A nonsmooth exponential

by

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Abstract. Let $\mathcal{M}$ be a type II$_1$ von Neumann algebra, $\tau$ a trace in $\mathcal{M}$, and $L^2(\mathcal{M}, \tau)$ the GNS Hilbert space of $\tau$. If $L^2(\mathcal{M}, \tau)_+$ is the completion of the set $\mathcal{M}_{sa}$ of selfadjoint elements, then each element $\xi \in L^2(\mathcal{M}, \tau)_+$ gives rise to a selfadjoint unbounded operator $L_\xi$ on $L^2(\mathcal{M}, \tau)$. In this note we show that the exponential $\exp : L^2(\mathcal{M}, \tau)_+ \to L^2(\mathcal{M}, \tau)$, $\exp(\xi) = e^{iL_\xi}$, is continuous but not differentiable. The same holds for the Cayley transform $C(\xi) = (L_\xi - i)(L_\xi + i)^{-1}$. We also show that the unitary group $U_\mathcal{M} \subset L^2(\mathcal{M}, \tau)$ with the strong operator topology is not an embedded submanifold of $L^2(\mathcal{M}, \tau)$, in any way which makes the product $(u, w) \mapsto uw$ ($u, w \in U_\mathcal{M}$) a differentiable map.

1. Introduction. Let $\mathcal{M}$ be a type II$_1$ von Neumann algebra with a faithful and normal tracial state $\tau$. Let $L^2(\mathcal{M}, \tau)$ be the Hilbert space obtained by completion of $\mathcal{M}$ with respect to the norm $\|x\|_2 = \tau(x^*x)^{1/2}$. By Segal’s theory of abstract integration [3], any element $\xi \in L^2(\mathcal{M}, \tau)$ can be regarded as a (possibly unbounded) operator $L_\xi$ on $L^2(\mathcal{M}, \tau)$, affiliated to $\mathcal{M}$, as follows (see [1]). Let $J$ be the antiunitary involution of $L^2(\mathcal{M}, \tau)$, which on the dense subspace $\mathcal{M} \subset L^2(\mathcal{M}, \tau)$ is just the involution $^*$ of $\mathcal{M}$, and for $m \in \mathcal{M}$ consider the linear map $m \mapsto Jm^*J\xi$. This map is a closable operator, and $L_\xi$ is its closure.

The elements $m \in \mathcal{M}$ will be considered as operators acting by left multiplication on $L^2(\mathcal{M}, \tau)$; when regarded as elements of $L^2(\mathcal{M}, \tau)$, they will be denoted by $\tilde{m}$.

An interesting fact [3] is that if $\xi$ satisfies $J\xi = \xi$, then the associated operator $L_\xi$ is selfadjoint. Define $L^2(\mathcal{M}, \tau)_+ = \{\xi \in L^2(\mathcal{M}, \tau) : J\xi = \xi\}$. Clearly $L^2(\mathcal{M}, \tau)_+$ is a real Hilbert space, and the inner product of $L^2(\mathcal{M}, \tau)$ is real and symmetric when restricted to $L^2(\mathcal{M}, \tau)_+$. Indeed, $L^2(\mathcal{M}, \tau)_+$ is the completion of the set $\mathcal{M}_{sa}$ of selfadjoint elements of $\mathcal{M}$, and if $\tilde{m}_1, \tilde{m}_2 \in \mathcal{M}_{sa}$, then $\langle \tilde{m}_1, \tilde{m}_2 \rangle = \tau(m_2m_1) = \tau(m_1m_2) = \langle \tilde{m}_2, \tilde{m}_1 \rangle$.

In this note we consider the map
$$\exp : L^2(\mathcal{M}, \tau)_+ \to U_\mathcal{M} \tilde{1} \subset L^2(\mathcal{M}, \tau), \quad \exp(\xi) = e^{iL_\xi} \tilde{1},$$

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where $U_{\mathcal{M}}$ is the unitary group of $\mathcal{M}$, and $U_{\mathcal{M}} \bar{1}$ is just the same set regarded as a subset of $L^2(\mathcal{M}, \tau)$, which induces on $U_{\mathcal{M}}$ a metric topology equivalent to the strong operator topology. In what follows we identify $d$ierentiable. We consider the Cayley transform as a subset of $\mathcal{M}$ invariant under the one-parameter group $D$ then $U_{\mathcal{M}} \bar{1}$.

We prove that the map $\exp$ is continuous but not smooth, in fact, not differentiable. We consider the Cayley transform

$$ C : L^2(\mathcal{M}, \tau)_+ \to U_{\mathcal{M}} \subset L^2(\mathcal{M}, \tau), \quad C(\xi) = (L_\xi - i)(L_\xi + i)^{-1} \bar{1}, $$

which is also continuous and nondierentiable.

The unitary group $U_{\mathcal{M}}$ in the strong operator topology can be embedded in $L^2(\mathcal{M}, \tau)$ as a complete topological group. The group operations are clearly continuous in the $L^2$-metric, and $U_{\mathcal{M}} \subset L^2(\mathcal{M}, \tau)$ is closed. We finish this note by proving that it cannot be embedded as a differentiable Banach–Lie group.

2. Nonregularity of $\exp$ and $C$. Let us see first that these maps are continuous. The following lemma will be useful. It relates the $L^2$ topology to the generalization of the strong topology to unbounded operators. Our reference on this subject is [2].

**Lemma 2.1.** If a sequence $\xi_n$ converges in $L^2(\mathcal{M}, \tau)_+$ to $\xi$, then the operators $L_{\xi_n}$ converge to $L_\xi$ in the strong resolvent sense.

**Proof.** Suppose that $\xi_n \in L^2(\mathcal{M}, \tau)_+$ converges to $\xi$. Then for $m \in \mathcal{M}$, $L_{\xi_n} m \bar{1}$ converges to $L_\xi m \bar{1}$. Indeed, $L_{\xi_n} m \bar{1} = Jm^* J \xi_n \to Jm^* J \xi$ because $Jm^* J$ is bounded, and this last vector equals $L_\xi m \bar{1}$. We claim that $\mathcal{M} \bar{1}$ is a common core for all (selfadjoint) operators of the form $L_\eta$, $\eta \in L^2(\mathcal{M}, \tau)_+$. In that case, from [2, VIII.25] it follows that $L_{\xi_n}$ converges to $L_\xi$ in the strong resolvent sense. Our claim follows by using another result in [2, VIII.11]. If $A$ is a selfadjoint operator and $D \subset D(A)$ is a dense subspace which is invariant under the one-parameter group $e^{itA}$, i.e. $e^{itA}(D) \subset D$ for all $t \in \mathbb{R}$, then $D$ is a core for $A$. Now clearly $e^{itL_\eta} m \bar{1} \in \mathcal{M} \bar{1}$, because $e^{itL_\eta} \in \mathcal{M}$ for all $t$ if $\eta \in L^2(\mathcal{M}, \tau)_+$. It follows that $\mathcal{M} \bar{1}$ is a core for $L_\eta$. •

It will be useful to have an alternative formula for $C$. Note that $(L_\xi - i)^{-1}$ and $(L_\xi + i)^{-1}$ commute, and $(L_\xi - i)^{-1}(\xi - i \bar{1}) = \bar{1}$. Therefore $(L_\xi + i)^{-1} \bar{1} = (L_\xi - i)^{-1}(L_\xi + i)^{-1}(\xi - i \bar{1})$, and then

$$ C(\xi) = (L_\xi + i)^{-1}(\xi - i \bar{1}), \quad \xi \in L^2(\mathcal{M}, \tau)_+. $$

**Proposition 2.2.** The maps $\exp$ and $C$ are continuous.

**Proof.** If $\xi_n \to \xi$ in $L^2(\mathcal{M}, \tau)_+$, then the resolvents $(L_{\xi_n} + i)^{-1}$ converge strongly to the resolvent $(L_\xi + i)^{-1}$. Note also that these operators are contractions. On the other hand $\xi_n - i \bar{1} \to \xi - i \bar{1}$ in $L^2(\mathcal{M}, \tau)_+$. It follows that $C(\xi_n) = (L_{\xi_n} - i)^{-1}(\xi_n - i \bar{1})$ converges to $C(\xi)$. The same type of argument shows that the function $\exp$ is continuous. Indeed, if $L_{\xi_n}$ converges
to $L_\xi$ in the strong resolvent sense, and $f$ is a bounded continuous function on $\mathbb{R}$, then $f(L_{\xi_n}) \to f(L_\xi)$ strongly ([2, VII.20]). Therefore $\exp(\xi) = e^{iL_\xi \mathbb{I}}$ is continuous. ■

Although these maps are not differentiable, they do have directional derivatives at the origin.

**Lemma 2.3.** For all $\eta, v \in L^2(\mathcal{M}, \tau)_+$, the curve $t \mapsto C(\eta + tv)$ is differentiable at $t = 0$, and

$$\frac{d}{dt} C(\eta + tv)|_{t=0} = \frac{dC}{dv}(\eta) = -2i(L_\eta + i)^{-1}J(L_\eta - i)^{-1}Jv.$$

**Proof.** Note that

$$C(\eta + tv) - C(\eta) = (L_{\eta + tv} + i)^{-1}(\eta + tv - i\mathbb{I}) - (L_\eta + i)^{-1}(\eta - i\mathbb{I}).$$

The above sum can be decomposed into the following terms:

$$(L_{\eta + tv} + i)^{-1}(\eta + tv - i\mathbb{I}) - (L_{\eta + tv} + i)^{-1}(\eta - i\mathbb{I})$$

and

$$(L_{\eta + tv} + i)^{-1}(\eta - i\mathbb{I}) - (L_\eta + i)^{-1}(\eta - i\mathbb{I}).$$

We deal first with the first term, which equals

$$ (L_{\eta + tv} + i)^{-1}(tv) = t(L_{\eta + tv} + i)^{-1}(v).$$

The second term equals

$$((L_{\eta + tv} + i)^{-1} - (L_\eta + i)^{-1})(\eta - i\mathbb{I})$$

$$= (L_{\eta + tv} + i)^{-1}[L_\eta + i - (L_{\eta + tv} + i)](L_\eta + i)^{-1}(\eta - i\mathbb{I}).$$

Note that this expression is well defined, since the vector $(L_\eta + i)^{-1}(\eta - i\mathbb{I}) = C(\eta) \in \mathcal{M}\mathbb{I}$ lies in the domain of any combination of the operators $L_\nu$, $\nu \in L^2(\mathcal{M}, \tau)_+$. Moreover, it apparently equals

$$ -t(L_{\eta + tv} + i)^{-1}L_\nu(L_\eta + i)^{-1}(L_\eta - i)\mathbb{I} = t(L_{\eta + tv} + i)^{-1}J(L_\eta + i)(L_\eta - i)^{-1}Jv.$$

Putting both terms together yields

$$\frac{C(\eta + tv) - C(\eta)}{t} = (L_{\eta + tv} + i)^{-1}(v - J(L_\eta + i)(L_\eta - i)^{-1}Jv).$$

If we let $t$ tend to 0, then $\eta + tv \to \eta$ in $L^2(\mathcal{M}, \tau)_+$ and $(L_{\eta + tv} + i)^{-1} \to (L_\eta + i)^{-1}$ strongly. Therefore the derivative of $C(\eta + tv)$ at $t = 0$ exists and equals

$$(L_{\eta + tv} + i)^{-1}(v - J(L_\eta + i)(L_\eta - i)^{-1}Jv).$$

Finally, the vector $v - J(L_\eta + i)(L_\eta - i)^{-1}Jv$ can be written as

$$v - J(L_\eta + i)(L_\eta - i)^{-1}Jv = J(1 - (L_\eta + i)(L_\eta - i)^{-1})Jv = -2iJ(L_\eta - i)^{-1}Jv.$$

If one replaces this last expression in the result obtained for the derivative, one obtains the desired formula. ■
Lemma 2.4. For any $v \in L^2(\mathcal{M}, \tau)_+$, the curve $t \mapsto \exp(tv)$ is differentiable at $t = 0$ and
\[
\frac{d}{dt} \exp(tv) \bigg|_{t=0} = \frac{d}{dv} \exp(0) = iv.
\]

Proof. The one-parameter unitary group $t \mapsto e^{itL_v}$ is strongly differentiable at $t = 0$ on the domain of $L_v$ (see [2]), i.e. if $\xi \in D(L_v)$, then $\frac{d}{dt} e^{itL_v} \xi \big|_{t=0}$ exists and equals $iL_v \xi$. Taking $\xi = 1 \in D(L_v)$ proves the result.

Theorem 2.5. The maps $\exp : L^2(\mathcal{M}, \tau)_+ \to U_{\mathcal{M}} \subset L^2(\mathcal{M}, \tau)$ and $C : L^2(\mathcal{M}, \tau)_+ \to U_{\mathcal{M}} \subset L^2(\mathcal{M}, \tau)$ are not differentiable on any neighbourhood of $0 \in L^2(\mathcal{M}, \tau)_+$.

Proof. Suppose that $\exp$ is differentiable on a neighbourhood $0 \in \mathcal{V} \subset L^2(\mathcal{M}, \tau)_+$. For any $\xi \in L^2(\mathcal{M}, \tau)$ let $\xi = \xi_+ + \xi_-$ be the decomposition of $\xi$ in $L^2(\mathcal{M}, \tau) = L^2(\mathcal{M}, \tau)_+ \oplus L^2(\mathcal{M}, \tau)_-$. We shall construct a local diffeomorphism on $L^2(\mathcal{M}, \tau)$ which restricted to $L^2(\mathcal{M}, \tau)_+$ will provide a local homeomorphism onto a neighbourhood of $\overline{I}$ in $U_{\mathcal{M}}$. Afterwards we shall prove that this fact leads to contradiction. Consider the map
\[
\theta : L^2(\mathcal{M}, \tau) \to L^2(\mathcal{M}, \tau), \quad \theta(\xi) = \exp(\xi_+) + i\xi_-.
\]
The projections $\xi \mapsto \xi_+$ and $\xi \mapsto \xi_-$ are (real) linear and bounded, therefore they are $C^\infty$. It follows that $\theta$ is differentiable in $\mathcal{V}$. Note that $\theta(0) = \overline{I}$ and $\frac{d}{d\xi} \exp(0) = i\xi$. Therefore
\[
d\theta_0(\xi) = i\xi_+ + i\xi_- = i\xi.
\]
By the inverse function theorem it follows that there exists a ball $0 \in \mathcal{B}_\varepsilon(0) \subset \mathcal{V}$ in the $\| \|_2$-metric and an open set $\overline{I} \in \mathcal{W}$ of $L^2(\mathcal{M}, \tau)$ such that $\theta : \mathcal{B}_\varepsilon(0) \to \mathcal{W}$ is a diffeomorphism. Note that $\theta$ maps $\mathcal{B}_\varepsilon(0) \cap L^2(\mathcal{M}, \tau)_+$ onto $\mathcal{W} \cap U_{\mathcal{M}}$, i.e. $\theta|L^2(\mathcal{M}, \tau)_+$ is a local homeomorphism between $\mathcal{B}_\varepsilon(0)$ and a neighbourhood of $\overline{I}$ in $U_{\mathcal{M}}$ in the $L^2$-topology.

Fix $\delta^{1/2} < \varepsilon$, and for each integer $n \geq 1$ choose a projection $p_n \in \mathcal{M}$ such that $\tau(p_n) = \delta/n^2$. Put $a_n = np_n$. Note that $\|a_n\|_2 = \delta^{1/2}$. Indeed, $\tau(a_n^*a_n) = n^2\tau(p_n) = \delta$. Therefore $a_n \in \mathcal{B}_\varepsilon(0)$ and $a_n$ does not tend to 0. On the other hand
\[
\|\exp(a_n) - 1\|_2^2 = 2 - \tau(e^{ia_n}) - \tau(e^{-ia_n}).
\]
Clearly
\[
\tau(e^{ia_n}) = 1 + \sum_{k \geq 1} \tau\left(\frac{(in)^k}{k!} p_n\right) = 1 + \frac{\delta}{n^2} (e^{in} - 1).
\]
Analogously $\tau(e^{-ia_n}) = 1 + (\delta/n^2)(e^{-in} - 1)$. It follows that $\exp(a_n) \to \overline{I}$ but $\theta^{-1}(\exp(a_n))$ does not tend to 0, a contradiction.
To prove the same result for $C$, one proceeds analogously. Using the fact that if $C$ were differentiable, then by 2.3, $dC_0(\xi) = -2i\xi$, also in this case one can construct a local homeomorphism between a ball centred at $0 \in L^2(\mathcal{M}, \tau)_+$ and an $L^2$-neighbourhood of $-\mathbf{1}$ in $U_{\mathcal{M}}$ (note that $C(0) = -\mathbf{1}$). Let $p_n \neq 0$ be projections in $\mathcal{M}$ such that $\tau(p_n) \to 0$. Then, as above, we have $\|1 - \exp(p_n)\|_2 \to 0$. Note also that $1 - \exp(p_n)$ has nontrivial kernel, indeed, $1 - \exp(p_n) = (e^i - 1)p_n$. On the other hand, if $C$ were a local homeomorphism, then there would be a neighbourhood $-1 \in \mathcal{U} \subset U_{\mathcal{M}}$ where all $v \in \mathcal{U}$ would be such that $v-1$ has trivial kernel, because unitaries in the range of the Cayley transform have this property. ■

For the Cayley transform one has the following weaker regularity conditions.

**PROPOSITION 2.6.** The Cayley transform $C$ is weakly $C^1$, i.e. for any fixed $\nu \in L^2(\mathcal{M}, \tau)$, the complex-valued map $\xi \mapsto (C(\xi), \nu)$ is $C^1$. If we regard $C$ as a map from $L^2(\mathcal{M}, \tau)_+$ to $L^1(\mathcal{M}, \tau)$, it is differentiable.

**Proof.** For any $\eta, \nu \in L^2(\mathcal{M}, \tau)_+$, using 2.3 one has

$$\frac{d}{d\nu} \langle C, \nu \rangle (\eta) = \left\langle \frac{dC}{d\nu}(\eta), \nu \right\rangle = -2i((L_{\eta} + i)^{-1}J(L_{\eta} - i)^{-1}J, \nu).$$

Recall that if $\eta_n \to \eta$ in $L^2(\mathcal{M}, \tau)_+$, then the resolvents $(L_{\eta_n} - i)^{-1}$ and $(L_{\eta_n} + i)^{-1}$ are contractions which converge strongly to $(L_{\eta} - i)^{-1}$ and $(L_{\eta} + i)^{-1}$. It follows that $\frac{d}{d\nu} \langle C, \nu \rangle (\eta)$ is continuous in both parameters $\nu, \nu \in L^2(\mathcal{M}, \tau)_+$, and $C$ is weakly $C^1$. Let us prove that

$$C : L^2(\mathcal{M}, \tau)_+ \to L^1(\mathcal{M}, \tau)$$

is differentiable. Using the computations done in 2.3, one sees that

$$C(\eta + \nu) - C(\eta) = -2i((L_{\eta + \nu} + i)^{-1}J(L_{\eta} - i)^{-1}J\nu$$

and

$$\frac{dC}{d\nu}(\eta) = -2i(L_{\eta} + i)^{-1}J(L_{\eta} - i)^{-1}J\nu,$$

therefore $C(\eta + \nu) - C(\eta) - \frac{dC}{d\nu}(\eta)$ equals

$$-2i[(L_{\eta + \nu} + i)^{-1} - (L_{\eta} + i)^{-1}]J(L_{\eta} - i)^{-1}J\nu.$$

We must show that the $L^1$ norm of this expression divided by $\|\nu\|_2$ tends to zero if $\nu$ tends to zero in $L^2(\mathcal{M}, \tau)$. Define $\Delta = (L_{\eta + \nu} + i)^{-1} - (L_{\eta} + i)^{-1}$ and $\psi = J(L_{\eta} - i)^{-1}J\nu$. Note that $\Delta \in \mathcal{M}$ and $\psi \in L^2(\mathcal{M}, \tau)$ with $\|\psi\|_2 \leq \|\nu\|_2$. Also

$$\|\Delta \psi\|_1 \leq \|\Delta\|_2 \|\psi\|_2.$$

Indeed, let $\bar{x}_n$ be a sequence in $\mathcal{M}\mathcal{T}$ converging to $\psi$ in $L^2(\mathcal{M}, \tau)$. Then $\|\Delta \bar{x}_n\|_1 = \tau((\Delta \bar{x}_n)) = \tau(u^* \Delta x_n)$ where $u$ is the partial isometry in the polar decomposition of $\Delta x_n \in \mathcal{M}$, which can be chosen unitary since $\mathcal{M}$ is finite.
By the Cauchy–Schwarz inequality \( \tau(u \Delta x_n) \leq \tau(\Delta^* \Delta)^{1/2} \tau(x_n^* x_n)^{1/2} \). Since \( x_n \to \psi, \Delta x_n \to \Delta \psi \) in \( L^2(\mathcal{M}, \tau) \), and the inequality follows. Therefore

\[
\frac{|C(\eta + v) - C(\eta) - \frac{dC}{dv}(\eta)||_1}{\|v\|_2} = 2 \frac{\|\Delta \psi\|_1}{\|v\|_2} \leq 2\|\Delta\|_2.
\]

The proof ends by showing that \( \|\Delta\|_2 \to 0 \) as \( v \) tends to zero. Clearly \( \Delta \) tends to zero in the strong operator topology, because \( L_{\eta + v} \to L_{\eta} \) in the strong resolvent sense. Then

\[
\tau(\Delta^* \Delta) = \langle \Delta \mathbf{1}, \Delta \mathbf{1} \rangle \to 0. \quad \blacksquare
\]

We now state the result on the nonembeddability of \( U_\mathcal{M} \) in \( L^2(\mathcal{M}, \tau) \) as a Lie group.

**Theorem 2.7.** The unitary group \( U_\mathcal{M} \) of \( \mathcal{M} \) with the \( L^2 \) metric is not an embedded submanifold of \( L^2(\mathcal{M}, \tau) \) with differentiable multiplication map \( (u, w) \mapsto uw \) (\( u, w \in U_\mathcal{M} \)).

**Proof.** The proof consists in showing that if \( U_\mathcal{M} \subset L^2(\mathcal{M}, \tau) \) were an embedded submanifold, then it would be a Lie group, with Lie algebra identified with \( L^2(\mathcal{M}, \tau)_- := \{ \xi \in L^2(\mathcal{M}, \tau) : J\xi = -\xi \} \). Moreover, the Lie bracket would extend the commutator of (antiselfadjoint) elements of \( \mathcal{M} \), \([x, y] = xy - yx\). This is clearly not possible: the commutators of elements of \( L^2(\mathcal{M}, \tau) \) lie in \( L^1(\mathcal{M}, \tau) \), possibly outside of \( L^2(\mathcal{M}, \tau) \) (see [3]).

Suppose that \( U_\mathcal{M} \subset L^2(\mathcal{M}, \tau) \) is a submanifold. If \( u(t) \) is a smooth curve of unitaries with \( u(0) = 1 \) and \( u'(0) = \xi \), differentiating \( u^*(t)u(t) = 1 \) at \( t = 0 \) yields \( J\xi + \xi = 0 \), i.e. \( \xi \in L^2(\mathcal{M}, \tau)_- \). Also any element \( \xi \in L^2(\mathcal{M}, \tau)_- \) can be obtained as the velocity vector of a curve in \( U_\mathcal{M} \). It was shown above that the curve \( u(t) = \exp(tv) \) is differentiable at \( t = 0 \) for any \( v \in L^2(\mathcal{M}, \tau)_+ \); if we put \( v = -i\xi \), then \( u'(0) = \xi \). The tangent space of \( U_\mathcal{M} \) at a point \( u \in U_\mathcal{M} \) clearly identifies with \( uL^2(\mathcal{M}, \tau)_- \).

The multiplication is differentiable by hypothesis. The inversion \( u \mapsto u^{-1} = u^* \) is continuous (\( \mathcal{M} \) is finite). It can be regarded as the restriction of a real linear map of \( L^2(\mathcal{M}, \tau) \), namely \( J \), and therefore is differentiable.

Therefore \( U_\mathcal{M} \) is a Lie group, and its Lie algebra identifies with \( L^2(\mathcal{M}, \tau)_- \). Let us compute the bracket under this identification. The left action of the group \( U_\mathcal{M} \) on itself, \( \ell^u : U_\mathcal{M} \to U_\mathcal{M}, \ell^u(w) = uw \), extends to a bounded linear operator on \( L^2(\mathcal{M}, \tau) \). Then if \( \xi \in L^2(\mathcal{M}, \tau)_- \), the left invariant vector field induced by \( \xi \) is \( X_\xi(u) = u\xi \) (\( u \in U_\mathcal{M} \)). If \( f \) is a smooth function on a neighbourhood of \( \mathbf{1} \in U_\mathcal{M} \), then the derivative \( X_\xi f \) can be computed as follows:

\[
X_\xi f(u) = \left. \frac{d}{dt} f(ue^{t\xi}) \right|_{t=0} = df_u(u\xi),
\]

where \( df_u \) is the tangent map of \( f \) at \( u \in U_\mathcal{M} \). Note that in the above
A nonsmooth exponential computation again one only uses the fact that \( t \mapsto e^{tL}\xi \) is differentiable at \( t = 0 \) (\( \xi \in L^2(\mathcal{M}, \tau)_- \)). Let \( \bar{x}, \bar{y} \in i\mathcal{M}_{sa} \subset L^2(\mathcal{M}, \tau)_- \) be two antiselfadjoint elements on \( \mathcal{M} \) (note that \( i\mathcal{M}_{sa} \) is dense in \( L^2(\mathcal{M}, \tau)_- \)). Let us compute \( X_{\bar{x}}X_{\bar{y}}f \):

\[
X_{\bar{x}}X_{\bar{y}}f(u) = \frac{d}{dt} d_{ue^{tx}}(ue^{tx} \bar{y}) \bigg|_{t=0} = d^2 f_u(u\bar{x}, u\bar{y}) + df_u(u\bar{x}\bar{y}).
\]

Since \( d^2 f_u \) is a symmetric bilinear form, it follows that the bracket \( [X_{\bar{x}}, X_{\bar{y}}] \) is given by

\[
[X_{\bar{x}}, X_{\bar{y}}]f(u) = df_u(u\bar{xy} - u\bar{yx}) = df_u(u(xy - yx)),
\]

which coincides with the left invariant derivation \( X_{\bar{x}}X_{\bar{y}}f(u) \). This says that the bracket of \( x, y \) regarded as an element of the Lie algebra of \( U_{\mathcal{M}} \) is the usual commutator \( xy - yx \). 

It would be interesting to know if the result holds without the assumption on the differentiability of the multiplication, that is, if \( U_{\mathcal{M}} \subset L^2(\mathcal{M}, \tau) \) is never an embedded submanifold.

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