_i)A_{k_{i+1}} + \bigcap_{n=i+2}^{\infty} |\gamma_i^{n+1} - \gamma_i^n|(A_{k_i} \vee A_{k_{i+1}}) \\ &= \gamma_i A_{k_i} + (1-\gamma_i)A_{k_{i+1}} \quad \text{(see [12, Lemma 3.10]).}\end{aligned}$$

In this way we prove that

$$(A, B) \le (\gamma_i A_{k_i} + (1-\gamma_i)A_{k_{i+1}}, \gamma_i B_{k_i} + (1-\gamma_i)B_{k_{i+1}}).$$

We know that $A = A_m + x$, $B = B_m + x$ for some $m \in \mathbb{N}$, $x \in X$. According to the definition of k_{i+2} we have $m \ge k_{i+2}$ for all $i \in \mathbb{N}$, which leads to a contradiction with our assumption. ■

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