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Power means and the reverse Hölder inequality

by

Victor D. Didenko (Brunei) and Anatolii A. Korenovskyi (Odessa)

Abstract. Let w be a non-negative measurable function defined on the positive semi-axis and satisfying the reverse Hölder inequality with exponents $0 < \alpha < \beta$. In the present paper, sharp estimates of the compositions of the power means $\mathcal{P}_\alpha w(x) := ((1/x)\int_0^x w^\alpha(t)\,dt)^{1/\alpha}$, $x > 0$, are obtained for various exponents α. As a result, for the function w a property of self-improvement of summability exponents is established.

1. Introduction. Let $\mathbb{R}_+$ denote the set of all non-negative real numbers, and let $w : \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function. The power mean (transform) $\mathcal{P}_\alpha$ of order $\alpha > 0$ is defined by

$$\mathcal{P}_\alpha w(x) := \left(\frac{1}{x}\int_0^x w^\alpha(t)\,dt\right)^{1/\alpha}, \quad x > 0. \tag{1.1}$$

The operators $\mathcal{P}_\alpha$ are widely used in various problems of analysis. Thus $\mathcal{P}_1$ is the usual Hardy transform, and the famous Hardy inequality for the transform $\mathcal{P}_1$ [HLP] has an enormous amount of applications.

If $0 < \alpha < \beta$, then an immediate consequence of the Hölder inequality is that $\mathcal{P}_\alpha w \le \mathcal{P}_\beta w$ for any function w. In addition, if x is a fixed number, then the equality

$$\mathcal{P}_\beta w(x) = \mathcal{P}_\alpha w(x)$$

is satisfied if and only if the function f is equivalent to a constant on the interval $[0, x]$.

Let $0 < \alpha < \beta$ and $B > 1$. A function w is said to satisfy the *reverse Hölder inequality*, denoted $w \in RH^{\alpha,\beta}(B)$, if for any $x > 0$ the inequality

$$\mathcal{P}_\beta w(x) \le B \cdot \mathcal{P}_\alpha w(x) \tag{1.2}$$

holds. If $\alpha = 1$, this inequality represents a particular case of Gehring's condition [G]. Starting with the works of Muckenhoupt [M], Gehring [G], and Coifman and Fefferman [CF], the classes of functions satisfying the reverse

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Hölder inequality have found various applications in the theory of weighted spaces, quasiconformal mappings, PDEs, and in other fields of analysis. The reason for such popularity is a remarkable property of functions belonging to $RH^{\alpha,\beta}$ classes. This property, called self-improvement of summability exponents, draw close attention and soon became an object of independent investigations.

The main result of the present paper concerns the sharp two-sided estimates of the compositions of $\mathcal{P}_\alpha$ transforms for functions from the classes $RH^{\alpha,\beta}(B)$; see Theorem 2.4 below. It is worth mentioning that our proof of Theorem 2.4 is extremely simple. Nevertheless, it gives the sharp estimates (2.6)–(2.7).

In Section 3 we show how to obtain the property of self-improvement of summability exponents for both power means and monotone functions from $RH^{\alpha,\beta}$.

Remark 1.1. Using the substitution $v = w^\alpha$ or $v = w^\beta$, one can reduce the problem under consideration to one of the cases: $\alpha = 1 < \beta$ or $0 < \alpha < 1 = \beta$. Nevertheless, for the sake of symmetry of the conditions involved, here we consider arbitrary parameters $0 < \alpha < \beta$. In our opinion, such an approach may help the reader in a better understanding of the conditions obtained.

2. Estimates of power means. Let α be a positive real number. In the present section we derive certain estimates for the transform (1.1). These estimates can be conveniently expressed in terms of a special function φ_α defined on the set $E_\alpha := (-\infty, 0) \cup [\alpha, \infty)$ by

$$\varphi_\alpha(\gamma) := \left(1 - \frac{\alpha}{\gamma}\right)^{1/\alpha}, \quad \gamma \in E_\alpha.$$

Note that φ_α is continuous on E_α. Moreover, it increases on each interval $(-\infty, 0)$ and $[\alpha, \infty)$ with $\varphi_\alpha((-\infty, 0)) = (1, \infty)$ and $\varphi_\alpha([\alpha, \infty)) = [0, 1)$, and for any $\gamma \in E_\alpha \setminus \{\alpha\}$ it satisfies the relation

$$\varphi_\alpha(\gamma) = \frac{1}{\varphi_\alpha(\alpha - \gamma)}. \tag{2.1}$$

For the sake of convenience, let us agree that

$$\frac{1}{0\pm} := \pm\infty, \quad \frac{1}{\infty\pm} := 0\pm, \quad \varphi(0-) := \infty, \quad \varphi(\pm\infty) := 1 \mp 0.$$

Remark 2.1. The function φ_α is closely connected with the transform (1.1). Namely, if $w_0(x) := x^\varepsilon$, then identity (2.1) implies that

$$\mathcal{P}_\alpha w_0 = \frac{w_0}{\varphi_\alpha(-1/\varepsilon)} = \varphi_\alpha\left(\alpha + \frac{1}{\varepsilon}\right) w_0, \quad \varepsilon > -1/\alpha, \tag{2.2}$$

which means that the operator $\mathcal{P}_\alpha$ acts on the function w_0 as the operator of multiplication by the constant $\varphi_\alpha(\alpha+1/\varepsilon)$.

Let $0<\alpha<\beta$, and let $\Phi_{\alpha,\beta}$ be the function defined on the set $E_\beta\setminus\{\beta\}$ by

$$\Phi_{\alpha,\beta}(\gamma) := \frac{\varphi_\alpha(\gamma)}{\varphi_\beta(\gamma)}.$$

Then $\Phi_{\alpha,\beta}$ is continuous on $E_\beta\setminus\{\beta\}$ and is increasing on $(-\infty,0)$ and decreasing on (β,∞) with $\Phi_{\alpha,\beta}((-\infty,0)) = \Phi_{\alpha,\beta}((\beta,\infty)) = (1,\infty)$. Therefore, for any $B>1$ the equation

$$\Phi_{\alpha,\beta}(\gamma) = B \tag{2.3}$$

possesses exactly two roots $\gamma_-\in(-\infty,0)$ and $\gamma_+\in(\beta,\infty)$, and the identity $\Phi_{\alpha,\beta}(\gamma_+) = \Phi_{\alpha,\beta}(\gamma_-)$ implies

$$\frac{\varphi_\alpha(\gamma_-)}{\varphi_\alpha(\gamma_+)} = \frac{\varphi_\beta(\gamma_-)}{\varphi_\beta(\gamma_+)}. \tag{2.4}$$

Equation (2.3) is used to define the range of parameters in various inequalities below. Note that modifications of this equation have already been mentioned in the literature [DS, K1, K2, P].

Remark 2.2. If $w \not\sim 0$ on $\mathbb{R}_+$, then the condition $w\in RH^{\alpha,\beta}(B)$ implies that $\mathcal{P}_\alpha w(x)>0$ for all $x>0$. Indeed, if

$$y = \sup\{x : \mathcal{P}_\alpha w(x) = 0\} \in (0,\infty),$$

then for any $h>0$ the Hölder inequality leads to the estimate

$$\begin{aligned} &\frac{\mathcal{P}_\beta w(y+h)}{\mathcal{P}_\alpha w(y+h)} \\ &\quad = \left(\frac{h}{y+h}\right)^{1/\beta-1/\alpha} \frac{\left(\frac{1}{h}\int_y^{y+h} w^\beta(x)\,dx\right)^{1/\beta}}{\left(\frac{1}{h}\int_y^{y+h} w^\alpha(x)\,dx\right)^{1/\alpha}} \ge \left(\frac{h}{y+h}\right)^{1/\beta-1/\alpha}. \end{aligned} \tag{2.5}$$

The expression on the right-hand side of (2.5) tends to ∞ as h tends to $0+$, and this contradicts the assumption $w\in RH^{\alpha,\beta}(B)$.

Remark 2.3. If $0<\alpha<\beta$ and if w is a function such that w^β is locally summable on $\mathbb{R}_+$, then

$$t^{1-\alpha/\beta}\Big(\int_0^t w^\beta(\tau)\,d\tau\Big)^{\alpha/\beta} = o(t^{1-\alpha/\beta}), \quad t\to 0+.$$

Moreover, using the Hölder inequality with exponent $\beta/\alpha > 1$ one gets

$$t^{1-\beta/\alpha}\Big(\int_0^t w^\alpha(\tau)\,d\tau\Big)^{\beta/\alpha} \le t\bigg(\frac{1}{t}\int_0^t w^\beta(\tau)\,d\tau\bigg) = o(1), \quad t \to 0+.$$

In the following, we will often use the notation $\mathbf{P}_\alpha := \mathcal{P}_\alpha w = \mathcal{P}_\alpha w(x)$. Accordingly, $\mathbf{P}_\alpha^\beta := (\mathcal{P}_\alpha w(x))^\beta$, $\mathbf{P}_\alpha\mathbf{P}_\beta := \mathcal{P}_\alpha\mathcal{P}_\beta w(x)$ and so on.

Theorem 2.4. *Let $0 < \alpha < \beta$, $B > 1$, and let $\gamma_\pm$ be the roots of equation (2.3). Then for any function $w \in RH^{\alpha,\beta}(B)$ the inequalities*

$$\varphi_\alpha(\gamma_+) \le \varphi_\alpha\bigg(\frac{\alpha\mathbf{P}_\alpha^\alpha\mathbf{P}_\beta}{\mathbf{P}_\alpha^\alpha\mathbf{P}_\beta - \mathbf{P}_\beta^\alpha}\bigg) \le \varphi_\alpha(\gamma_-), \quad x > 0, \tag{2.6}$$

$$\varphi_\beta(\gamma_+) \le \varphi_\beta\bigg(\frac{\beta\mathbf{P}_\beta^\beta\mathbf{P}_\alpha}{\mathbf{P}_\beta^\beta\mathbf{P}_\alpha - \mathbf{P}_\alpha^\beta}\bigg) \le \varphi_\beta(\gamma_-), \quad x > 0, \tag{2.7}$$

hold. Moreover, the constants on the left- and the right-hand sides of (2.6) *and* (2.7) *are sharp.*

Proof. Setting

$$\gamma = \alpha\,\frac{\mathbf{P}_\alpha^\alpha\mathbf{P}_\beta}{\mathbf{P}_\alpha^\alpha\mathbf{P}_\beta - \mathbf{P}_\beta^\alpha}, \tag{2.8}$$

and taking into account the properties of the function φ_α, one can see that the left inequality in (2.6) is satisfied if and only if

$$\gamma \in E_+ := (-\infty, 0) \cup [\gamma_+, \infty),$$

whereas the right inequality of (2.6) is valid if and only if

$$\gamma \in E_- := (-\infty, \gamma_-] \cup [\alpha, \infty),$$

so (2.6) is equivalent to

$$\gamma \in E_- \cap E_+ = (-\infty, \gamma_-] \cup [\gamma_+, \infty).$$

However, the last relation is equivalent to

$$\Phi_{\alpha,\beta}(\gamma) \le B. \tag{2.9}$$

To prove (2.9) we will use the identity

$$\begin{aligned} &\frac{d}{dt}\Big[t^{1-\alpha/\beta}\Big(\int_0^t w^\beta(\tau)\,d\tau\Big)^{\alpha/\beta}\Big] \\ &\quad = \Big(1 - \frac{\alpha}{\beta}\Big)\Big(\frac{1}{t}\int_0^t w^\beta(\tau)\,d\tau\Big)^{\alpha/\beta} + \frac{\alpha}{\beta}\Big(\frac{1}{t}\int_0^t w^\beta(\tau)\,d\tau\Big)^{\alpha/\beta-1} w^\beta(t), \end{aligned} \tag{2.10}$$

which is valid for almost all $t > 0$. Fix an $x > 0$. It follows from Remark 2.3 that the integral of the left-hand side of (2.10) over $[0, x]$ is equal to

$x(x^{-1}\int_0^x w^\beta(t)\,dt))^{\alpha/\beta}$, so integrating (2.10) over $[0,x]$ we derive the identity

$$(2.11)\qquad \left(\frac{1}{x}\int_0^x w^\beta(\tau)\,d\tau\right)^{\alpha/\beta} = \left(1-\frac{\alpha}{\beta}\right)\frac{1}{x}\int_0^x (\frac{1}{t}\int_0^t w^\beta(\tau)\,d\tau)^{\alpha/\beta}\,dt + \frac{\alpha}{\beta}\frac{1}{x}\int_0^x\left(\frac{1}{t}\int_0^t w^\beta(\tau)\,d\tau\right)^{\alpha/\beta-1} w^\beta(t)\,dt.$$

In order to estimate the second term on the right-hand side of (2.11) one can use the Hölder inequality with the exponent $\alpha/\beta\in(0,1)$ and the condition $w\in RH^{\alpha,\beta}(B)$. Thus

$$\frac{1}{x}\int_0^x\left(\frac{1}{t}\int_0^t w^\beta(\tau)\,d\tau\right)^{\alpha/\beta-1} w^\beta(t)\,dt \geq \left(\frac{1}{x}\int_0^x\left(\frac{1}{t}\int_0^t w^\beta(\tau)\,d\tau\right)^{\alpha/\beta}dt\right)^{1-\beta/\alpha}\left(\frac{1}{x}\int_0^x w^\alpha(t)\,dt\right)^{\beta/\alpha} \geq B^{-\beta}\left(\frac{1}{x}\int_0^x\left(\frac{1}{t}\int_0^t w^\beta(\tau)\,d\tau\right)^{\alpha/\beta}dt\right)^{1-\beta/\alpha}\frac{1}{x}\int_0^x w^\beta(t)\,dt.$$

Hence the identity (2.11) implies that

$$\left(\frac{1}{x}\int_0^x w^\beta(t)\,dt\right)^{\alpha/\beta} \geq \left(1-\frac{\alpha}{\beta}\right)\frac{1}{x}\int_0^x\left(\frac{1}{t}\int_0^t w^\beta(\tau)\,d\tau\right)^{\alpha/\beta}dt + B^{-\beta}\frac{\alpha}{\beta}\left(\frac{1}{x}\int_0^x\left(\frac{1}{t}\int_0^t w^\beta(\tau)\,d\tau\right)^{\alpha/\beta}dt\right)^{1-\beta/\alpha}\frac{1}{x}\int_0^x w^\beta(t)\,dt.$$

The last inequality can be written as

$$(2.12)\qquad \mathbf{P}_\beta^\alpha \geq \left(1-\frac{\alpha}{\beta}\right)\mathbf{P}_\alpha^\alpha\mathbf{P}_\beta + B^{-\beta}\,\frac{\alpha}{\beta}\,\mathbf{P}_\alpha^{\alpha-\beta}\mathbf{P}_\beta\cdot\mathbf{P}_\beta^\beta,$$

and simple transformations of (2.12) lead to the inequality

$$(2.13)\qquad \left(1-\alpha\frac{\mathbf{P}_\alpha^\alpha\mathbf{P}_\beta-\mathbf{P}_\beta^\alpha}{\alpha\mathbf{P}_\alpha^\alpha\mathbf{P}_\beta}\right)^{1/\alpha} \leq B\left(1-\beta\frac{\mathbf{P}_\alpha^\alpha\mathbf{P}_\beta-\mathbf{P}_\beta^\alpha}{\alpha\mathbf{P}_\alpha^\alpha\mathbf{P}_\beta}\right)^{1/\beta}.$$

Recalling now the notation (2.8), one notes that (2.13) is equivalent to the inequality (2.9), and (2.6) is proved.

Reciprocally changing parameters α and β in (2.8), i.e. setting

$$(2.14)\qquad \gamma = \beta\frac{\mathbf{P}_\beta^\beta\mathbf{P}_\alpha}{\mathbf{P}_\beta^\beta\mathbf{P}_\alpha-\mathbf{P}_\beta^\alpha},$$

we note that inequality (2.7) is equivalent to (2.9) but with γ defined now by

(2.14). It is clear that the same interchange of α and β does not influence the validity of the identities (2.10) and (2.11). Therefore, the Hölder inequality, this time with the exponent $\beta/\alpha > 1$, can be exploited once again, and the rest of the proof is similar to the corresponding steps in the proof of (2.9).

It remains to show that the constants $\varphi_\alpha(\gamma_\pm)$ and $\varphi_\beta(\gamma_\pm)$ in (2.6) and (2.7) are sharp. Choosing $w_0 = w_0(x) = x^{-1/\gamma_\pm}$ and taking into account Remark 2.1 along with (2.1) and (2.2), one obtains

$$\begin{aligned}\alpha\frac{\mathcal{P}^\alpha_\alpha\mathcal{P}_\beta w_0}{\mathcal{P}^\alpha_\alpha\mathcal{P}_\beta w_0-\mathcal{P}^\alpha_\beta w_0} &= \alpha\frac{\varphi^\alpha_\alpha(\alpha-\gamma_\pm)\cdot\varphi^\alpha_\beta(\beta-\gamma_\pm)}{\varphi^\alpha_\alpha(\alpha-\gamma_\pm)\cdot\varphi^\alpha_\beta(\beta-\gamma_\pm)-\varphi^\alpha_\beta(\beta-\gamma_\pm)}\\ &= \alpha\frac{\varphi^{-\alpha}_\alpha(\gamma_\pm)}{\varphi^{-\alpha}_\alpha(\gamma_\pm)-1}=\frac{\alpha}{1-(1-\alpha/\gamma_\pm)}=\gamma_\pm.\end{aligned}$$

Analogously,

$$\beta\frac{\mathcal{P}^\beta_\beta\mathcal{P}_\alpha w_0}{\mathcal{P}^\beta_\beta\mathcal{P}_\alpha w_0-\mathcal{P}^\beta_\alpha w_0}=\gamma_\pm.$$

This means that the values $\gamma_\pm$, present in (2.6) and (2.7), cannot be improved, and the proof of Theorem 2.4 is complete. ∎

Remark 2.5. Using the obvious equalities

$$\varphi_\alpha\left(\frac{\alpha\mathbf{P}^\alpha_\alpha\mathbf{P}_\beta}{\mathbf{P}^\alpha_\alpha\mathbf{P}_\beta-\mathbf{P}^\alpha_\beta}\right)=\frac{\mathbf{P}_\beta}{\mathbf{P}_\alpha\mathbf{P}_\beta},\quad \varphi_\beta\left(\frac{\beta\mathbf{P}^\beta_\beta\mathbf{P}_\alpha}{\mathbf{P}^\beta_\beta\mathbf{P}_\alpha-\mathbf{P}^\beta_\alpha}\right)=\frac{\mathbf{P}_\alpha}{\mathbf{P}_\beta\mathbf{P}_\alpha},$$

one can rewrite inequalities (2.6) and (2.7) as

$$\varphi_\alpha(\gamma_+)\le\frac{\mathcal{P}_\beta w}{\mathcal{P}_\alpha\mathcal{P}_\beta w}\le\varphi_\alpha(\gamma_-), \tag{2.15}$$

$$\varphi_\beta(\gamma_+)\le\frac{\mathcal{P}_\alpha w}{\mathcal{P}_\beta\mathcal{P}_\alpha w}\le\varphi_\beta(\gamma_-), \tag{2.16}$$

respectively.

To proceed we need an auxiliary result.

Lemma 2.6 ([M, W]). *Let m and M be positive numbers. If a function $f:\mathbb{R}_+\to\mathbb{R}_+$ satisfies the condition*

$$m\le\frac{f(\tau)}{\frac{1}{\tau}\int_0^\tau f(\xi)\,d\xi}\le M,\quad \tau>0, \tag{2.17}$$

then

$$(2.18)\qquad \left(\frac{x}{t}\right)^{m-1} \le \frac{\frac{1}{x}\int_0^x f(\tau)\,d\tau}{\frac{1}{t}\int_0^t f(\tau)\,d\tau} \le \left(\frac{x}{t}\right)^{M-1}, \quad 0 < t \le x.$$

Proof. To make the paper self-contained we give a short proof of this result. Rewrite condition (2.17) as

$$(2.19)\qquad \frac{m}{\tau} \le \frac{f(\tau)}{\int_0^\tau f(\xi)\,d\xi} \le \frac{M}{\tau}, \quad \tau > 0,$$

and note that

$$\frac{f(\tau)}{\int_0^\tau f(\xi)\,d\xi} = \frac{d}{d\tau}\Big[\ln\int_0^\tau f(\xi)\,d\xi\Big].$$

Integrating (2.19) over $[t, x]$, where $0 < t \le x$, we get

$$m\ln\frac{x}{t} \le \ln\frac{\int_0^x f(\tau)\,d\tau}{\int_0^\tau f(\tau)\,d\tau} \le M\ln\frac{x}{t},$$

and inequality (2.18) follows. ■

Combining Theorem 2.4 and Lemma 2.6 leads to the following result.

THEOREM 2.7. *Let $0 < \alpha < \beta$, $B > 1$, and let $\gamma_\pm$ be the roots of equation* (2.3)*. Then for any $w \in RH^{\alpha,\beta}(B)$ the functions $x^{1/\gamma_+}\mathcal{P}_\alpha\mathcal{P}_\beta w(x)$ and $x^{1/\gamma_+}\mathcal{P}_\beta\mathcal{P}_\alpha w(x)$ are non-decreasing whereas $x^{1/\gamma_-}\mathcal{P}_\alpha\mathcal{P}_\beta w(x)$ and $x^{1/\gamma_-}\mathcal{P}_\beta\mathcal{P}_\alpha w(x)$ are non-increasing, and the exponents $1/\gamma_-$ and $1/\gamma_+$ are sharp.*

Note that in this case, "sharp" means that if an exponent passes the value under consideration, then the corresponding function may lose the monotonicity indicated. See the proof for more details.

Proof. Let $x > 0$. Set

$$f(x) := \left(\frac{1}{x}\int_0^x w^\beta(t)\,dt\right)^{\alpha/\beta}.$$

Raise inequality (2.15) to the power α and note that the function f satisfies condition (2.17) with $m = 1 - \alpha/\gamma_+$ and $M = 1 - \alpha/\gamma_-$. Therefore, by

Lemma 2.6, for all t, $0 < t < x$, one has

$$\left(\frac{x}{t}\right)^{-\alpha/\gamma_+} \le \frac{\frac{1}{x}\int_0^x \left(\frac{1}{\tau}\int_0^\tau w^\beta(\xi)\,d\xi\right)^{\alpha/\beta} d\tau}{\frac{1}{t}\int_0^t \left(\frac{1}{\tau}\int_0^\tau w^\beta(\xi)\,d\xi\right)^{\alpha/\beta} d\tau} \le \left(\frac{x}{t}\right)^{-\alpha/\gamma_-}.$$

If raised to the power $-1/\alpha$, this inequality is equivalent to the monotonicity of the function $x^{1/\gamma_\pm}\mathcal{P}_\alpha\mathcal{P}_\beta w(x)$. Analogously, the inequality (2.16) and Lemma 2.6 imply the monotonicity of the function $x^{1/\gamma_\pm}\mathcal{P}_\beta\mathcal{P}_\alpha w(x)$.

The proof of the sharpness of the exponents $1/\gamma_\pm$ is similar to that in Theorem 2.4. Consider the function $w_0(x) = x^\varepsilon$. Using Remark 2.1, one can see that for $\varepsilon \in [-1/\gamma_+, -1/\gamma_-]$ the function w_0 belongs to the class $RH^{\alpha,\beta}(B)$, and

$$\mathcal{P}_\alpha\mathcal{P}_\beta w_0(x) = \mathcal{P}_\beta\mathcal{P}_\alpha w_0(x) = \varphi_\alpha(\alpha + 1/\varepsilon)\varphi_\beta(\beta + 1/\varepsilon)\cdot x^\varepsilon.$$

Choose $\varepsilon = -1/\gamma_+$. If $0 < \gamma \le \gamma_+$, then the function

$$x^{1/\gamma}\mathcal{P}_\alpha\mathcal{P}_\beta w_0(x) = x^{1/\gamma}\mathcal{P}_\beta\mathcal{P}_\alpha w_0(x) = \varphi_\alpha(\alpha - \gamma_+)\varphi_\beta(\beta - \gamma_+)\cdot x^{1/\gamma - 1/\gamma_+}$$

is non-decreasing. On the other hand, for $\gamma > \gamma_+$ it is decreasing.

Analogously, setting $\varepsilon = -1/\gamma_-$, one can see that if $0 > \gamma \ge \gamma_-$ then the function

$$x^{1/\gamma}\mathcal{P}_\alpha\mathcal{P}_\beta w_0(x) = x^{1/\gamma}\mathcal{P}_\beta\mathcal{P}_\alpha w_0(x) = \varphi_\alpha(\alpha - \gamma_-)\varphi_\beta(\beta - \gamma_-)\cdot x^{1/\gamma - 1/\gamma_-}$$

is non-increasing whereas for $\gamma < \gamma_-$ it is decreasing, and we are done. ∎

3. The self-improvement property. As was already mentioned, Gehring's class possesses the property of self-improvement of the summability exponent. For the class $RH^{\alpha,\beta}(B)$ this property means that any function $w \in RH^{\alpha,\beta}(B)$ is locally summable with a power $\gamma > \beta$. The considerations of the previous section allow us to establish the reverse Hölder inequality for the power means $\mathcal{P}_\alpha w$ and $\mathcal{P}_\beta w$, as well. More precisely, the following result is true.

Theorem 3.1. *Let $0 < \alpha < \beta$, $B > 1$, let $\gamma_\pm$ be the roots of equation (2.3), and let $w \in RH^{\alpha,\beta}(B)$. If $\gamma < \gamma_+$, then*

$$\mathcal{P}_\beta w \in RH^{\alpha,\gamma}\left(\frac{\varphi_\alpha(\gamma_-)}{\varphi_\gamma(\gamma_+)}\right), \tag{3.1}$$

$$\mathcal{P}_\alpha w \in RH^{\beta,\gamma}\left(\frac{\varphi_\beta(\gamma_-)}{\varphi_\gamma(\gamma_+)}\right). \tag{3.2}$$

On the other hand, if $\gamma > \gamma_-$, then

$$\mathcal{P}_\beta w \in RH^{\gamma,\alpha}\left(\frac{\varphi_\gamma(\gamma_-)}{\varphi_\alpha(\gamma_+)}\right), \tag{3.3}$$

$$\mathcal{P}_\alpha w \in RH^{\gamma,\beta}\left(\frac{\varphi_\gamma(\gamma_-)}{\varphi_\beta(\gamma_+)}\right), \tag{3.4}$$

and the conditions $\gamma < \gamma_+$ and $\gamma > \gamma_-$ for the parameter γ are not improvable.

Proof. Let $0 < t \le x$. If $\gamma > 0$, then raising (2.15) to the power γ, one obtains

$$\varphi_\alpha^\gamma(\gamma_+)\, t^{-\gamma/\gamma_-}(t^{1/\gamma_-}\mathcal{P}_\alpha\mathcal{P}_\beta w(t))^\gamma \le \mathcal{P}_\beta^\gamma w(t) \le \varphi_\alpha^\gamma(\gamma_-)\, t^{-\gamma/\gamma_+}(t^{1/\gamma_+}\mathcal{P}_\alpha\mathcal{P}_\beta w(t))^\gamma. \tag{3.5}$$

By Theorem 2.7 the functions $t^{1/\gamma_-}\mathcal{P}_\alpha\mathcal{P}_\beta w(t)$ and $t^{1/\gamma_+}\mathcal{P}_\alpha\mathcal{P}_\beta w(t)$ are monotone, hence

$$\varphi_\alpha^\gamma(\gamma_+)\, t^{-\gamma/\gamma_-}(x^{1/\gamma_-}\mathcal{P}_\alpha\mathcal{P}_\beta w(x))^\gamma \le \mathcal{P}_\beta^\gamma w(t) \le \varphi_\alpha^\gamma(\gamma_-)\, t^{-\gamma/\gamma_+}(x^{1/\gamma_+}\mathcal{P}_\alpha\mathcal{P}_\beta w(x))^\gamma. \tag{3.6}$$

On the other hand, if $\gamma < 0$, then the inequality signs in (3.5) and (3.6) should be reversed.

Let $\gamma < \gamma_+$, $\gamma \ne 0$. If $\gamma > 0$, then we integrate the right inequality in (3.6) in variable t from 0 to x and raise the result to the $(1/\gamma)$th power. If $\gamma < 0$, the same should be done for the opposite inequality. As a result, one obtains the inequality

$$\frac{\mathcal{P}_\gamma\mathcal{P}_\beta w(x)}{\mathcal{P}_\alpha\mathcal{P}_\beta w(x)} \le \frac{\varphi_\alpha(\gamma_-)}{\varphi_\gamma(\gamma_+)}, \tag{3.7}$$

no matter what the sign of γ is.

Analogously, using the left inequality in (3.6), one notes that for $\gamma > \gamma_-$, $\gamma \ne 0$, the inequality

$$\frac{\mathcal{P}_\gamma\mathcal{P}_\beta w(x)}{\mathcal{P}_\alpha\mathcal{P}_\beta w(x)} \ge \frac{\varphi_\alpha(\gamma_+)}{\varphi_\gamma(\gamma_-)} \tag{3.8}$$

holds. It is clear that inequalities (3.7) and (3.8) are equivalent to (3.1) and (3.3), respectively.

Similarly, inequality (2.16) implies (3.2) and (3.4).

It remains to show that in (3.1)–(3.4) the conditions involving the parameter γ cannot be improved. This can be done as in the proof of Theorem 2.4. Thus if $w_0(x) := x^{-1/\gamma_+}$, then by Remark 2.1,

$$\mathcal{P}_\alpha\mathcal{P}_\beta w_0(x) = \mathcal{P}_\beta\mathcal{P}_\alpha w_0(x) = \frac{x^{-1/\gamma_+}}{\varphi_\alpha(\gamma_+)\varphi_\beta(\gamma_+)} < \infty, \quad x > 0.$$

On the other hand, $\mathcal{P}_\gamma \mathcal{P}_\beta w_0(x) = \mathcal{P}_\beta \mathcal{P}_\gamma w_0(x) = \infty$ for $\gamma \geq \gamma_+$. This means that the condition $\gamma < \gamma_+$ from (3.1)–(3.2) cannot be improved. Analogously, for $w_1(x) := x^{-1/\gamma_-}$ one has

$$\mathcal{P}_\alpha \mathcal{P}_\beta w_1(x) = \mathcal{P}_\beta \mathcal{P}_\alpha w_1(x) = \frac{x^{-1/\gamma_-}}{\varphi_\alpha(\gamma_-)\varphi_\beta(\gamma_-)} < \infty, \quad x > 0,$$

and if $\gamma \leq \gamma_-$, then $\mathcal{P}_\gamma \mathcal{P}_\beta w_1(x) = \mathcal{P}_\beta \mathcal{P}_\gamma w_1(x) = \infty$, so the condition $\gamma > \gamma_-$ from (3.3)–(3.4) is also unimprovable, and the proof is complete. ∎

Note that up to now we have not imposed any additional conditions on the function $w \in RH^{\alpha,\beta}(B)$. However, if w is monotone, then the property of self-improvement of summability exponents can be specified as follows.

THEOREM 3.2. *Let* $0 < \alpha < \beta$, $B > 1$, *let* $\gamma_\pm$ *be the roots of equation* (2.3), *and let* $w \in RH^{\alpha,\beta}(B)$. *If* w *is non-increasing, then*

$$\text{(a)} \quad w \in RH^{\alpha,\gamma}\left(\frac{\varphi_\alpha(\gamma_-)}{\varphi_\alpha(\gamma_+)\varphi_\gamma(\gamma_+)}\right), \quad \gamma \in (\alpha, \gamma_+);$$

$$\text{(b)} \quad w \in RH^{\beta,\gamma}\left(\frac{\varphi_\beta(\gamma_-)}{\varphi_\beta(\gamma_+)\varphi_\gamma(\gamma_+)}\right), \quad \gamma \in (\beta, \gamma_+).$$

On the other hand, if w *is non-decreasing, then*

$$\text{(c)} \quad w \in RH^{\gamma,\alpha}\left(\frac{\varphi_\alpha(\gamma_-)\varphi_\gamma(\gamma_-)}{\varphi_\alpha(\gamma_+)}\right), \quad \gamma \in (\gamma_-, 0);$$

$$\text{(d)} \quad w \in RH^{\gamma,\beta}\left(\frac{\varphi_\beta(\gamma_-)\varphi_\gamma(\gamma_-)}{\varphi_\beta(\gamma_+)}\right), \quad \gamma \in (\gamma_-, 0),$$

and the upper bound for γ *in* (a) *and* (b) *and the lower bound in* (c) *and* (d) *cannot be improved.*

Proof. Using (3.2) and the left inequality in (2.16) one obtains

$$\mathcal{P}_\gamma \mathcal{P}_\alpha w \leq \frac{\varphi_\beta(\gamma_-)}{\varphi_\gamma(\gamma_+)} \mathcal{P}_\beta \mathcal{P}_\alpha w \leq \frac{\varphi_\beta(\gamma_-)}{\varphi_\gamma(\gamma_+)} \cdot \frac{1}{\varphi_\beta(\gamma_+)} \mathcal{P}_\alpha w.$$

If w is non-decreasing, then $\mathcal{P}_\gamma \mathcal{P}_\alpha w \geq \mathcal{P}_\gamma w$, hence

$$\mathcal{P}_\gamma w \leq \frac{\varphi_\beta(\gamma_-)}{\varphi_\beta(\gamma_+)} \cdot \frac{1}{\varphi_\gamma(\gamma_+)} \mathcal{P}_\alpha w.$$

Applying (2.4), one obtains assertion (a). Analogously, assertions (b)–(d) can be derived from Theorems 2.4 and 2.7.

The sharpness of the condition $\gamma < \gamma_+$ in (a) and (b), as well as the condition $\gamma > \gamma_-$ in (c) and (d), can be verified by using the power function $w = x^\varepsilon$ with an appropriate ε. ∎

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References

[CF] R. R. Coifman and Ch. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. 51 (1974), 241–250.

[DS] L. D'Apuzzo and C. Sbordone, *Reverse Hölder inequalities. A sharp result*, Rend. Mat. 10 (1990), 357–366.

[G] F. W. Gehring, *The L^p-integrability of the partial derivatives of a quasiconformal mapping*, Acta Math. 130 (1973), 265–273.

[HLP] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, 2nd ed., Cambridge Univ. Press, Cambridge, 1952.

[K1] A. A. Korenovskii, *Mean Oscillations and Equimeasurable Rearrangements of Functions*, Lecture Notes of Unione Mat. Ital. 4, Springer, Berlin, 2007.

[K2] —, *On the reverse Hölder inequality*, Math. Notes 181 (2007), 318–328.

[M] B. Muckenhoupt, *Weighted inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 533–565.

[P] A. Popoli, *Optimal integrability in B_p^q classes*, Matematiche (Catania) 52 (1974) 159–170.

[W] I. Wik, *On Muckenhoupt's classes of weight functions*, Studia Math. 94 (1989), 245–255.

Victor D. Didenko
University of Brunei Darussalam
BE1410 Bandar Seri Begawan, Brunei
E-mail: diviol@gmail.com

Anatolii A. Korenovskyi
Odessa I. I. Mechnikov National University
Dvoryanskaya 2
65026 Odessa, Ukraine
E-mail: anakor@paco.net

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