STUDIA MATHEMATICA 172 (2) (2006)

Weighted norm estimates for the maximal operator of the Laguerre functions heat diffusion semigroup

by

R. Macías (Santa Fe), C. Segovia (Buenos Aires) and J. L. Torrea (Madrid)

Abstract. We obtain weighted L^p boundedness, with weights of the type y^δ, $\delta > -1$, for the maximal operator of the heat semigroup associated to the Laguerre functions, $\{\mathcal{L}_k^\alpha\}_k$, when the parameter α is greater than -1. It is proved that when $-1 < \alpha < 0$, the maximal operator is of strong type (p,p) if $p > 1$ and $2(1+\delta)/(2+\alpha) < p < 2(1+\delta)/(-\alpha)$, and if $\alpha \geq 0$ it is of strong type for $1 < p \leq \infty$ and $2(1+\delta)/(2+\alpha) < p$.

The behavior at the end points of the intervals where there is strong type is studied in detail and sharp results about the existence or not of strong, weak or restricted types are given.

1. Introduction. The *Laguerre polynomials* $L_k^\alpha(y)$ are given by

$$e^{-y} y^\alpha L_k^\alpha(y) = \frac{1}{k!} \frac{d}{dy^k}(e^{-y} y^{k+\alpha}),$$

where y is positive. We assume that $\alpha > -1$. The Laguerre polynomials $\{L_k^\alpha(y)\}_{k=0}^\infty$ form an orthogonal system with respect to the measure $e^{-y} y^\alpha dy$. More precisely,

$$\int_0^\infty L_k^\alpha(y) L_j^\alpha(y) e^{-y} y^\alpha \, dy = \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)} \delta_{kj}.$$

The *Laguerre functions* $\mathcal{L}_k^\alpha(y)$ are defined by

$$\mathcal{L}_k^\alpha(y) = \left(\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} \right)^{1/2} e^{-y/2} y^{\alpha/2} L_k^\alpha(y).$$

2000 *Mathematics Subject Classification*: 42A45, 42B15, 42B20, 42B25, 42C10.

Key words and phrases: heat and Poisson semigroups, Laguerre functions.

Research partially supported by Dirección General de Investigación, Ministerio de Ciencia y Tecnología, BFM2002-04013-C02-02 and Proyecto IALE (UAM-Banco Santander Central-Hispano).

The first author was partially supported by Universidad Nacional del Litoral, Argentina.

The second author was partially supported by CONICET, Argentina.

Standard references for Laguerre functions and polynomials are [1], [9] and [10].

We define the heat diffusion kernel $W^\alpha(t,y,z)$ for $\alpha > -1$, $t>0$, $y>0$, and $z>0$ as

$$W^\alpha(t,y,z) = \sum_{n=0}^{\infty} e^{-t(n+(\alpha+1)/2)} \mathcal{L}_n^\alpha(y)\mathcal{L}_n^\alpha(z),$$

and the heat diffusion integral $W^\alpha f(t,y)$ as

$$W^\alpha f(t,y) = \int_0^\infty W^\alpha(t,y,z) f(z)\,dz.$$

The heat diffusion integral $W^\alpha f(t,y)$ satisfies the semigroup property

$$W^\alpha f(t_1+t_2,y) = \int_0^\infty W^\alpha(t_1,y,z) W^\alpha f(t_2,z)\,dz.$$

The maximal operator $W^{\alpha,*}$ associated to the heat diffusion integral $W^\alpha f(t,y)$ is given by

$$W^{\alpha,*} f(y) = \sup_{t>0} |W^\alpha f(t,y)|.$$

We define the fractional maximal function $M_\theta f(y)$ for $0 \le \theta < 1$ as

$$M_\theta f(y) = \sup_{h>0} \frac{1}{(2h)^{1-\theta}} \int_{|z|\le h} |f(y-z)|\,dz.$$

If $\theta = 0$, $M_0 f(y)$ is the Hardy–Littlewood centered maximal function. It is well known that if y^δ is a weight with $-1<\delta<p-1$, then M_0 is of strong type (p,p) for $p>1$ and of weak type $(1,1)$ if $p=1$ for the measure $y^\delta dy$. We will also need the right-sided maximal function

$$M^+ f(y) = \sup_{h>0} \frac{1}{h} \int_y^{y+h} |f(z)|\,dz$$

We denote by A_p the class of all weights $\omega(y)$ such that M_0 is of strong type (p,p) for $p>1$, and of weak type for $p=1$, for the measure $\omega(y)dy$, and by A_p^+ the class of all weights $\omega(y)$ such that M^+ is of strong type (p,p) for $p>1$, and of weak type for $p=1$, for the measure $\omega(y)dy$. It is well known that $A_1 \subset A_p$ and $A_1^+ \subset A_p^+$ for every $p>1$. For M^+ we need to know that it is of weak type $(1,1)$ for the measure $y^\delta dy$ for any $\delta > -1$. This is true because for $\delta \ge 0$ the weight is a non-decreasing function, and for $-1<\delta\le 0$, because $M^+ f(y) \le 2M_0 f(y)$. As references see [4]–[6].

The purpose of this paper is to study the action of the maximal operator $W^{\alpha,*}$ just defined on the spaces $L^p((0,\infty), y^\delta dy)$ for $\delta > -1$. For

$\alpha \geq 0$ and $\delta = 0$ the results we give here were obtained by Stempak in [8], and for $-1 < \alpha < 0$ and $\delta = 0$ by Macías, Segovia and Torrea in [3]. Even if not explicitly stated, estimates obtained in [8, Section 3] give L^p weighted results, $1 < p < \infty$, with the power weights y^δ, where $-1 < \delta < p - 1$. However, Theorem 1 in the present paper shows that for $\alpha \geq 0$ the δ-range can be enlarged to $-1 < \delta < (1 + \alpha/2)p - 1$. For the case when $\alpha \geq 0$ and $\delta > 0$ we can majorize $W^{\alpha,*}f(y)$ by a constant times $W^{0,*}f(y)$ and thus we obtain the strong type (p, p) of $W^{\alpha,*}$ whenever $p > 1 + \delta > 0$. However, we can do better; in fact, in Theorem 1 we show that $W^{\alpha,*}$ is of strong type (p, p) for the possibly greater interval $p > 1$ and $p > 2(1 + \delta)/(\alpha + 2)$.

2. Statement of the results. Let N_α denote the interval

$$N_\alpha = \begin{cases} \left(\dfrac{2(1+\delta)}{2+\alpha}, \dfrac{2(1+\delta)}{-\alpha}\right) \cap (1,\infty) & \text{if } -1 < \alpha < 0, \\ \left(\dfrac{2(1+\delta)}{2+\alpha}, \infty\right] \cap (1,\infty] & \text{if } \alpha \geq 0. \end{cases}$$

We will assume that N_α is not empty. This implies that $1 + \delta + \alpha/2 > 0$; if not otherwise stated, we assume this throughout. With this notation, we have

THEOREM 1. *Let $-1 < \alpha < \infty$ and $-1 < \delta < \infty$. If $p \in N_\alpha$, then the maximal operator $W^{\alpha,*}$ is of strong type (p, p) with respect to the measure $y^\delta dy$, that is,*

$$\int_0^\infty W^{\alpha,*}f(y)^p y^\delta\, dy \leq C_{\alpha,\delta,p} \int_0^\infty |f(y)|^p y^\delta\, dy$$

with a constant $C_{\alpha,\delta,p}$ depending on α, p and δ only.

The following theorem gives the behavior of $W^{\alpha,*}$ at the end points of N_α. We set $a_\alpha = \max(1, 2(1 + \delta)/(2 + \alpha))$ and $b_\alpha = 2(1 + \delta)/(-\alpha)$ if $-1 < \alpha < 0$, and $b_\alpha = \infty$ if $\alpha \geq 0$.

THEOREM 2. *Let $\delta > -1$. At the end points of N_α, we have*:

(a) *If $-1 < \alpha < 0$, then the operator $W^{\alpha,*}$ is of weak type and not of strong type (b_α, b_α) with respect to the measure $y^\delta dy$.*
(b) *If $\alpha \geq 0$, then $W^{\alpha,*}$ is of strong type (∞, ∞) with respect to $y^\delta dy$.*
(c) *If $\alpha > -1$ and $a_\alpha = 2(1 + \delta)/(2 + \alpha)$, then $W^{\alpha,*}$ is of restricted weak type and not of weak type (a_α, a_α) with respect to $y^\delta dy$.*
(d) *If $\alpha > -1$ and $a_\alpha = 1$, then $W^{\alpha,*}$ is of weak type and not of strong type $(1, 1)$ with respect to $y^\delta dy$.*

Remark 1. If $-1 < \alpha < 0$ and $2(1+\delta)/(-\alpha) = 1$, then the interval N_α is empty. However, since $2(1+\delta)/(2+\alpha) < 2(1+\delta)/(-\alpha) = 1$, by the proof of part (d) of Theorem 2, the operator $W^{\alpha,*}$ is *of weak type and not of strong type* $(1,1)$ *with respect to the measure* $y^\delta dy$.

Remark 2. The results obtained in Theorem 2 do not depend on Theorem 1, and can be used to give a proof of Theorem 1 by interpolation (see [7] and [2]).

3. Lemmas. Throughout this paper we shall assume that f is a nonnegative function. The constants will not have the same value at each occurrence.

Definition 1. Let f be a locally integrable function on $(0,\infty)$. We define the maximal function $M^R f$ for $0 < y < \infty$ by

$$M^R f(y) = \sup_{J_y \subset (y/4,3y)} \frac{1}{|J_y|} \int_{J_y} f(z)\,dz, \tag{3.1}$$

where J_y runs over all intervals containing y. Obviously, $M^R f(y) \le CM_0 f(y)$.

Lemma 1. *The maximal function M^R is of weak type (p,p), $1 \le p \le \infty$, with respect to the measure $y^\delta dy$ for any real δ.*

Proof. The case $p=\infty$ is obvious. Let us represent $(0,\infty)$ as the union of the intervals $\{(8^k,8^{k+1})\}_{k=-\infty}^{\infty}$. If $y \in \{y : \lambda < M^R f(y)\} \cap (8^k,8^{k+1}]$, then there exists an interval J_y such that $y \in J_y \subset (y/4,3y)$ and

$$M^R f(y) \le 2\,\frac{1}{|J_y|} \int_{J_y} f(z)\,dz.$$

This interval J_y is contained in $(8^{k-1},8^{k+2})$. Then, by Hölder's inequality,

$$\lambda^p < M^R f(y)^p \le \Big(2\,\frac{1}{|J_y|}\int_{J_y} f(z)\,dz\Big)^p \le 2^p\,\frac{1}{|J_y|}\int_{J_y} f(z)^p\,dz.$$

Given a compact subset K of $\{y : \lambda < M^R f(y)\} \cap (8^k,8^{k+1}]$, we can find a finite sequence $\{J_{y_i}\}$ that covers K and is such that no point of K belongs to more than three intervals of the sequence. Then

$$\begin{aligned}\int_K y^\delta\,dy &\le \sum_i \int_{J_{y_i}} y^\delta\,dy \le c_\delta 8^{k\delta} \sum_i |J_{y_i}| \le c_\delta 2^p 8^{k\delta}\,\frac{1}{\lambda^p}\sum_i \int_{J_{y_i}} f(z)^p\,dz \\ &\le 3c_\delta 2^p 8^{k\delta}\,\frac{1}{\lambda^p} \int_{(8^{k-1},8^{k+2})} f(z)^p\,dz \le c_{\delta,p}\,\frac{1}{\lambda^p}\int_{(8^{k-1},8^{k+2})} f(z)^p z^\delta\,dz.\end{aligned}$$

Thus,

$$\int\limits_{\{y\,:\,\lambda<M^R f(y)\}\cap(8^k,8^{k+1}]} y^\delta\,dy \le c_{\delta,p}\,\frac{1}{\lambda^p}\int\limits_{(8^{k-1},8^{k+2})} f(z)^p z^\delta\,dz.$$

Hence,

$$\int\limits_{\{y\,:\,\lambda<M^R f(y)\}} y^\delta\,dy \le c_{\delta,p}\,\frac{1}{\lambda^p}\int\limits_0^\infty f(z)^p z^\delta\,dz,$$

and Lemma 1 is proved. ∎

LEMMA 2 ([3, Lemma 1]). *Given* $0\le\beta<1$, *there exists a constant* C_β *such that for every* $y>0$,

$$\begin{aligned}(3.2)\quad y^{-\beta/2}M_\beta(f(z)z^{-\beta/2})(y) \le C_\beta\{&y^{\beta/2}M_0(f(z)z^{-\beta/2})(y)\\ &+y^{-\beta/2}M_0(f(z)z^{\beta/2})(y)+M_0f(y)\}.\end{aligned}$$

We shall introduce some notation. Let us consider the generating function for the Laguerre polynomials

$$\begin{aligned}(3.3)\quad &\sum_{n=0}^\infty \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\,L_n^\alpha(y)L_n^\alpha(z)r^n\\ &\qquad= \frac{1}{1-r}\,e^{-r(z+y)/(1-r)}(ryz)^{-\alpha/2}I_\alpha\bigg(2\,\frac{(ryz)^{1/2}}{1-r}\bigg),\end{aligned}$$

where $0\le r<1$ and $I_\alpha(y)=e^{-i\alpha\pi/2}J_\alpha(iy)$ is the modified Bessel function (see [1, p. 189, (20)]. Let

$$Q_\alpha(y,z,r)=\sum_{n=0}^\infty \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\,e^{-y/2}y^{\alpha/2}L_n^\alpha(y)e^{-z/2}z^{\alpha/2}L_n^\alpha(z)r^{n+(\alpha+1)/2};$$

then, by (3.3), $Q_\alpha(y,z,r)$ is equal to

$$\sum_{n=0}^\infty \mathcal{L}_n^\alpha(y)\mathcal{L}_n^\alpha(z)r^{n+(\alpha+1)/2} = \frac{r^{1/2}}{1-r}\,e^{-(z+y)/2}e^{-r(z+y)/(1-r)}I_\alpha\bigg(2\,\frac{(ryz)^{1/2}}{1-r}\bigg).$$

This shows that $Q_\alpha(y,z,e^{-t})=W^\alpha(t,y,z)$. Let

$$e^{-t}=\bigg(\frac{1-s}{1+s}\bigg)^2;$$

then $0<s\le1$ if and only if $0<t\le\infty$. If we define

$$R_\alpha(y,z,s)=Q_\alpha\bigg(y,z,\bigg(\frac{1-s}{1+s}\bigg)^2\bigg),$$

then we get the expression

$$\begin{aligned}(3.4)\quad &R_\alpha(y,z,s)\\ &\quad=\frac12\,\frac{1-s^2}{2s}\,e^{-\frac14(s+1/s)(y^{1/2}-z^{1/2})^2}e^{-\frac12(s+1/s)(yz)^{1/2}}I_\alpha\bigg(\frac{1-s^2}{2s}\,(yz)^{1/2}\bigg).\end{aligned}$$

Observe also that

$$W^{\alpha}f(t,y)=\int_0^{\infty} R_\alpha(y,z,s)f(z)\,dz \qquad \text{for } s=(1-e^{-t/2})/(1+e^{-t/2})=\tanh(t/4). \tag{3.5}$$

Moreover,

$$1-s^2=4e^{-t/2}/(1+e^{-t/2})^2\le 4e^{-t/2}. \tag{3.6}$$

We shall need the following estimates for $I_\alpha(y)$: for $\alpha>-1$, there exist two constants c_α and C_α such that (see [1, p. 5, (12)] and [1, p. 86, (5)])

$$\begin{aligned} &(1)\ \text{If } 0\le y\le 1, \quad \text{then} \quad c_\alpha y^\alpha \le I_\alpha(y)\le C_\alpha y^\alpha.\\ &(2)\ \text{If } y\ge 1, \quad \text{then} \quad c_\alpha \frac{1}{y^{1/2}}\,e^y \le I_\alpha(y)\le C_\alpha \frac{1}{y^{1/2}}\,e^y. \end{aligned} \tag{3.7}$$

Let

$$D_s=\left\{y:\left(\frac{1-s^2}{2s}\right)^2 y\ge 1\right\}.$$

By (3.7) and (3.4) we have

$$\begin{aligned} &\chi_{D_s}(yz)R_\alpha(y,z,s) \\ &\le C\,\frac{1}{2}\,\frac{1-s^2}{2s}\,e^{-\frac14(s+1/s)(z^{1/2}-y^{1/2})^2-\frac12(s+1/s)(zy)^{1/2}}\chi_{D_s}(yz)\frac{e^{(\frac{1-s^2}{2s})(zy)^{1/2}}}{\left(\frac{1-s^2}{2s}\right)^{1/2}(zy)^{1/4}} \\ &\le \frac12\,\frac{1-s^2}{2s}\,e^{-\frac{1}{4s}(z^{1/2}-y^{1/2})^2}\chi_{D_s}(yz)\frac{1}{(\frac{1-s^2}{2s})^{1/2}(zy)^{1/4}}. \end{aligned} \tag{3.8}$$

Here we have used the fact that

$$-\frac12\left(s+\frac1s\right)+\frac{1-s^2}{2s}=-s.$$

Analogously, by (3.7), $\chi_{D_s^{\mathrm c}}(yz)R_\alpha(y,z,s)$ is bounded by a constant times

$$\frac12\,\frac{1-s^2}{2s}\,e^{-\frac{1}{4s}(z^{1/2}-y^{1/2})^2}\chi_{D_s^{\mathrm c}}(yz)\left(\frac{1-s^2}{2s}\,(yz)^{1/2}\right)^{\alpha}. \tag{3.9}$$

We define

$$H_{\alpha,1}(s,y)=\int_0^\infty \chi_{D_s}(yz)R_\alpha(y,z,s)f(z)\,dz,$$

$$H_{\alpha,2}(s,y)=\int_0^\infty \chi_{D_s^{\mathrm c}}(yz)R_\alpha(y,z,s)f(z)\,dz.$$

Given $y,s>0$, for every integer k we define

$$B_k(y)=\{z:2^k s^{1/2}<|z^{1/2}-y^{1/2}|\le 2^{k+1}s^{1/2}\}.$$

Let k_0 be an integer to be fixed later. Then

$$\begin{aligned}(3.10)\quad H_{\alpha,1}(s,y) &\le C_\alpha \sum_{k=-\infty}^{k_0} \frac{1-s^2}{2s}\, e^{-2^{2k}/4} \int\limits_{B_k(y)} \chi_{D_s}(yz)\, \frac{f(z)\,dz}{\left(\frac{1-s^2}{2s}\right)^{1/2}(zy)^{1/4}} \\ &\quad + C_\alpha \sum_{k=k_0+1}^{\infty} \frac{1-s^2}{2s}\, e^{-2^{2k}/4} \int\limits_{B_k(y)} \chi_{D_s}(yz)\, \frac{f(z)\,dz}{\left(\frac{1-s^2}{2s}\right)^{1/2}(zy)^{1/4}} \\ &= H_{\alpha,11}(s,y) + H_{\alpha,12}(s,y).\end{aligned}$$

For the same k_0 and $B_k(y)$, $H_{\alpha,2}(s,y)$ is bounded by a constant times

$$\begin{aligned}(3.11)\quad & \sum_{k=-\infty}^{k_0} \frac{1-s^2}{2s}\, e^{-2^{2k}/4} \int\limits_{B_k(y)} \chi_{D_s^c}(yz) \left(\frac{1-s^2}{2s}\,(yz)^{1/2}\right)^{\alpha} f(z)\,dz \\ &\quad + \sum_{k=k_0+1}^{\infty} \frac{1-s^2}{2s}\, e^{-2^{2k}/4} \int\limits_{B_k(y)} \chi_{D_s^c}(yz) \left(\frac{1-s^2}{2s}\,(yz)^{1/2}\right)^{\alpha} f(z)\,dz \\ &= H_{\alpha,21}(s,y) + H_{\alpha,22}(s,y).\end{aligned}$$

Given $y, s > 0$, let k_0 be the unique integer satisfying

$$2^{k_0+2}s^{1/2} < y^{1/2} \le 2^{k_0+3}s^{1/2}.$$

If $k \le k_0$ and $z \in B_k(y)$ then, since $|z^{1/2} - y^{1/2}| \le 2^{k+1}s^{1/2}$, we get

$$(3.12)\quad y/4 \le (y^{1/2} - 2^{k+1}s^{1/2})^2 \le z \le (y^{1/2} + 2^{k+1}s^{1/2})^2 \le 3y.$$

In particular,

$$(3.13)\quad y/4 \le z \le 3y.$$

If $k \ge k_0$ and $z \in B_k(y)$, since $|z^{1/2} - y^{1/2}| \le 2^{k+1}s^{1/2}$, we get

$$(3.14)\quad 0 < z \le 136 \cdot 2^{2k}s \quad \text{and} \quad 0 < y \le 100 \cdot 2^{2k}s.$$

LEMMA 3. *Let $\alpha > -1$. We have the following estimates for the heat diffusion integral $W^\alpha f(t,y)$:*

(a) *If $-1 < \alpha \le 0$, we set $\beta = -\alpha$. Then*

$$\begin{aligned}(3.15)\quad & W^\alpha f(t,y) \\ &\quad \le C_\alpha \{e^{-t/4} M_0 f(y) + e^{-t(1-\beta)/2} y^{-\beta/2} M_\beta(z^{-\beta/2} f(z))(y)\}.\end{aligned}$$

(b) *If $\alpha \ge 0$, then*

$$\begin{aligned}(3.16)\quad & W^\alpha f(t,y) \\ &\quad \le C_\alpha e^{-t/4} \left\{ M^R f(y) + M^+ f(y) + y^{-\alpha/2}\, \frac{1}{y} \int_0^y z^{\alpha/2} f(z)\,dz \right\}.\end{aligned}$$

Proof. We will estimate $H_{\alpha,11}(s,y)$, $H_{\alpha,12}(s,y)$, $H_{\alpha,21}(s,y)$, $H_{\alpha,22}(s,y)$ for $\alpha > -1$. We observe that

$$\sum_{k=-\infty}^{\infty} e^{-2^{2k}/4} 2^{\varrho k} < \infty \quad \text{if } \varrho > 0.$$

Estimate of $H_{\alpha,11}(s,y)$ for $\alpha > -1$. By (3.10), (3.13) and (3.12), $H_{\alpha,11}(s,y)$ is less than or equal to a constant times the sum over $k \le k_0$ of the terms

$$\left(\frac{1-s^2}{2s}\right)^{1/2} e^{-2^{2k}/4} y^{-1/2} \int_{(y^{1/2}-2^{k+1}s^{1/2})^2}^{(y^{1/2}+2^{k+1}s^{1/2})^2} f(z)\,dz.$$

Clearly the above expression is bounded by a constant times

$$(1-s^2)^{1/2} e^{-2^{2k}/4} 2^k \,\frac{1}{4y^{1/2}2^{k+1}s^{1/2}} \int_{(y^{1/2}-2^{k+1}s^{1/2})^2}^{(y^{1/2}+2^{k+1}s^{1/2})^2} f(z)\,dz. \tag{3.17}$$

Then, considering (3.12), (3.17) and (3.1), and the fact $(y^{1/2}+2^{k+1}s^{1/2})^2 - (y^{1/2}-2^{k+1}s^{1/2})^2 = 4y^{1/2}2^{k+1}s^{1/2}$, we get

$$H_{\alpha,11}(s,y) \le C_\alpha (1-s^2)^{1/2} M^R f(y). \tag{3.18}$$

Estimate of $H_{\alpha,12}(s,y)$ for $\alpha > -1$. By (3.10), $H_{\alpha,12}(s,y)$ is bounded by a constant times the sum over $k > k_0$ of the terms

$$\frac{1-s^2}{2s}\, e^{-2^{2k}/4} \int_{B_k(y)} \chi_{D_s}(yz)\, \frac{f(z)\,dz}{\left(\frac{1-s^2}{2s}\right)^{1/2}(zy)^{1/4}}. \tag{3.19}$$

The condition $\chi_{D_s}(yz) = 1$ is equivalent to $z \ge \frac{1}{y}\left(\frac{2s}{1-s^2}\right)^2$, and by (3.14), $y \le 100 \cdot 2^{2k} s$.

Let $\gamma \ge 0$. Then (3.19) is bounded by a constant times

$$\begin{aligned}
&\frac{1-s^2}{2s}\, e^{-2^{2k}/4} \int_{\frac{1}{y}(\frac{2s}{1-s^2})^2}^{100\cdot 2^{2k}s} \chi_{D_s}(yz)\, \frac{f(z)\,dz}{\left(\frac{1-s^2}{2s}\right)^{1/2}(zy)^{1/4}} \\
&\quad = \left(\frac{1-s^2}{2s}\right) e^{-2^{2k}/4} \int_{\frac{1}{y}(\frac{2s}{1-s^2})^2}^{y} \chi_{D_s}(yz)\, \frac{z^\gamma f(z)\,dz}{\left(\frac{1-s^2}{2s}\right)^{1/2}(zy)^{1/4} z^\gamma} \\
&\qquad + \left(\frac{1-s^2}{2s}\right) e^{-2^{2k}/4} \int_{y}^{100\cdot 2^{2k}s} \chi_{D_s}(yz)\, \frac{f(z)\,dz}{\left(\frac{1-s^2}{2s}\right)^{1/2}(zy)^{1/4}}
\end{aligned} \tag{3.20}$$

$$\le C(1-s^2)^{1+2\gamma} e^{-2^{2k}/4} 2^{2(1+2\gamma)k} y^{-\gamma} \frac{1}{y} \int_0^y z^\gamma f(z)\,dz$$
$$+ C(1-s^2) e^{-2^{2k}/4} 2^{2k} \frac{1}{100\,2^{2k} s} \int_y^{100\cdot 2^{2k} s} f(z)\,dz.$$

Thus, for any $\gamma \ge 0$, $H_{\alpha,12}(s,y)$ is bounded by a constant times

$$(1-s^2)^{1+2\gamma} y^{-\gamma} \frac{1}{y} \int_0^y z^\gamma f(z)\,dz + (1-s^2) M^+ f(y).$$

This implies that for $\gamma = 0$, we get

$$H_{\alpha,12}(s,y) \le C_\alpha (1-s^2) M_0 f(y), \tag{3.21}$$

and for $\alpha \ge 0$, taking $\gamma = \alpha/2$, we get

$$\begin{aligned} H_{\alpha,12}(s,y) \le C_\alpha (1-s^2)^{1+\alpha} y^{-\alpha/2} \frac{1}{y} \int_0^y z^{\alpha/2} f(z)\,dz \\ + (1-s^2) M^+ f(y). \end{aligned} \tag{3.22}$$

Estimate of $H_{\alpha,21}(s,y)$ *for* $-1<\alpha<0$. Let $\beta = -\alpha$. By (3.11)–(3.13), $H_{\alpha,21}(s,y)$ is bounded by a constant times the sum over $k \le k_0$ of the terms

$$\frac{1-s^2}{2s} e^{-2^{2k}/4} \int_{B_k(y)} \chi_{D_s^c}(yz) \left(\frac{1-s^2}{2s} (yz)^{1/2} \right)^\alpha f(z)\,dz$$
$$\le C\left(\frac{1-s^2}{2s}\right)^{1+\alpha} e^{-2^{2k}/4} y^{\alpha/2} \int_{(y^{1/2}-2^{k+1}s^{1/2})^2}^{(y^{1/2}+2^{k+1}s^{1/2})^2} \chi_{D_s^c}(yz) z^{\alpha/2} f(z)\,dz.$$

If $\frac{1-s^2}{2s}(yy/4)^{1/2} > 1$, then for $z \ge y/4$ we have $\frac{1-s^2}{2s}(yz)^{1/2} > 1$, thus $\chi_{D_s^c}(yz) = 0$ and the integral above is zero. Therefore, we can assume that $\frac{1-s^2}{2s}(yy/4)^{1/2} \le 1$, which implies that

$$(1-s^2) y \le 4s. \tag{3.23}$$

Therefore $H_{\alpha,21}(s,y)$ is bounded by a constant times the sum over $k \le k_0$ of the terms

$$\left(\frac{1-s^2}{2s}\right)^{1+\alpha} e^{-2^{2k}/4} y^{\alpha/2} \frac{(4y^{1/2}2^{k+1}s^{1/2})^{1+\alpha}}{(4y^{1/2}2^{k+1}s^{1/2})^{1+\alpha}} \int_{(y^{1/2}-2^{k+1}s^{1/2})^2}^{(y^{1/2}+2^{k+1}s^{1/2})^2} f(z) z^{\alpha/2}\,dz$$
$$\le C_\alpha (1-s^2)^{(1-\beta)/2} e^{-2^{2k}/4} 2^{2k(1+\alpha)} y^{-\beta/2} M_\beta(f(z) z^{-\beta/2})(y).$$

Thus, summing up over $k \le k_0$, we get

$$H_{\alpha,21}(s,y) \le C_\alpha (1-s^2)^{(1-\beta)/2} y^{-\beta/2} M_\beta(f(z) z^{-\beta/2})(y) \tag{3.24}$$

for $-1<\alpha<0$.

Estimate of $H_{\alpha,21}(s,y)$ for $\alpha \geq 0$. By (3.11)–(3.13), $H_{\alpha,21}(s,y)$ is bounded by a constant times the sum over $k \leq k_0$ of the terms

$$\frac{1-s^2}{2s}e^{-2^{2k}/4}\int_{B_k(y)}\chi_{D_s^c}(yz)\left(\frac{1-s^2}{2s}(yz)^{1/2}\right)^{\alpha}f(z)\,dz \tag{3.25}$$

$$\leq C_\alpha\left(\frac{1-s^2}{2s}\right)^{1+\alpha}e^{-2^{2k}/4}y^{\alpha}\int_{(y^{1/2}-2^{k+1}s^{1/2})^2}^{(y^{1/2}+2^{k+1}s^{1/2})^2}\chi_{D_s^c}(yz)f(z)\,dz$$

$$= C_\alpha\left(\frac{1-s^2}{2s}\right)^{1+\alpha}e^{-2^{2k}/4}y^{\alpha}\,\frac{4y^{1/2}2^{k+1}s^{1/2}}{4y^{1/2}2^{k+1}s^{1/2}}\int_{(y^{1/2}-2^{k+1}s^{1/2})^2}^{(y^{1/2}+2^{k+1}s^{1/2})^2}\chi_{D_s^c}(yz)f(z)\,dz.$$

By using (3.23) and the fact that $\alpha \geq 0$, we see that (3.25) is bounded by a constant times

$$(1-s^2)^{1/2}\frac{e^{-2^{2k}/4}2^k}{4y^{1/2}2^{k+1}s^{1/2}}\int_{(y^{1/2}-2^{k+1}s^{1/2})^2}^{(y^{1/2}+2^{k+1}s^{1/2})^2}f(z)\,dz \leq C(1-s^2)^{1/2}e^{-2^{2k}/4}2^kM^Rf(y).$$

Thus,

$$H_{\alpha,21}(s,y) \leq C_\alpha(1-s^2)^{1/2}M^Rf(y). \tag{3.26}$$

Estimate of $H_{\alpha,22}(s,y)$ for $-1 < \alpha < 0$. Let $\beta = -\alpha$. By (3.11) and (3.14), $H_{\alpha,22}(s,y)$ is bounded by a constant times the sum over $k > k_0$ of the terms

$$\frac{1-s^2}{2s}\,e^{-2^{2k}/4}\int_{B_k(y)}\chi_{D_s^c}(zy)\left(\frac{1-s^2}{2s}\,(zy)^{1/2}\right)^{-\beta}f(z)\,dz$$

$$\leq\left(\frac{1-s^2}{2s}\right)^{1-\beta}e^{-2^{2k}/4}y^{-\beta/2}\left(\frac{100\cdot 2^{2k}s}{100\cdot 2^{2k}s}\right)^{1-\beta}\int_0^{100\cdot 2^{2k}s}\chi_{D_s^c}(zy)z^{-\beta/2}f(z)\,dz.$$

The above expression is smaller than or equal to a constant times

$$(1-s^2)^{1-\beta}e^{-2^{2k}/4}2^{2k(1-\beta)}y^{-\beta/2}\frac{1}{(100\cdot 2^{2k}s)^{1-\beta}}\int_0^{100\cdot 2^{2k}s}z^{-\beta/2}f(z)\,dz$$
$$\leq (1-s^2)^{1-\beta}e^{-2^{2k}/4}2^{2k(1-\beta)}y^{-\beta/2}M_\beta(z^{-\beta/2}f(z))(y).$$

Hence, for $-1 < \alpha < 0$, $H_{\alpha,22}(s,y)$ is bounded by a constant times

$$(1-s^2)^{1-\beta}y^{-\beta/2}M_\beta(z^{-\beta/2}f(z))(y). \tag{3.27}$$

Estimate of $H_{\alpha,22}(s,y)$ for $\alpha \geq 0$. By (3.11) and (3.14), $H_{\alpha,22}(s,y)$ is bounded by a constant times the sum over $k > k_0$ of the terms

$$\frac{1-s^2}{2s}\,e^{-2^{2k}/4}\int\limits_{B_k(y)}\chi_{D_s^c}(yz)\left(\frac{1-s^2}{2s}\,(yz)^{1/2}\right)^{\alpha}f(z)\,dz \tag{3.28}$$
$$\leq \left(\frac{1-s^2}{2s}\right)^{1+\alpha}e^{-2^{2k}/4}y^{\alpha/2}\int_0^y\chi_{D_s^c}(yz)z^{\alpha/2}f(z)\,dz$$
$$+\left(\frac{1-s^2}{2s}\right)^{1+\alpha}e^{-2^{2k}/4}y^{\alpha/2}\int_y^{100\cdot 2^{2k}s}\chi_{D_s^c}(yz)z^{\alpha/2}f(z)\,dz$$
$$\leq \left(\frac{1-s^2}{2s}\right)^{1+\alpha}e^{-2^{2k}/4}y^{\alpha/2}\int_0^y f(z)z^{\alpha/2}\,dz$$
$$+\left(\frac{1-s^2}{2s}\right)^{1+\alpha}e^{-2^{2k}/4}y^{\alpha/2}(100\cdot 2^{2k}s)^{\alpha/2}\int_y^{100\cdot 2^{2k}s}f(z)\,dz.$$

Since $y \leq 100\cdot 2^{2k}s$, we find that (3.28) is bounded by a constant times

$$(1-s^2)^{1+\alpha}e^{-2^{2k}/4}2^{2(1+\alpha)k}y^{-\alpha/2}\,\frac{1}{y}\int_0^y z^{\alpha/2}f(z)\,dz$$
$$+(1-s^2)^{1+\alpha}e^{-2^{2k}/4}2^{2(1+\alpha)k}\,\frac{1}{100\cdot 2^{2k}s}\int_y^{100\cdot 2^{2k}s}f(z)\,dz.$$

Thus, we have shown that, for $\alpha \geq 0$, $H_{\alpha,22}(s,y)$ is bounded by a constant times

$$(1-s^2)^{1+\alpha}\left(y^{-\alpha/2}\,\frac{1}{y}\int_0^y z^{\alpha/2}f(z)\,dz + M^+f(y)\right). \tag{3.29}$$

Now, taking into account (3.5) and (3.6), part (a) of Lemma 3 follows from (3.18), (3.20), (3.24), (3.27), and part (b) follows from (3.18), (3.22), (3.26), and (3.29). Thus, Lemma 3 is proved. ■

4. Proof of the main results

Proof of Theorem 1. As usual, set $\beta = -\alpha$. By Lemma 3, we have

$$W^{\alpha,*}f(y) \leq C_\alpha\{M_0f(y) + y^{-\beta/2}M_\beta(z^{-\beta/2}f(z))(y)\}.$$

Thus, applying Lemma 2, we get

$$W^{\alpha,*}f(y) \leq C_\beta\{y^{\beta/2}M_0(f(z)z^{-\beta/2})(y) + y^{-\beta/2}M_0(f(z)z^{\beta/2})(y) + M_0f(y)\}.$$

The hypothesis "if $-1 < \alpha < 0$, then $p \in (a_\alpha, b_\alpha)$" is equivalent to $-1 < \delta - p\beta/2 \leq \delta + p\beta/2 < p-1$, and $p > 1$. Under these conditions, the weights

$y^{\delta+p\beta/2}$, $y^{\delta-p\beta/2}$ and y^{δ} belong to the class A_p of Muckenhoupt, thus

$$\int_0^\infty (y^{\beta/2}M(f(z)z^{-\beta/2})(y))^p y^\delta \, dy \le C_{\alpha,p} \int_0^\infty f(y)^p y^\delta \, dy,$$

$$\int_0^\infty (y^{-\beta/2}M(f(z)z^{\beta/2})(y))^p y^\delta \, dy \le C_{\alpha,p} \int_0^\infty f(y)^p y^\delta \, dy,$$

$$\int_0^\infty M_0 f(y)^p y^\delta \, dy \le C_{\alpha,p} \int_0^\infty f(y)^p y^\delta \, dy,$$

proving that $W^{\alpha,*}$ is of strong type (p,p) for $p \in N_\alpha$ with respect to the measure $y^\delta dy$ if $-1 < \alpha < 0$.

Now, let $\alpha \ge 0$. By (3.16) of Lemma 3, we have

$$W^\alpha f(t,y) \le C_\alpha \left\{ M^R f(y) + M^+ f(y) + y^{-\alpha/2} \frac{1}{y} \int_0^y z^{\alpha/2} f(z) \, dz \right\}.$$

For $M^R f(y)$, by Lemma 1, for any $p > 1$ and any $\delta > -1$, we have

$$\int_0^\infty M^R f(y)^p y^\delta \, dy \le C_{p,\delta} \int_0^\infty f(y)^p y^\delta \, dy.$$

For $M^+ f(y)$, since $y^\delta \in A_1^+ \subset A_p^+$ for any $\delta > -1$, as mentioned in the introduction, we have

$$\int_0^\infty M^+ f(y)^p y^\delta \, dy \le C_{p,\delta} \int_0^\infty f(y)^p y^\delta \, dy.$$

Finally,

$$y^{-\alpha/2} \frac{1}{y} \int_0^y z^{\alpha/2} f(z) \, dz \le 2 y^{-\alpha/2} M_0(f(z) z^{\alpha/2})(y).$$

Thus, if $-1 < \delta - p\,\alpha/2 < p - 1$ and $p > 1$ we have

$$\begin{aligned} \int_0^\infty \left(y^{-\alpha/2} \frac{1}{y} \int_0^y z^{\alpha/2} f(z) \, dz \right)^p y^\delta \, dy &\le \int_0^\infty (y^{-\alpha/2} M_0(f(z) z^{\alpha/2})(y))^p y^\delta \, dy \\ &\le C_{p,\alpha,\delta} \int_0^\infty f(y)^p y^\delta \, dy. \end{aligned}$$

The conditions $-1 < \delta - p\,\alpha/2 < p - 1$, $p > 1$ are equivalent to $p > 2(1+\delta)/(\alpha+2)$, $p > 1$, and $p < 2(1+\delta)/\alpha$. In order to finish the proof we need to show that the condition $p < 2(1+\delta)/\alpha$ can be removed.

Observe that $W^{0,*} f(y)$ is bounded by a constant times $M_0 f(y)$, so $W^{0,*}$ is of strong type (p,p) for $-1 < \delta < p - 1$ and $p > 1$ with respect to the measure $y^\delta dy$.

Assume that $0 \le \alpha < 2$ and $p \ge 2(1+\delta)/\alpha$. Then $p \ge 2(1+\delta)/\alpha > 2(1+\delta)/(0+2) = 1+\delta$. Since by (3.7) we have $W^{\alpha,*}f(y) \le C_\alpha W^{0,*}f(y)$, it follows that $W^{\alpha,*}$ is of strong type (p,p) for $p \ge 2(1+\delta)/\alpha$. We have showed the result for α in the range $0 \le \alpha < 0+2$. Now the result follows by induction on $[j, j+2)$. ■

Proof of Theorem 2. (a) If $-1 < \alpha < 0$ and $2(1+\delta)/(-\alpha) > 1$, the upper end point of N_α is $2(1+\delta)/(-\alpha)$. For s fixed, $0 < s < 1$, consider points y and z satisfying

$$\frac{1-s^2}{2s}\, y \le 1 \quad \text{and} \quad \frac{1-s^2}{2s}\, z \le 1.$$

By (3.4), using (3.7), we have

$$R_\alpha(y,z,s) \ge C_{\alpha,s} y^{\alpha/2} z^{\alpha/2}.$$

Thus, setting $a = 2s/(1-s^2)$, we get

$$W^\alpha(\chi_{(0,a)})(s,y) \ge C_{\alpha,s} y^{\alpha/2} \int_0^a z^{\alpha/2}\, dz = C_{\alpha,s} y^{\alpha/2}$$

for every $0 \le y \le a$. Since

$$\int_0^a (y^{\alpha/2})^{2(1+\delta)/(-\alpha)} y^\delta\, dy = \int_0^a y^{-1}\, dy = \infty,$$

it follows that the operator $W^{\alpha,*}$ is not of strong type $(2(1+\delta)/(-\alpha), 2(1+\delta)/(-\alpha))$ with respect to the measure $y^\delta dy$. However, it is of weak type. In fact, let $\beta = -\alpha$; it will be enough to show that the three terms on the right hand side of (3.2) satisfy the weak type condition. Since $-1 < \alpha < 0$ implies $-1 < \delta < 2(1+\delta)/(-\alpha) - 1$, the third term of (3.2) is of strong type $(2(1+\delta)/(-\alpha), 2(1+\delta)/(-\alpha))$ with respect to the measure $y^\delta dy$.

The first term is bounded by $y^{(-\alpha)/2} M_0(f(z)z^{\alpha/2})(y)$ and since

$$-1 < ((-\alpha)/2)2(1+\delta)/(-\alpha) + \delta < 2(1+\delta)/(-\alpha) - 1,$$

the weight $y^{((-\alpha)/2)2(1+\delta)/(-\alpha)+\delta}$ is in $A_{2(1+\delta)/(-\alpha)}$. This shows that

$$\int_0^\infty (y^{(-\alpha)/2} M_0(f(z)z^{\alpha/2})(y))^{2(1+\delta)/(-\alpha)} y^\delta\, dy \le C_{\alpha,\delta} \int f(y)^{2(1+\delta)/(-\alpha)} y^\delta\, dy,$$

which implies the strong type $(2(1+\delta)/(-\alpha), 2(1+\delta)/(-\alpha))$ of the first term of (3.2) with respect to the measure $y^\delta dy$.

Consider now the second term of (3.2). If we denote $2(1+\delta)/(-\alpha)$ by p, then $p' = 2(1+\delta)/(2(1+\delta)+\alpha)$. By Hölder's inequality, we obtain

$$\begin{aligned}\frac{1}{(2h)^{1+\alpha}} \int_y^{y+h} & f(z) z^{\alpha/2}\, dz \\ & \le \frac{1}{(2h)^{1+\alpha}} \|f\|_{L^p((y,y+h),z^\delta\, dz)} \|z^{\alpha/2-\delta}\|_{L^{p'}((y,y+h),z^\delta\, dz)}.\end{aligned}$$

In order to estimate $\|z^{\alpha/2-\delta}\|_{L^{p'}((y,y+h),z^\delta\,dz)}$ we observe that

$$\delta+(\alpha/2-\delta)p'>-1 \quad \text{and} \quad (\delta+(\alpha/2-\delta)p'+1)/p'=1+\alpha.$$

Then $\|z^{\alpha/2-\delta}\|_{L^{p'}((y,y+h),z^\delta\,dz)} \le c_{\delta,\beta}(y+h)^{1+\alpha}$. Thus, since $y\le 2h$, we have

$$\begin{aligned}\frac{1}{(2h)^{1+\alpha}}\int_y^{y+h} f(z)z^{\alpha/2}\,dz &\le C_{\delta,\alpha}\left(\frac{y+h}{h}\right)^{1+\alpha}\|f\|_{L^p((y,y+h),z^\delta\,dz)}\\ &\le C_{\delta,\alpha}\|f\|_{L^p((0,\infty),z^\delta\,dz)}.\end{aligned}$$

Multiplying by $y^{-\beta/2}$ and taking the supremum over $h\ge y/2$, we obtain

$$\sup_{h\ge y/2} y^{\alpha/2}\frac{1}{(2h)^{1+\alpha}}\int_y^{y+h} f(z)z^{-\beta/2}\,dz \le C_{\delta,\beta}y^{\alpha/2}\|f\|_{L^p((0,\infty),z^\delta\,dz)}.$$

From this inequality the weak type (p,p) for $p=2(1+\delta)/(-\alpha)$ with respect to the measure $y^\delta dy$ is readily obtained.

(b) If $\alpha\ge 0$, the upper end point of N_α is ∞, and by (3.7) and (3.16), we have $W^{\alpha,*}f(y)\le C_\alpha W^{0,*}f(y)\le C_\alpha M_0 f(y)$. Therefore since $L^\infty((0,\infty),y^\delta dy) = L^\infty((0,\infty),dy)$ for $\delta>-1$, the operator $W^{\alpha,*}$ is of strong type (∞,∞) with respect to the measure $y^\delta dy$.

(c) If the lower end point of N_α is greater than 1, then it coincides with $2(1+\delta)/(2+\alpha)$. This implies that $2\delta-\alpha>0$. If for a given $a>0$ the integral $\int_0^a f(z)z^{\alpha/2}\,dz = \int_0^a f(z)z^{\alpha/2-\delta}z^\delta\,dz$ is finite for every $f\in L^{2(1+\delta)/(2-\beta)}((0,a),z^\delta\,dz)$, then since

$$\left(\frac{2(1+\delta)}{2+\alpha}\right)' = \frac{2(1+\delta)}{2\delta-\alpha},$$

by uniform boundedness, it follows that $z^{\alpha/2-\delta}\in L^{2(1+\delta)/(2\delta-\alpha)}((0,a),z^\delta dz)$. This is a contradiction since $z^{(\alpha/2-\delta)2(1+\delta)/(2\delta-\alpha)+\delta}=z^{-1}$. Therefore, there exists $f\in L^{2(1+\delta)/(2+\alpha)}((0,a),z^\delta dz)$ such that $\int_0^a f(z)z^{\alpha/2}\,dz=\infty$. For this f, if $a=2s/(1-s^2)$, then

$$\int_0^a R(s,y,z)f(z)\,dz \ge C_{\alpha,s}y^{\alpha/2}\int_0^a z^{\alpha/2}f(z)\,dz=\infty,$$

showing that $W^{\alpha,*}f(y)=\infty$ for every $y\le a$. This tells us that the operator $W^{\alpha,*}$ cannot be of weak type at the lower end point $2(1+\delta)/(2-\beta)>1$ with respect to the measure $y^\delta dy$.

Now we shall prove the restricted type. Let $-1<\alpha<0$ and $\beta=-\alpha$. By (3.15) and Lemma 2, we have

$$\begin{aligned}(4.1)\qquad & W^{\alpha,*}f(y)\\ &\le C_\beta\left\{M_0 f(y)+y^{\beta/2}\frac{1}{y}\int_0^y f(z)z^{-\beta/2}\,dz + y^{-\beta/2}M_0(f(z)z^{\beta/2})(y)\right\}.\end{aligned}$$

It is easy to see that

$$-1 < \delta < 2(1+\delta)/(2-\beta) - 1,$$
$$-1 < \delta - (\beta/2)2(1+\delta)/(2-\beta) < 2(1+\delta)/(2-\beta) - 1.$$

These inequalities imply that the weights y^δ and $y^{\delta-(\beta/2)2(1+\delta)/(2-\beta)}$ belong to $A_{2(1+\delta)/(2-\beta)}$. Therefore, the operators defined by

$$M_0 f(y) \quad \text{and} \quad y^{-\beta/2} M_0(f(z)z^{\beta/2})(y)$$

are of strong type $(2(1+\delta)/(2-\beta), 2(1+\delta)/(2-\beta))$ with respect to the measure $y^\delta dy$. We have not considered the second term of (4.1) yet. If $\alpha \geq 0$, by (3.16), we have

$$(4.2)\quad W^{\alpha,*}f(y) \leq C_\alpha \Big\{ M^R f(y) + y^{-\alpha/2}\frac{1}{y}\int_0^y z^{\alpha/2} f(z)\,dz + M^+ f(y) \Big\}.$$

By Lemma 1, the first term on the right hand side of (4.2) is of weak type for any $p \geq 1$ and any $\delta > -1$. As mentioned in the introduction, the weight y^δ belongs to the class $A_1^+ \subset A_p^+$ of Sawyer for $\delta > -1$, so the operator M^+ is of weak type (p,p) for any $p \geq 1$ with respect to the measure $y^\delta dy$ for any $\delta > -1$.

Now we are going to consider the second terms on the right hand side of both (4.1) and (4.2). They are of the form $y^{-\alpha/2}(1/y)\int_0^y z^{\alpha/2} f(z)\,dz$ allowing $\alpha > -1$. Let E be a measurable set contained in $(0,\infty)$ and F the set defined by $\chi_E(u^{1/(1+\delta)}) = \chi_F(u)$. By the change of variables $z = u^{1/(1+\delta)}$, we have

$$(4.3)\qquad \int_0^\infty \chi_E(z) z^\delta\,dz = \frac{1}{1+\delta}\int_0^\infty \chi_E(u^{1/(1+\delta)})\,du = \frac{1}{1+\delta}\,|F|,$$

and

$$\begin{aligned}\int_0^y \chi_E(z) z^{\alpha/2}\,dz &= \frac{1}{1+\delta}\int_0^{y^{1+\delta}} \chi_E(u^{1/(1+\delta)})u^{(\alpha/2-\delta)/(1+\delta)}du \\ &= \frac{1}{1+\delta}\int_0^{y^{1+\delta}} \chi_F(u)u^{(\alpha/2-\delta)/(1+\delta)}du.\end{aligned}$$

Since $2(1+\delta)/(2+\alpha) > 1$ implies $\alpha/2 - \delta < 0$, it follows that

$$\begin{aligned}\int_0^{y^{1+\delta}} \chi_F(u)u^{(\alpha/2-\delta)/(1+\delta)}\,du &\leq \int_0^\infty \chi_F(u)u^{(\alpha/2-\delta)/(1+\delta)}\,du \\ &\leq \int_0^{|F|} u^{(\alpha/2-\delta)/(1+\delta)}du.\end{aligned}$$

Taking into account that $\alpha > -1$ implies $(\alpha/2-\delta)/(1+\delta) > -1$, we can compute the last integral above, obtaining

$$\int_0^{|F|} u^{(\alpha/2-\delta)/(1+\delta)} du = \frac{2(1+\delta)}{\alpha+2}\,|F|^{(\alpha+2)/2(1+\delta)}.$$

Then, by (4.3), we get

$$\begin{aligned}\int_0^{|F|} u^{\alpha/2-\delta/(1+\delta)}\,du &= \frac{2(1+\delta)}{2-\beta}\Big((1+\delta)\int_0^\infty \chi_E(z)z^\delta\,dz\Big)^{(\alpha+2)/2(1+\delta)}\\ &= c_{\alpha,\delta}\Big(\int_0^\infty \chi_E(z)z^\delta\,dz\Big)^{(\alpha+2)/2(1+\delta)}.\end{aligned}$$

In consequence,

$$y^{-\alpha/2}\,\frac{1}{y}\int_0^y \chi_E(z)z^{\alpha/2}\,dz \le c_{\alpha,\delta}y^{-\alpha/2}\,\frac{1}{y}\Big(\int_0^\infty \chi_E(u)u^\delta\,du\Big)^{(\alpha+2)/2(1+\delta)}.$$

From this, the restricted weak type $(2(1+\delta)/(2+\alpha), 2(1+\delta)/(2+\alpha))$ for the operator $W^{\alpha,*}f$ with respect to the measure $y^\delta dy$ is readily obtained.

(d) Let us show that if the lower end point of N_α is 1, then the operator $W^{\alpha,*}$ cannot be of strong type $(1,1)$ with respect to the measure $y^\alpha dy$. In fact, by (3.7), we have

$$\begin{aligned}&\chi_{D_s}(yz)R_\alpha(y,z,s)\\ &\quad\ge C_\alpha\Big(\frac{1-s^2}{2s}\Big)^{1/2} e^{-\frac{1}{4s}(y^{1/2}-z^{1/2})^2}e^{-\frac{s}{4}(y^{1/2}-z^{1/2})^2}e^{-s(yz)^{1/2}}\chi_{D_s}(yz)\,\frac{1}{(yz)^{1/4}}.\end{aligned}$$

Take $0<\varepsilon\le 1$. Assume that $1<z\le 1+\varepsilon$, $1+2\varepsilon\le y\le 2$, and $s=(y-1)^2/4$. Then $s\le 1/4$, $\frac{1-s^2}{2s}\ge 1$, and $\big(\frac{1-s^2}{2s}\big)^{1/2}(yz)^{1/4}\ge 1$. Thus $\chi_{D_s}(yz)=1$ and since

$$\begin{aligned}\frac{1}{4s}(y^{1/2}-z^{1/2})^2 = \frac{(y^{1/2}-z^{1/2})^2}{(y-1)^2} &\le \Big(\frac{(y-z)}{2(y-1)}\Big)^2\\ &\le \Big(\frac12+\frac{|1-z|}{2(y-1)}\Big)^2 \le \Big(\frac12+\frac{\varepsilon}{4\varepsilon}\Big)^2\le 1,\end{aligned}$$

we get $R_\alpha(y,z,s)\ge C_\alpha/(y-1)$, and therefore

$$W^{\alpha,*}(\chi_{(1,1+\varepsilon)})(y) \ge \frac{C_\alpha}{y-1}\int_0^\infty \chi_{(1,1+\varepsilon)}(z)\,dz = C_\alpha\,\frac{\varepsilon}{y-1}$$

for $1+2\varepsilon\le y\le 2$. Thus, if $W^{\alpha,*}$ were of strong type $(1,1)$ with respect to

$y^\delta dy$, and recalling that $\delta > -1$, we would have

$$\int_0^\infty W^{\alpha,*}(\chi_{(1,1+\varepsilon)})(y) y^\delta \, dy \le A_\alpha \int_0^\infty \chi_{(1,1+\varepsilon)}(y) y^\delta \, dy = A_\alpha \frac{(1+\varepsilon)^{1+\delta}-1}{1+\delta} \le A_{\alpha,\delta}\, \varepsilon \tag{4.4}$$

for a finite constant $A_{\alpha,\delta}$ depending on α and δ only. On the other hand, we get

$$\int_{1+2\varepsilon}^{2} W^{\alpha,*}(\chi_{(1,1+\varepsilon)})(y) y^\delta \, dy \ge C_\alpha \int_{1+2\varepsilon}^{2} \frac{\varepsilon}{y-1} y^\delta \, dy \ge C_{\alpha,\delta} \int_{1+2\varepsilon}^{2} \frac{\varepsilon}{y-1} \, dy = C_{\alpha,\delta}\, \varepsilon \log(1/2\varepsilon). \tag{4.5}$$

In consequence, from (4.4) and (4.5), it follows that $C_{\alpha,\delta}\, \varepsilon \log(1/2\varepsilon) \le A_\alpha\, \varepsilon$, or also, that $C_{\alpha,\delta} \log(1/2\varepsilon) \le A_{\alpha,\delta}$. This is a contradiction since the left hand side of this inequality tends to ∞ as ε tends to 0, proving that $W^{\alpha,*}$ is not of strong type $(1,1)$ with respect to $y^\delta dy$.

However, as we are going to show, $W^{\alpha,*}$ is of weak type $(1,1)$ with respect to $y^\delta dy$. Since $2(1+\delta)/(2+\alpha) \le 1$, it follows that $2\delta - \alpha \le 0$. Notice that since N_α is not empty we always have $2(1+\delta)+\alpha \ge 0$, which is equivalent to $1+\alpha+\delta-\alpha/2 \ge 0$. Assume $-1 < \alpha < 0$, and let $\beta = -\alpha$. By (3.2) and (3.15) (Lemma 2), $W^{\alpha,*} f(y)$ is bounded by a constant times

$$M_0 f(y) + \sup_{y \le 2h} \left(\frac{y^{-\beta/2}}{(2h)^{1-\beta}} \int_y^{y+h} f(z) z^{-\beta/2} \, dz \right) + y^{\beta/2} \frac{1}{y} \int_0^y f(z) z^{-\beta/2} \, dz. \tag{4.6}$$

Since $2\delta + \beta = 2\delta - \alpha \le 0$ it follows that $-1 < \delta < -\beta/2 < 0$. Thus, M_0 is of weak type $(1,1)$ with respect to $y^\delta dy$. For the second term of (4.6), since $y \le 2h$ and $2\delta + \beta \le 0$, we have

$$\begin{aligned} \frac{y^{-\beta/2}}{(2h)^{1-\beta}} \int_y^{y+h} f(z) z^{-\beta/2} \, dz &\le \frac{y^{-\beta/2}}{(2h)^{1-\beta}} \int_0^{3h} f(t) z^\delta z^{-(\delta+\beta/2)} \, dz \\ &\le y^{-\beta/2} \frac{(3h)^{-(\delta+\beta/2)}}{(2h)^{1-\beta}} \int_0^{3h} f(z) z^\delta \, dz \\ &= c_{\alpha,\delta} y^{-\beta/2} \frac{1}{h^{1-\beta+(\delta+\beta/2)}} \int_0^{3h} f(z) z^\delta \, dz \\ &\le c_{\beta,\delta} \frac{1}{y^{1+\delta}} \int_0^\infty f(z) z^\delta \, dz, \end{aligned} \tag{4.7}$$

which clearly implies the weak type $(1,1)$ of the second term.

We still have to estimate the third term of (4.6). For $\alpha \geq 0$, from (3.16) we see that $W^{\alpha,*}f(y)$ is bounded by

$$C_\alpha\left\{M^R f(y) + M^+ f(y) + y^{-\alpha/2}\frac{1}{y}\int_0^y z^{\alpha/2} f(z)\,dz\right\}. \tag{4.8}$$

By Lemma 1, the first term of (4.8) is of weak type for any $1 \leq p \leq \infty$ for the measure $y^\delta dy$ for any δ. As mentioned before, $y^\delta \in A_1^+$ for any $\delta > -1$, therefore M^+ is of weak type $(1,1)$ with respect to the same measure $y^\delta dy$. For the third terms of (4.6) and (4.8), for $\alpha > -1$ we have

$$y^{-\alpha/2}\frac{1}{y}\int_0^y z^{\alpha/2} f(z)\,dz = y^{-\alpha/2}\frac{1}{y}\int_0^y z^{\alpha/2-\delta} f(z) z^\delta\,dz.$$

Since $\alpha/2 - \delta \geq 0$, this expression is bounded by

$$y^{-\alpha/2}\frac{y^{\alpha/2-\delta}}{y}\int_0^y f(z)z^\delta\,dz \leq \frac{1}{y^{1+\delta}}\int_0^\infty f(z)z^\delta\,dz.$$

This inequality and (4.8) imply the $(1,1)$ weak type of the operator $W^{\alpha,*}$ with respect to the measure $y^\delta dy$. ■

Acknowledgements. We thank the referee for his suggestions, which have contributed to clarify and improve the paper. We also thank B. Viviani for her enlightening comments.

References

[1] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Volume II, McGraw-Hill, 1953.

[2] M. de Guzmán, *Real Variable Methods in Fourier Analysis*, North-Holland Math. Stud. 46, North-Holland, 1981.

[3] R. A. Macías, C. Segovia and J. L. Torrea, *Heat diffusion maximal operators for Laguerre semigroups with negative parameters*, J. Funct. Anal., to appear.

[4] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207–226.

[5] E. Sawyer, *Weighted inequalities for the one-sided Hardy Litllewood maximal functions*, ibid. 297 (1986), 53–61.

[6] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Math. Ser. 30, Princeton Univ. Press, 1970.

[7] E. M. Stein and G. Weiss, *Fourier Analysis on Euclidean Spaces*, Princeton Math. Ser. 32, Princeton Univ. Press, 1971.

[8] K. Stempak, *Heat-diffusion and Poisson integrals for Laguerre expansions*, Tohoku Math. J. 46 (1994) , 83–104.

[9] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. 23, Amer. Math. Soc., 1939.

[10] S. Thangavelu, *Lectures on Hermite and Laguerre Expansions*, Math. Notes 24, Princeton Univ. Press, 1993.

IMAL-FIQ
CONICET–Universidad Nacional del Litoral
Güemes 3450
3000 Santa Fe, Argentina
E-mail: rmacias@ceride.gov.ar

Instituto Argentino de Matemática
(IAM)–CONICET
Saavedra 15
1083 Buenos Aires, Argentina
E-mail: segovia@iamba.edu.ar

Departamento de Matemáticas
Facultad de Ciencias
Universidad Autónoma de Madrid
28049 Madrid, Spain
E-mail: joseluis.torrea@uam.es

Received October 11, 2004
Revised version October 21, 2005 (5511)