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Robustness with respect to small time-varied delay for linear dynamical systems on Banach spaces

by

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Abstract. Under suitable conditions we prove the wellposedness of small time-varied delay equations and then establish the robust stability for such systems on the phase space of continuous vector-valued functions.

1. Introduction. The robustness of delay equations has been studied by many authors (see cf. [Ba1, Ba2, Da, EN, Hu, FN, JGH, Liu]).

In this paper we consider the time-varied delay equation of the form

$$\begin{cases} x'(t) = Ax(t) + Bx(t-\tau(t)), & t \geq 0, \\ x(\theta) = \xi(\theta), & -r \leq \theta \leq 0, \end{cases} \tag{1.1}$$

where A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X, B is a closed densely defined linear operator on X, $\tau(t)$ is continuous and ξ is taken from some phase space.

Huang ([Hu]) proved the robust stability of the delay equation (1.1) on the phase space $C(-r,0;X)$ in the case that B is a bounded operator. Dropping the assumption that B is bounded, Liu ([Liu]) showed that if A generates a holomorphic semigroup and B is $(-A)^\alpha$-bounded, then the exponential stability of (1.1) (with $\tau(t) \equiv r$) on the phase space $C(-r,0;D(A))$ is robust. Bátkai *et al.* ([Ba1, Ba2]) proved a similar result on the phase space $X \times L^p(-r,0;D(B))$.

Our goal in this paper is to study the robust stability of the time-varied delay equation (1.1) in the case that B is unbounded. The organization of the paper is as follows: in Section 2, we will prove the wellposedness of (1.1) under some general assumptions on A and B, and that the solution operators are given by Dyson–Phillips series. In Section 3, we prove the robust stability of the equation with time-varied delay on the phase space of continuous functions under the assumption that $BT(t)$ is norm continuous

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for $t > 0$. In addition, we will give an example to show that under this condition, the semigroup $T(t)$ is not necessarily holomorphic. So our results in this section generalize that of [Liu]. Moreover, our results show that on the phase space of continuous functions, the robust stability of the system without delay persists in the system with time-varied delay. However, the time-varied delay on the phase space $X \times L^p(-r, 0; D(B))$ will greatly affect the robustness and even the wellposedness of the delay equation. This will be taken up in a subsequent paper.

2. Preliminaries and wellposedness. Let X be a Banach space with norm $\|\cdot\|$ and let $\mathbf{B}(X)$ be the Banach algebra of all bounded linear operators on X. If A is a linear operator on X, we write $D(A)$ for its domain. We denote by $(f * g)(t) = \int_0^t f(t-s)g(s)\,ds$ the convolution of f and g. Throughout this paper the following assumptions will be in force:

General Assumptions. (a1) A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X.

(a2) B is a closed linear operator on X, $D(A) \subset D(B)$ and there is a non-negative measurable function $k \in L^1_{\mathrm{loc}}(0,\infty)$ such that

$$\|BT(t)x\| \leq k(t)\|x\|, \quad t \geq 0,\ x \in D(A). \tag{2.1}$$

Since $k \in L^1_{\mathrm{loc}}(0,\infty)$, from [DS, pp. 631, Theorem 19] one knows that $A + B$ with domain $D(A)$ generates a C_0-semigroup $(T_B(t))_{t\geq 0}$ on X.

Let $\omega_0(T)$ be the *growth bound* of $(T(t))_{t\geq 0}$, that is, for $\omega > \omega_0(T)$ and $0 < \delta < \omega - \omega_0(T)$, there is a constant $M \geq 1$ such that $\|T(t)\| \leq Me^{(\omega-\delta)t}$ for $t \geq 0$. Let $t_0 > 0$ by such that $k(t_0)$ is finite. Then by (2.1), for $t \geq t_0$ and $x \in D(A)$, we have

$$\begin{aligned}\|BT(t)x\| &= \|BT(t_0)T(t-t_0)x\| \\ &\leq k(t_0)\|T(t-t_0)x\| \leq k(t_0)Me^{(\omega-\delta)(t-t_0)}\|x\|.\end{aligned} \tag{2.2}$$

This shows that $BT(t)$ extends to a bounded operator on X for $t \geq t_0$ since $D(A)$ is dense. We will also denote this extension by $BT(t)$ in the rest of this paper. Moreover, since there is a sequence $\{t_n\} \subset \mathbb{R}^+$ such that $t_n \to 0$ and $k(t_n)$ is finite, we know that $BT(t) \in \mathbf{B}(X)$ for all $t > 0$. Let $k_0(t) = \|BT(t)\|$. By (2.2), we have $k^0(t) := k_0(t)e^{-\omega t} \in L^1(0,\infty)$ and

$$k^0(t) \leq k_0(t_0)Me^{-\delta(t-t_0)}, \quad t \geq t_0. \tag{2.3}$$

Furthermore, we have

Lemma 2.1. *For all $t > 0$, $BT_B(t) \in \mathbf{B}(X)$ and $k_1(t) := \|BT_B(t)\| \in L^1_{\mathrm{loc}}(0,\infty)$ satisfies*

$$k_1(t) \leq k_1(t_0)Me^{(\omega-\delta)(t-t_0)}, \quad t \geq t_0 > 0, \tag{2.4}$$

where $\omega > \max\{0, \omega_0(T), \omega_0(T_B)\}$ *is large enough such that* $\|T(t)\|, \|T_B(t)\| \le Me^{(\omega-\delta)t}$ *for* $t \ge 0$ *and some constant* $M \ge 1$, *and*

$$\beta := M \int_0^\infty k^0(t)\, dt < 1. \tag{2.5}$$

Proof. Choose $\omega > \max\{0, \omega_0(T), \omega_0(T_B)\}$ large enough such that (2.5) holds. Then for $x \in X$ and $t > 0$, multiplying the equation

$$BT_B(t)x = BT(t)x + \int_0^t BT(t-s)BT_B(s)x\, ds \tag{2.6}$$

by $e^{-\omega t}$ yields

$$e^{-\omega s}\|BT_B(s)x\| \le k^0(s)\|x\| + \int_0^s k^0(s-\tau)e^{-\omega\tau}\|BT_B(\tau)x\|\, d\tau, \quad s > 0. \tag{2.7}$$

Integrating (2.7) from 0 to t gives

$$\begin{aligned}\int_0^t e^{-\omega s}\|BT_B(s)x\|\, ds &\le \int_0^t k^0(s)\|x\|\, ds + \int_0^t\int_0^s k^0(s-\tau)e^{-\omega\tau}\|BT_B(\tau)x\|\, d\tau\, ds \\ &\le \beta\|x\| + \int_0^t e^{-\omega\tau}\|BT_B(\tau)x\| \int_\tau^t k^0(s-\tau)\, ds\, d\tau \\ &\le \beta\|x\| + \beta\int_0^t e^{-\omega\tau}\|BT_B(\tau)x\|\, d\tau.\end{aligned}$$

It follows that

$$\int_0^t e^{-\omega s}\|BT_B(s)x\|\, ds \le \beta(1-\beta)^{-1}\|x\|, \quad t > 0,\ x \in X. \tag{2.8}$$

By induction, from (2.7) using (2.8) we have

$$e^{-\omega t}\|BT_B(t)x\| \le k_2(t)\|x\|, \quad t > 0,\ x \in X, \tag{2.9}$$

where $k_2(t) = \sum (k^0)^{*n}(t)$ and $(k^0)^{*n} = k^0 * \cdots * k^0$ is the n-fold convolution of the kernel k^0. Since $\|k^0\|_{L^1(0,\infty)} \le \beta$, we have $\|k_2\|_{L^1(0,\infty)} \le (1-\beta)^{-1}\beta$. Thus from (2.9) we have $\|BT_B(t)\| \le e^{\omega t}\beta(1-\beta)^{-1}$, and similarly to the proof of (2.2) and (2.3), one can show that (2.4) holds. ∎

Next we consider the norm continuity of $BT(t)$ and $BT_B(t)$.

LEMMA 2.2. *If* $BT(t)$ *is norm continuous for* $t > 0$, *then so is* $BT_B(t)$.

Proof. Let $t > 0$ and $0 < \delta < t/2$. By (2.6), for $|h| < \delta$ and $x \in X$ satisfying $\|x\| \le 1$,

$$\begin{aligned}
&\|BT_B(t+h)x - BT_B(t)x\| \\
&\quad = \Big\| B(T(t+h)x - T(t))x + \int_{t-\delta}^{t+h} BT(t+h-s)BT_B(s)x\,ds \\
&\qquad + \int_0^{t-\delta} [BT(t+h-s) - BT(t-s)]BT_B(s)x\,ds \\
&\qquad - \int_{t-\delta}^{t} BT(t-s)BT_B(s)x\,ds \Big\| \\
&\quad \le \|BT(t+h) - BT(t)\| + \int_0^{t-\delta} \|BT(t+h-s) - BT(t-s)\| k_1(s)\,ds \\
&\qquad + M_t \Big[\int_{t-\delta}^{t+h} k_0(t+h-s)\,ds + \int_{t-\delta}^{t} k_0(t-s)\,ds \Big],
\end{aligned}$$

where $M_t := \max\{k_1(s) : t/2 \le s \le 3t/2\}$; (2.4) implies that M_t is finite for $t > 0$. Since $BT(t)$ is norm continuous for $t > 0$ and $k_0 \in L^1_{\mathrm{loc}}(0,\infty)$, for every $\varepsilon > 0$ there is a $\delta_1 \in (0, t/4)$ such that when $|h| < \delta/2$ and $\delta \le \delta_1$,

$$\|BT(t+h) - BT(t)\| + M_t \Big(\int_{t-\delta}^{t+h} k_0(t+h-s)\,ds + \int_{t-\delta}^{t} k_0(t-s)\,ds \Big) < \varepsilon/2.$$

Moreover, for given $0 < \delta \le \delta_1$, since $BT(t)$ is uniformly continuous on $[\delta/2, t+\delta/2]$, there exists $\delta_\varepsilon \in (0, \delta/2)$ such that for $s \in [0, t-\delta]$ and $|h| \le \delta_\varepsilon$,

$$\|BT(t+h-s) - BT(t-s)\| < \frac{1}{2}\Big(\int_0^t k_1(s)\,ds \Big)^{-1} \varepsilon.$$

Combining all these inequalities, for $x \in X$ with $\|x\| \le 1$ and $|h| < \delta_\varepsilon$ we obtain

$$\|BT_B(t+h)x - BT_B(t)x\| < \varepsilon,$$

which implies the norm continuity of $BT_B(t)$ on $(0,\infty)$. ∎

The continuity of $BT(t)x$ for some point x is equivalent to that of $BT_B(t)x$:

LEMMA 2.3. *Let $x \in D(B)$. Then $BT(t)x$ is continuous for $t \ge 0$ if and only if $BT_B(t)x$ is continuous for $t \ge 0$.*

Proof. Suppose that $BT(t)x$ is continuous for $t \geq 0$. Since for $0 \leq t \leq 1$,

$$\begin{aligned}\|BT_B(t)x - Bx\| &= \Big\|BT(t)x - Bx + \int_0^t BT_B(t-s)BT(s)x\,ds\Big\| \\ &\leq \|BT(t)x - Bx\| + \int_0^t k_1(t-s)\,ds \max_{0\leq\tau\leq 1}\|BT(\tau)x\|,\end{aligned}$$

$BT_B(t)x$ is right-continuous at 0. Now let $t > 0$ and $|h| < \delta < \min\{1, t/2\}$. Then

$$\begin{aligned}&\|BT_B(t+h)x - BT_B(t)x\| \\ &\quad= \Big\|BT(t+h)x - BT(t)x + \int_0^{t+h} BT_B(s)B(t+h-s)x\,ds \\ &\qquad - \int_0^t BT_B(s)BT(t-s)x\,ds\Big\| \\ &\quad\leq \|BT(t+h)x - BT(t)x\| \\ &\qquad + \int_0^{t-\delta} k_1(s)\|BT(t+h-s)x - BT(t-s)x\|\,ds \\ &\qquad + \Big[\int_{t-\delta}^{t+h} k_1(s)\,ds + \int_{t-\delta}^{t} k_1(s)\,ds\Big] \max_{0\leq\tau\leq 2}\|BT(\tau)x\|.\end{aligned}$$

Since $BT(s)x$ is uniformly continuous for $s \in [0, t+1]$ and $k_1 \in L^1_{\mathrm{loc}}(0,\infty)$, for every $\varepsilon > 0$ one can find a constant $\delta_\varepsilon \in (0, \min\{1, t/2\})$ such that for $|h| < \delta < \delta_\varepsilon$,

$$\|BT(s+h)x - BT(s)x\| < \frac{1}{2}\Big(1 + \int_0^t k_1(s)\,ds\Big)^{-1}\varepsilon, \quad s \in [0, t+1],$$

and

$$\int_{t-\delta}^{t+h} k_1(s)\,ds + \int_{t-\delta}^{t} k_1(s)\,ds < \frac{1}{2}\big(\max_{0\leq\tau\leq 2}\|BT(\tau)x\|\big)^{-1}\varepsilon.$$

By the above estimates, we have

$$\|BT_B(t+h)x - BT_B(t)x\| < \varepsilon, \quad |h| < \delta_\varepsilon,$$

which means that $BT_B(t)x$ is continuous for $t \geq 0$. Conversely, if $BT_B(t)x$ is continuous for $t \geq 0$, then from

$$BT(t)x = BT_B(t)x - \int_0^t BT(s)BT_B(t-s)x\,ds, \quad t \geq 0,$$

by a similar argument one can show that $BT(t)x$ is continuous for $t \geq 0$. ∎

We are particularly interested in the subspace of X on which $BT(t)$ (and also $BT_B(t)$ by Lemma 2.3) is strongly continuous.

LEMMA 2.4. *Let X_b be the subspace of X defined by*

$$X_b = \{x \in D(B) : BT(t)x \text{ is continuous for } t \geq 0\}.$$

Then $D(A) \subset X_b \subset D(B)$ and X_b is a Banach space with norm

$$\|x\|_b = \|x\| + \sup_{s\geq 0} \|e^{-\omega s} BT(s)x\|, \quad x \in X_b, \tag{2.10}$$

where $\omega > \max\{0, \omega_0(T), \omega_0(T_B)\}$ is large enough such that $\|T(t)\| + \|T_B(t)\| \leq Me^{(\omega-\delta)t}$ for $t \geq 0$ and some constant $M \geq 1$, and

$$\gamma := M \int_0^\infty e^{-\omega t}(k_0(t) + k_1(t))\, dt < 1.$$

Moreover, the norm

$$\|x\|_{b'} := \|x\| + \sup_{s\geq 0} \|e^{-\omega s} BT_B(s)x\|$$

on X_b is equivalent to $\|\cdot\|_b$. Finally, if $T_B(t)$ is exponentially stable, that is, there are constants $M_b \geq 1$ and $\omega_b > 0$ such that $\|T_B(t)\| \leq M_b e^{-\omega_b t}$ for $t \geq 0$, then the norm

$$\|x\|_s := \|x\| + \sup_{s\geq 0} \|BT_B(s)x\| \tag{2.11}$$

on X_b is also equivalent to $\|\cdot\|_b$.

Proof. If $x \in D(A)$, then for $t \geq 0$ and $h > 0$,

$$\|BT(t+h)x - BT(t)x\| = \Big\|B \int_t^{t+h} T(s)Ax\, ds\Big\| \leq \int_t^{t+h} k_0(s)\, ds \cdot \|Ax\|,$$

so $BT(t)x$ is continuous for $t \geq 0$ since $k_0(\cdot) \in L^1_{\mathrm{loc}}(0,\infty)$. Hence $D(A) \subset X_b \subset D(B)$.

Next we show that $(X_b, \|\cdot\|_b)$ is a Banach space. Let $\{x_n\} \subset X_b$ be a Cauchy sequence in X_b. Then from the definition of the norm, both $\{x_n\}$ and $\{Bx_n\}$ are Cauchy sequences in X and thus converge. Suppose that $x_n \to x$ and $Bx_n \to y$ in X. Then from the closedness of B we have $x \in D(B)$ and $Bx = y$. Now the strong continuity of $BT(t)x$ follows from the facts that x_n converges to x and the convergence of $BT(t)x_n$ to $BT(t)x$ is uniform in compact intervals. Similarly one can show that $(X_b, \|\cdot\|_{b'})$ is also a Banach space by using Lemma 2.3.

To see the equivalence of the two norms, by the Inverse Mapping Theorem, we only need to show that one norm is stronger than the other. Let $x \in X_b$. By the definition of b'-norm we have

$$e^{-\omega t}\|BT_B(t)x\| \leq \|x\|_{b'}, \quad t \geq 0,$$

thus

$$\begin{aligned}\|x\|_{b'} &= \|x\| + \sup_{s\ge 0}\|e^{-\omega s}BT_B(s)x\| \\ &\le \|x\| + \sup_{s\ge 0}\|e^{-\omega s}BT(s)x\| \\ &\quad + \sup_{s\ge 0}\Big\|\int_0^s e^{-\omega(s-\tau)}BT(s-\tau)e^{-\omega\tau}BT_B(\tau)x\,d\tau\Big\| \\ &\le \|x\|_b + \sup_{s\ge 0}\int_0^s e^{-\omega(s-\tau)}k_0(s-\tau)e^{-\omega\tau}\|BT_B(\tau)x\|\,d\tau \\ &\le \|x\|_b + \sup_{s\ge 0}\int_0^s e^{-\omega(s-\tau)}k_0(s-\tau)\|x\|_{b'}\,d\tau \\ &\le \|x\|_b + \gamma\|x\|_{b'}.\end{aligned}$$

It follows that $\|x\|_{b'} \le (1-\gamma)^{-1}\|x\|_b$ for $x \in X_b$, and therefore, the $\|\cdot\|_b$-norm is stronger than the $\|\cdot\|_{b'}$-norm.

If $T_B(t)$ is exponentially stable, then by Lemma 2.1, $BT_B(t) \in \mathbf{B}(X)$ for all $t > 0$ and

$$\|BT_B(t)\| = \|BT_B(t_0)T_B(t-t_0)\| \le k_1(t_0)M_b e^{-\omega_b(t-t_0)}, \quad t \ge t_0.$$

So $\|\cdot\|_s$ is a norm on X_b and $(X_b, \|\cdot\|_s)$ is a Banach space. Moreover, for $x \in X_b$,

$$\|x\|_{b'} = \|x\| + \sup_{s\ge 0}\|e^{-\omega s}BT_B(s)x\| \le \|x\| + \sup_{s\ge 0}\|BT_B(s)x\| = \|x\|_s,$$

and again by the Inverse Mapping Theorem, the norms $\|\cdot\|_s$ and $\|\cdot\|_b$ on X_b are equivalent. ■

After these preparations, we now consider the delay equation

$$\begin{cases} x'(t) = Ax(t) + Bx(t-\tau(t)), & t \ge 0, \\ x(\theta) = \xi(\theta), & -r \le \theta \le 0, \end{cases} \tag{2.12}$$

where $0 \le \tau \le r$, $\tau(t)$ is continuous for $t \ge 0$ and $\xi(\cdot) \in C(-r,0;X_b)$. In the rest of this paper we will denote by $\mathcal{X} = C(-r,0;X_b)$ the *phase space*. The solution of (2.12) also satisfies

$$\begin{cases} x(t) = T(t)\xi(0) + \displaystyle\int_0^t T(t-s)Bx(s-\tau(s))\,ds, & t \ge 0, \\ x(\theta) = \xi(\theta), & -r \le \theta \le 0. \end{cases} \tag{2.13}$$

We call $x(t)$ a *solution* of (2.13) if $x(t) \in C(-r,\infty;X_b)$ satisfies (2.13) and $x_t(\cdot) \in \mathcal{X}$ is continuous for $t \ge 0$, where $x_t(\theta) := x(t+\theta)$ for $t \ge 0$ and $-r \le \theta \le 0$. In the following we will denote the solution of (2.13) at ξ by $x(t,\xi)$ and call it the *mild solution* of (2.12).

THEOREM 2.5. *For any $r > 0$ and $\xi \in \mathcal{X}$, (2.13) has a unique solution $x(t,\xi)$. Let*

$$(T_r(t)\xi)(\theta) := x_t(\theta,\xi), \quad t \ge 0,\ -r \le \theta \le 0,$$

be the solution operator. Then there exist positive constants M_0 and ω_0, independent of r, such that

$$\|T_r(t)\xi\|_{\mathcal{X}} \le M_0 e^{\omega_0 t}\|\xi\|_{\mathcal{X}}, \quad t \ge 0,\ r > 0,\ \xi \in \mathcal{X}. \tag{2.14}$$

Proof. We will choose the $\|\cdot\|_b$-norm on X_b given by (2.10), with the constant ω so large that $\|T(t)\| \le M e^{(\omega-\delta)t}$ for all $t \ge 0$, and $\omega > \delta > M$ such that

$$\beta_0 := M \int_0^\infty e^{-\omega t} k_0(t)\, dt < 1 - M\delta^{-1}.$$

For $r > 0$ and $\xi \in \mathcal{X}$, define

$$x^{(0)}(t) = \begin{cases} T(t)\xi(0), & t \ge 0, \\ \xi(t), & -r \le t < 0, \end{cases}$$

and for $n = 1, 2, \ldots,$

$$x^{(n)}(t) = \begin{cases} \int_0^t T(t-s)Bx^{(n-1)}(s-\tau(s))\, ds, & t \ge 0, \\ 0, & -r \le t < 0. \end{cases} \tag{2.15}$$

It is clear from the definition of X_b that $x^{(0)}(t)$ is continuous for $t \ge -r$ in X_b, and from

$$x_t^{(0)}(\theta) = \begin{cases} T(t+\theta)\xi(0), & t \ge r, -r \le \theta \le 0 \text{ or } 0 \le t \le r, -t \le \theta \le 0, \\ \xi(t+\theta), & 0 \le t \le r, -r \le \theta \le -t, \end{cases}$$

we have for $t \ge r, -r \le \theta \le 0$ or $0 \le t \le r, -t \le \theta \le 0$,

$$\begin{aligned} \|x_t^{(0)}(\theta)\|_b &= \|T(t+\theta)\xi(0)\| + \sup_{s\ge 0}\|e^{-\omega s}BT(s+t+\theta)\xi(0)\| \\ &\le Me^{\omega(t+\theta)}\|\xi(0)\| + e^{\omega(t+\theta)}\sup_{s\ge 0}\|e^{-\omega(s+t+\theta)}BT(s+t+\theta)\xi(0)\| \\ &\le Me^{\omega t}\|\xi(0)\|_b + e^{\omega t}\|\xi(0)\|_b \le (1+M)e^{\omega t}\|\xi\|_{\mathcal{X}}, \end{aligned}$$

and for $0 \le t \le r$, $-r \le \theta \le -t$,

$$\|x_t^{(0)}(\theta)\|_b = \|\xi(t+\theta)\|_b \le \|\xi\|_{\mathcal{X}}.$$

It follows that

$$\|x_t^{(0)}(\cdot)\|_{\mathcal{X}} \le (1+M)e^{\omega t}\|\xi\|_{\mathcal{X}}, \quad t \ge 0. \tag{2.16}$$

Moreover, from (2.15) it is easy to see that $x^{(1)}(t)$ is continuous for $t \ge 0$ in X_b and by using (2.16) one can show that

$$\|x_t^{(1)}(\cdot)\|_{\mathcal{X}} \le (1+M)\beta_1 e^{\omega t}\|\xi\|_{\mathcal{X}}, \quad t \ge 0,\ \xi \in \mathcal{X},$$

where $\beta_1 := M\delta^{-1} + \beta_0 < 1$. Then by induction on n we find that $x^{(n)}(t)$ is continuous in X_b and

$$\|x_t^{(n)}(\cdot)\|_{\mathcal{X}} \leq (1+M)\beta_1^n e^{\omega t}\|\xi\|_{\mathcal{X}}, \quad t \geq 0,\ n = 0,1,2,\ldots. \tag{2.17}$$

Set $x(t) = \sum_{n=0}^{\infty} x^{(n)}(t)$ for $t \geq -r$. By (2.17) the series $\sum_{n=0}^{\infty} x^{(n)}(t)$ is absolutely convergent on compact intervals in X_b and

$$\begin{aligned}\|x_t(\cdot)\|_{\mathcal{X}} &\leq \sum_{n=0}^{\infty} \|x_t^{(n)}(\cdot)\|_{\mathcal{X}} \leq \sum_{n=0}^{\infty} (1+M)\beta_1^n e^{\omega t}\|\xi\|_{\mathcal{X}} \\ &= (1+M)(1-\beta_1)^{-1} e^{\omega t}\|\xi\|_{\mathcal{X}}.\end{aligned} \tag{2.18}$$

Thus $x(t)$ is continuous for $t \geq -r$ in X_b and

$$\begin{aligned} x(t) &= \begin{cases} T(t)\xi(0) + \sum_{n=0}^{\infty} \int_0^t T(t-s)Bx^{(n)}(s-\tau(s))\,ds, & t \geq 0, \\ \xi(t), & -r \leq t \leq 0, \end{cases} \\ &= \begin{cases} T(t)\xi(0) + \int_0^t T(t-s)B\sum_{n=0}^{\infty} x^{(n)}(s-\tau(s))\,ds, & t \geq 0, \\ \xi(t), & -r \leq t \leq 0, \end{cases} \\ &= \begin{cases} T(t)\xi(0) + \int_0^t T(t-s)Bx(s-\tau(s))\,ds, & t \geq 0, \\ \xi(t), & -r \leq t \leq 0, \end{cases} \end{aligned}$$

that is, $x(t)$ satisfies (2.13) and by (2.18),

$$\|x_t(\cdot)\|_{\mathcal{X}} \leq (1+M)(1-\beta_1)^{-1} e^{\omega t}\|\xi\|_{\mathcal{X}}, \quad t \geq 0,\ \xi \in \mathcal{X}. \tag{2.19}$$

To show the uniqueness of the solutions, let $x(t)$ be a solution of (2.13) with initial value $\xi(t) \equiv 0$ ($t \in [-r,0]$). Then $x(t) = 0$ for $-r \leq t \leq 0$, while for $t \geq 0$,

$$x(t) = \int_0^t T(t-s)Bx(s-\tau(s))\,ds.$$

It is easy to show that for $t \geq r, -r \leq \theta \leq 0$ or $0 \leq t \leq r, -t \leq \theta \leq 0$,

$$\|x_t(\theta)\|_b \leq M \int_0^{t+\theta} [e^{(\omega-\delta)(t+\theta-s)} + k_0(t+\theta-s)]\|x(s-\tau(s))\|_b\,ds, \tag{2.20}$$

which implies that

$$\|x_t(\cdot)\|_{\mathcal{X}} \leq \beta_1 e^{\omega t}\|x_t(\cdot)\|_{\mathcal{X}};$$

by using this inequality on the right-hand side of (2.20) and by induction one obtains

$$\|x_t(\cdot)\|_{\mathcal{X}} \leq \beta_1^n e^{\omega t}\|x_t(\cdot)\|_{\mathcal{X}}, \quad n = 1,2,\ldots.$$

Since n is arbitrary and $\beta_1 < 1$, we have $x_t \equiv 0$, which proves the uniqueness of the solutions. So we can define

$$(T_r(t)\xi)(\theta) = x(t+\theta, \xi), \quad t \geq 0,\ -r \leq \theta \leq 0,\ \xi \in \mathcal{X},$$

where $x(t,\xi)$ is the solution of (2.13) at $\xi \in \mathcal{X}$. Moreover, (2.19) implies that (2.14) holds for $M_0 = (1+M)(1-\beta_0)^{-1}$ and $\omega_0 = \omega$. Finally, since $x^{(n)}(t)$ are uniformly continuous on $[-r, t_0]$ for every $t_0 > -r$ and $x^{(n)}(\cdot)$ is continuous for $t \geq 0$ in $\mathcal{X}$, by (2.18), we know that $x_t(\cdot)$ is continuous for $t \geq 0$ in $\mathcal{X}$. ∎

3. Robustness with respect to small time-varied delay. In this section we will investigate the stability of the solution of (2.12). To this end, we rewrite (2.12) as

$$\begin{cases} x'(t) = (A+B)x(t) + B(x(t-\tau(t)) - x(t)), & t \geq 0, \\ x(\theta) = \xi(\theta), & -r \leq \theta \leq 0, \end{cases} \tag{3.1}$$

where $\xi(\cdot) \in \mathcal{X}$, $0 \leq \tau(t) \leq r$ and $\tau(t)$ is continuous for $t \geq 0$. The solution of (3.1) is related to the integrated equation

$$\begin{cases} x(t) = T_B(t)\xi(0) \\ \qquad\qquad + \int_0^t T_B(t-s)B(x(s-\tau(s)) - x(s))\,ds, & t \geq 0, \\ x(\theta) = \xi(\theta), & -r \leq \theta \leq 0. \end{cases} \tag{3.2}$$

LEMMA 3.1. *The space X_b is $T_B(t)$-invariant, i.e., $T_B(t)X_b \subset X_b$ for $t \geq 0$, and $(T_B(t))_{t\geq 0}$ is a C_0-semigroup on X_b. Moreover, if $T_B(t)$ is exponentially stable on X, then so is $T_B(t)$ on X_b and*

$$\|T_B(t)x\|_s \leq (3 + k_1(t_0))M_b e^{\omega_b t_0} e^{-\omega_b t}\|x\|_s, \quad t \geq 0,\ x \in X_b, \tag{3.3}$$

where $t_0 > 0$ is arbitrary, M_b and ω_b are positive constants such that $\|T_B(t)\| \leq M_b e^{-\omega_b t}$ for $t \geq 0$, and $\|\cdot\|_s$ is given by (2.11).

Proof. It is easy to see that X_b is $T_B(t)$-invariant. Now we suppose that $T_B(t)$ is exponentially stable on X. Let $x \in X_b$ and $t_0 > 0$. Then for every $\varepsilon > 0$, there is a $T_\varepsilon \geq t_0$ such that for $s \geq T_\varepsilon$ and $t \geq 0$,

$$\begin{aligned} \|BT_B(t+s)x - BT_B(s)x\| &= \|BT_B(t_0)(T_B(t+s-t_0)x - T_B(s-t_0)x)\| \\ &\leq k_1(t_0)M_b(e^{-\omega_b(t+s-t_0)} + e^{-\omega_b(s-t_0)})\|x\| < \varepsilon/2. \end{aligned}$$

On the other hand, by Lemma 2.3, $BT_B(s)x$ is continuous for $s \geq 0$, and therefore uniformly continuous on $[0, T_\varepsilon + 1]$. So we can find $\delta_\varepsilon \in (0,1)$ such that when $t \in [0, \delta_\varepsilon]$,

$$\|BT_B(t+s)x - BT_B(s)x\| < \varepsilon/2, \quad s \in [0, T_\varepsilon].$$

Therefore, for $t \in [0, \delta_\varepsilon]$, we have

$$\begin{aligned}\|T_B(t)x - x\|_s &= \|T_B(t)x - x\| + \sup_{s\geq 0} \|BT_B(s)(T_B(t)x - x)\| \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon,\end{aligned}$$

which proves the strong continuity of $T_B(t)$ on $(X_b, \|\cdot\|_s)$.

Next we show that $T_B(t)$ is exponentially stable on $(X_b, \|\cdot\|_s)$ and (3.3) holds. In fact, for $x \in X_b$ and $t \geq t_0 > 0$, we have

$$\begin{aligned}\|T_B(t)x\|_s &= \|T_B(t)x\| + \sup_{s\geq 0} \|BT_B(s)T_B(t)x\| \\ &= \|T_B(t)x\| + \sup_{s\geq 0} \|BT_B(s+t)x\| \\ &= \|T_B(t)x\| + \sup_{s\geq 0} \|BT_B(t_0)T_B(t+s-t_0)x\| \\ &\leq M_b e^{-\omega_b t}\|x\| + \sup_{s\geq 0} k_1(t_0) M_b e^{-\omega_b(t+s-t_0)}\|x\| \\ &\leq (1 + k_1(t_0)e^{\omega_b t_0}) M_b e^{-\omega_b t}\|x\|_s,\end{aligned}$$

and for $0 \leq t \leq t_0$,

$$\begin{aligned}\|T_B(t)x\|_s &= \|T_B(t)x\| + \sup_{s\geq 0} \|BT_B(s+t)x\| \\ &= M_b e^{-\omega_b t}\|x\| + \|x\|_s \leq (M_b + e^{\omega_b t_0})e^{-\omega_b t}\|x\|_s.\end{aligned}$$

This implies (3.3) since $M_b, e^{\omega_b t_0} \geq 1$. ■

In the following we will assume that $T_B(t)$ is exponentially stable on X, and adopt the $\|\cdot\|_s$-norm on X_b. Note that by Lemma 2.4, this norm is equivalent to the $\|\cdot\|_b$-norm.

Definition 3.2. We say that *the exponential stability* of $T_B(t)$ *with small time-varied delay* on the phase space $\mathcal{X}$ *is robust* or the solutions of (3.2) in $\mathcal{X}$ are *uniformly exponentially stable with small time-varied delay* if there are positive constants r_0, M_0, and ω_0 such that for $t \geq 0,\ 0 \leq \tau(t) \leq r \leq r_0$ continuous and $\xi \in \mathcal{X}$,

$$\|T_r(t)\xi\|_{\mathcal{X}} \leq M_0 e^{-\omega_0 t}\|\xi\|_{\mathcal{X}}.$$

Remark 3.3. The robustness defined above has some kind of uniformity since the constants M_0 and ω_0 (depend on r_0) are independent of r.

Our main result is

Theorem 3.4. *If $BT(t)$ is norm continuous for $t > 0$, then the exponential stability of $T_B(t)$ with small time-varied delay on the phase space $\mathcal{X}$ is robust.*

Proof. Suppose that $\|T_B(t)\| \le M_b e^{-\omega_b t}$ for $t \ge 0$. By Lemma 3.1, $T_B(t)$ is exponentially stable on X_b and (3.3) holds. Since $BT(t)$ is norm continuous for $t > 0$, so is $BT_B(t)$ by Lemma 2.2. For $r > 0$ and $\xi \in \mathcal{X}$, by Theorem 2.5, (3.2) has a unique solution $x_t(\cdot) = x_t(\cdot, \xi) = x(t + \cdot, \xi) \in \mathcal{X}$ and

$$\|x_t(\cdot)\|_{\mathcal{X}} \le N_0 e^{\sigma_0 t} \|\xi\|_{\mathcal{X}}, \tag{3.4}$$

where N_0 and σ_0 are independent of r. For $\omega_1 \in (0, \omega_b)$ and $t_0 > 0$, note that

$$\begin{aligned} e^{\omega_1 t} k_1(t) = e^{\omega_1 t} \|BT_B(t)\| &= e^{\omega_1 t} \|BT_B(t_0) T_B(t - t_0)\| \\ &\le k_1(t_0) M_b e^{\omega_1 t} e^{-\omega_b (t - t_0)}. \end{aligned}$$

For $t \ge t_0$ and $k_1 \in L^1_{\mathrm{loc}}(0, \infty)$, we have

$$\beta_2 := \int_0^\infty e^{\omega_1 t} k_1(t)\, dt < \infty$$

$$\eta(t) := \sup_{s \ge 0} \int_s^{s+t} e^{\omega_1 \tau} k_1(\tau)\, d\tau \to 0 \qquad \text{as } t \to 0+.$$

Choose $\tau_0 \in (0, 1]$ small enough such that

$$\begin{aligned} &(e^{\omega_b} + 1)\eta(\tau_0) < 1, \\ &e^{\omega_b} \left[\tau_0 M_b \left(\frac{1}{M_1} + \frac{1}{\omega_b - \omega_1} \right) + 2\eta(\tau_0) \right] (1 - (e^{\omega_b} + 1)\eta(\tau_0))^{-1} < 1. \end{aligned} \tag{3.5}$$

Since $BT_B(t)$ is norm continuous for $t > 0$, for $r_1 = t_0/2$ there exists $r_0 \in (0, r_1)$ such that

$$\|BT_B(r_1 - r) - BT_B(r_1)\| < r_1, \qquad 0 \le r \le r_0. \tag{3.6}$$

Now we estimate $\|Bx(t - \tau(t)) - Bx(t)\|$ for $t \ge 0$, where $0 \le \tau(t) \le r \le r_0$ and $\tau(t)$ is continuous for $t \ge 0$. For $t \in [0, \tau_0]$, since $\tau_0 \le 1$, by (3.4) we have

$$\begin{aligned} \|Bx(t - \tau(t)) - Bx(t)\| &\le \|x(t - \tau(t)) - x(t)\|_s \le 2\|x_t(\cdot)\|_{\mathcal{X}} \\ &\le 2N_0 e^{\sigma_0 t} \|\xi\|_{\mathcal{X}} \le 2N_0 e^{\sigma_0} \|\xi\|_{\mathcal{X}}. \end{aligned} \tag{3.7}$$

Let $M_1 = 2N_0 e^{2\sigma_0}$. We will prove that

$$\|Bx(t - \tau(t)) - Bx(t)\| \le M_1 e^{\omega_1 t} \|\xi\|_{\mathcal{X}}, \qquad t \ge 0,\ \xi \in \mathcal{X}. \tag{3.8}$$

For $t \in [0, \tau_0]$, we know from (3.7) that (3.8) holds. Next, suppose that (3.8) holds for $t \in [0, n\tau_0]$, where n is any positive integer, and let $t \in [n\tau_0, (n+1)\tau_0]$. If $t - \tau(t) > n\tau_0$, then by (3.6) and (3.8), we have

$$
\begin{aligned}
&\|Bx(t-\tau(t)) - Bx(t)\| \\
&\qquad = \Big\| B(T_B(t-\tau(t)) - T_B(t))\xi(0) \\
&\qquad\quad + \int_0^{t-\tau(t)} BT_B(t-\tau(t)-s)B(x(s-\tau(s)) - x(s))\,ds \\
&\qquad\quad - \int_0^{t} BT_B(t-s)B(x(s-\tau(s)) - x(s))\,ds \Big\| \\
&\qquad \le \|B(T_B(r_1-\tau(t)) - T_B(r_1))T_B(t-r_1)\xi(0)\| \\
&\qquad\quad + \int_0^{n\tau_0 - r_1} \|B(T_B(r_1-\tau(t)) - T_B(r_1))\| \\
&\qquad\quad \cdot \|T_B(t-r_1-s)\| \cdot \|B(x(s-\tau(s)) - x(s))\|\,ds \\
&\qquad\quad + \int_{n\tau_0 - r_1}^{n\tau_0} [k_1(t-\tau(t)-s) + k_1(t-s)]\|B(x(s-\tau(s)) - x(s))\|\,ds \\
&\qquad\quad + \int_{n\tau_0}^{t-\tau(t)} k_1(t-\tau(t)-s)\|B(x(s-\tau(s)) - x(s))\|\,ds \\
&\qquad\quad + \int_{n\tau_0}^{t} k_1(t-s)\|B(x(s-\tau(s)) - x(s))\|\,ds,
\end{aligned}
$$

and the first three terms on the right-hand side are bounded by

$$
\begin{aligned}
&r_1 M_b e^{-\omega_b(t-r_1)}\|\xi\|_{\mathcal{X}} + \int_0^{n\tau_0 - r_1} r_1 M_b e^{-\omega_b(t-r_1)}\|\xi\|_{\mathcal{X}}\,ds \\
&\qquad + \int_{n\tau_0 - r_1}^{n\tau_0} [k_1(t-\tau(t)-s) + k_1(t-s)] M_1 e^{-\omega_1 s}\|\xi\|_{\mathcal{X}}\,ds \\
&\quad \le r_1 M_b e^{-\omega_b(t-r_1)}\|\xi\|_{\mathcal{X}} + r_1 M_b M_1 \int_{t-n\tau_0}^{t-r_1} e^{-\omega_b \tau} e^{-\omega_1(t-r_1-\tau)}\,d\tau\,\|\xi\|_{\mathcal{X}} \\
&\qquad + \Big[\int_{t-n\tau_0-\tau(t)}^{t-n\tau_0+r_1-\tau(t)} k_1(\tau) e^{-\omega_1(t-\tau(t)-\tau)}\,d\tau \\
&\qquad + \int_{t-n\tau_0}^{t-n\tau_0+r_1} k_1(\tau) e^{-\omega_1(t-\tau)}\,d\tau \Big] M_1\|\xi\|_{\mathcal{X}} \\
&\quad \le [r_1 M_b e^{\omega_b}(1 + M_1(\omega_b - \omega_1)^{-1}) + 2M_1 e^{\omega_b}\eta(r_1)] e^{-\omega_1 t}\|\xi\|_{\mathcal{X}} \\
&\quad \le M_2 e^{-\omega_1 t}\|\xi\|_{\mathcal{X}},
\end{aligned}
$$

where $M_2 := M_1\varrho$, $\varrho := e^{\omega_b}[\tau_0 M_b(M_1^{-1} + (\omega_b - \omega_1)^{-1}) + 2\eta(\tau_0)]$. Therefore,

$$\begin{aligned}
(3.9)\qquad &\|Bx(t-\tau(t)) - Bx(t)\| \\
&\le M_2 e^{-\omega_1 t}\|\xi\|_{\mathcal{X}} + \int_{n\tau_0}^{t-\tau(t)} k_1(t-\tau(t)-s)\|B(x(s-\tau(s)) - x(s))\|\,ds \\
&\quad + \int_{n\tau_0}^{t} k_1(t-s)\|B(x(s-\tau(s)) - x(s))\|\,ds.
\end{aligned}$$

Then by the generalized Gronwall inequality or by induction, from (3.9), we have for $t - \tau(t) > n\tau_0$, $t \in [n\tau_0, (n+1)\tau_0]$,

$$\|B(x(t-\tau(t)) - x(t))\| \le \sum_{n=0}^{\infty} y^{(n)}(t), \tag{3.10}$$

where $y^{(0)}(t) := M_2 e^{-\omega_1 t}\|\xi\|_{\mathcal{X}}$ and for $n = 1, 2, \ldots,$

$$y^{(n)}(t) := \int_{n\tau_0}^{t-\tau(t)} k_1(t-\tau(t)-s) y^{(n-1)}(s)\,ds + \int_{n\tau_0}^{t} k_1(t-s) y^{(n-1)}(s)\,ds.$$

Hence,

$$\begin{aligned}
y^{(1)}(t) &= \int_{n\tau_0}^{t-\tau(t)} k_1(t-\tau(t)-s) y^{(0)}(s)\,ds + \int_{n\tau_0}^{t} k_1(t-s) y^{(0)}(s)\,ds \\
&= \Big[\int_0^{t-n\tau_0-\tau(t)} k_1(\tau) e^{-\omega_1(t-\tau(t)-\tau)}\,d\tau \\
&\quad + \int_0^{t-n\tau_0} k_1(\tau) e^{-\omega_1(t-\tau)}\,d\tau\Big] M_2\|\xi\|_{\mathcal{X}} \\
&\le M_2(e^{\omega_1 r} + 1)\eta(\tau_0) e^{-\omega_1 t}\|\xi\|_{\mathcal{X}};
\end{aligned}$$

and then by induction,

$$y^{(n)}(t) \le M_2(e^{\omega_1 r} + 1)^n \eta(\tau_0)^n e^{-\omega_1 t}\|\xi\|_{\mathcal{X}}, \quad n = 0, 1, 2, \ldots. \tag{3.11}$$

Now (3.10) and (3.11) imply

$$\begin{aligned}
(3.12)\qquad \|B(x(t-\tau(t)) - x(t))\| &\le \sum_{n=0}^{\infty} M_2(e^{\omega_1 r} + 1)^n \eta(\tau_0)^n e^{-\omega_1 t}\|\xi\|_{\mathcal{X}} \\
&= M_2[1 - (e^{\omega_1 r} + 1)\eta(\tau_0)]^{-1} e^{-\omega_1 t}\|\xi\|_{\mathcal{X}} \\
&\le M_2[1 - (e^{\omega_b} + 1)\eta(\tau_0)]^{-1} e^{-\omega_1 t}\|\xi\|_{\mathcal{X}}.
\end{aligned}$$

Note that by (3.5), $M_2[1-(e^{\omega_b}+1)\eta(\tau_0)]^{-1} < M_1$, and thus (3.8) holds for $t \in [n\tau_0, (n+1)\tau_0]$ and $t-\tau(t) \geq n\tau_0$. But from the calculations above it is easy to see that (3.12) is also valid for $t \in [n\tau_0, (n+1)\tau_0]$ with $t-\tau(t) \leq n\tau_0$. Therefore, (3.8) holds for all $t \geq 0$.

Finally, we estimate $\|x_t(\cdot)\|_{\mathcal{X}}$. For $t \geq \tau_0,\ -r \leq \theta \leq 0$ and $0 \leq \tau(t) \leq r \leq r_0$, by (3.3) and (3.8) we have

$$\begin{aligned}
\|x_t(\theta)\|_s &= \Big\|T_B(t+\theta)\xi(0) + \int_0^{t+\theta} T_B(t+\theta-s)B(x(s-\tau(s))-x(s))\,ds\Big\|_s \\
&\leq \|T_B(t+\theta)\xi(0)\|_s + \Big\|\int_0^{t+\theta} T_B(t+\theta-s)B(x(s-\tau(s))-x(s))\,ds\Big\|_s \\
&\leq (3+k_1(1))e^{\omega_b}M_b e^{-\omega_b t}\|\xi(0)\|_s \\
&\quad + \Big\|\int_0^{t+\theta} T_B(t+\theta-s)B(x(s-\tau(s))-x(s))\,ds\Big\| \\
&\quad + \sup_{\sigma\geq 0}\Big\|\int_0^{t+\theta} BT_B(t+\theta-s)T_B(\sigma)B(x(s-\tau(s))-x(s))\,ds\Big\| \\
&\leq (3+k_1(1))e^{\omega_b}M_b e^{-\omega_b t}\|\xi\|_{\mathcal{X}} \\
&\quad + \int_0^{t+\theta} M_b e^{-\omega_b(t+\theta-s)} M_1 e^{-\omega_1 s} M_1 e^{-\omega_1 s}\|\xi\|_{\mathcal{X}}\,ds \\
&\quad + M_b \int_0^{t+\theta} k_1(t+\theta-s)M_1 e^{-\omega_1 s}\|\xi\|_{\mathcal{X}}\,ds \\
&\leq M_b M_1 e^{\omega_b}[(3+k_1(1))M_1^{-1} + (\omega_b-\omega_1)^{-1} + \beta_2]e^{-\omega_1 t}\|\xi\|_{\mathcal{X}},
\end{aligned}$$

which proves

$$\|x_t(\cdot)\|_{\mathcal{X}} \leq M_3 e^{-\omega_1 t}\|\xi\|_{\mathcal{X}}, \quad t \geq \tau_0,\ r \leq r_0,$$

where $M_3 := M_b M_1 e^{\omega_b}[(3+k_1(1))M_1^{-1} + (\omega_b-\omega_1)^{-1} + \beta_2]$. Moreover, for $t \in [0,\tau_0]$, by (3.4), we have

$$\begin{aligned}
\|x_t(\cdot)\|_{\mathcal{X}} &\leq N_0 e^{\sigma_0\tau_0}\|\xi\|_{\mathcal{X}} \leq N_0 e^{\sigma_0\tau_0}e^{\omega_1\tau_0}e^{-\omega_1 t}\|\xi\|_{\mathcal{X}} \\
&\leq N_0 e^{\sigma_0+\omega_b}e^{-\omega_1 t}\|\xi\|_{\mathcal{X}}.
\end{aligned}$$

Therefore, for $t \geq 0,\ r \in [0, r_0]$, and $\xi \in \mathcal{X}$,

$$\|x_t(\cdot)\|_{\mathcal{X}} \leq (M_3 + N_0 e^{\sigma_0+\omega_b})e^{-\omega_1 t}\|\xi\|_{\mathcal{X}}. \ \blacksquare$$

EXAMPLE 3.5. Let H_1, H_2 be Hilbert spaces. Suppose that A_j generates a C_0-semigroup $T_j(t)$ on H_j for $j = 1, 2$ respectively, and $T_2(\cdot)$ is holomorphic. Moreover, suppose that $B_1 : D(B_1) \subset H_1 \to H_2$ is a closed linear operator satisfying $D(B_1) \supset D((-A_2)^r)$, where $0 < r < 1$. Since $T_2(t)$ is holomorphic, by [EN], $B_1T_2(t) \in \mathbf{B}(H_2, H_1)$ and there exist constants M and ω such that $\|B_1T_2(t)\| \le Me^{\omega t}/t^r =: k(t)$ for $t > 0$. Let $H = H_1 \times H_2$,

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & B_1 \\ 0 & 0 \end{pmatrix}.$$

Then A generates a C_0-semigroup

$$T(t) = \begin{pmatrix} T_1(t) & 0 \\ 0 & T_2(t) \end{pmatrix}, \quad t \ge 0,$$

on H and

$$BT(t) = \begin{pmatrix} 0 & B_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T_1(t) & 0 \\ 0 & T_2(t) \end{pmatrix} = \begin{pmatrix} 0 & B_1T_2(t) \\ 0 & 0 \end{pmatrix}$$

is norm continuous for $t > 0$ with $\|BT(t)\| = \|B_1T_2(t)\|_{\mathbf{B}(H_2,H_1)} \le k(t) \in L^1_{\mathrm{loc}}(0,\infty)$. So the operators A and B satisfy the assumptions of Theorems 2.5 and 3.4, but the C_0-semigroup $T(t)$ is not holomorphic.

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