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On the Rockafellar theorem for $\Phi^{\gamma(\cdot,\cdot)}$-monotone multifunctions

by

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Abstract. Let X be an arbitrary set, and $\gamma : X \times X \to \mathbb{R}$ any function. Let Φ be a family of real-valued functions defined on X. Let $\Gamma : X \to 2^{\Phi}$ be a cyclic $\Phi^{\gamma(\cdot,\cdot)}$-monotone multifunction with non-empty values. It is shown that the following generalization of the Rockafellar theorem holds. There is a function $f : X \to \mathbb{R}$ such that Γ is contained in the $\Phi^{\gamma(\cdot,\cdot)}$-subdifferential of f, $\Gamma(x) \subset \partial_{\Phi}^{\gamma(\cdot,\cdot)} f|_x$.

Rockafellar (1970b) proved the following theorem:

Rockafellar Theorem. *Let $(X, \|\cdot\|)$ be a normed space and let X^* be its dual. Let Γ be a cyclic monotone multifunction mapping X into subsets of X^*, $\Gamma : X \to 2^{X^*}$, i.e. for any $n \in \mathbb{N}$ and $x_0, x_1, \ldots, x_n = x_0 \in X$ and $x_i^* \in \Gamma(x_i)$, $i = 1, \ldots, n$, we have*

$$\sum_{i=1}^{n} [x_{i-1}^*(x_{i-1}) - x_{i-1}^*(x_i)] \geq 0. \tag{1}$$

Suppose that $\Gamma(x) \neq \emptyset$ for all $x \in X$. Then there is a convex function f such that Γ is contained in the subdifferential of f,

$$\Gamma(x) \subset \partial f|_x. \tag{2}$$

If Γ is a maximal cyclic monotone multifunction, we have equality in (2).

In Pallaschke–Rolewicz (1997) (Proposition I.1.11) (see also Levin (1999), (2003)) it is shown that the construction of Rockafellar can give the more general

Theorem 0. *Let X be an arbitrary set. Let Φ be a family of real-valued functions defined on X. Let $\Gamma : X \to 2^{\Phi}$ be a cyclic Φ-monotone multifunction, i.e. for any $n \in \mathbb{N}$ and $x_0, x_1, \ldots, x_n = x_0 \in X$ and $\phi_{x_i} \in \Gamma(x_i)$,*

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$i = 1, \dots, n$, we have

$$\sum_{i=1}^{n} [\phi_{x_{i-1}}(x_{i-1}) - \phi_{x_{i-1}}(x_i)] \geq 0. \tag{3}$$

Suppose that $\mathit{\Gamma}(x) \neq \emptyset$ for all $x \in X$. Then there is a $\mathit{\Phi}$-convex function $f : X \to \mathbb{R}$ such that $\mathit{\Gamma}$ is contained in its $\mathit{\Phi}$-subdifferential,

$$\mathit{\Gamma}(x) \subset \partial^{\mathit{\Phi}} f|_x. \tag{4}$$

In this note we show that the construction of Rockafellar can give a similar result for a larger class of multifunctions.

Let X be an arbitrary set. Let $\mathit{\Phi}$ be a family of real-valued functions defined on X. Let $\gamma : X \times X \to \mathbb{R}$ and $f : X \to \mathbb{R}$. We say that a function $\phi_0 \in \mathit{\Phi}$ is a $\mathit{\Phi}^{\gamma(\cdot,\cdot)}$-*subgradient* of f at a point x_0 if

$$f(x) - f(x_0) \geq \phi_0(x) - \phi_0(x_0) + \gamma(x, x_0) \tag{5}$$

for all $x \in X$.

The set of all $\mathit{\Phi}^{\gamma(\cdot,\cdot)}$-subgradients of f at x_0 will be called the $\mathit{\Phi}^{\gamma(\cdot,\cdot)}$-*subdifferential* of f at x_0 and denoted by $\partial_{\mathit{\Phi}}^{\gamma(\cdot,\cdot)} f|_{x_0}$. Of course $\partial_{\mathit{\Phi}}^{\gamma(\cdot,\cdot)} f|_x$ is a multifunction mapping X into subsets of $\mathit{\Phi}$, $\partial_{\mathit{\Phi}}^{\gamma(\cdot,\cdot)} f|_x : X \to 2^{\mathit{\Phi}}$.

Example 1. Let $(X, \|\cdot\|)$ be a normed space and let $\mathit{\Phi} = X^*$ be its conjugate. Let $\gamma(\cdot,\cdot) \equiv 0$. Then a $\mathit{\Phi}^{\gamma(\cdot,\cdot)}$-subgradient is a subgradient in the sense of convex analysis (see for example Rockafellar (1970a)).

Example 2. Let $(X, \|\cdot\|)$ be a normed space and let $\mathit{\Phi} = X^*$. Let $\gamma(x, y) = -\varepsilon\|x - y\|$, where $\varepsilon > 0$. Then a $\mathit{\Phi}^{\gamma(\cdot,\cdot)}$-subgradient is an ε-subgradient in the sense of Ekeland–Lebourg (1975).

Example 3. Let $(X, \|\cdot\|)$ be a normed space and let $\mathit{\Phi} = X^*$. Suppose that

$$\liminf_{x \to x_0} \frac{\gamma(x, x_0)}{\|x - x_0\|} \geq 0.$$

Then a $\mathit{\Phi}^{\gamma(\cdot,\cdot)}$-subgradient is a Fréchet (approximate) subgradient of f at x_0 (see Ioffe (1984), (1986), (1989), (1990), (2000), Mordukhovich (1976), (1980), (1988), Borwein and Zhu (2005)).

Example 4. Let X be an arbitrary set. Let $\mathit{\Phi}$ be a family of real-valued functions defined on X. Let $\gamma(\cdot,\cdot) \equiv 0$. Then a $\mathit{\Phi}^{\gamma(\cdot,\cdot)}$-subgradient is a $\mathit{\Phi}$-subgradient in the sense of $\mathit{\Phi}$-convex analysis (see for example Pallaschke–Rolewicz (1997), Rubinov (2000), Singer (1997)).

Example 5. Let (X, d_X) be a metric space. Let $\mathit{\Phi}$ be a family of real-valued continuous functions defined on X. Let $\gamma(x, y) = \alpha(d_X(x, y))$, where

$\alpha(\cdot)$ is a real-valued function. Then a $\Phi^{\gamma(\cdot,\cdot)}$-subgradient is a strong Φ-subgradient with modulus $\alpha(\cdot)$ if $\alpha(\cdot) \geq 0$ (Rolewicz (1998), (2003)), and it is a weak Φ-subgradient with modulus $\alpha(\cdot)$ if $\alpha(\cdot) \leq 0$ (Rolewicz (2000a)).

If $\partial_{\Phi}^{\gamma(\cdot,\cdot)} f|_{x_0} \neq \emptyset$ we say that f is $\Phi^{\gamma(\cdot,\cdot)}$-*subdifferentiable* at x_0. If $\partial_{\Phi}^{\gamma(\cdot,\cdot)} f|_{x} \neq \emptyset$ for all $x \in X$ we say that f is $\Phi^{\gamma(\cdot,\cdot)}$-*subdifferentiable.*

Putting $x = x_0$ in (1) we trivially get

PROPOSITION 6. *If a function f is $\Phi^{\gamma(\cdot,\cdot)}$-subdifferentiable at x_0, then $\gamma(x_0, x_0) \leq 0$. If f is $\Phi^{\gamma(\cdot,\cdot)}$-subdifferentiable, then $\gamma(x,x) \leq 0$ for all $x \in X$.*

A multifunction Γ mapping X into 2^{Φ} is called n-*cyclic* $\Phi^{\gamma(\cdot,\cdot)}$-*monotone* if, for any $x_0, x_1, \ldots, x_n = x_0 \in X$ and $\phi_{x_i} \in \Gamma(x_i)$, $i = 1, \ldots, n$, we have

$$\sum_{i=1}^{n} [\phi_{x_{i-1}}(x_{i-1}) - \phi_{x_{i-1}}(x_i) - \gamma(x_i, x_{i-1})] \geq 0. \tag{6}$$

2-monotone multifunctions are simply called $\Phi^{\gamma(\cdot,\cdot)}$-*monotone.*

A multifunction Γ mapping X into 2^{Φ} is called *cyclic* $\Phi^{\gamma(\cdot,\cdot)}$-*monotone* if it is n-cyclic $\Phi^{\gamma(\cdot,\cdot)}$-monotone for $n = 2, 3, \ldots$. The definition immediately yields

PROPOSITION 7. *If $\gamma_1(x,y) \leq \gamma(x,y)$ for all $x, y \in X$ then an n-cyclic* (*resp. cyclic*) *$\Phi^{\gamma(\cdot,\cdot)}$-monotone multifunction Γ is n-cyclic* (*resp. cyclic*) *$\Phi^{\gamma_1(\cdot,\cdot)}$-monotone.*

It is not difficult to show

PROPOSITION 8. *Let X be an arbitrary set. Let Φ be a family of real-valued functions defined on X. Let $\gamma : X \times X \to \mathbb{R}$. Let f be a $\Phi^{\gamma(\cdot,\cdot)}$-subdifferentiable function. Then its $\Phi^{\gamma(\cdot,\cdot)}$-subdifferential, $\partial_{\Phi}^{\gamma(\cdot,\cdot)} f|_{x}$, considered as a multifunction of x, is cyclic $\Phi^{\gamma(\cdot,\cdot)}$-monotone.*

Proof. Since f is $\Phi^{\gamma(\cdot,\cdot)}$-subdifferentiable, for any $x_0, x_1, \ldots, x_n = x_0 \in X$ and $\phi_{x_i} \in \partial_{\Phi}^{\gamma(\cdot,\cdot)} f|_{x_i}$, $i = 1, \ldots, n$, we have

$$f(x_i) - f(x_{i-1}) \geq \phi_{x_{i-1}}(x_i) - \phi_{x_{i-1}}(x_{i-1}) + \gamma(x_i, x_{i-1}). \tag{5i}$$

Adding (5i), $i = 1, \ldots, n$, we get

$$0 \geq \sum_{i=1}^{n} [\phi_{x_{i-1}}(x_i) - \phi_{x_{i-1}}(x_{i-1}) + \gamma(x_i, x_{i-1})], \tag{7}$$

which trivially implies (6). ∎

EXAMPLE 1m. Let $(X, \|\cdot\|)$ be a normed space and let $\Phi = X^*$. Let $\gamma(\cdot,\cdot) \equiv 0$. Then each n-cyclic (resp. cyclic) $\Phi^{\gamma(\cdot,\cdot)}$-monotone multifunction Γ is n-cyclic (resp. cyclic) monotone in the classical sense (Rockafellar (1967), (1970a)).

Example 2m. Let $(X, \|\cdot\|)$ be a normed space and let $\Phi = X^*$. Let $\gamma(x, y) = -\varepsilon\|x-y\|$, where $\varepsilon > 0$. Then each $\Phi^{\gamma(\cdot,\cdot)}$-monotone multifunction Γ is ε-monotone (Jofré–Luc–Théra (1998), Luc–Ngai–Théra (1999)).

Example 4m. Let X be an arbitrary set. Let Φ be a family of real-valued functions defined on X. Let $\gamma(\cdot,\cdot) \equiv 0$. Then each n-cyclic (resp. cyclic) $\Phi^{\gamma(\cdot,\cdot)}$-monotone multifunction Γ is n-cyclic (resp. cyclic) monotone in the sense of Φ-convex analysis (see for example Pallaschke–Rolewicz (1997)).

Example 5m. Let (X, d_X) be a metric space. Let Φ be a family of real-valued continuous functions defined on X. Let $\gamma(x, y) = \alpha(d_X(x, y))$, where $\alpha(\cdot)$ is a real-valued function. Then each n-cyclic (resp. cyclic) $\Phi^{\gamma(\cdot,\cdot)}$-monotone multifunction Γ is n-cyclic (resp. cyclic) strongly $\alpha(\cdot)$-monotone if $\alpha(\cdot) \geq 0$ (Rolewicz (1998)) and weakly $\alpha(\cdot)$-monotone if $\alpha(\cdot) \leq 0$ (Rolewicz (2000a)).

A cyclic $\Phi^{\gamma(\cdot,\cdot)}$-monotone multifunction Γ is called *maximal cyclic $\Phi^{\gamma(\cdot,\cdot)}$-monotone* if for each cyclic $\Phi^{\gamma(\cdot,\cdot)}$-monotone multifunction Γ_1 such that $\Gamma(x) \subset \Gamma_1(x)$ for all x (in other words, the graph of Γ, $G(\Gamma)$, is contained in $G(\Gamma_1)$), we have $\Gamma(x) = \Gamma_1(x)$ for all $x \in X$.

Theorem 9. *Let X be an arbitrary set. Let Φ be a family of real-valued functions defined on X. Let $\gamma : X \times X \to \mathbb{R}$. Let Γ be a cyclic $\Phi^{\gamma(\cdot,\cdot)}$-monotone multifunction. Suppose that $\Gamma(x) \neq \emptyset$ for all $x \in X$. Then there is a $\Phi^{\gamma(\cdot,\cdot)}$-subdifferentiable function f such that Γ is contained in the $\Phi^{\gamma(\cdot,\cdot)}$-subdifferential of f,*

$$\Gamma(x) \subset \partial_{\Phi}^{\gamma(\cdot,\cdot)} f|_x. \tag{8}$$

Proof. Fix $x_0 \in X$ and $\phi_{x_0} \in \Gamma(x_0)$. We define

$$\begin{aligned} f(x) = \sup\{&(\phi_{x_n}(x) - \phi_{x_n}(x_n) + \gamma(x, x_n)) \\ &+ (\phi_{x_{n-1}}(x_n) - \phi_{x_{n-1}}(x_{n-1}) + \gamma(x_n, x_{n-1})) \\ &+ \cdots + (\phi_{x_0}(x_1) - \phi_{x_0}(x_0) + \gamma(x_1, x_0))\}, \end{aligned} \tag{9}$$

where the supremum is taken over all $x_1, \ldots, x_n \in X$, $\phi_{x_1} \in \Gamma(x_1), \ldots, \phi_{x_n} \in \Gamma(x_n)$. Observe that $f(x_0) \leq 0$ by cyclic $\Phi^{\gamma(\cdot,\cdot)}$-monotonicity of $\Gamma(\cdot)$.

Take any $x \in X$ and $\phi_x \in \Gamma(x)$. Let λ be an arbitrary number smaller than $f(x)$. By the definition of $f(x)$, there are $x_1, \ldots, x_n \in X$, $\phi_{x_1} \in \Gamma(x_1), \ldots, \phi_{x_n} \in \Gamma(x_n)$ such that

$$\begin{aligned} \lambda < &(\phi_{x_n}(x) - \phi_{x_n}(x_n) + \gamma(x, x_n)) \\ &+ (\phi_{x_{n-1}}(x_n) - \phi_{x_{n-1}}(x_{n-1}) + \gamma(x_n, x_{n-1})) \\ &+ \cdots + (\phi_{x_0}(x_1) - \phi_{x_0}(x_0) + \gamma(x_1, x_0)). \end{aligned} \tag{10}$$

Put $x_{n+1} = x$ and $\phi_{x_{n+1}} = \phi_x$. Then for all $y \in X$,

$$f(y) \geq (\phi_x(y) - \phi_x(x) + \gamma(x, y)) + \lambda.$$

Since this holds for any $\lambda < f(x)$, we trivially obtain

$$f(y) \geq f(x) + \phi_x(y) - \phi_x(x) + \gamma(x, y). \tag{11}$$

Therefore $f(x_0) \leq 0$ implies that $f(x) < \infty$ for all $x \in X$. Moreover from (11) we have

$$f(y) - f(x) \geq \phi_x(y) - \phi_x(x) + \gamma(\cdot, \cdot), \tag{12}$$

i.e. ϕ_x is a $\Phi^{\gamma(\cdot,\cdot)}$-subgradient of f at x.

Since ϕ_x was an arbitrary element of $\Gamma(x)$ we get

$$\Gamma(x) \subset \partial_{\Phi}^{\gamma(\cdot,\cdot)} f|_x. \ \blacksquare \tag{8}$$

COROLLARY 10. *Let X be an arbitrary set. Let Φ be a family of real-valued functions defined on X. Let $\gamma : X \times X \to \mathbb{R}$. Let Γ be a maximal cyclic $\Phi^{\gamma(\cdot,\cdot)}$-monotone multifunction. Suppose that $\Gamma(x) \neq \emptyset$ for all $x \in X$. Then there is a $\Phi^{\gamma(\cdot,\cdot)}$-subdifferentiable function f such that Γ is equal to the $\Phi^{\gamma(\cdot,\cdot)}$-subdifferential of f,*

$$\Gamma(x) = \partial_{\Phi}^{\gamma(\cdot,\cdot)} f|_x. \tag{13}$$

Proof. By Theorem 9 the graph $G(\Gamma)$ is contained in $G(\partial_{\Phi}^{\gamma(\cdot,\cdot)} f|_x)$. By Proposition 8 the multifunction $\partial_{\Phi}^{\gamma(\cdot,\cdot)} f|_x$ is cyclic $\Phi^{\gamma(\cdot,\cdot)}$-monotone. Since Γ is maximal cyclic $\Phi^{\gamma(\cdot,\cdot)}$-monotone, this implies that

$$\Gamma(x) = \partial_{\Phi}^{\gamma(\cdot,\cdot)} f|_x. \ \blacksquare \tag{13}$$

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