STUDIA MATHEMATICA 172 (2) (2006)

Completely multi-positive linear maps between locally C^*-algebras and representations on Hilbert modules

by

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Abstract. A KSGNS (Kasparov, Stinespring, Gel'fand, Naimark, Segal) type construction for strict (respectively, covariant non-degenerate) completely multi-positive linear maps between locally C^*-algebras is described.

1. Introduction. It is well known that a positive linear functional on a C^*-algebra A induces a $*$-representation of A on a Hilbert space by the GNS (Gel'fand, Naimark, Segal) construction. Stinespring [12] extended this construction to completely positive linear maps from A to $L(H)$, the C^*-algebra of all bounded linear operators on a Hilbert space H. On the other hand, Paschke [9] (respectively, Kasparov [7]) showed that a completely positive linear map from A to another C^*-algebra B (respectively, from A to the C^*-algebra of all adjointable operators on the Hilbert C^*-module H_B) induces a $*$-representation of A on a Hilbert B-module. Kaplan [6] introduced the notion of n-positive linear functional on a C^*-algebra A, an $n \times n$ matrix of linear functionals on A which induces a positive linear map from $M_n(A)$ to $M_n(\mathbb{C})$, and showed that an n-positive linear functional on a C^*-algebra induces a $*$-representation of this C^*-algebra on a Hilbert space in terms of the GNS construction. On the other hand, Heo [1] generalized Kaplan's construction to a completely multi-positive linear map from a unital C^*-algebra A to another unital C^*-algebra B, an $n \times n$ matrix of linear maps from A to B which induces a completely positive linear map from $M_n(A)$ to $M_n(B)$, showing that a completely multi-positive linear map from A to B induces a $*$-representation of A on a Hilbert B-module.

Locally C^*-algebras are generalizations of C^*-algebras. Instead of being given by a single norm, the topology on a locally C^*-algebra is defined

2000 *Mathematics Subject Classification*: 46L05, 46L08, 46L40.

Key words and phrases: completely multi-positive linear maps, covariant completely multi-positive linear maps, locally C^*-algebras, Hilbert modules, representations of locally C^*-algebras.

by a directed family of C^*-seminorms. Such important concepts as Hilbert C^*-modules, adjointable operators, (completely) positive linear maps, (completely) multi-positive linear maps, C^*-dynamical systems can be defined with obvious modifications in the framework of locally C^*-algebras. The proofs are not always straightforward. In [2], the author extended the KSGNS (Kasparov, Stinespring, Gel'fand, Naimark, Segal) construction to a strict continuous, completely positive linear map from a locally C^*-algebra A to $L_B(E)$, the locally C^*-algebra of all adjointable operators on a Hilbert module E over a locally C^*-algebra B. A covariant version of this construction is proved in [5].

In this paper we extend the KSGNS construction to strict continuous, completely multi-positive linear maps from a locally C^*-algebra A to $L_B(E)$ and prove a covariant version of this construction. That is, we consider multi-positive linear maps between locally C^*-algebras, not necessarily unital, and show that a strict completely multi-positive linear map $[\varrho] = [\varrho_{ij}]_{i,j=1}^n$ from a locally C^*-algebra A to $L_B(E)$ induces a non-degenerate continuous $*$-representation $\Phi_{[\varrho]}$ of A on a Hilbert B-module $E_{[\varrho]}$, called the KSGNS representation of A associated with $[\varrho]$ (Theorem 3.4). Moreover, this representation is unique up to unitary equivalence. In particular, when A and B are unital C^*-algebras and $E = B$ we obtain Theorem 2.1 of [1]. Then we consider the covariant version of this construction: given a locally C^*-dynamical system (G, A, α), a covariant non-degenerate completely multi-positive linear map from A to $L_B(E)$ induces a non-degenerate, covariant representation of (G, A, α) on a Hilbert B-module which is unique up to unitary equivalence (Theorem 4.3). This construction extends Heo's construction associated with a covariant completely multi-positive linear map with respect to a unital C^*-dynamical system. Finally, as an application of this construction we show that given a locally C^*-dynamical system (G, A, α) such that α is a continuous inverse limit action, a covariant non-degenerate, completely multi-positive linear map from A to $L_B(E)$ extends to a non-degenerate, completely multi-positive linear map from $A \times_\alpha G$ to $L_B(E)$ (Proposition 4.5).

2. Preliminaries. A *locally C^*-algebra* is a complete complex Hausdorff topological $*$-algebra whose topology is determined by a directed family of C^*-seminorms. If A is a locally C^*-algebra and $S(A)$ is the set of all continuous C^*-seminorms on A, then for each $p \in S(A)$, the quotient algebra $A/\ker p$, denoted by A_p, is a C^*-algebra with the norm induced by p. The canonical map from A onto A_p is denoted by π_p. For $p, q \in S(A)$ with $p \geq q$, there is a canonical map π_{pq} from A_p onto A_q such that $\pi_{pq}(\pi_p(a)) = \pi_q(a)$ for all $a \in A$. Then $\{A_p; \pi_{pq}\}_{p,q \in S(A),\, p \geq q}$ is an inverse system of C^*-algebras and A can be identified with $\varprojlim_p A_p$.

An *approximate unit* of A is an increasing net $\{e_\lambda\}_{\lambda\in\Lambda}$ of positive elements of A such that $p(e_\lambda) \leq 1$ for all $p \in S(A)$ and for all $\lambda \in \Lambda$, $p(ae_\lambda - a) \to 0$ and $p(e_\lambda a - a) \to 0$ for all $p \in S(A)$ and $a \in A$. Any locally C^*-algebra has an approximate unit [10, Proposition 3.11].

A *morphism of locally C^*-algebras* is a continuous $*$-morphism Φ from a locally C^*-algebra A to another locally C^*-algebra B. An *isomorphism of locally C^*-algebras* is a bijective linear map Φ from A to B such that Φ and Φ^{-1} are morphisms of locally C^*-algebras.

Let $M_n(A)$ denote the $*$-algebra of all $n\times n$ matrices over A with the algebraic operations and the topology obtained by regarding it as a direct sum of n^2 copies of A. Then $\{M_n(A_p); \pi_{pq}^{(n)}\}_{p,q\in S(A),\, p\geq q}$, where $\pi_{pq}^{(n)}([\pi_p(a_{ij})]_{i,j=1}^n) = [\pi_q(a_{ij})]_{i,j=1}^n$, is an inverse system of C^*-algebras and $M_n(A)$ can be identified with $\varprojlim_p M_n(A_p)$.

A linear map $\varrho : A \to B$ between two locally C^*-algebras is *completely positive* if the linear maps $\varrho^{(n)} : M_n(A) \to M_n(B)$, $\varrho^{(n)}([a_{ij}]_{i,j=1}^n) = [\varrho(a_{ij})]_{i,j=1}^n$, $n = 1, 2, \ldots$, are all positive.

Definition 2.1. A *pre-Hilbert A-module* is a complex vector space E which is also a right A-module, compatible with the complex algebra structure, equipped with an A-valued inner product $\langle\cdot,\cdot\rangle : E \times E \to A$ which is $\mathbb{C}$- and A-linear in its second variable and satisfies the following relations:

(i) $\langle\xi,\eta\rangle^* = \langle\eta,\xi\rangle$ for every $\xi,\eta \in E$;
(ii) $\langle\xi,\xi\rangle \geq 0$ for every $\xi \in E$;
(iii) $\langle\xi,\xi\rangle = 0$ if and only if $\xi = 0$.

We say that E is a *Hilbert A-module* if E is complete with respect to the topology determined by the family $\{\|\cdot\|_p\}_{p\in S(A)}$ of seminorms, where $\|\xi\|_p = \sqrt{p(\langle\xi,\xi\rangle)}$ for $\xi \in E$ [10, Definition 4.1].

Let E be a Hilbert A-module. For $p \in S(A)$, $\mathcal{E}_p = \{\xi \in E;\, p(\langle\xi,\xi\rangle) = 0\}$ is a closed submodule of E and $E_p = E/\mathcal{E}_p$ is a Hilbert A_p-module with $(\xi + \mathcal{E}_p)\pi_p(a) = \xi a + \mathcal{E}_p$ and $\langle\xi + \mathcal{E}_p, \eta + \mathcal{E}_p\rangle = \pi_p(\langle\xi,\eta\rangle)$. The canonical map from E onto E_p is denoted by σ_p. For $p, q \in S(A)$ with $p \geq q$ there is a canonical morphism of vector spaces σ_{pq} from E_p onto E_q such that $\sigma_{pq}(\sigma_p(\xi)) = \sigma_q(\xi)$ for $\xi \in E$. Then $\{E_p; A_p; \sigma_{pq}\}_{p,q\in S(A),\, p\geq q}$ is an inverse system of Hilbert C^*-modules in the following sense: $\sigma_{pq}(\xi_p a_p) = \sigma_{pq}(\xi_p)\pi_{pq}(a_p)$ for $\xi_p \in E_p$ and $a_p \in A_p$; $\langle\sigma_{pq}(\xi_p), \sigma_{pq}(\eta_p)\rangle = \pi_{pq}(\langle\xi_p,\eta_p\rangle)$ for $\xi_p, \eta_p \in E_p$; $\sigma_{pp}(\xi_p) = \xi_p$ for $\xi_p \in E_p$ and $\sigma_{qr} \circ \sigma_{pq} = \sigma_{pr}$ if $p \geq q \geq r$, and $\varprojlim_p E_p$ is a Hilbert A-module which can be identified with E [10, Proposition 4.4].

Let E and F be Hilbert A-modules. We say that an A-module morphism $T : E \to F$ is *adjointable* if there is an A-module morphism $T^* : F \to E$ such that $\langle T\xi,\eta\rangle = \langle\xi, T^*\eta\rangle$ for every $\xi \in E$ and $\eta \in F$. Any adjointable A-module morphism is continuous. The set $L_A(E,F)$ of all adjointable A-module morphisms from E into F becomes a locally convex space with topology defined by the family $\{\widetilde{p}\}_{p\in S(A)}$ of seminorms, where $\widetilde{p}(T) = \|(\pi_p)_*(T)\|_{L_{A_p}(E_p,F_p)}$ for $T \in L_A(E,F)$ and $(\pi_p)_*(T)(\xi + \mathcal{E}_p) = T\xi + \mathcal{F}_p$ for $\xi \in E$. Moreover, $\{L_{A_p}(E_p,F_p);(\pi_{pq})_*\}_{p,q\in S(A),\,p\geq q}$ is an inverse system of Banach spaces, where $(\pi_{pq})_* : L_{A_p}(E_p,F_p) \to L_{A_q}(E_q,F_q)$, $(\pi_{pq})_*(T_p)(\sigma_q(\xi)) = \chi_{pq}(T_p(\sigma_p(\xi)))$, and χ_{pq}, $p,q \in S(A)$, $p \geq q$, are the connecting maps of the inverse system $\{F_p\}_{p\in S(A)}$; the limit $\varprojlim_p L_{A_p}(E_p,F_p)$ can be identified with $L_A(E,F)$ [10, Proposition 4.7]. Thus topologized, $L_A(E,E)$ becomes a locally C^*-algebra, and we write $L_A(E)$ for $L_A(E,E)$.

The *strict topology* on $L_A(E)$ is defined by the family $\{\|\cdot\|_{p,\xi}\}_{(p,\xi)\in S(A)\times E}$ of seminorms, where $\|T\|_{p,\xi} = \|T\xi\|_p + \|T^*\xi\|_p$ for $T \in L_A(E)$.

Two Hilbert A-modules E and F are *unitarily equivalent* if there is a unitary element in $L_A(E,F)$.

Let G be a locally compact group and let A be a locally C^*-algebra. An *action* of G on A is a morphism α from G to $\mathrm{Aut}(A)$, the set of all isomorphisms of the locally C^*-algebra. The action α is *continuous* if the function $(t,a) \mapsto \alpha_t(a)$ from $G\times A$ to A is jointly continuous. An action α is called an *inverse limit action* if we can write A as an inverse limit $\varprojlim_{\delta\in\Delta} A_\delta$ of C^*-algebras in such a way that there are actions $\alpha^{(\delta)}$ of G on A_δ such that $\alpha_t = \varprojlim_{\delta\in\Delta} \alpha_t^{(\delta)}$ for all t in G [11, Definition 5.1]. An action α of G on A is a *continuous inverse limit action* if there is a cofinal subset $S_G(A,\alpha)$ of G-invariant continuous C^*-seminorms on A (a continuous C^*-seminorm p on A is *G-invariant* if $p(\alpha_t(a)) = p(a)$ for all a in A and for all t in G). So if α is a continuous inverse limit action of G on A we can suppose that $S(A) = S_G(A,\alpha)$.

A *unitary representation* of G on a Hilbert module E over a locally C^*-algebra B is a map u from G to $L_B(E)$ such that

(i) u_g is a unitary element in $L_B(E)$ for all $g \in G$;
(ii) $u_{gt} = u_g u_t$ for all $g,t \in G$;
(iii) the map $g \mapsto u_g\xi$ from G to E is continuous for all $\xi \in E$.

If u is a unitary representation of G on E, then for each $q \in S(B)$, $g \mapsto (\pi_q)_*\circ u$ is a unitary representation of G on E_q. Moreover, $u_g = \varprojlim_q u_g^{(q)}$, where

$$u_g^{(q)} = (\pi_q)_*(u_g) \quad \text{for all } g \in G.$$

A *locally C^*-dynamical system* is a triple (G,A,α), where G is a locally compact group, A is a locally C^*-algebra and α is a continuous action of G on A.

A *non-degenerate*, *covariant representation* of (G,A,α) on a Hilbert B-module E is a triple (Φ,v,E), where Φ is a non-degenerate, continuous

$*$-representation of A on E, v is a unitary representation of G on E and

$$\Phi(\alpha_g(a)) = v_g \Phi(a) v_g^*$$

for all $g \in G$ and $a \in A$.

Let α be a continuous inverse limit action of G on A. The set $C_{\mathrm{c}}(G, A)$ of all continuous functions from G to A with compact support becomes a $*$-algebra with convolution of two functions

$$(f \times h)(s) = \int_G f(t)\alpha_t(h(t^{-1}s))\, dt$$

as product and involution defined by

$$f^\sharp(t) = \Delta(t)^{-1}\alpha_t(f(t^{-1})^*)$$

where Δ is the modular function on G. The Hausdorff completion of $C_{\mathrm{c}}(G, A)$ with respect to the topology defined by the family $\{N_p\}_{p \in S(A)}$ of submultiplicative $*$-seminorms, where

$$N_p(f) = \int_G p(f(s))\, ds,$$

is denoted by $L^1(G, A, \alpha)$, and the enveloping locally C^*-algebra $A \times_\alpha G$ of $L^1(G, A, \alpha)$ is called the *crossed product* of A by α [4, Definition 3.14]. Moreover, the C^*-algebras $(A \times_\alpha G)_p$ are isomorphic with $A \times_{\alpha^{(p)}} G$ for all $p \in S(A)$ and so $A \times_\alpha G$ can be identified with $\varprojlim_p A \times_{\alpha^{(p)}} G$ [4, Remark 3.15].

3. Representations associated with a completely multi-positive linear map. Let A and B be two locally C^*-algebras. An $n \times n$ matrix $[\varrho_{ij}]_{i,j=1}^n$ of continuous linear maps from A to B can be regarded as a linear map from $M_n(A)$ to $M_n(B)$ defined by

$$[\varrho_{ij}]_{i,j=1}^n([a_{ij}]_{i,j=1}^n) = [\varrho_{ij}(a_{ij})]_{i,j=1}^n.$$

Moreover, $[\varrho_{ij}]_{i,j=1}^n$ is continuous.

We say that $[\varrho_{ij}]_{i,j=1}^n$ is a *multi-positive* (respectively, *completely multi-positive*) linear map from A to B if it is a positive (respectively, completely positive) linear map from $M_n(A)$ to $M_n(B)$.

DEFINITION 3.1. Let E be a Hilbert B-module. A continuous completely multi-positive linear map $[\varrho_{ij}]_{i,j=1}^n$ from A to $L_B(E)$ is *strict* (respectively, *non-degenerate*) if the nets $\{\varrho_{ii}(e_\lambda)\}_{\lambda \in \Lambda}$, $i = 1, \ldots, n$, are strictly Cauchy (respectively, strictly convergent to the identity map on E) in $L_B(E)$, for some approximate unit $\{e_\lambda\}_{\lambda \in \Lambda}$ of A.

Recall that a *continuous $*$-representation* of A on E is a continuous $*$-morphism Φ from A to $L_B(E)$. A continuous $*$-representation Φ of A on E is *non-degenerate* if $\Phi(A)E$ is dense in E.

Proposition 3.2. *Let A and B be two locally C^*-algebras, let E and F be Hilbert B-modules, let V_i, $i = 1,\dots,n$, be n elements in $L_B(E,F)$, and let Φ be a non-degenerate, continuous $*$-representation of A on F. Then $[\varrho_{ij}]_{i,j=1}^n$, where*

$$\varrho_{ij}(a) = V_i^*\Phi(a)V_j \quad \text{for all } a \in A \text{ and } i,j \in \{1,\dots,n\},$$

is a strict completely multi-positive linear map from A to $L_B(E)$.

Proof. It is a simple verification. ∎

Construction 3.3. Let A and B be locally C^*-algebras, let E be a Hilbert B -module and let $[\varrho]= [\varrho_{ij}]_{i,j=1}^n$ be a completely multi-positive linear map from A to $L_B(E)$.

Let $q \in S(B)$. Then $[\varrho_q] = [(\pi_q)_* \circ \varrho_{ij}]_{i,j=1}^n$ is a completely multi-positive linear map from A to $L_{B_q}(E_q)$. We denote by $(A \otimes_{\text{alg}} E_q)^n$ the direct sum of n copies of the algebraic tensor product $A \otimes_{\text{alg}} E_q$ of A and E_q. Using the fact that $[\varrho_q]$ is a completely multi-positive linear map from A to $L_{B_q}(E_q)$, and the same arguments as in the proof of Theorem 2.1 in [1], it is not difficult to check that $(A \otimes_{\text{alg}} E_q)^n$ becomes a right B_q-module with the action of B_q on $(A \otimes_{\text{alg}} E_q)^n$ defined by

$$\Big(\sum_{s=1}^m \bigoplus_{i=1}^n (a_{i,s} \otimes \xi_{i,s})\Big)b = \sum_{s=1}^m \bigoplus_{i=1}^n (a_{i,s} \otimes \xi_{i,s}b)$$

and the map $\langle\cdot,\cdot\rangle_0$ from $(A \otimes_{\text{alg}} E_q)^n \times (A \otimes_{\text{alg}} E_q)^n$ to B_q defined by

$$\Big\langle \sum_{s=1}^m \bigoplus_{i=1}^n (a_{i,s}\otimes\xi_{i,s}), \sum_{t=1}^l \bigoplus_{j=1}^n (b_{j,t}\otimes\eta_{j,t})\Big\rangle_0 = \sum_{s,t=1}^{m,l} \sum_{i,j=1}^n \langle \xi_{i,s}, (\pi_q)_*(\varrho_{ij}(a_{i,s}^* b_{j,t}))\eta_{j,t}\rangle$$

is $\mathbb{C}$- and B_q-linear in its second variable and satisfies conditions (i) and (ii) of Definition 2.1.

Let $\mathcal{N}_q = \{x \in (A \otimes_{\text{alg}} E_q)^n;\ \langle x,x\rangle_0 = 0\}$. By the Cauchy–Schwarz inequality, $\mathcal{N}_q$ is a B_q-submodule of $(A \otimes_{\text{alg}} E_q)^n$. Then $(A \otimes_{\text{alg}} E_q)^n/\mathcal{N}_q$ becomes a pre-Hilbert B_q-module with the action of B_q on $(A \otimes_{\text{alg}} E_q)^n/\mathcal{N}_q$ defined by

$$(x + \mathcal{N}_q)b = xb + \mathcal{N}_q$$

and the inner product defined by

$$\langle x + \mathcal{N}_q, y + \mathcal{N}_q\rangle = \langle x, y\rangle_0.$$

The completion of $(A \otimes_{\text{alg}} E_q)^n/\mathcal{N}_q$ is denoted by $E_{[\varrho_q]}$. Let $q, r \in S(B)$ be such that $q \geq r$. Since

$$\langle \bigoplus_{i=1}^n (a_i \otimes \sigma_r(\xi_i)), \bigoplus_{i=1}^n (a_i \otimes \sigma_r(\xi_i))\rangle_0 = \pi_{qr}(\langle \bigoplus_{i=1}^n (a_i \otimes \sigma_q(\xi_i)), \bigoplus_{i=1}^n (a_i \otimes \sigma_q(\xi_i))\rangle_0)$$

for all $a_i \in A$ and $\xi_i \in E$, $i = 1, \dots, n$, we can define a linear map $\widetilde{\sigma}_{qr}$ from $(A \otimes_{\mathrm{alg}} E_q)^n/\mathcal{N}_q$ to $(A \otimes_{\mathrm{alg}} E_r)^n/\mathcal{N}_r$ by

$$\widetilde{\sigma}_{qr}\Big(\bigoplus_{i=1}^{n}(a_i \otimes \sigma_q(\xi_i)) + \mathcal{N}_q\Big) = \bigoplus_{i=1}^{n}(a_i \otimes \sigma_r(\xi_i)) + \mathcal{N}_r.$$

Moreover, $\widetilde{\sigma}_{qr}$ extends by continuity and linearity to a surjective linear map $\widetilde{\sigma}_{qr}$ from $E_{[\varrho_q]}$ to $E_{[\varrho_r]}$. One can check that $\{E_{[\varrho_q]}; B_q; \widetilde{\sigma}_{qr}\}_{q \geq r,\, q,r \in S(B)}$ is an inverse system of Hilbert C^*-modules. We denote by $E_{[\varrho]}$ the Hilbert B-module $\varprojlim_q E_{[\varrho_q]}$. Also it is not difficult to check that $(E_{[\varrho]})_q$ can be identified with $E_{[\varrho_q]}$ for all $q \in S(B)$. Then by Proposition 4.7 of [10], the locally C^*-algebras $L_B(E_{[\varrho]})$ and $\varprojlim_q L_{B_q}(E_{[\varrho_q]})$ are isomorphic, as are the locally convex spaces $L_B(F, E_{[\varrho]})$ and $\varprojlim_q L_{B_q}(F_q, E_{[\varrho_q]})$, where F is an arbitrary Hilbert B-module.

The following theorem extends the KSGNS construction for strict continuous completely positive linear maps between locally C^*-algebras [2, Theorem 4.6] to the case of strict completely multi-positive linear maps between locally C^*-algebras.

THEOREM 3.4. *Let A and B be locally C^*-algebras, let E be a Hilbert B-module and let $[\varrho] = [\varrho_{ij}]_{i,j=1}^n$ be a strict completely multi-positive linear map from A to $L_B(E)$.*

(1) *There is a continuous $*$-representation $\Phi_{[\varrho]}$ of A on $E_{[\varrho]}$ and n elements $V_{[\varrho],i}$, $i = 1, \dots, n$, in $L_B(E, E_{[\varrho]})$ such that*
   (a) *$\varrho_{ij}(a) = V_{[\varrho],i}^* \Phi_{[\varrho]}(a) V_{[\varrho],j}$ for all a in A and $i, j \in \{1, \dots, n\}$;*
   (b) *$\{\Phi_{[\varrho]}(a) V_{[\varrho],i}\xi;\ a \in A,\ \xi \in E,\ i = 1, \dots, n\}$ spans a dense submodule of $E_{[\varrho]}$.*

(2) *If F is a Hilbert B-module, Φ is a continuous $*$-representation of A on F, and W_i, $i = 1, \dots, n$, are n elements in $L_B(E, F)$ such that*
   (a) *$\varrho_{ij}(a) = W_i^* \Phi(a) W_j$ for all a in A and $i, j \in \{1, \dots, n\}$;*
   (b) *$\{\Phi(a) W_i \xi;\ a \in A,\ \xi \in E,\ i = 1, \dots, n\}$ spans a dense submodule of F,*

   then there is a unitary operator U in $L_B(E_{[\varrho]}, F)$ such that
   (i) *$\Phi(a) U = U \Phi_{[\varrho]}(a)$ for all a in A;*
   (ii) *$W_i = U V_{[\varrho],i}$ for all $i \in \{1, \dots, n\}$.*

DEFINITION 3.5. The representation $(\Phi_{[\varrho]}; V_{[\varrho],i},\ i = 1, \dots, n; E_{[\varrho]})$ constructed in Theorem 3.4 is called the *KSGNS construction* associated with $[\varrho] = [\varrho_{ij}]_{i,j=1}^n$.

REMARK 3.6. The KSGNS construction associated with a strict completely multi-positive map $[\varrho_{ij}]_{i,j=1}^n$ is unique up to unitary equivalence.

Remark 3.7. In particular, we obtain the GNS construction for continuous completely multi-positive linear functionals on locally C^*-algebras [3, Theorem 4.1] as well as Stinespring's construction for completely multi-positive maps between unital C^*-algebras [1, Corollary 2.4].

Proof of Theorem 3.4. We partition the proof into three steps.

Step 1. Suppose that A and B are C^*-algebras. Let $\{e_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit of A such that the nets $\{\varrho_{ii}(e_\lambda)\}_{\lambda\in\Lambda}$, $i=1,\dots,n$, are strictly Cauchy in $L_B(E)$.

(1) Let $a\in A$. It is not difficult to check that the linear map $\Phi_{[\varrho]}(a)$ from $(A\otimes_{\mathrm{alg}}E)^n$ to $(A\otimes_{\mathrm{alg}}E)^n$ defined by

$$\Phi_{[\varrho]}(a)(\bigoplus_{i=1}^n(a_i\otimes\xi_i))=\bigoplus_{i=1}^n(aa_i\otimes\xi_i)$$

extends to a bounded linear operator $\Phi_{[\varrho]}$ from $E_{[\varrho]}$ to $E_{[\varrho]}$. Moreover, $\Phi_{[\varrho]}(a)$ is adjointable and $(\Phi_{[\varrho]}(a))^*=\Phi_{[\varrho]}(a^*)$. Thus we have obtained a map $\Phi_{[\varrho]}$ from A to $L_B(E_{[\varrho]})$. It is easy to verify that $\Phi_{[\varrho]}$ is a continuous $*$-representation of A on $E_{[\varrho]}$.

Let $i\in\{1,\dots,n\}$. Since

$$\begin{aligned}\Big\|\sum_{s=1}^m\sum_{k=1}^n\varrho_{ik}(a_{k,s})\xi_{k,s}\Big\|^2&=\Big\|\Big\langle\sum_{s=1}^m\sum_{k=1}^n\varrho_{ik}(a_{k,s})\xi_{k,s},\sum_{s=1}^m\sum_{k=1}^n\varrho_{ik}(a_{k,s})\xi_{k,s}\Big\rangle\Big\|\\&\le\Big\|\sum_{i=1}^n\Big\langle\sum_{s=1}^m\sum_{k=1}^n\varrho_{ik}(a_{k,s})\xi_{k,s},\sum_{s=1}^m\sum_{k=1}^n\varrho_{ik}(a_{k,s})\xi_{k,s}\Big\rangle\Big\|\\&=\Big\|\sum_{s,t=1}^m\sum_{k,j=1}^n\Big\langle\xi_{k,s},\sum_{i=1}^n\varrho_{ki}(a_{k,s}^*)\varrho_{ij}(a_{j,t})\xi_{j,t}\Big\rangle\Big\|\\&\le\|[\varrho_{ij}]_{i,j=1}^n\|\Big\|\sum_{s,t=1}^m\sum_{k,j=1}^n\langle\xi_{k,s},\varrho_{kj}(a_{k,s}^*a_{j,t})\xi_{j,t}\rangle\Big\|\\&\qquad\qquad\text{(Lemma 5.4 in [8])}\\&=\|[\varrho_{ij}]_{i,j=1}^n\|\Big\|\sum_{s=1}^m\bigoplus_{k=1}^n(a_{k,s}\otimes\xi_{k,s})+\mathcal{N}\Big\|^2\end{aligned}$$

for all $\sum_{s=1}^m\bigoplus_{k=1}^n(a_{k,s}\otimes\xi_{k,s})\in(A\otimes_{\mathrm{alg}}E)^n$, the linear map $\bigoplus_{k=1}^n(a_k\otimes\xi_k)+\mathcal{N}\mapsto\sum_{k=1}^n\varrho_{ik}(a_k)\xi_k$ from $(A\otimes_{\mathrm{alg}}E)^n/\mathcal{N}$ to E extends by continuity and linearity to a bounded linear operator $\widetilde{V}_i$ from $E_{[\varrho]}$ to E.

Let $\lambda\in\Lambda$ and $\xi\in E$. We denote by ξ_i^λ the element in $(A\otimes_{\mathrm{alg}}E)^n$ whose ith component is $e_\lambda\otimes\xi$ and all the other components are 0. Since

$$\|(\xi_i^\lambda+\mathcal{N})-(\xi_i^\mu+\mathcal{N})\|^2=\|\langle\xi,\varrho_{ii}((e_\lambda-e_\mu)^2)\xi\rangle\|\le2\|\langle\xi,\varrho_{ii}(e_\lambda-e_\mu)\xi\rangle\|$$

for all $\lambda,\mu\in\Lambda$ with $\lambda\ge\mu$, and since the net $\{\varrho_{ii}(e_\lambda)\xi\}_{\lambda\in\Lambda}$ is convergent in E, the net $\{\xi_i^\lambda + \mathcal{N}\}_{\lambda\in\Lambda}$ is convergent in $E_{[\varrho]}$. Define a map $V_{[\varrho],i}$ from E to $E_{[\varrho]}$ by

$$V_{[\varrho],i}\xi = \lim_{\lambda\in\Lambda}(\xi_i^\lambda + \mathcal{N}).$$

To show that $V_{[\varrho],i}$ is an element in $L_B(E, E_{[\varrho]})$ it is sufficient to show that

$$\langle V_{[\varrho],i}\xi, \bigoplus_{k=1}^{n}(a_k\otimes\xi_k) + \mathcal{N}\rangle = \langle \xi, \widetilde{V}_i(\bigoplus_{k=1}^{n}(a_k\otimes\xi_k) + \mathcal{N})\rangle$$

for all $\xi\in E$ and $\bigoplus_{k=1}^{n}(a_k\otimes\xi_k) \in (A\otimes_{\mathrm{alg}} E)^n$.

Let $\xi\in E$ and $\bigoplus_{k=1}^{n}(a_k\otimes\xi_k) \in (A\otimes_{\mathrm{alg}} E)^n$. Then we have

$$\begin{aligned}\langle V_{[\varrho],i}\xi, \bigoplus_{k=1}^{n}(a_k\otimes\xi_k) + \mathcal{N}\rangle &= \lim_{\lambda\in\Lambda}\langle \xi_i^\lambda, \bigoplus_{k=1}^{n}(a_k\otimes\xi_k) + \mathcal{N}\rangle\\ &= \lim_{\lambda\in\Lambda}\Big\langle \xi, \sum_{k=1}^{n}\varrho_{ik}(e_\lambda a_k)\xi_k\Big\rangle\\ &= \Big\langle \xi, \sum_{k=1}^{n}\varrho_{ik}(a_k)\xi_k\Big\rangle\\ &= \langle \xi, \widetilde{V}_i(\bigoplus_{k=1}^{n}(a_k\otimes\xi_k) + \mathcal{N})\rangle.\end{aligned}$$

Hence $V_{[\varrho],i}\in L_B(E, E_{[\varrho]})$.

Let $a\in A$ and $\xi\in E$. We denote by $\xi_{i,a}$ the element in $(A\otimes_{\mathrm{alg}} E)^n$ whose ith component is $a\otimes\xi$ and all the other components are 0. It is not difficult to check that $\Phi_{[\varrho]}(a)V_{[\varrho],i}\xi = \xi_{i,a} + \mathcal{N}$. Therefore the submodule of $E_{[\varrho]}$ generated by $\{\Phi_{[\varrho]}(a)V_{[\varrho],i}\xi;\ a\in A,\ \xi\in E,\ i = 1,\dots,n\}$ is exactly $(A\otimes_{\mathrm{alg}} E)^n/\mathcal{N}$ and thus condition (b) is satisfied.

Let $a\in A$ and $i,j\in\{1,\dots,n\}$. Then we have

$$V_{[\varrho],i}^{*}\Phi_{[\varrho]}(a)V_{[\varrho],j}\xi = V_{[\varrho],i}^{*}(\xi_{j,a} + \mathcal{N}) = \varrho_{ij}(a)\xi$$

for all $\xi\in E$ and so condition (a) is also satisfied.

(2) Using the fact that

$$\varrho_{ij}(a) = V_{[\varrho],i}^{*}\Phi_{[\varrho]}(a)V_{[\varrho],j} = W_i^{*}\Phi(a)W_j$$

for all $a\in A$ and $i,j\in\{1,\dots,n\}$, it is not difficult to check that

$$\Big\|\sum_{s=1}^{m}\sum_{i=1}^{n}\alpha_i\Phi_{[\varrho]}(a_s)V_{[\varrho],i}\xi_s\Big\| = \Big\|\sum_{s=1}^{m}\sum_{i=1}^{n}\alpha_i\Phi(a_s)W_i\xi_s\Big\|$$

for all $\alpha_1,\dots,\alpha_n\in\mathbb{C}$, $a_1,\dots,a_m\in A$ and $\xi_1,\dots,\xi_m\in E$. Therefore the linear map $\Phi_{[\varrho]}(a)V_{[\varrho],i}\xi\mapsto\Phi(a)W_i\xi$ from the submodule of $E_{[\varrho]}$ generated by $\{\Phi_{[\varrho]}(a)V_{[\varrho],i}\xi;\ a\in A,\ \xi\in E,\ i = 1,\dots,n\}$ to the submodule of F generated by $\{\Phi(a)W_i\xi;\ a\in A,\ \xi\in E,\ i = 1,\dots,n\}$ extends to a surjective isometric B-linear map U from $E_{[\varrho]}$ onto F. Then, by Theorem 3.5 of [8], U is unitary.

Let $a \in A$. From

$$\begin{aligned}\Phi(a)U(\Phi_{[\varrho]}(b)V_{[\varrho],i}\xi) &= \Phi(a)\Phi(b)W_i\xi = U(\Phi_{[\varrho]}(ab)V_{[\varrho],i}\xi)\\ &= U\Phi_{[\varrho]}(a)(\Phi_{[\varrho]}(b)V_{[\varrho],i}\xi)\end{aligned}$$

for all $b \in A$, $\xi \in E$ and $i \in \{1, \dots, n\}$, we conclude that $\Phi(a)U = U\Phi_{[\varrho]}(a)$.

Since Φ and $\Phi_{[\varrho]}$ are non-degenerate, by Proposition 4.2 of [2], we have

$$UV_{[\varrho],i}\xi = \lim_{\lambda\in\Lambda} U\Phi_{[\varrho]}(e_\lambda)V_{[\varrho],i}\xi = \lim_{\lambda\in\Lambda} \Phi(e_\lambda)W_i\xi = W_i\xi$$

for all $\xi \in E$ and $i \in \{1, \dots, n\}$. Therefore $W_i = UV_{[\varrho],i}$ for all $i \in \{1, \dots, n\}$.

Step 2. Suppose that A is a locally C^*-algebra and B is a C^*-algebra. Then there is $p \in S(A)$, a strict completely multi-positive map $[\varrho_p] = [\varrho_{ij}^p]_{i,j=1}^n$ from A_p to $L_B(E)$ such that $[\varrho_{ij}]_{i,j=1}^n = [\varrho_{ij}^p \circ \pi_p]_{i,j=1}^n$, and a $*$-representation Φ_p of A_p on F such that $\Phi = \Phi_p \circ \pi_p$.

It is not difficult to check that the linear map $\widetilde{U}$ from $(A \otimes_{\text{alg}} E)^n$ to $(A_p \otimes_{\text{alg}} E)^n$ defined by

$$\widetilde{U}(\bigoplus_{i=1}^n (a_i \otimes \xi_i)) = \bigoplus_{i=1}^n (\pi_p(a_i) \otimes \xi_i)$$

extends to a bounded linear map $\widetilde{U}$ from $E_{[\varrho]}$ to $E_{[\varrho_p]}$. Moreover, $\widetilde{U}$ is unitary. Therefore, the Hilbert B-modules $E_{[\varrho]}$ to $E_{[\varrho_p]}$ are unitarily equivalent.

By Step 1, there is a $*$-representation $\Phi_{[\varrho_p]}$ of A_p on $E_{[\varrho_p]}$, and n elements $V_{[\varrho_p],i}$, $i = 1, \dots, n$, in $L_B(E, E_{[\varrho_p]})$ such that

$$\varrho_{ij}^p(a) = V_{[\varrho_p],i}^* \Phi_{[\varrho_p]}(a) V_{[\varrho_p],j}$$

for all $a \in A_p$ and $i, j \in \{1, \dots, n\}$, and $\{\Phi_{[\varrho_p]}(a)V_{[\varrho_p],i}\xi;\ a \in A_p,\ \xi \in E,\ i = 1, \dots, n\}$ spans a dense submodule of $E_{[\varrho_p]}$. Also there is a unitary operator U^p in $L_B(E_{[\varrho_p]}, F)$ such that $\Phi_p(a)U^p = U^p\Phi_{[\varrho_p]}(a)$ for all a in A_p, and $W_i = U^pV_{[\varrho_p],i}$ for all $i = 1, \dots, n$.

Let $\Phi_{[\varrho]} = \Phi_{[\varrho_p]} \circ \pi_p$, $V_{[\varrho],i} = \widetilde{U}^*V_{[\varrho_p],i}$, $i = 1, \dots, n$, and $U = U^p\widetilde{U}$. A simple calculation shows $(\Phi_{[\varrho]}; V_{[\varrho],i},\ i = 1, \dots, n; E_{[\varrho]})$ is the KSGNS construction associated with $[\varrho_{ij}]_{i,j=1}^n$, and U is a unitary operator in $L_B(E_{[\varrho]}, F)$ such that $\Phi(a)U = U\Phi_{[\varrho]}(a)$ for all a in A and $W_i = UV_{[\varrho],i}$ for all $i = 1, \dots, n$.

Step 3 (The general case). For each $q \in S(B)$, $[\varrho_q] = [(\pi_q)_* \circ \varrho_{ij}]_{i,j=1}^n$ is a strict completely multi-positive linear map from A to $L_{B_q}(E_q)$, and $\Phi_q = (\pi_q)_* \circ \Phi$ is a continuous $*$-representation of A on E_q such that $(\pi_q)_*(\varrho_{ij}(a)) = (\pi_q)_*(W_i^*)\Phi_q(a)(\pi_q)_*(W_j)$ for all $a \in A$ and $\{\Phi_q(a)(\pi_q)_*(W_i)\xi;\ a \in A,\ \xi \in E,\ i = 1, \dots, n\}$ spans a dense submodule of F_q.

Let $(\Phi_{[\varrho_q]}; V_{[\varrho_q],i}, i = 1, \dots, n; E_{[\varrho_q]})$ be the KSGNS construction associated with $[\varrho_q]$. By Step 2, there is a unitary operator U_q in $L_{B_q}(E_{[\varrho_q]}, F_q)$

such that $\Phi_q(a)U_q = U_q\Phi_{[\varrho_q]}(a)$ for all a in A and $(\pi_q)_*(W_i) = U_qV_{[\varrho_q],i}$ for all $i = 1,\dots,n$.

It is not difficult to check that

$$(\pi_{qr})_* \circ \Phi_{[\varrho_q]} = \Phi_{[\varrho_r]}$$

for all $q,r \in S(A)$ with $q \geq r$. This implies that there is a continuous $*$-morphism $\Phi_{[\varrho]}$ from A to $L_B(E_{[\varrho]})$ such that $(\pi_q)_* \circ \Phi_{[\varrho]} = \Phi_{[\varrho_q]}$ for all $q \in S(B)$. Also it is not difficult to check that $(V_{[\varrho_q],i})_q$, $i = 1,\dots,n$, are coherent sequences in $L_{B_q}(E_q, E_{[\varrho_q]})$ and $(U_q)_q$ is a coherent sequence in $L_{B_q}(E_{[\varrho_q]}, F_q)$.

Let $V_{[\varrho],i} = (V_{[\varrho_q],i})_q$, $i = 1,\dots,n$, and $U = (U_q)_q$. A simple calculation shows that $(\Phi_{[\varrho]}; V_{[\varrho],i},\ i = 1,\dots,n; E_{[\varrho]})$ is the KSGNS construction associated with $[\varrho]$, $\Phi(a)U = U\Phi_{[\varrho]}(a)$ for all a in A, and $W_i = UV_{[\varrho],i}$ for all $i \in \{1,\dots,n\}$. ∎

4. Covariant representations associated with a covariant completely multi-positive linear map

Definition 4.1. Let (G, A, α) be a locally C^*-dynamical system and let u be a unitary representation of G on a Hilbert B-module E. We say that a completely multi-positive linear map $[\varrho_{ij}]_{i,j=1}^n$ from A to $L_B(E)$ is *u-covariant* with respect to the locally C^*-dynamical system (G, A, α) if

$$\varrho_{ij}(\alpha_g(a)) = u_g\varrho_{ij}(a)u_g^* \quad \text{for all } a \in A, g \in G \text{ and } i,j \in \{1,\dots,n\}.$$

Proposition 4.2. *Let (G, A, α) be a locally C^*-dynamical system, let u be a unitary representation of G on a Hilbert module E over a locally C^*-algebra B, let (Φ, v, F) be a covariant non-degenerate representation of (G, A, α) on a Hilbert B-module F, and let V_i, $i = 1,\dots,n$, be n partial isometries in $L_B(E,F)$ (that is, $V_i^*V_i = \mathrm{id}_E$ for all $i = 1,\dots,n$) such that $V_iu_g = v_gV_i$ for all $g \in G$ and $i = 1,\dots,n$. Then there is a u-covariant non-degenerate completely multi-positive linear map $[\varrho] = [\varrho_{ij}]_{i,j=1}^n$ from A to $L_B(E)$ such that*

$$\varrho_{ij}(a) = V_i^*\Phi(a)V_j \quad \textit{for all } a \in A \textit{ and } i,j = 1,\dots,n.$$

Proof. It is a simple verification. ∎

The following theorem is a covariant version of Theorem 3.4.

Theorem 4.3. *Let (G, A, α) be a locally C^*-dynamical system, let u be a unitary representation of G on a Hilbert module E over a locally C^*-algebra B, and let $[\varrho] = [\varrho_{ij}]_{i,j=1}^n$ be a u-covariant, non-degenerate, completely multi-positive linear map from A to $L_B(E)$.*

(1) *There is a covariant representation $(\Phi_{[\varrho]}, v^{[\varrho]}, E_{[\varrho]})$ of (G, A, α) and n elements $V_{[\varrho],i}$, $i = 1,\dots,n$, in $L_B(E, E_{[\varrho]})$ such that*

(a) $\varrho_{ij}(a) = V^*_{[\varrho],i}\Phi_{[\varrho]}(a)V_{[\varrho],j}$ *for all* $a \in A$ *and* $i,j \in \{1,\dots,n\}$;

(b) $\{\Phi_{[\varrho]}(a)V_{[\varrho],i}\xi;\ a \in A,\ \xi \in E,\ i = 1,\dots,n\}$ *spans a dense submodule of* $E_{[\varrho]}$;

(c) $v^{[\varrho]}_g V_{[\varrho],i} = V_{[\varrho],i}u_g$ *for all* $g \in G$ *and* $i \in \{1,\dots,n\}$.

(2) *If* F *is a Hilbert* B*-module,* (Φ, v, F) *is a covariant representation of* (G, A, α) *and* W_i, $i = 1,\dots,n$, *are* n *elements in* $L_B(E, F)$ *such that*

(a) $\varrho_{ij}(a) = W^*_i\Phi(a)W_j$ *for all* $a \in A$ *and* $i,j \in \{1,\dots,n\}$;

(b) $\{\Phi(a)W_i\xi;\ a \in A,\ \xi \in F,\ i = 1,\dots,n\}$ *spans a dense submodule of* F;

(c) $v_gW_i = W_iu_g$ *for all* $g \in G$ *and for all* $i \in \{1,\dots,n\}$,

then there is a unitary operator U *in* $L_B(E_{[\varrho]}, F)$ *such that*

(i) $\Phi(a)U = U\Phi_{[\varrho]}(a)$ *for all* $a \in A$;

(ii) $v_gU = Uv^{[\varrho]}_g$ *for all* $g \in G$;

(iii) $W_i = UV_{[\varrho],i}$ *for all* $i \in \{1,\dots,n\}$.

Proof. We partition the proof into two steps.

STEP 1. Suppose that B is a C^*-algebra.

(1) Let $\{e_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit of A such that the nets $\{\varrho_{ii}(e_\lambda)\}_{\lambda\in\Lambda}$, $i = 1,\dots,n$, are strictly convergent to the identity operator on E, and let $(\Phi_{[\varrho]}; V_{[\varrho],i},\ i = 1,\dots,n; E_{[\varrho]})$ be the KSGNS construction associated with $[\varrho_{ij}]^n_{i,j=1}$. Then $V_{[\varrho],i}$ is a partial isometry for each $i \in \{1,\dots,n\}$. For each $g \in G$, we define a linear map $v^{[\varrho]}_g$ from $(A \otimes_{\mathrm{alg}} E)^n$ to $(A \otimes_{\mathrm{alg}} E)^n$ by

$$v^{[\varrho]}_g\Big(\bigoplus_{i=1}^n(a_i \otimes \xi_i)\Big) = \bigoplus_{i=1}^n(\alpha_g(a_i) \otimes u_g\xi_i).$$

Using the fact that $[\varrho_{ij}]^n_{i,j=1}$ is u-covariant, it is not difficult to check that $v^{[\varrho]}_g$ extends to a bounded linear map $v^{[\varrho]}_g$ from $E_{[\varrho]}$ to $E_{[\varrho]}$, and since

$$\Big\langle v^{[\varrho]}_g\Big(\bigoplus_{i=1}^n(a_i \otimes \xi_i) + \mathcal{N}\Big), \bigoplus_{i=1}^n(b_i \otimes \eta_i) + \mathcal{N}\Big\rangle = \Big\langle \bigoplus_{i=1}^n(a_i \otimes \xi_i) + \mathcal{N}, v^{[\varrho]}_{g^{-1}}\Big(\bigoplus_{i=1}^n(b_i \otimes \eta_i) + \mathcal{N}\Big)\Big\rangle$$

for all $\bigoplus_{i=1}^n(a_i \otimes \xi_i), \bigoplus_{i=1}^n(b_i \otimes \eta_i) \in (A \otimes_{\mathrm{alg}} E)^n$, $v^{[\varrho]}_g \in L_B(E_{[\varrho]})$ and moreover, $(v^{[\varrho]}_g)^* = v^{[\varrho]}_{g^{-1}}$. Also it is not difficult to check that the map $g \mapsto v^{[\varrho]}_g$ is a unitary representation of G on $E_{[\varrho]}$.

To show that $(\Phi_{[\varrho]}, v^{[\varrho]}, E_{[\varrho]})$ is a covariant representation of (G, A, α) it remains to prove that $\Phi_{[\varrho]}(\alpha_g(a)) = v^{[\varrho]}_g\Phi_{[\varrho]}(a)v^{[\varrho]}_{g^{-1}}$ for all $g \in G$ and $a \in A$.

Let $g \in G$ and $a \in A$. We have

$$\begin{aligned}(v_g^{[\varrho]}\Phi_{[\varrho]}(a)v_{g^{-1}}^{[\varrho]})(\bigoplus_{i=1}^{n}(a_i \otimes \xi_i) + \mathcal{N})\\ &= (v_g^{[\varrho]}\Phi_{[\varrho]}(a))(\bigoplus_{i=1}^{n}(\alpha_{g^{-1}}(a_i) \otimes u_{g^{-1}}\xi_i) + \mathcal{N})\\ &= v_g^{[\varrho]}(\bigoplus_{i=1}^{n}(a\alpha_{g^{-1}}(a_i) \otimes u_{g^{-1}}\xi_i) + \mathcal{N})\\ &= \bigoplus_{i=1}^{n}(\alpha_g(a)a_i \otimes \xi_i) + \mathcal{N}\\ &= (\Phi_{[\varrho]}(\alpha_g(a)))(\bigoplus_{i=1}^{n}(a_i \otimes \xi_i) + \mathcal{N})\end{aligned}$$

for all $\bigoplus_{i=1}^{n}(a_i \otimes \xi_i) \in (A \otimes_{\mathrm{alg}} E)^n$. Hence $\Phi_{[\varrho]}(\alpha_g(a)) = v_g^{[\varrho]}\Phi_{[\varrho]}(a)v_{g^{-1}}^{[\varrho]}$.

By Theorem 3.4(1) conditions (a) and (b) are satisfied. To show that (c) is satisfied, let $\xi \in E$, $g \in G$ and $i \in \{1, \ldots, n\}$. Then we have

$$\begin{aligned}&\|v_g^{[\varrho]}V_{[\varrho],i}\xi - V_{[\varrho],i}u_g\xi\|^2 = \lim_{\lambda\in\Lambda}\|v_g^{[\varrho]}\xi_i^{\lambda} - V_{[\varrho],i}u_g\xi\|^2\\ &= \lim_{\lambda\in\Lambda}\|\langle\xi, \varrho_{ii}(e_\lambda^2)\xi\rangle + \langle\xi,\xi\rangle - \langle\varrho_{ii}(\alpha_g(e_\lambda))u_g\xi, u_g\xi\rangle - \langle u_g\xi, \varrho_{ii}(\alpha_g(e_\lambda))u_g\xi\rangle\|\\ &\le \lim_{\lambda\in\Lambda}\|\langle\xi, \varrho_{ii}(e_\lambda)\xi\rangle + \langle\xi,\xi\rangle - \langle\varrho_{ii}(e_\lambda)\xi, \xi\rangle - \langle\xi, \varrho_{ii}(e_\lambda)\xi\rangle\|\\ &= \lim_{\lambda\in\Lambda}\|\langle\xi - \varrho_{ii}(e_\lambda)\xi, \xi\rangle\| = 0.\end{aligned}$$

Hence condition (c) is also satisfied.

(2) By Theorem 3.4(2), there is a unitary operator U in $L_B(E_{[\varrho]}, F)$ defined by $U(\sum_{s=1}^{m}\sum_{i=1}^{n}\alpha_i\Phi_{[\varrho]}(a_s)V_{[\varrho],i}\xi_s) = \sum_{s=1}^{m}\sum_{i=1}^{n}\alpha_i\Phi(a_s)W_i\xi_s$ such that $\Phi(a)U = U\Phi_{[\varrho]}(a)$ for all $a \in A$, and $W_i = UV_{[\varrho],i}$ for all $i \in \{1, \ldots, n\}$.

Let $g \in G$, $i \in \{1, \ldots, n\}$, $a \in A$, $\xi \in E$. We have

$$\begin{aligned}(v_gU)(\Phi_{[\varrho]}(a)V_{[\varrho],i}\xi) &= v_g(\Phi(a)W_i\xi) = \Phi(a)v_gW_i\xi\\ &= \Phi(a)W_iu_g\xi = U(\Phi_{[\varrho]}(a)V_{[\varrho],i}u_g\xi)\\ &= U(\Phi_{[\varrho]}(a)v_g^{[\varrho]}V_{[\varrho],i}\xi) = (Uv_g^{[\varrho]})(\Phi_{[\varrho]}(a)V_{[\varrho],i}\xi).\end{aligned}$$

This implies that $v_gU = Uv_g^{[\varrho]}$ and thus assertion (2) is proved.

Step 2 (The general case). Let $q \in S(B)$. Then $[\varrho_q] = [(\pi_q)_* \circ \varrho_{ij}]_{i,j=1}^{n}$ is a u^q-covariant, non-degenerate, completely multi-positive linear map from A to $L_{B_q}(E_q)$, $((\pi_q)_* \circ \Phi, v^q, F_q)$ is a covariant representation of (G, A, α) and $(\pi_q)_*(W_i)$, $i = 1, \ldots, n$, are n elements in $L_{B_q}(E_q, F_q)$ such that conditions (a)–(c) of (2) are satisfied.

By Step 1, there is a covariant representation $(\Phi_{[\varrho_q]}, v^{[\varrho_q]}, E_{[\varrho_q]})$ of (G, A, α) and n elements $V_{[\varrho^q],i}$, $i = 1, \dots, n$, in $L_{B_q}(E_q, E_{[\varrho_q]})$ which satisfy conditions (a)–(c) of (1) and there is a unitary operator U_q in $L_{B_q}(E_{[\varrho_q]}, F_q)$ which satisfies conditions (i)–(iii) of (2). Moreover,

$$v_g^{[\varrho_q]}\Big(\bigoplus_{i=1}^{n}(a_i \otimes \sigma_q(\xi_i)) + \mathcal{N}_q\Big) = \bigoplus_{i=1}^{n}(\alpha_g(a_i) \otimes u_g^q \sigma_q(\xi_i)) + \mathcal{N}_q$$

for all $\bigoplus_{i=1}^{n}(a_i \otimes \sigma_q(\xi_i)) \in (A \otimes_{\mathrm{alg}} E_q)^n$ and $g \in G$.

Let $(\Phi_{[\varrho]}; V_{[\varrho],i}, i = 1, \dots, n; E_{[\varrho]})$ be the KSGNS construction associated with $[\varrho_{ij}]_{i,j=1}^n$. According to the proof of Theorem 3.4, $(\pi_q)_* \circ \Phi_{[\varrho]} = \Phi_{[\varrho_q]}$, $(\pi_q)_*(V_{[\varrho],i}) = V_{[\varrho_q],i}$, $i = 1, \dots, n$, and $(E_{[\varrho]})_q = E_{[\varrho_q]}$ for all $q \in S(B)$.

It is not difficult to check that for each $g \in G$, $(v_g^{[\varrho_q]})_q$ is a coherent sequence in $L_{B_q}(E_{[\varrho_q]})$, and the map $g \mapsto v_g^{[\varrho]}$, where $v_g^{[\varrho]} = (v_g^{[\varrho_q]})_q$, is a unitary representation of G on $E_{[\varrho]}$. Also one can check that $(\Phi_{[\varrho]}, v^{[\varrho]}, E_{[\varrho]})$ is a covariant representation of (G, A, α) which satisfies conditions (a)–(c) of (1).

Since $U_q(\Phi_{[\varrho_q]}(a) V_{[\varrho_q],i} \sigma_q(\xi)) = (\pi_q)_*(\Phi(a)) W_i \sigma_q(\xi)$ for all $a \in A$, $\xi \in E$, $i \in \{1, \dots, n\}$ and $q \in S(B)$, it is not hard to verify that $(U_q)_q$ is a coherent sequence in $L_{B_q}(E_{[\varrho_q]}, F_q)$. Then $U = (U_q)_q$ is a unitary element in $L_B(E_{[\varrho]}, F)$ which satisfies conditions (i)–(iii) of (2), and the theorem is proved. ∎

Remark 4.4. In the particular case when (G, A, α) is a unital C^*-dynamical system, B is a unital C^*-algebra and $E = B$, the statements of Theorem 4.3 are given in Theorem 3.1 of [1].

In [1], Heo showed that given a unital C^*-dynamical system (G, A, α), a covariant completely multi-positive linear map $[\varrho_{ij}]_{i,j=1}^n$ from A to B extends to a completely multi-positive linear map on the crossed product $A \times_\alpha G$. We generalize this result to the case of locally C^*-dynamical systems, not necessarily unital, in the following proposition.

Proposition 4.5. *Let (G, A, α) be a locally C^*-dynamical system such that α is a continuous inverse limit action, let B be a locally C^*-algebra, let E be a Hilbert B-module and let u be a unitary representation of G on E. If $[\varrho] = [\varrho_{ij}]_{i,j=1}^n$ is a u-covariant, non-degenerate, completely multi-positive linear map from A to $L_B(E)$, then there is a unique completely multi-positive linear map $[\varphi_{ij}]_{i,j=1}^n$ from $A \times_\alpha G$ to $L_B(E)$ such that*

$$\varphi_{ij}(f) = \int_G \varrho_{ij}(f(g)) u_g \, dg$$

for all $f \in C_c(G, A)$ and $i, j \in \{1, \dots, n\}$. Moreover, $[\varphi_{ij}]_{i,j=1}^n$ is non-degenerate.

Proof. By Theorem 4.3, there is a covariant representation $(\Phi_{[\varrho]}, v^{[\varrho]}, E_{[\varrho]})$ of (G, A, α) and n elements $V_{[\varrho],i}$, $i = 1, \ldots, n$, in $L_B(E, E_{[\varrho]})$ such that $\varrho_{ij}(a) = V^*_{[\varrho],i}\Phi_{[\varrho]}(a)V_{[\varrho],j}$ and $v_g^{[\varrho]}V_{[\varrho],i} = V_{[\varrho],i}u_g$ for all $a \in A$, $g \in G$ and $i, j \in \{1, \ldots, n\}$.

Let $\Phi_{[\varrho]} \times v^{[\varrho]}$ be the representation of $A \times_\alpha G$ associated with $(\Phi_{[\varrho]}, v^{[\varrho]}, E_{[\varrho]})$ [5, Proposition 3.4]. For each $i, j \in \{1, \ldots, n\}$, define $\varphi_{ij} : A \times_\alpha G \to L_B(E)$ by

$$\varphi_{ij}(x) = V^*_{[\varrho],i}(\Phi_{[\varrho]} \times v^{[\varrho]})(x)V_{[\varrho],j}.$$

Clearly $[\varphi_{ij}]_{i,j=1}^n$ is a completely multi-positive linear map from $A \times_\alpha G$ to $L_B(E)$. Let $\{e_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit for A and let $\xi \in E$. Then, since $\Phi_{[\varrho]} \times v^{[\varrho]}$ and $[\varrho]$ are non-degenerate,

$$\lim_\lambda \varphi_{ii}(e_\lambda)\xi = \lim_\lambda V^*_{[\varrho],i}(\Phi_{[\varrho]} \times v^{[\varrho]})(e_\lambda)V_{[\varrho],i}\xi = V^*_{[\varrho],i}V_{[\varrho],i}\xi = \xi$$

for all $i = 1, \ldots, n$. Therefore $[\varphi_{ij}]_{i,j=1}^n$ is non-degenerate. Moreover, if $f \in C_c(G, A)$, then

$$\begin{aligned}\varphi_{ij}(f) = V^*_{[\varrho],i}(\Phi_{[\varrho]} \times v^{[\varrho]})(f)V_{[\varrho],j} &= \int_G V^*_{[\varrho],i}\Phi_{[\varrho]}(f(g))v_g^{[\varrho]}V_{[\varrho],j}\,dg \\ &= \int_G V^*_{[\varrho],i}\Phi_{[\varrho]}(f(g))V_{[\varrho],j}u_g\,dg = \int_G \varrho_{ij}(f(g))u_g\,dg,\end{aligned}$$

and since $C_c(G, A)$ is dense in $A \times_\alpha G$, $[\varphi_{ij}]_{i,j=1}^n$ is unique. ∎

Using the fact that any continuous action of a compact group on a locally C^*-algebra is an inverse limit action [11, Lemma 5.2], from Proposition 4.5 we obtain the following corollary.

COROLLARY 4.6. *Let (G, A, α) be a locally C^*-dynamical system, B a locally C^*-algebra, E a Hilbert B-module, u a unitary representation of G on E, and $[\varrho] = [\varrho_{ij}]_{i,j=1}^n$ a u-covariant, non-degenerate, completely multi-positive linear map from A to $L_B(E)$. If G is a compact group, then there is a unique completely multi-positive linear map $[\varphi_{ij}]_{i,j=1}^n$ from $A \times_\alpha G$ to $L_B(E)$ such that*

$$\varphi_{ij}(f) = \int_G \varrho_{ij}(f(g))u_g\,dg$$

for all $f \in C_c(G, A)$ and $i, j \in \{1, \ldots, n\}$. Moreover, $[\varphi_{ij}]_{i,j=1}^n$ is non-degenerate.

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Received May 11, 2005 (5647)