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Metric spaces with the small ball property

by

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Abstract. A metric space (M,d) is said to have the *small ball property* (sbp) if for every $\varepsilon_0 > 0$ it is possible to write M as the union of a sequence $(B(x_n, r_n))$ of closed balls such that the r_n are smaller than ε_0 and $\lim r_n = 0$. We study permanence properties and examples of sbp. The main results of this paper are the following: 1. Bounded convex closed sets in Banach spaces have sbp only if they are compact. 2. Precisely the finite-dimensional Banach spaces have sbp. (More generally: a complete metric group has sbp iff it is separable and locally compact.) 3. Let B be a boundary in the bidual of an infinite-dimensional Banach space. Then B does not have sbp. In particular the set of extreme points in the unit ball of an infinite-dimensional reflexive Banach space fails to have sbp.

1. Introduction. There are various notions to express the fact that a certain class of objects is "small"; they refer to different structures of the underlying set. "Small" can mean that the measure is zero, that the set under consideration is of first category, that the Hausdorff dimension is zero or something else. For an account of some classical notions of smallness we refer the reader to [12]; more recent results are discused in Chapter 6 of [1]. Here we study another notion of this kind. The definition can be found in the abstract, it applies to arbitrary metric spaces.

We are not aware of any systematic study of the small ball property (sbp) in the literature. Implicitly, however, this notion occurs occasionally; for an example we refer the reader to the proof of Lemma 11, Chapter 9, in Diestel's book [2].

We start our investigations in Section 2 where we collect some general facts concerning sbp in metric spaces. In Section 3 we deal with spaces with a special structure, in particular with normed spaces $(X, \|\cdot\|)$. One of our main results is that bounded convex closed sets in Banach spaces only very rarely have sbp (only if they are compact). In particular it follows that only finite-dimensional Banach spaces have this property. In fact we show more:

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if X admits a Banach space norm $|\cdot|$ such that the associated topology is finer than the original topology then $(X, \|\cdot\|)$ has sbp iff the unit ball with respect to $|\cdot|$ is $\|\cdot\|$-precompact. (Recall that a metric space (M,d) is called *precompact* if for every positive ε there are finitely many points $x_1, \ldots, x_n$ in M such that the closed balls $B(x_i, \varepsilon)$ with centre x_i and radius ε cover M.)

In Section 4 we investigate *boundaries*. (A boundary for a Banach space X is a subset B of the dual unit ball such that for every $x \in X$ there is an $x' \in B$ such that $x'(x) = \|x\|$. The most important example of a boundary is the set of extreme functionals.)

Boundaries in infinite-dimensional reflexive Banach spaces are, in a sense, always "large". In [10] it has been shown that the collection of extreme points in the unit ball of such spaces is always uncountable, and this has been generalized in a number of other papers (see [3]–[7], [9], [13]). Here we provide a further theorem in this direction: boundaries in infinite-dimensional bidual spaces never have sbp, a result which applies in particular to the set of extreme points in the unit ball of a reflexive space.

It is easy to see that σ-precompact metric spaces have sbp, and by the results of Section 3 the converse also holds for closed convex sets in Banach spaces. We show in Section 5, however, that the converse is not generally true: there is a complete metric space with the small ball property which fails to be σ-precompact. This section also contains some *other counterexamples*: the concepts of category and sbp are independent, and products of spaces with the small ball property might fail to have it.

2. Metric spaces with the small ball property: basic results. First we note that σ-precompact metric spaces, i.e., spaces which can be written as a countable union of precompact subsets $M_1, M_2, \ldots$, have the small ball property: given ε_0, simply cover M_n by finitely many balls with radius ε_0/n and arrange these countably many balls as a sequence. Therefore any subset of any finite-dimensional normed space has the small ball property; other examples are the ranges of compact operators between Banach spaces.

Sometimes it will be convenient to work with suitable reformulations of sbp: for a metric space (M,d) it is equivalent to each of the following properties:

• For every $\varepsilon_0 > 0$ it is possible to write M as the union of a sequence of subsets whose diameters tend to zero and are bounded by ε_0.

• For every sequence (ε_n) of positive numbers there are finite subsets Δ_n of M such that

$$M \subset \bigcup_n \bigcup_{x \in \Delta_n} B(x, \varepsilon_n).$$

The next proposition provides some permanence results:

PROPOSITION 2.1. *The small ball property passes to*

(i) *subsets*;
(ii) *countable unions*;
(iii) *images under uniformly continuous maps.*

Proof. (i) This follows immediately from the first of the preceding two observations.

(ii) Let $M_1, M_2, \ldots$ be subsets of M and suppose that each M_n has the small ball property. Fix $\varepsilon_0 > 0$. We cover M_n by a sequence $(B(x_m^n, r_m^n)_m)$ of balls, where $r_m^n \le \varepsilon_0/n$ and $r_m^n \to 0$ as $m \to \infty$ for each n. Then each arrangement of the countable family

$$\{B(x_m^n, r_m^n) \mid n, m \in \mathbb{N}\}$$

as a sequence will cover $\bigcup M_n$ in the desired way.

(iii) can easily be established. ∎

Whereas sbp passes to subsets it does not pass to closures. For a counterexample consider any separable infinite-dimensional Banach space and note that countable sets have sbp but infinite-dimensional Banach spaces fail to have it (see Corollary 3.9 below). Even the closure of a *convex* set with the small ball property might not have it: let (x_n) be dense in the infinite-dimensional Banach space X and denote by K the convex hull of this sequence. Then K is σ-compact and thus a set with sbp, its closure, however, is the whole space, a set without sbp.

3. The case of normed spaces.

Let $(X, \|\cdot\|)$ be a normed space and M a subset of X. It is then obvious that together with M also any multiple λM and any translate $M + x_0$ have the small ball property. In what follows we will be interested in situations where the whole space belongs to this class.

PROPOSITION 3.1. *The following conditions are equivalent*:

(i) *X has the small ball property.*
(ii) *The unit ball of X has the small ball property.*
(iii) *X can be written as the union of a sequence of balls whose radii tend to zero.*
(iv) *Every open subset O of X has the "open" small ball property*: *for $\varepsilon_0 > 0$, O can be written as the union of open balls $B^o(x_n, r_n)$ such that $r_n \le \varepsilon_0$ and $r_n \to 0$.*
(v) *There is a basis of the topology which is a sequence $(B^o(x_n, r_n))$ of open balls such that* $\lim r_n = 0$.

Proof. It is easy to show that (i)⇔(ii)⇔(iii) as well as (iv)⇒(i)⇒(v)⇒(iv); the details are left to the reader. ■

Now we are going to show that convex sets in Banach spaces rarely have the small ball property. As we will see, this property is a consequence of the fact that convex sets have locally the same structure as the whole set. We start with the following

DEFINITION 3.2. Let K be a convex set in a linear space X. We will say that K has the *intersection property* if the following holds: whenever one defines sets $K_n := x_n + \varepsilon_n K$ in such a way that $\varepsilon_n \to 0$ and $K_1 \supset K_2 \supset K_3 \supset \ldots$, then $\bigcap K_n \neq \emptyset$.

PROPOSITION 3.3. *Suppose that there exists a norm* $|\cdot|$ *on* X *such that* $(X, |\cdot|)$ *is a Banach space and* $K \subset X$ *is closed and bounded. Then* K *has the intersection property.*

Proof. This is an immediate consequence of the Hausdorff intersection theorem. ■

THEOREM 3.4. *Let* X *be a normed space and* $K \subset X$ *a convex bounded set with the intersection property. Then* K *has the small ball property iff* K *is precompact.*

Proof. Only one implication needs a proof. We assume that K is not precompact and we will show that K fails to have the small ball property.

Since K is not precompact we may fix a positive δ with the following property: whenever $x_1, \ldots, x_n$ are finitely many points in X, there exists an $x_0 \in K$ such that $\|x_0 - x_i\| > 2\delta$ for all i. With $d :=$ "the diameter of K" and $a := \delta/d$ we consider the set

$$K' := \{(1-a)x_0 + ax \mid x \in K\}.$$

Then K' has the following properties:

- $K' \subset K$ (by convexity).
- $K' \subset B(x_0, \delta)$ (by construction); in particular, all $B(x_i, \delta) \cap K'$ are empty.
- K' is of the form $x_0' + aK$.

Now let $\Delta_1, \Delta_2, \ldots$ be finite subsets of X and $\varepsilon_1, \varepsilon_2, \ldots$ positive numbers such that $\varepsilon_k \leq \delta a^{k-1}$.

As noted above we may find a subset K_1 of K such that K_1 is of the form $x_0' + aK$ and

$$K_1 \cap \bigcup_{x \in \Delta_1} B(x, \varepsilon_1) = \emptyset.$$

Next we apply the above construction with K replaced by K_1: up to the factor a everything is as before, we get a set K_2, a subset of K_1, which is of the form $x_0'' + a^2 K$ and which does not meet $\bigcup_{x \in \Delta_2} B(x, \varepsilon_2)$.

It should be clear how to continue. It remains to choose an x in the intersection of the K_i: it will not be an element of $\bigcup_k \bigcup_{x\in\Delta_k} B(x,\varepsilon_k)$. Hence K fails to have the small ball property. ∎

Corollary 3.5. *Let X be a Banach space and K a closed convex subset of X. Then K has the small ball property iff K is σ-compact.*

Proof. Let K have sbp. By the preceding theorem the sets $K_n :=$ "the intersection of K with the ball with radius n" are compact. Since $K = \bigcup K_n$, K is precompact. The converse is true by the results of Section 2. ∎

As a further corollary we obtain

Theorem 3.6. *Let $(X, \|\cdot\|)$ be a normed space. Suppose that there exists a norm $|\cdot|$ such that $(X, |\cdot|)$ is complete, and $\|\cdot\| \le |\cdot|$. The unit ball of X with respect to $|\cdot|$ will be denoted by K. Then the following are equivalent*:

(i) $(X, \|\cdot\|)$ *has the small ball property.*
(ii) K, *as a subset of* $(X, \|\cdot\|)$, *is precompact.*

Remarks. 1. Let Y and Z be Banach spaces and $T : Y \to Z$ a one-to-one continuous operator such that $\|T\| \le 1$. Then $X := T(Y)$, provided with the Z-norm, satisfies the condition of the theorem: simply put $|Ty| := \|y\|_Y$. It follows from the theorem that X has the small ball property iff T is a compact operator.

As a special case consider the l^p-spaces $(1 \le p < \infty)$ as subspaces of c_0 (here T means the embedding operator). The norm $|\cdot|$ on l^p can be chosen to be the ordinary l^p-norm, and since the associated unit balls are not precompact in c_0 it follows that the l^p-spaces, considered as subspaces of c_0, fail to have the small ball property.

2. In particular it follows that—under the assumptions of the theorem—X has sbp iff X is σ-precompact. It would be interesting to know whether this is true for arbitrary normed spaces.

Proof of Theorem 3.6. First suppose that K is precompact. Then $X = \bigcup nK$ is σ-precompact, and thus it has the small ball property.

If, conversely, $(X, \|\cdot\|)$ has the small ball property, then also the subset K will have it. By Proposition 3.3, K has the intersection property, and it follows from Theorem 3.4 that K is $\|\cdot\|$-precompact. ∎

Corollary 3.7. *No infinite-dimensional Banach space has the small ball property.*

We close this section with a similar result for spaces without a linear structure which admit "sufficiently many" isometries. It obviously generalizes the preceding corollary.

Proposition 3.8. *Let (M, d) be a complete metric space such that for arbitrary $x, y \in M$ there is an isometry T on M with $Tx = y$.*

(i) *If* (M,d) *has the small ball property*, *then* M *is locally compact.*

(ii) *If* M *is separable and has a compact ball*, *then* M *is* σ*-precompact. Consequently*, M *has the small ball property iff* M *is* σ*-precompact.*

Proof. (i) The idea is to modify the construction from the proof of Theorem 3.4. Suppose that there are no compact balls; we will show that (M,d) fails to have sbp. The assumption implies that the balls are "uniformly noncompact": for every $r>0$ there is an $\varepsilon=\varepsilon(r)>0$ such that no ball with radius r has a finite 2ε-net.

Now let $(B(x_n,r_n))_n$ be a sequence of balls such that $r_n\le\varepsilon(1)$ and $r_n\to 0$; we will show that there is an x which is contained in no $B(x_n,r_n)$.

Denote by B_1 the ball with radius 1 and centre x_1. We choose n_1 such that $r_n\le\varepsilon(\varepsilon(1))$ for $n\ge n_1$. Since B_1 has no finite $2\varepsilon(1)$-net we find y such that the ball B_2 with radius $\varepsilon(1)$ and centre y is contained in B_1 and does not meet $\bigcup_{n=1}^{n_1}B(x_n,r_n)$. Next we choose n_2 in such a way that $r_n\le\varepsilon(\varepsilon(2))$ for $n\ge n_2$. We obtain B_3, a ball with radius $\varepsilon(2)$, such that $B_3\subset B_2$ as well as $B_3\cap\bigcup_{n=1}^{n_2}B(x_n,r_n)=\emptyset$.

It should be clear how to proceed; the unique x in the intersection of the B_n is contained in no $B(x_n,r_n)$.

(ii) Let $x_1,x_2,\ldots$ be dense in M and $B(x,r)$ be a compact ball. The assumption implies that all $B(x_n,r)$ are compact as well, and it remains to note that $M=\bigcup_n B(x_n,r)$. ∎

COROLLARY 3.9. *A complete metric group has sbp iff it is separable and locally compact.*

4. Boundaries in biduals never have the small ball property. In this section we investigate boundaries in the unit ball of biduals; our main result will be Theorem 4.6. (As noted by V. P. Fonf, the assertion also follows from a characterization theorem for polyhedral Banach spaces in [4]; the connection will be discussed below.)

The structure of this section is as follows. We start with the construction of "large" subsets of the unit ball of a Banach space where certain prescribed functionals do *not* attain their norm. This is the content of Lemma 4.1; it will be crucial in what follows. A first and rather easy application is Theorem 4.2: boundaries in infinite-dimensional reflexive spaces fail to have sbp.

One has to argue more subtly in order to show that the same technique can be modified to cover the more general case of boundaries in bidual spaces. This generalization is prepared in Lemmas 4.3–4.5; the main result—which contains Theorem 4.2 as a special case—can then be found in Theorem 4.6.

LEMMA 4.1. *Let* X *be a Banach space and* $Y\subset X$ *an infinite-dimensional closed affine subspace* (*this means that* Y *is of the form* x_0+Z *with an* *infinite-dimensional closed linear subspace Z). Suppose that there are given a finite subset Δ of $B_{X'}$ (= the closed unit ball of the dual space X') and two numbers a, ε with $0 < a, \varepsilon < 1$. Then, if $d(0,Y)$ ($:= \inf_{y\in Y} \|y\|$) is smaller than a, there exists $W \subset Y$ with the following properties:*

(i) *W is an infinite-dimensional closed affine subspace of X.*

(ii) *$d(0,W) < a + \varepsilon$.*

(iii) *For every $y' \in \bigcup_{x'\in\Delta} B(x',\varepsilon)$ and every $y \in B_X \cap W$ one has $\|y\| > y'(y)$.*

Proof. Choose $y_0 \in Y$ with $\|y_0\| < a$ and consider

$$Z := \{y \in Y \mid x'(y) = x'(y_0) \text{ for all } x' \in \Delta\}.$$

This is a closed infinite-dimensional affine subspace of Y, and $d(0,Z) < a$ (since $y_0 \in Z$).

Next we select $y_1 \in Z$ with $\|y_0\| + \varepsilon < \|y_1\| < a + \varepsilon$ and a normalized $x'_0 \in X'$ such that $x'_0(y_1) = \|y_1\|$. We claim that

$$W := \{y \in Z \mid x'_0(y) = x'_0(y_1)\}$$

behaves as desired.

(i) is obvious, the codimension of W in Y is even finite. (ii) is true since $y_1 \in W$; it remains to prove (iii). Let $y \in W \cap B_X$, $x' \in \Delta$ and $\|z'\| \le \varepsilon$ be given. Then

$$(x' + z')(y) = x'(y_0) + z'(y) \le \|y_0\| + \varepsilon.$$

On the other hand we have

$$\|y\| \ge x'_1(y) = x'_1(y_1) = \|y_1\| > \|y_0\| + \varepsilon,$$

and this proves (iii). ∎

THEOREM 4.2. *Let X be an infinite-dimensional reflexive Banach space. Then there exists no boundary for X in X' with sbp. In particular the set of extreme points fails to have sbp.*

Proof. Let $\Delta_1, \Delta_2, \ldots$ be finite subsets of $B_{X'}$ and $(\varepsilon_n)_{n\ge 0}$ a sequence of positive numbers such that $\sum \varepsilon_n < 1$. The theorem will be proved as soon as we have found an $x \in B_X$ such that $y'(x) < \|x\|$ for all y' in $\bigcup_n \bigcup_{x'\in\Delta_n} B(x',\varepsilon_n)$.

We start with $Y_1 := X$ and apply the lemma with $Y := Y_1$, $a := \varepsilon_0$, $\varepsilon := \varepsilon_1$, and $\Delta := \Delta_1$. The lemma provides $Y_2 \subset Y_1$ such that the norm of no $y \in B_X \cap Y_2$ is attained at any $y'' \in \bigcup_{x'\in\Delta_1} B(x',\varepsilon_1)$. We continue with $Y := Y_2$, $a := \varepsilon_0 + \varepsilon_1$, $\varepsilon := \varepsilon_2$, and $\Delta := \Delta_2$. Then every x in the intersection of the sets $B_X \cap Y_n$will have the claimed properties, and it remains to note that this intersection is nonempty by the weak compactness of the unit ball of X and our choice $\sum \varepsilon_n < 1$. ∎

In order to generalize this result to boundaries in biduals it is necessary to replace the use of compactness in the original space (which was possible by reflexivity) by an application of the w^*-compactness of the dual unit ball. Our Lemma 4.1, however, only provides *closed* subspaces whereas we would need w^*-*closed* subspaces in order to argue similarly. To overcome this difficulty we need some preparations.

LEMMA 4.3. *Let X be a separable Banach space and Y an infinite-dimensional subspace of X'. Then there is a sequence (x'_n) in the unit sphere of Y such that the w^*-limit of (x'_n) is zero.*

Proof. Fix a dense sequence (x_n) in the unit sphere of X. For each n, the set

$$Y_n := \{x' \in Y \mid x'(x_1) = \ldots = x'(x_n) = 0\}$$

is a subspace of Y with finite codimension. Choose any $x'_n \in Y_n$ with $\|x'_n\| = 1$; it is then obvious that these functionals tend to zero in the w^*-topology. ∎

Let us recall that a sequence (x'_n) in a dual Banach space X' is said to be a w^*-*basic sequence* if there exist $x_1, x_2, \ldots \in X$ with $x'_n(x_m) = \delta_{n,m}$ and such that for every x' in the w^*-closed linear span of (x'_n) the sequence $\sum_{i=1}^{n} x'(x_i)x'_i$ is w^*-convergent to x' (cf. [11], Definition 1.b.8). The following fact is contained in the proof of Proposition 1.b.12 of [11]:

LEMMA 4.4. *Let X be a Banach space such that both X and X' are separable. Then for every sequence (x'_n) of normalized functionals with $w^*\text{-}\lim x'_n = 0$ there is a subsequence which forms a boundedly complete w^*-basic sequence.* (*For the definition of a boundedly complete basis see* [11], *Definition* 1.b.3.)

LEMMA 4.5. *Let X be an infinite-dimensional Banach space such that X and X' are separable. Then for every infinite-dimensional closed subspace Y of X' there exists an infinite-dimensional subspace W of Y such that W is w^*-closed in X'.*

Proof. A combination of the preceding lemmas provides a boundedly complete w^*-basic sequence in the unit sphere of Y. Let W be the (norm)-closed linear span of Y; it remains to show that W is w^*-closed.

Let x' be an element of the w^*-closure of W and (x_n) be as in the definition of a w^*-basic sequence. Then x' is the w^*-limit of $\sum_{i=1}^{n} x'(x_i)x'_i$, and since these partial sums are bounded we even have convergence in norm (this follows from the definition of a boundedly complete basic sequence). Thus $x' \in W$ as desired. ∎

THEOREM 4.6. *Let X be an infinite-dimensional Banach space and $B \subset B_{X''}$ be a boundary for X'. Then B fails to have sbp.*

Proof. If a boundary is not separable it fails to have sbp, and thus we may start with a norm separable boundary for X' in the unit ball of X''. Then X'' (and thus a fortiori X and X') are separable; this follows from the Rodé–Godefroy theorem (cf. Theorem 77 of [8]). Therefore we may assume that we are dealing with separable X and X' only.

Now it is easy to modify the proof of Theorem 4.2. First, by Lemma 4.5, we may replace "closed" by "w^*-closed" in Lemma 4.1 if we work with separable dual spaces.

Some care is needed to assure that the distance condition is met. If Y is closed and affine with $d(0,Y) < a$ choose $y_0 \in Y$ with $\|y_0\| < a$ and write $Y = y_0 + W$ with a closed subspace W. Pass to a w^*-closed subspace Z of W and continue with $y_0 + Z$.

Then we argue similarly to the above: for finite $\Delta_1, \Delta_2, \ldots$ in $B_{X''}$ and positive $\varepsilon_0, \varepsilon_1, \ldots$ with $\varepsilon_0 + \varepsilon_1 + \ldots < 1$ we find infinite-dimensional affine subspaces Y_n of X' with $d(0, Y_n) < \varepsilon_0 + \ldots + \varepsilon_n$ which are decreasing and w^*-*closed* such that for no $x' \in Y_n \cap B_{X'}$ and no $y'' \in \bigcup_{x'' \in \Delta_n} B(x'', \varepsilon_n)$ does one have $\|x'\| = y''(x')$. The $B_{X'} \cap Y_n$ are nonempty since $d(0, Y_n) < \sum \varepsilon_n < 1$, and by the w^*-compactness of $B_{X'}$ we get an x' for which the norm is not attained for any

$$y'' \in \bigcup_n \bigcup_{x'' \in \Delta_n} B(x'', \varepsilon_n).$$

Therefore there are no boundaries in $B_{X''}$ with sbp. ∎

Remark. We close this section by indicating another way to prove the theorem; the idea is due to V. P. Fonf. Let X'' be an infinite-dimensional bidual Banach space with a boundary B with sbp. By the above preparation we may assume that X' is separable.

Choose a sequence of balls $B_n := B(x''_n, r_n)$ with $\|x''_n\| \le 1$, $r_n \to 0$, $r_n < 1/2$ which cover B. Denote by $A_n \subset X'$ the collection of all x' with $\|x'\| = 1$ such that $x''(x') = 1$ for some $x'' \in B_n$. By assumption, the A_n cover the sphere of X', and A_n is contained in the slice $\{x' \mid \|x'\| \le 1,\ x''_n(x') \ge 1 - r_n\}$. By [4], this slice condition implies that X' is polyhedral.

But polyhedral spaces contain a copy of c_0 ([3]), and we arrive at a contradiction since there are no separable dual spaces with this property (see Prop. 2.e.8 of [11]).

5. Counterexamples. At the beginning of Section 2 we have pointed out that σ-precompact spaces have sbp. In Section 3 we have shown that there are situations where the converse holds: this happens for closed convex subsets of Banach spaces and also for normed spaces with a finer Banach space norm.

However, being σ-precompact and having sbp are different concepts. The following proposition shows that counterexamples can be found in *every* infinite-dimensional Banach space:

PROPOSITION 5.1. *Let X be an infinite-dimensional separable Banach space. Then X can be written as the disjoint union of two subsets A and B, where*

(i) *A is nowhere dense*;
(ii) *B is a dense G_δ-subset with sbp.*
(iii) *B fails to be σ-precompact.*

Proof. Let (x_n) be a dense sequence and (a_n) an arbitrary sequence of positive numbers tending to zero. We put

$$B := \bigcap_k \bigcup_n B^o(x_n, a_{k+n}),$$

where $B^o(x,r)$ denotes the open ball with centre x and radius r. With $A := X \setminus B$ it is clear that (i) and (ii) hold. Since precompact subsets of X are nowhere dense it follows that σ-compact sets are of first category. But B is of second category by Baire's theorem, and this proves (iii). ∎

REMARKS. 1. The proof shows that the proposition holds in the slightly more general setting of separable complete metric spaces where all compact sets have empty interior.

2. Let B be a non-σ-precompact sbp-subset of a Banach space. Then the closed convex hull of B cannot have sbp (this follows from Theorem 3.4).

It is more demanding to provide an example of a non-σ-precompact sbp-set which is at the same time *complete* (the preceding sets B never are). Recall that, for a Banach space X, the space of bounded sequences from $\mathbb{N}$ to X is denoted by $l^\infty(X)$; this space is provided with the supremum norm. Note that $l^\infty(X)$ can be identified with l^∞ in the case $X = l^\infty$. Our counterexample will be a suitable subset of this space.

Denote by e_n the canonical nth unit vector in l^∞. A map Φ from $\mathbb{N}^\mathbb{N}$ to $l^\infty(l^\infty)$ is defined by

$$(a_1, a_2, \ldots) \mapsto \left(e_{a_1}, \frac{1}{2^{a_1}} e_{a_2}, \frac{1}{2^{a_1+a_2}} e_{a_3}, \frac{1}{2^{a_1+a_2+a_3}} e_{a_4}, \ldots \right).$$

The range of Φ will be called K.

It will be convenient to introduce some more notation. For $a_1, \ldots, a_r \in \mathbb{N}$ we put

$$K_{a_1,\ldots,a_r} := \{\Phi(a_1, \ldots, a_r, b_{r+1}, b_{r+2}, \ldots) \mid b_{r+1}, b_{r+2}, \ldots \in \mathbb{N}\}.$$

Then the following properties hold:

• $K_{a_1,\ldots,a_r}$ is the disjoint union of the $K_{a_1,\ldots,a_r,n}$ with $n \in \mathbb{N}$.

• For $n \neq m$, the distance from any element of $K_{a_1,\ldots,a_r,n}$ to any element of $K_{a_1,\ldots,a_r,m}$ is $1/2^{a_1+\ldots+a_r}$, and this distance is one if elements of K_n and K_m are concerned.

- The diameter of $K_{a_1,\dots,a_r}$ is $1/2^{a_1+\dots+a_r}$.
- K and all $K_{a_1,\dots,a_r}$ are complete metric spaces.
- No point of K has any precompact neighbourhood.

These facts easily follow from the definition. (For the last one note that, if $k = \Phi(a_1, a_2, \dots)$ is arbitrary, the $1/2^{a_1+\dots+a_r}$-neighbourhood of k contains all $K_{a_1,\dots,a_r,n}$, and these sets have mutual distance $1/2^{a_1+\dots+a_r}$.)

THEOREM 5.2. *K has the small ball property, but it fails to be σ-precompact.*

Proof. We indicate how K can be written as the union of a sequence of subsets whose diameters are bounded by ε_0 and tend to zero (cf. Proposition 2.1):

- $\varepsilon_0 = 1/2$: Consider $K = K_1 \cup K_2 \cup \dots$
- $\varepsilon_0 = 1/4$: K is the union of the $K_{11}, K_{12}, \dots$ and the $K_2, K_3, \dots$ It remains to arrange these countably many subsets as a sequence; the diameters will have the desired property.
- $\varepsilon_0 = 1/8$: This time one works with $K_{111}, K_{112}, K_{113}, \dots$ together with $K_{12}, K_{13}, \dots$ and $K_3, K_4, \dots$

It should now be clear how to proceed for $\varepsilon_0 = 1/16, 1/32, \dots$

It remains to show that K is not σ-precompact. We suppose that $K = \bigcup L_n$ with L_n precompact, and we will derive a contradiction.

As K is a complete metric space, the closures L_n^- of the L_n are compact. The union of these closures is K and thus, by Baire's theorem, there must be an n such that L_n^- has nonvoid interior. But this would imply that there are points with a compact neighbourhood, contrary to the above observation. ∎

We now turn to *finite products*; they will be provided with the maximum metric. Clearly the product of two σ-precompact spaces is also σ-precompact, and it is easy to check that the product of a σ-precompact space with a space with sbp also has sbp. However, in general this property does *not* pass to products:

THEOREM 5.3. *There is a complete metric space K such that K has the small ball property but $K \times K$ does not.*

Proof. We work with the space K of the preceding counterexample. Let $B_m = B(x_m, r_m)$, $m = 1, 2, \dots$, be a sequence of balls in $K \times K$ such that

$$r_m \le 1/3, \qquad \lim r_m = 0.$$

We will show that there is an $x \in K \times K$ which is not in the union of these balls.

Since—in the above notation—the mutual distance of the $K_1 \times K_1$, $K_2 \times K_1, \dots$ is 1, each B_m will meet at most one $K_n \times K_1$. Choose an

m_1 such that $r_m \leq 1/(2\cdot 3)$ for $m \geq m_1$ and then an a_1 such that none of the balls $B_1, \ldots, B_{m_1}$ meets $K_{a_1} \times K_1$. (In fact, there are infinitely many possible choices for a_1.)

$K_{a_1} \times K_1$ is the disjoint union of the $K_{a_1} \times K_{1n}$, $n = 1, 2, \ldots$, and the mutual distance of the $K_{a_1} \times K_{1n}$ is $1/2$. Thus a ball B_m with $m \geq m_1$ can meet at most one of them. Choose $m_2 > m_1$ in such a way that $r_n < 1/(2^{a_1}\cdot 3)$ for $m > m_2$ and then $a_2 \in \mathbb{N}$ such that

$$(K_{a_1} \times K_{1a_2}) \cap \bigcup_{m_1 \leq m \leq m_2} B_m = \emptyset;$$

note that in fact even

$$(K_{a_1} \times K_{1a_2}) \cap \bigcup_{m \leq m_2} B_m = \emptyset.$$

Similarly we find m_3 and a_3 such that $r_m \leq 1/(2^{1+a_2}\cdot 3)$ for $m \geq m_3$ and

$$(K_{a_1a_3} \times K_{1a_2}) \cap \bigcup_{m \leq m_3} B_m = \emptyset.$$

It should be clear how this construction has to be continued; finally we arrive at

$$x = (\Phi(a_1a_3a_5\ldots), \Phi(1a_2a_4\ldots)),$$

which is not in $\bigcup_m B_m$. ∎

Remark. Since a product $K \times L$ of two subsets of a Banach space X is just the sum of $K \times \{0\}$ and $\{0\} \times L$ in $X \times X$ the preceding counterexample also shows that sums of sets with sbp do not necessarily have it.

Let us point out *two open problems*:

1. Let the metric space (M, d) be σ-precompact. We have noted that then the product of M with every sbp-space has sbp. Does, conversely, σ-precompactness follow from this property?

2. We have provided essentially one class of spaces which have sbp but fail to be σ-precompact. These spaces are disconnected.

Is this property important for such examples? Are there, e.g., contractible spaces with sbp which are not σ-precompact?

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