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An Atkinson-type theorem for B-Fredholm operators

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Abstract. Let X be a Banach space and let T be a bounded linear operator acting on X. Atkinson's well known theorem says that T is a Fredholm operator if and only if its projection in the algebra $L(X)/F_0(X)$ is invertible, where $F_0(X)$ is the ideal of finite rank operators in the algebra $L(X)$ of bounded linear operators acting on X. In the main result of this paper we establish an Atkinson-type theorem for B-Fredholm operators. More precisely we prove that T is a B-Fredholm operator if and only if its projection in the algebra $L(X)/F_0(X)$ is Drazin invertible. We also show that the set of Drazin invertible elements in an algebra A with a unit is a regularity in the sense defined by Kordula and Müller [8].

1. Introduction. Let A be an algebra with a unit e. An element x of A is called *regular* if there is an element b of A such that $xbx = x$. In this case b is called a *generalized inverse* of x. Following [10] we say that an element x of A is *Drazin invertible* of degree k if there is an element b of A such that

$$x^k bx = x^k, \qquad bxb = b, \qquad xb = bx. \tag{1}$$

Recall that the concept of Drazin invertibility was originally considered by M. P. Drazin in [5] where elements satisfying (1) are called pseudo-invertible elements.

In [10] an element x of A satisfying (1) for $k = 1$ is called *group invertible*. It follows from [11, Theorem 3.3 and Proposition 3.9] that an element of A is group invertible if and only if it has a commuting generalized inverse. In [7] and [11] group invertible elements are called respectively simply polar elements and "generalized invertible" elements. It follows also from [11, Theorem 3.3 and Proposition 3.9] that an element x of A is group invertible if and only if there is a generalized inverse b of x such that $e-xb-bx$ is invertible in the algebra A.

In the first part of this paper we show that the set of Drazin invertible elements in the algebra A is a regularity in the sense of Kordula and Müller (Definition 2.2).

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So if A is a Banach algebra with unit, the spectrum associated to this regularity satisfies the spectral mapping theorem. We also show that if a is a Drazin invertible element in a Banach algebra A and if b is small in norm and is an invertible element of A commuting with a, then $a+b$ is invertible. Moreover we show that the product of two commuting Drazin invertible elements is also Drazin invertible.

In the second part of this paper we consider a Banach space X and the Banach algebra $L(X)$ of bounded linear operators acting on X. In Theorem 3.4 we show that an operator $T \in L(X)$ is a B-Fredholm operator if and only if its projection in the algebra $L(X)/F_0(X)$ is Drazin invertible, where $F_0(X)$ is the ideal of finite rank operators in the algebra $L(X)$. As an application we show that the set of B-Fredholm operators is stable under finite rank perturbations, and we prove that the product of two commuting B-Fredholm operators is a B-Fredholm operator. At the end of this paper, we give two open questions.

2. On Drazin invertibility. Let X be a vector space and let T be a linear operator acting on X.

Definition 2.1. For $n \in \mathbb{N}$, let $c_n(T) = \dim R(T^n)/R(T^{n+1})$ and $c'_n(T) = \dim N(T^{n+1})/N(T^n)$. Then the *descent* of T is defined by $\delta(T) = \inf\{n : c_n(T) = 0\} = \inf\{n : R(T^n) = R(T^{n+1})\}$ and the *ascent* $a(T)$ of T is defined by $a(T) = \inf\{n : c'_n(T) = 0\} = \inf\{n : N(T^n) = N(T^{n+1})\}$.

For an element x in an algebra A with unit, let L_x and R_x denote the left and right multiplication operators by x. Following [11, p. 835] we set $a_l(x) = a(L_x)$, $\delta_l(x) = \delta(L_x)$, $a_r(x) = a(R_x)$, $\delta_r(x) = \delta(R_x)$. From [11, Proposition 3.1], for each integer $n \geq 0$ we know that

$$a_l(x) = \delta_l(x) = n \;\Leftrightarrow\; a_r(x) = \delta_r(x) = n.$$

Moreover we know from [12, Proposition 3.2] that if $\delta_l(x)$ and $\delta_r(x)$ are both finite then $a_l(x) = \delta_l(x) = a_r(x) = \delta_r(x)$.

In [8], V. Kordula and V. Müller defined the concept of regularity as follows:

Definition 2.2. A non-empty subset $\mathbf{R} \subset A$ is called a *regularity* if it satisfies the following conditions:

(i) If $a \in A$ and $n \geq 1$ is an integer then $a \in \mathbf{R}$ if and only if $a^n \in \mathbf{R}$.

(ii) If $a, b, c, d \in A$ are mutually commuting elements satisfying $ac + bd = e$, then $ab \in \mathbf{R}$ if and only if $a, b \in \mathbf{R}$.

A regularity $\mathbf{R}$ defines in a natural way a spectrum by $\sigma_{\mathbf{R}}(a) = \{\lambda \in \mathbb{C} : a - \lambda I \notin \mathbf{R}\}$ for every $a \in A$. Moreover in the case of a Banach algebra A, the spectrum $\sigma_{\mathbf{R}}$ satisfies the spectral mapping theorem.

THEOREM 2.3. *Let A be an algebra with unit. Then the set $\mathbf{DR}(A)$ of Drazin invertible elements in the algebra A is a regularity.*

Proof. (i) Since every invertible element in A is Drazin invertible, $\mathbf{DR}(A)$ is a non-empty set. Let $a \in A$ and $n \geq 1$ an integer. If a is Drazin invertible, then from [5, Theorem 2], a^n is Drazin invertible. Conversely suppose that a^n is Drazin invertible. Then from [10, Lemma 2] there is an integer $k \geq 1$ such that a^{nk} is group invertible. Again from [10, Lemma 2], a is Drazin invertible.

(ii) Let a, b, c, d be mutually commuting elements of A such that $ac + bd = e$. As proved in [5, Theorem 4], an element a of A is Drazin invertible if and only there are elements c, d of A and positive integers p, q such that $a^p = a^{p+1}c$ and $a^q = da^{q+1}$. Hence a is Drazin invertible if and only if $\delta_l(a)$ and $\delta_r(a)$ are both finite. Since $ac + bd = e$, we have $L_aL_c + L_bL_d = I$ and $R_aR_c + R_bR_d = I$. From [9, Lemma 4] it follows that $\delta_l(ab)$ is finite if and only if $\delta_l(a)$ and $\delta_l(b)$ are finite, and $\delta_r(ab)$ is finite if and only if $\delta_r(a)$ and $\delta_r(b)$ are finite. Hence ab is Drazin invertible if and only if a and b are Drazin invertible.

If A is a Banach algebra with unit e and if $x \in A$, we define the *Drazin spectrum* of x by $\sigma_{\mathbf{DR}}(x) = \{\lambda \in A : x - \lambda e \notin \mathbf{DR}(A)\}$. Using the properties of regularities [8], we immediately obtain the following corollary:

COROLLARY 2.4. *Let A be a Banach algebra with unit, let $x \in A$ and let f be an analytic function in a neighborhood of the usual spectrum $\sigma(x)$ of x which is non-constant on any connected component of $\sigma(x)$. Then $f(\sigma_{\mathbf{DR}}(x)) = \sigma_{\mathbf{DR}}(f(x))$.*

PROPOSITION 2.5. *Let A be a Banach algebra with unit, let $a \in A$. Suppose that $b \in A$ is invertible and commutes with a. If a is Drazin invertible, and b is sufficiently small in norm, then $a + b$ is invertible.*

Proof. Suppose that a is Drazin invertible. Then the bounded linear operator L_a acting on the Banach algebra A is also Drazin invertible. Hence L_a has a finite ascent and descent. So L_a is an operator of topological uniform descent in the sense of Grabiner [6, Definition 2.5]. Using Grabiner's punctured neighborhood theorem [2, Theorem 4.5] it follows that if T is an invertible bounded linear operator commuting with L_a and having small norm, then $c_0(T + L_a) = c_p(L_a)$, $c'_0(T + L_a) = c'_p(L_a)$, for p large enough. Since a is Drazin invertible, for $n \geq \delta_l(a)$ we have $c_n(L_a) = c'_n(L_a) = 0$. So $c_0(T + L_a) = c'_0(T + L_a) = 0$ for $n \geq \delta_l(a)$ and this shows that $T + L_a$ is an invertible operator. Now if b is an invertible element of A commuting with a and having a small norm, and if we set $T = L_b$, then T is invertible and its norm is $\|T\| = \|L_b\| = \|b\|$. By the preceding argument we see that $L_b + L_a = L_{a+b}$ is invertible. So $a + b$ is invertible.

PROPOSITION 2.6. *Let a, b be two commuting Drazin invertible elements of an algebra A with unit. Then ab is Drazin invertible.*

Proof. It follows from [10, Lemma 2] that there is an integer n such that a^n and b^n are group invertible. So there are x and y such that $xa^nx = x$, $a^nxa^n = a^n$, $xa^n = a^nx$ and $yb^ny = y$, $b^nyb^n = b^n$, $yb^n = b^ny$. From [5, Theorem 1], we know that a^n, b^n, x, y are commuting elements. So

$$xy(ab)^nxy = xy, \quad (ab)^nxy(ab)^n = (ab)^n, \quad (ab)^nxy = xy(ab)^n.$$

Hence $(ab)^n$ is group invertible and so ab is Drazin invertible.

3. An Atkinson-type theorem for B-Fredholm operators. In this part we consider the Banach algebra $L(X)$ of bounded linear operators acting on a Banach space X. For $T \in L(X)$, we denote by $N(T)$ the null space of T, by $\alpha(T)$ the nullity of T, by $R(T)$ the range of T and by $\beta(T)$ its defect. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called a *Fredholm operator* and the *index* of T is defined by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. In this case it is well known that the range $R(T)$ of T is closed in X.

For each integer n, define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_0 = T$). If for some integer n the space $R(T^n)$ is closed and T_n is a Fredholm operator, then T is called a *B-Fredholm operator* [2, Definition 2.2]. In this case from [1, Proposition 2.1], T_m is a Fredholm operator and $\operatorname{ind}(T_m) = \operatorname{ind}(T_n)$ for each $m \geq n$. This remark leads to the following definition:

DEFINITION 3.1. Let $T \in L(X)$ be a B-Fredholm operator and let n be any integer such that T_n is a Fredholm operator. Then the *index* $\operatorname{ind}(T)$ of T is defined as the index of the Fredholm operator T_n.

In particular if T is a Fredholm operator we get the usual definition of the index.

Let $\mathrm{BF}(X)$ be the class of all B-Fredholm operators. In [1] the first author has studied this class of operators and he has proved [1, Theorem 2.1] that $T \in L(X)$ is a B-Fredholm operator if and only if $T = Q \oplus F$, where Q is a nilpotent operator and F a Fredholm operator. Let us recall that an operator $T \in L(X)$ *has a generalized inverse* if there is an operator $S \in L(X)$ such that $TST = T$. In this case T is also called a *regular operator* and S is called a generalized inverse of T. It is well known that T has a generalized inverse if and only if $R(T)$ and $N(T)$ are closed and complemented subspaces of X. In [4], S. R. Caradus has defined the following class of operators:

DEFINITION 3.2. $T \in L(X)$ is called a *generalized Fredholm operator* if T is regular and there is a generalized inverse S of T such that $I - ST - TS$ is a Fredholm operator.

Let $\Phi_{\mathrm{g}}(X)$ be the class of all generalized Fredholm operators. In [11], [12], C. Schmoeger has studied this class of operators and he has proved [13, Theorem 1.1] that $T \in L(X)$ is a generalized Fredholm operator if and only if $T = Q \oplus F$, where Q is a finite rank nilpotent operator and F is a Fredholm operator. Hence a generalized Fredholm operator is a B-Fredholm operator, but the converse is not true, for example a nilpotent operator with a non-closed range is a B-Fredholm operator but not a generalized Fredholm operator, since a non-closed range operator is not regular. Moreover the class $\mathrm{BF}(X)$ of B-Fredholm operators satisfies the spectral mapping theorem while the class $\Phi_{\mathrm{g}}(X)$ does not.

Let $A = L(X)/F_0(X)$ where $F_0(X)$ is the ideal of finite rank operators in $L(X)$ and let $\pi : L(X) \to A$ be the canonical projection. Atkinson's well known theorem [7, Theorem 6.4.3] says that $T \in L(X)$ is a Fredholm operator if and only if its projection $\pi(T)$ in the algebra A is invertible. In the following result we establish an Atkinson-type theorem for B-Fredholm operators. More precisely, in a first step we prove the following important relation between B-Fredholm operators and generalized Fredholm operators in the sense of Caradus:

Proposition 3.3. *Let $T \in L(X)$. Then T is a B-Fredholm operator if and only if there exists a positive integer $p \in \mathbb{N}$ such that T^p is a generalized Fredholm operator.*

Proof. If T is a generalized Fredholm operator, then T is a B-Fredholm operator. Conversely if T is a B-Fredholm operator, then from [1, Theorem 2.1], $T = Q \oplus F$, where Q is a nilpotent operator and F a Fredholm operator. Let n be an integer such that $Q^n = 0$. Then $T^n = Q^n \oplus F^n = 0 \oplus F^n$. Since F^n is a Fredholm operator, from [13, Theorem 1.1] we see that T^n is a generalized Fredholm operator.

Theorem 3.4. *Let $T \in L(X)$. Then T is a B-Fredholm operator if and only if $\pi(T)$ is Drazin invertible in the algebra $L(X)/F_0(X)$.*

Proof. From [11, Theorem 3.3] it follows that T is a generalized Fredholm operator if and only $\pi(T)$ is group invertible in the algebra $L(X)/F_0(X)$. Using the preceding proposition we see that T is a B-Fredholm operator if and only if there exists $p \in \mathbb{N}$ such that $\pi(T^p)$ is group invertible. Since $\pi(T^p) = \pi(T)^p$, using [10, Lemma 2] we see that T is a B-Fredholm operator if and only if $\pi(T)$ is Drazin invertible in $L(X)/F_0(X)$.

Corollary 3.5. (i) *Let T_1, T_2 be B-Fredholm operators such that T_1T_2 and T_2T_1 are finite rank operators. Then $T_1 + T_2$ is a B-Fredholm operator.*

(ii) *Let T_1, T_2 be commuting B-Fredholm operators. Then T_1T_2 is a B-Fredholm operator.*

(iii) *Let T be a B-Fredholm operator and let F be a finite rank operator. Then $T+F$ is a B-Fredholm operator.*

Proof. Let $A = L(X)/F_0(X)$ and let $\pi : L(X) \to L(X)/F_0(X)$ be the canonical projection. Then π is an algebra homomorphism and:

(i) We have $\pi(T_1)\pi(T_2) = \pi(T_2)\pi(T_1) = 0$. From [5, Corollary 1] it follows that $\pi(T_1+T_2) = \pi(T_1)+\pi(T_2)$ is Drazin invertible in A. So T_1+T_2 is a B-Fredholm operator.

(ii) We have $\pi(T_1T_2) = \pi(T_1)\pi(T_2) = \pi(T_2)\pi(T_1)$. From Proposition 2.6 it follows that $\pi(T_1T_2)$ is Drazin invertible. Hence T_1T_2 is a B-Fredholm operator.

(iii) If F is a finite rank operator and T is a B-Fredholm operator then $\pi(T+F) = \pi(T)$. So $T+F$ is a B-Fredholm operator.

Remark. The class of B-Fredholm operators is not stable under compact perturbations, that is, $\mathrm{BF}(X) + K(X) \not\subseteq \mathrm{BF}(X)$ where $K(X)$ is the closed ideal of all compact operators in $L(X)$. For example let $(\lambda_n)_n$ be a sequence in $\mathbb{C}$ such that $\lambda_n \neq 0$ for all n and $\lambda_n \to 0$ as $n \to \infty$, and consider the operator T defined on the Hilbert space $l^2(\mathbb{N})$ by

$$T(\xi_1, \xi_2, \xi_3, \ldots) = (\lambda_1\xi_1, \lambda_2\xi_2, \lambda_3\xi_3, \ldots).$$

Then

$$T^n(\xi_1, \xi_2, \xi_3, \ldots) = ((\lambda_1)^n\xi_1, (\lambda_2)^n\xi_2, (\lambda_3)^n\xi_3, \ldots).$$

Since $(\lambda_m)^n \neq 0$ for all $m \geq 0$ and $(\lambda_m)^n \to 0$ as $m \to \infty$ for all $n \geq 0$ we see that $T^n \in K(X)$ and T^n is not a finite rank operator for all $n \geq 1$. Hence $R(T^n)$ is not closed for all $n \geq 1$. Thus $T \notin \mathrm{BF}(X)$. Since $0 \in \mathrm{BF}(X)$ it follows that $\mathrm{BF}(X) + K(X) \not\subseteq \mathrm{BF}(X)$.

As a consequence of this remark, if $\mathbf{C}(X) = L(X)/K(X)$ is the Calkin algebra and if $\Pi : L(X) \to \mathbf{C}(X)$ is the canonical projection, then $\Pi(T) = 0$ is Drazin invertible in $\mathbf{C}(X)$ but T is not a B-Fredholm operator.

Open Questions. We finish this paper by the following open questions, suggested by a comparison between Fredholm operators and B-Fredholm operators:

1. It is well known that if S, T are Fredholm operators, then ST is a Fredholm operator and $\mathrm{ind}(ST) = \mathrm{ind}(S)+\mathrm{ind}(T)$, where ind is the index. Now if S, T are commuting B-Fredholm operators, we know from Corollary 3.5 that ST is a B-Fredholm operator. Is it still true that $\mathrm{ind}(ST) = \mathrm{ind}(S)+\mathrm{ind}(T)$?

2. Let T be a Fredholm operator and K a compact operator. It is known that $T+K$ is a Fredholm operator and $\mathrm{ind}(T+K) = \mathrm{ind}(T)$. Now if T is a B-Fredholm operator and F a finite rank operator, then $T+F$ is a B-Fredholm operator. Do we have $\mathrm{ind}(T+F) = \mathrm{ind}(T)$?

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