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A nonsmooth exponential

by

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Abstract. Let $\mathcal{M}$ be a type II_1 von Neumann algebra, τ a trace in $\mathcal{M}$, and $L^2(\mathcal{M},\tau)$ the GNS Hilbert space of τ. If $L^2(\mathcal{M},\tau)_+$ is the completion of the set $\mathcal{M}_{\mathrm{sa}}$ of selfadjoint elements, then each element $\xi \in L^2(\mathcal{M},\tau)_+$ gives rise to a selfadjoint unbounded operator L_ξ on $L^2(\mathcal{M},\tau)$. In this note we show that the exponential $\exp : L^2(\mathcal{M},\tau)_+ \to L^2(\mathcal{M},\tau)$, $\exp(\xi) = e^{iL_\xi}$, is continuous but not differentiable. The same holds for the Cayley transform $C(\xi) = (L_\xi - i)(L_\xi + i)^{-1}$. We also show that the unitary group $U_{\mathcal{M}} \subset L^2(\mathcal{M},\tau)$ with the strong operator topology is not an embedded submanifold of $L^2(\mathcal{M},\tau)$, in any way which makes the product $(u,w) \mapsto uw$ $(u,w \in U_{\mathcal{M}})$ a differentiable map.

1. Introduction. Let $\mathcal{M}$ be a type II_1 von Neumann algebra with a faithful and normal tracial state τ. Let $L^2(\mathcal{M},\tau)$ be the Hilbert space obtained by completion of $\mathcal{M}$ with respect to the norm $\|x\|_2 = \tau(x^*x)^{1/2}$. By Segal's theory of abstract integration [3], any element $\xi \in L^2(\mathcal{M},\tau)$ can be regarded as a (possibly unbounded) operator L_ξ on $L^2(\mathcal{M},\tau)$, affiliated to $\mathcal{M}$, as follows (see [1]). Let J be the antiunitary involution of $L^2(\mathcal{M},\tau)$, which on the dense subspace $\mathcal{M} \subset L^2(\mathcal{M},\tau)$ is just the involution * of $\mathcal{M}$, and for $m \in \mathcal{M}$ consider the linear map $m \mapsto Jm^*J\xi$. This map is a closable operator, and L_ξ is its closure.

The elements $m \in \mathcal{M}$ will be considered as operators acting by left multiplication on $L^2(\mathcal{M},\tau)$; when regarded as *elements* of $L^2(\mathcal{M},\tau)$, they will be denoted by $\vec{m}$.

An interesting fact [3] is that if ξ satisfies $J\xi = \xi$, then the associated operator L_ξ is selfadjoint. Define $L^2(\mathcal{M},\tau)_+ = \{\xi \in L^2(\mathcal{M},\tau) : J\xi = \xi\}$. Clearly $L^2(\mathcal{M},\tau)_+$ is a real Hilbert space, and the inner product of $L^2(\mathcal{M},\tau)$ is real and symmetric when restricted to $L^2(\mathcal{M},\tau)_+$. Indeed, $L^2(\mathcal{M},\tau)_+$ is the completion of the set $\mathcal{M}_{\mathrm{sa}}$ of selfadjoint elements of $\mathcal{M}$, and if $\vec{m}_1, \vec{m}_2 \in \mathcal{M}_{\mathrm{sa}}$, then $\langle \vec{m}_1, \vec{m}_2\rangle = \tau(m_2m_1) = \tau(m_1m_2) = \langle \vec{m}_2, \vec{m}_1\rangle$.

In this note we consider the map

$$\exp : L^2(\mathcal{M},\tau)_+ \to U_{\mathcal{M}}\vec{1} \subset L^2(\mathcal{M},\tau), \qquad \exp(\xi) = e^{iL_\xi}\vec{1},$$

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where $U_{\mathcal{M}}$ is the unitary group of $\mathcal{M}$, and $U_{\mathcal{M}}\vec{1}$ is just the same set regarded as a subset of $L^2(\mathcal{M},\tau)$, which induces on $U_{\mathcal{M}}$ a metric topology equivalent to the strong operator topology. In what follows we identify $U_{\mathcal{M}}$ and $U_{\mathcal{M}}\vec{1}$.

We prove that the map exp is continuous but not smooth, in fact, not differentiable. We consider the Cayley transform

$$C : L^2(\mathcal{M},\tau)_+ \to U_{\mathcal{M}} \subset L^2(\mathcal{M},\tau), \quad C(\xi) = (L_\xi - i)(L_\xi + i)^{-1}\vec{1},$$

which is also continuous and nondifferentiable.

The unitary group $U_{\mathcal{M}}$ in the strong operator topology can be embedded in $L^2(\mathcal{M},\tau)$ as a complete topological group. The group operations are clearly continuous in the L^2-metric, and $U_{\mathcal{M}} \subset L^2(\mathcal{M},\tau)$ is closed. We finish this note by proving that it cannot be embedded as a differentiable Banach–Lie group.

2. Nonregularity of exp **and** C**.** Let us see first that these maps are continuous. The following lemma will be useful. It relates the L^2 topology to the generalization of the strong topology to unbounded operators. Our reference on this subject is [2].

LEMMA 2.1. *If a sequence ξ_n converges in $L^2(\mathcal{M},\tau)_+$ to ξ, then the operators L_{ξ_n} converge to L_ξ in the strong resolvent sense.*

Proof. Suppose that $\xi_n \in L^2(\mathcal{M},\tau)_+$ converges to ξ. Then for $m \in \mathcal{M}$, $L_{\xi_n}m\vec{1}$ converges to $L_\xi m\vec{1}$. Indeed, $L_{\xi_n}m\vec{1} = Jm^*J\xi_n \to Jm^*J\xi$ because Jm^*J is bounded, and this last vector equals $L_\xi m\vec{1}$. We claim that $\mathcal{M}\vec{1}$ is a common core for all (selfadjoint) operators of the form L_η, $\eta \in L^2(\mathcal{M},\tau)_+$. In that case, from [2, VIII.25] it follows that L_{ξ_n} converges to L_ξ in the strong resolvent sense. Our claim follows by using another result in [2, VIII.11]. If A is a selfadjoint operator and $D \subset D(A)$ is a dense subspace which is invariant under the one-parameter group e^{itA}, i.e. $e^{itA}(D) \subset D$ for all $t \in \mathbb{R}$, then D is a core for A. Now clearly $e^{itL_\eta}m\vec{1} \in \mathcal{M}\vec{1}$, because $e^{itL_\eta} \in \mathcal{M}$ for all t if $\eta \in L^2(\mathcal{M},\tau)_+$. It follows that $\mathcal{M}\vec{1}$ is a core for L_η. ∎

It will be useful to have an alternative formula for C. Note that $(L_\xi - i)^{-1}$ and $(L_\xi + i)^{-1}$ commute, and $(L_\xi - i)^{-1}(\xi - i\vec{1}) = \vec{1}$. Therefore $(L_\xi + i)^{-1}\vec{1}$ $= (L_\xi - i)^{-1}(L_\xi + i)^{-1}(\xi - i\vec{1})$, and then

$$C(\xi) = (L_\xi + i)^{-1}(\xi - i\vec{1}), \quad \xi \in L^2(\mathcal{M},\tau)_+.$$

PROPOSITION 2.2. *The maps* exp *and C are continuous.*

Proof. If $\xi_n \to \xi$ in $L^2(\mathcal{M},\tau)_+$, then the resolvents $(L_{\xi_n} + i)^{-1}$ converge strongly to the resolvent $(L_\xi + i)^{-1}$. Note also that these operators are contractions. On the other hand $\xi_n - i\vec{1} \to \xi - i\vec{1}$ in $L^2(\mathcal{M},\tau)_+$. It follows that $C(\xi_n) = (L_{\xi_n} - i)^{-1}(\xi_n - i\vec{1})$ converges to $C(\xi)$. The same type of argument shows that the function exp is continuous. Indeed, if L_{ξ_n} converges to L_ξ in the strong resolvent sense, and f is a bounded continuous function on $\mathbb{R}$, then $f(L_{\xi_n}) \to f(L_\xi)$ strongly ([2, VII.20]). Therefore $\exp(\xi) = e^{iL_\xi}\vec{1}$ is continuous. ∎

Although these maps are not differentiable, they do have directional derivatives at the origin.

Lemma 2.3. *For all $\eta, v \in L^2(\mathcal{M},\tau)_+$, the curve $t \mapsto C(\eta + tv)$ is differentiable at $t = 0$, and*

$$\frac{d}{dt}C(\eta + tv)|_{t=0} = \frac{dC}{dv}(\eta) = -2i(L_\eta + i)^{-1}J(L_\eta - i)^{-1}Jv.$$

Proof. Note that

$$C(\eta + tv) - C(\eta) = (L_{\eta+tv} + i)^{-1}(\eta + tv - i\vec{1}) - (L_\eta + i)^{-1}(\eta - i\vec{1}).$$

The above sum can be decomposed into the following terms:

$$(L_{\eta+tv} + i)^{-1}(\eta + tv - i\vec{1}) - (L_{\eta+tv} + i)^{-1}(\eta - i\vec{1})$$

and

$$(L_{\eta+tv} + i)^{-1}(\eta - i\vec{1}) - (L_\eta + i)^{-1}(\eta - i\vec{1}).$$

We deal first with the first term, which equals

$$(L_{\eta+tv} + i)^{-1}(tv) = t(L_{\eta+tv} + i)^{-1}(v).$$

The second term equals

$$\begin{aligned}((L_{\eta+tv} &+ i)^{-1} - (L_\eta + i)^{-1})(\eta - i\vec{1})\\ &= (L_{\eta+tv} + i)^{-1}[L_\eta + i - (L_{\eta+tv} + i)](L_\eta + i)^{-1}(\eta - i\vec{1}).\end{aligned}$$

Note that this expression is well defined, since the vector $(L_\eta+i)^{-1}(\eta-i\vec{1}) = C(\eta) \in \mathcal{M}\vec{1}$ lies in the domain of any combination of the operators L_ν, $\nu \in L^2(\mathcal{M},\tau)_+$. Moreover, it apparently equals

$$-t(L_{\eta+tv}+i)^{-1}L_v(L_\eta+i)^{-1}(L_\eta-i)\vec{1} = t(L_{\eta+tv}+i)^{-1}J(L_\eta+i)(L_\eta-i)^{-1}Jv.$$

Putting both terms together yields

$$\frac{C(\eta + tv) - C(\eta)}{t} = (L_{\eta+tv} + i)^{-1}(v - J(L_\eta + i)(L_\eta - i)^{-1}Jv).$$

If we let t tend to 0, then $\eta + tv \to \eta$ in $L^2(\mathcal{M},\tau)_+$ and $(L_{\eta+tv} + i)^{-1} \to (L_\eta + i)^{-1}$ strongly. Therefore the derivative of $C(\eta + tv)$ at $t = 0$ exists and equals

$$(L_{\eta+v} + i)^{-1}(v - J(L_\eta + i)(L_\eta - i)^{-1}Jv).$$

Finally, the vector $v - J(L_\eta + i)(L_\eta - i)^{-1}Jv$ can be written as

$$v - J(L_\eta+i)(L_\eta-i)^{-1}Jv = J(1-(L_\eta+i)(L_\eta-i)^{-1})Jv = -2iJ(L_\eta-i)^{-1}Jv.$$

If one replaces this last expression in the result obtained for the derivative, one obtains the desired formula. ∎

Lemma 2.4. *For any* $v \in L^2(\mathcal{M},\tau)_+$, *the curve* $t \mapsto \exp(tv)$ *is differentiable at* $t=0$ *and*

$$\frac{d}{dt}\exp(tv)\bigg|_{t=0} = \frac{d}{dv}\exp(0) = iv.$$

Proof. The one-parameter unitary group $t \mapsto e^{itL_v}$ is strongly differentiable at $t=0$ on the domain of L_v (see [2]), i.e. if $\xi \in D(L_v)$, then $\frac{d}{dt}e^{itL_v}\xi|_{t=0}$ exists and equals $iL_v\xi$. Taking $\xi = \vec{1} \in D(L_v)$ proves the result. ∎

Theorem 2.5. *The maps* $\exp : L^2(\mathcal{M},\tau)_+ \to U_{\mathcal{M}} \subset L^2(\mathcal{M},\tau)$ *and* $C : L^2(\mathcal{M},\tau)_+ \to U_{\mathcal{M}} \subset L^2(\mathcal{M},\tau)$ *are not differentiable on any neighbourhood of* $0 \in L^2(\mathcal{M},\tau)_+$.

Proof. Suppose that exp is differentiable on a neighbourhood $0 \in \mathcal{V} \subset L^2(\mathcal{M},\tau)_+$. For any $\xi \in L^2(\mathcal{M},\tau)$ let $\xi = \xi_+ + \xi_-$ be the decomposition of ξ in $L^2(\mathcal{M},\tau) = L^2(\mathcal{M},\tau)_+ \oplus L^2(\mathcal{M},\tau)_-$. We shall construct a local diffeomorphism on $L^2(\mathcal{M},\tau)$ which restricted to $L^2(\mathcal{M},\tau)_+$ will provide a local homeomorphism onto a neighbourhood of $\vec{1}$ in $U_{\mathcal{M}}$. Afterwards we shall prove that this fact leads to contradiction. Consider the map

$$\theta : L^2(\mathcal{M},\tau) \to L^2(\mathcal{M},\tau), \qquad \theta(\xi) = \exp(\xi_+) + i\xi_-.$$

The projections $\xi \mapsto \xi_+$ and $\xi \mapsto \xi_-$ are (real) linear and bounded, therefore they are C^∞. It follows that θ is differentiable in $\mathcal{V}$. Note that $\theta(0) = \vec{1}$ and $\frac{d}{d\xi}\exp(0) = i\xi$. Therefore

$$d\theta_0(\xi) = i\xi_+ + i\xi_- = i\xi.$$

By the inverse function theorem it follows that there exists a ball $0 \in \mathcal{B}_\varepsilon(0) \subset \mathcal{V}$ in the $\|\ \|_2$-metric and an open set $\vec{1} \in \mathcal{W}$ of $L^2(\mathcal{M},\tau)$ such that $\theta : \mathcal{B}_\varepsilon(0) \to \mathcal{W}$ is a diffeomorphism. Note that θ maps $\mathcal{B}_\varepsilon(0) \cap L^2(\mathcal{M},\tau)_+$ onto $\mathcal{W} \cap U_{\mathcal{M}}$, i.e. $\theta|_{L^2(\mathcal{M},\tau)_+}$ is a local homeomorphism between $\mathcal{B}_\varepsilon(0)$ and a neighbourhood of $\vec{1}$ in $U_{\mathcal{M}}$ in the L^2-topology.

Fix $\delta^{1/2} < \varepsilon$, and for each integer $n \ge 1$ choose a projection $p_n \in \mathcal{M}$ such that $\tau(p_n) = \delta/n^2$. Put $a_n = np_n$. Note that $\|a_n\|_2 = \delta^{1/2}$. Indeed, $\tau(a_n^*a_n) = n^2\tau(p_n) = \delta$. Therefore $a_n \in \mathcal{B}_\varepsilon(0)$ and a_n does not tend to 0. On the other hand

$$\|\exp(a_n) - 1\|_2^2 = 2 - \tau(e^{ia_n}) - \tau(e^{-ia_n}).$$

Clearly

$$\tau(e^{ia_n}) = 1 + \sum_{k\ge 1}\tau\left(\frac{(in)^k}{k!}p_n\right) = 1 + \frac{\delta}{n^2}(e^{in} - 1).$$

Analogously $\tau(e^{-ia_n}) = 1 + (\delta/n^2)(e^{-in} - 1)$. It follows that $\exp(a_n) \to \vec{1}$ but $\theta^{-1}(\exp(a_n))$ does not tend to 0, a contradiction.

To prove the same result for C, one proceeds analogously. Using the fact that if C were differentiable, then by 2.3, $dC_0(\xi) = -2i\xi$, also in this case one can construct a local homeomorphism between a ball centred at $0 \in L^2(\mathcal{M},\tau)_+$ and an L^2-neighbourhood of $-\vec{1}$ in $U_{\mathcal{M}}$ (note that $C(0) = -\vec{1}$). Let $p_n \neq 0$ be projections in $\mathcal{M}$ such that $\tau(p_n) \to 0$. Then, as above, we have $\|1 - \exp(p_n)\|_2 \to 0$. Note also that $1 - \exp(p_n)$ has nontrivial kernel, indeed, $1 - \exp(p_n) = (e^i - 1)p_n$. On the other hand, if C were a local homeomorphism, then there would be a neighbourhood $-1 \in \mathcal{U} \subset U_{\mathcal{M}}$ where all $v \in \mathcal{U}$ would be such that $v - 1$ has trivial kernel, because unitaries in the range of the Cayley transform have this property. ■

For the Cayley transform one has the following weaker regularity conditions.

PROPOSITION 2.6. *The Cayley transform C is weakly* C^1, *i.e. for any fixed $\nu \in L^2(\mathcal{M},\tau)$, the complex-valued map $\xi \mapsto \langle C(\xi),\nu\rangle$ is* C^1. *If we regard C as a map from $L^2(\mathcal{M},\tau)_+$ to $L^1(\mathcal{M},\tau)$, it is differentiable.*

Proof. For any $\eta, v \in L^2(\mathcal{M},\tau)_+$, using 2.3 one has

$$\frac{d}{dv}\langle C,\nu\rangle(\eta) = \left\langle \frac{dC}{dv}(\eta),\nu \right\rangle = -2i\langle (L_\eta + i)^{-1}J(L_\eta - i)^{-1}Jv,\nu\rangle.$$

Recall that if $\eta_n \to \eta$ in $L^2(\mathcal{M},\tau)_+$, then the resolvents $(L_{\eta_n} - i)^{-1}$ and $(L_{\eta_n} + i)^{-1}$ are contractions which converge strongly to $(L_\eta - i)^{-1}$ and $(L_\eta + i)^{-1}$. It follows that $\frac{d}{dv}\langle C,\nu\rangle(\eta)$ is continuous in both parameters $\nu, v \in L^2(\mathcal{M},\tau)_+$, and C is weakly C^1. Let us prove that

$$C : L^2(\mathcal{M},\tau)_+ \to L^1(\mathcal{M},\tau)$$

is differentiable. Using the computations done in 2.3, one sees that

$$C(\eta + v) - C(\eta) = -2i(L_{\eta+v} + i)^{-1}J(L_\eta - i)^{-1}Jv$$

and

$$\frac{dC}{dv}(\eta) = -2i(L_\eta + i)^{-1}J(L_\eta - i)^{-1}Jv,$$

therefore $C(\eta + v) - C(\eta) - \frac{dC}{dv}(\eta)$ equals

$$-2i[(L_{\eta+v} + i)^{-1} - (L_\eta + i)^{-1}]J(L_\eta - i)^{-1}Jv.$$

We must show that the L^1 norm of this expression divided by $\|v\|_2$ tends to zero if v tends to zero in $L^2(\mathcal{M},\tau)$. Define $\Delta = (L_{\eta+v}+i)^{-1} - (L_\eta+i)^{-1}$ and $\psi = J(L_\eta - i)^{-1}Jv$. Note that $\Delta \in \mathcal{M}$ and $\psi \in L^2(\mathcal{M},\tau)$ with $\|\psi\|_2 \le \|v\|_2$. Also

$$\|\Delta\psi\|_1 \le \|\Delta\|_2\|\psi\|_2.$$

Indeed, let $\vec{x}_n$ be a sequence in $\mathcal{M}\vec{1}$ converging to ψ in $L^2(\mathcal{M},\tau)$. Then $\|\Delta\vec{x}_n\|_1 = \tau(|\Delta x_n|) = \tau(u^*\Delta x_n)$ where u is the partial isometry in the polar decomposition of $\Delta x_n \in \mathcal{M}$, which can be chosen unitary since $\mathcal{M}$ is finite.

By the Cauchy–Schwarz inequality $\tau(u\Delta x_n) \le \tau(\Delta^*\Delta)^{1/2}\tau(x_n^* x_n)^{1/2}$. Since $x_n \to \psi$, $\Delta x_n \to \Delta\psi$ in $L^2(\mathcal{M},\tau)$, and the inequality follows. Therefore

$$\frac{\|C(\eta+v) - C(\eta) - \frac{dC}{dv}(\eta)\|_1}{\|v\|_2} = 2\frac{\|\Delta\psi\|_1}{\|v\|_2} \le 2\|\Delta\|_2.$$

The proof ends by showing that $\|\Delta\|_2 \to 0$ as v tends to zero. Clearly Δ tends to zero in the strong operator topology, because $L_{\eta+v} \to L_\eta$ in the strong resolvent sense. Then

$$\tau(\Delta^*\Delta) = \langle \Delta\vec{1}, \Delta\vec{1}\rangle \to 0. \ \blacksquare$$

We now state the result on the nonembeddability of $U_{\mathcal{M}}$ in $L^2(\mathcal{M},\tau)$ as a Lie group.

THEOREM 2.7. *The unitary group $U_{\mathcal{M}}$ of M with the L^2 metric is not an embedded submanifold of $L^2(\mathcal{M},\tau)$ with differentiable multiplication map $(u,w) \mapsto uw$ $(u,w \in U_{\mathcal{M}})$.*

Proof. The proof consists in showing that if $U_{\mathcal{M}} \subset L^2(\mathcal{M},\tau)$ were an embedded submanifold, then it would be a Lie group, with Lie algebra identified with $L^2(\mathcal{M},\tau)_- := \{\xi \in L^2(\mathcal{M},\tau) : J\xi = -\xi\}$. Moreover, the Lie bracket would extend the commutator of (antiselfadjoint) elements of $\mathcal{M}$, $[x,y] = xy - yx$. This is clearly not possible: the commutators of elements of $L^2(\mathcal{M},\tau)$ lie in $L^1(\mathcal{M},\tau)$, possibly outside of $L^2(\mathcal{M},\tau)$ (see [3]).

Suppose that $U_{\mathcal{M}} \subset L^2(\mathcal{M},\tau)$ is a submanifold. If $u(t)$ is a smooth curve of unitaries with $u(0) = 1$ and $u'(0) = \xi$, differentiating $u^*(t)u(t) = 1$ at $t = 0$ yields $J\xi + \xi = 0$, i.e. $\xi \in L^2(\mathcal{M},\tau)_-$. Also any element $\xi \in L^2(\mathcal{M},\tau)_-$ can be obtained as the velocity vector of a curve in $U_{\mathcal{M}}$. It was shown above that the curve $u(t) = \exp(tv)$ is differentiable at $t = 0$ for any $v \in L^2(\mathcal{M},\tau)_+$; if we put $v = -i\xi$, then $u'(0) = \xi$. The tangent space of $U_{\mathcal{M}}$ at a point $u \in U_{\mathcal{M}}$ clearly identifies with $uL^2(\mathcal{M},\tau)_-$.

The multiplication is differentiable by hypothesis. The inversion $u \mapsto u^{-1} = u^*$ is continuous ($\mathcal{M}$ is finite). It can be regarded as the restriction of a real linear map of $L^2(\mathcal{M},\tau)$, namely J, and therefore is differentiable.

Therefore $U_{\mathcal{M}}$ is a Lie group, and its Lie algebra identifies with $L^2(\mathcal{M},\tau)_-$. Let us compute the bracket under this identification. The left action of the group $U_{\mathcal{M}}$ on itself, $\ell^u : U_{\mathcal{M}} \to U_{\mathcal{M}}$, $\ell^u(w) = uw$, extends to a bounded linear operator on $L^2(\mathcal{M},\tau)$. Then if $\xi \in L^2(\mathcal{M},\tau)_-$, the left invariant vector field induced by ξ is $X_\xi(u) = u\xi$ $(u \in U_{\mathcal{M}})$. If f is a smooth function on a neighbourhood of $\vec{1} \in U_{\mathcal{M}}$, then the derivative $X_\xi f$ can be computed as follows:

$$X_\xi f(u) = \frac{d}{dt} f(ue^{tL_\xi})\Big|_{t=0} = df_u(u\xi),$$

where df_u is the tangent map of f at $u \in U_{\mathcal{M}}$. Note that in the above computation again one only uses the fact that $t \mapsto e^{tL_\xi}\vec{1}$ is differentiable at $t=0$ ($\xi \in L^2(\mathcal{M},\tau)_-$). Let $\vec{x},\vec{y} \in i\mathcal{M}_{\rm sa} \subset L^2(\mathcal{M},\tau)_-$ be two antiselfadjoint elements on $\mathcal{M}$ (note that $i\mathcal{M}_{\rm sa}$ is dense in $L^2(\mathcal{M},\tau)_-$). Let us compute $X_{\vec{x}}X_{\vec{y}}f$:

$$X_{\vec{x}}X_{\vec{y}}f(u) = \frac{d}{dt}df_{ue^{tx}}(ue^{tx}\vec{y})\Big|_{t=0} = d^2f_u(u\vec{x},u\vec{y}) + df_u(u\overrightarrow{xy}).$$

Since d^2f_u is a symmetric bilinear form, it follows that the bracket $[X_{\vec{x}},X_{\vec{y}}]$ is given by

$$[X_{\vec{x}},X_{\vec{y}}]f(u) = df_u(u\overrightarrow{xy} - u\overrightarrow{yx}) = df_u(u(\overrightarrow{xy-yx})),$$

which coincides with the left invariant derivation $X_{\overrightarrow{xy-yx}}f(u)$. This says that the bracket of x,y regarded as an element of the Lie algebra of $U_\mathcal{M}$ is the usual commutator $xy-yx$. ∎

It would be interesting to know if the result holds without the assumption on the differentiability of the multiplication, that is, if $U_\mathcal{M} \subset L^2(\mathcal{M},\tau)$ is never an embedded submanifold.

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