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Bergelson's theorem for weakly mixing C^*-dynamical systems

by

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Abstract. We study a nonconventional ergodic average for asymptotically abelian weakly mixing C^*-dynamical systems, related to a second iteration of Khinchin's recurrence theorem obtained by Bergelson in the measure-theoretic case. A noncommutative recurrence theorem for such systems is obtained as a corollary.

1. Introduction. In 1977 Furstenberg [10] published a very influential paper where he proved a recurrence theorem for measure preserving dynamical systems (X, Σ, ν, T), which followed from

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \nu(V\cap T^{-n}V\cap T^{-2n}V\cap\cdots\cap T^{-kn}V) > 0 \tag{1.1}$$

where $V\in\Sigma$ with $\nu(V)>0$, and led to an alternative proof of Szemerédi's Theorem in combinatorial number theory. This approach to Szemerédi's Theorem leads to various generalizations of the latter and a field of research now often called Ergodic Ramsey Theory. Recently Niculescu, Ströh and Zsidó [17] initiated a programme to extend Furstenberg's result to C^*-dynamical systems, and more generally to study "noncommutative recurrence"; see also [16] and [6].

Meanwhile, much research, e.g. [11, 2, 13], has been done for measure-theoretic dynamical systems to determine when the lim inf in (1.1), and generalizations thereof, is in fact a limit (the study of "nonconventional ergodic averages"), and to find lower bounds for these limits similar to the lower bound appearing in

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \nu(V\cap T^{-n}V) \geq \nu(V)^2, \tag{1.2}$$

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which follows from the mean ergodic theorem, and from which in turn Khinchin's recurrence theorem follows. In particular, in [13] it was shown that (1.1) is indeed a limit, but certain negative results regarding lower bounds were found in [2].

However, a very interesting theorem was proven by Bergelson [1], which was later significantly generalized by Host and Kra [12, 13]. Simply put, they consider averages along cubes in $\mathbb{Z}^q$, rather than along arithmetic progressions as in (1.1). In particular, Bergelson's Theorem covers the two-dimensional case, i.e. a square in $\mathbb{Z}^2$, which can also be viewed as a second iteration of (1.2), with the average being of the form

$$\begin{aligned}(1.3)\qquad \lim_{N\to\infty}\frac{1}{N^2}\sum_{m=1}^{N}\sum_{n=1}^{N}\nu(V\cap T^{-n}V\cap T^{-m}(V\cap T^{-n}V)) \\ = \lim_{N\to\infty}\frac{1}{N^2}\sum_{m=1}^{N}\sum_{n=1}^{N}\nu(V\cap T^{-n}V\cap T^{-m}V\cap T^{-(m+n)}V) \\ \geq \nu(V)^4.\end{aligned}$$

For simplicity the averages in (1.2) and (1.3) were taken over $[1,N]$ and $[1,N]\times[1,N]$ respectively. But in fact, the average in (1.2) can be taken over $[M,N]$, and as [1] shows, the average in (1.3) can be taken over $[M,N]\times[M,N]$, with the limit $N-M\to\infty$ being taken, and the results then still hold. This provides a uniformity which leads to the relative denseness in the resulting recurrence theorems, for example in Khinchin's case for any $\varepsilon>0$ the set

$$\{n:\nu(V\cap T^{-n}V)>\nu(V)^2-\varepsilon\}$$

is relatively dense (also said to be syndetic) in $\mathbb{N}=\{1,2,3,\dots\}$, i.e. the set has bounded gaps.

In this paper we study an extension of Bergelson's Theorem to C^*-dynamical systems. The main difference of course is that the probability space (X,Σ,ν) and some abelian algebra of functions on it, like $L^\infty(\nu)$, are replaced by a unital C^*-algebra A which need not be abelian, and a state ω on A. We also work with actions of more general abelian groups than $\mathbb{Z}$ (in particular, we have in mind the groups $\mathbb{Z}^n$ and $\mathbb{R}^n$, but for clarity and generality we will formulate and prove our results in a more abstract setting). We follow the basic structure of Bergelson's proof [1, Section 5]. However, we will only prove Bergelson's Theorem for asymptotically abelian (in the sense of Definition 2.5) weakly mixing C^*-dynamical systems with ω a trace, so a degree of abelianness is still present in the dynamical system. In order to get the uniformity mentioned above, we need to restrict further to countable groups, and use a stronger form of asymptotic abelianness which we call "uniform asymptotic abelianness"; see Definition 2.5.

The main results are Theorem 5.6 and its corollary at the end of the paper. The rest of the paper systematically builds up the required tools, including an appropriate "van der Corput lemma" in Section 5, to prove this theorem.

2. Definitions and notations. In this section we collect some of the definitions and notations that we will use in the rest of the paper.

The space of all bounded linear operators $X \to X$ on a normed space X will be denoted by $B(X)$. For a C^*-algebra A we will denote the group of all $*$-isomorphisms $A \to A$ by $\mathrm{Aut}(A)$.

For a state ω on a unital C^*-algebra A, i.e. a linear functional on A such that $\omega(a^*a) \geq 0$ and $\omega(1) = 1$, we will denote the GNS representation of (A, ω) by (H, ι), which is a Hilbert space H and a linear mapping $\iota : A \to H$ such that $\langle \iota(a), \iota(b) \rangle = \omega(a^*b)$ for all $a, b \in A$ and with $\iota(A)$ dense in H. Note that we use the convention that inner products are conjugate linear in the first slot. The mapping ι can be expressed in terms of a $*$-homomorphism $\pi : A \to B(H)$ and the formula $\iota(a) = \pi(a)\Omega$ where $\Omega := \iota(1) \in H$. More generally, π is given by $\pi(a)\iota(b) = \iota(ab)$. Furthermore, we note that $\|a\|_\omega := \sqrt{\omega(a^*a)} = \|\iota(a)\|$ defines a seminorm on A, and $\|a\|_\omega \leq \|a\|$. A state ω will be called *tracial* or a *trace* when $\omega(ab) = \omega(ba)$ for all $a, b \in A$.

In a group or semigroup G we will use the notations $Vg := \{vg : v \in V\}$, $VW := \{vw : v \in V, w \in W\}$, $V^{-1} := \{v^{-1} : v \in V\}$ (in groups), etc. for any $V, W \subset G$ and $g \in G$, and we will use multiplicative notation even though we will only work with abelian groups and semigroups. If Σ is a σ-algebra in some set, then for any $V \in \Sigma$ we have a σ-algebra $\Sigma|_V := \{W \cap V : W \in \Sigma\}$ in V, and we let $\Sigma \times \Sigma$ or Σ^2 denote the product σ-algebra of Σ with itself. In integrals with respect to the given measure μ we will often write dg with g being the variable involved, instead of $d\mu$ or $d\mu(g)$.

Throughout this paper G denotes an abelian second countable locally compact group with identity e, Borel σ-algebra Σ and (regular) Haar measure μ, and containing a semigroup $K \in \Sigma$ which possesses a Følner sequence (Λ_n) defined as follows:

DEFINITION 2.1. A sequence (Λ_n) in $\Sigma|_K$ is called a *Følner sequence* in K if $0 < \mu(\Lambda_n) < \infty$ for n large enough and

$$\lim_{n\to\infty} \frac{\mu(\Lambda_n \triangle (\Lambda_n g))}{\mu(\Lambda_n)} = 0 \quad \text{for all } g \in K.$$

(In some cases a Følner sequence (Λ'_n) will be constructed from another Følner sequence, and in such cases (Λ'_n) will only be required to be in Σ for

n large enough.) A Følner sequence (Λ_n) is called *uniform* if

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \sup_{g\in\Lambda_m} \mu(\Lambda_n \triangle (\Lambda_n g)) = 0$$

for m large enough.

We do not require the Følner sequence to exhaust the semigroup (so for example the sequence $\Lambda_n = \{1, \dots, n\}$ is a Følner sequence in $K = G = \mathbb{Z}$). Eventually in our final results (Section 5) a semigroup will not be sufficient, and we will take $K = G$. We refer to [8] for more details on Følner sequences in groups. In particular, [8, Theorem 4] considers the existence of Følner sequences, while [8, Theorem 3], together with [9, Theorems 1 and 2], give conditions for the existence of a uniform Følner sequence.

Note that since G is both locally compact and second countable, μ is σ-finite. Second countability and σ-finiteness are of importance, since we will be working with products of the measure space (G, Σ, μ). For example, second countability ensures that the product σ-algebra is equal to the Borel σ-algebra of the product topology. In Section 5 we will also require the following additional property:

Definition 2.2. A Følner sequence (Λ_n) in K is said to satisfy the *Tempel'man condition* if there is a real number $c > 0$ such that

$$\mu(\Lambda_n^{-1}\Lambda_n) \leq c\mu(\Lambda_n) \quad \text{for } n \text{ large enough}$$

(in particular $\Lambda_n^{-1}\Lambda_n \in \Sigma$ is required for n large enough).

Now we define the type of dynamical system with which we will be working:

Definition 2.3. Let ω be a state on a unital C^*-algebra A. Consider a function $\tau : G \to \mathrm{Aut}(A) : g \mapsto \tau_g$ such that τ_e is the identity on A, $\tau_g \circ \tau_h = \tau_{gh}$ and $\omega \circ \tau_g = \omega$ for all $g, h \in G$, and such that $K \to \mathbb{C} : g \mapsto \omega(a\tau_g(b))$ is $(\Sigma|_K)$-measurable for all $a, b \in A$. Then we call (A, ω, τ, K) a *C^*-dynamical system*. We will consider the following special cases:

(1) Ergodicity and weak mixing, with the latter implying the former:

(1a) If

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \omega(a\tau_g(b))\, dg = \omega(a)\omega(b) \quad \text{for all } a, b \in A,$$

then we will call (A, ω, τ, K) *ergodic with respect to* (Λ_n).

(1b) If

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |\omega(a\tau_g(b)) - \omega(a)\omega(b)|\, dg = 0 \quad \text{for all } a, b \in A,$$

then we will call (A, ω, τ, K) *weakly mixing with respect to* (Λ_n).

(2) If $K \to \mathbb{R} : g \mapsto \|[a, \tau_g(b)]\|$ is $(\Sigma|_K)$-measurable for all $a, b \in A$, where $[a, b] := ab - ba$, then:

(2a) We call (A, ω, τ, K) *asymptotically abelian with respect to* (Λ_n) if

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \|[a, \tau_g(b)]\|\, dg = 0 \quad \text{for all } a, b \in A.$$

(2b) We call (A, ω, τ, K) *uniformly asymptotically abelian with respect to* (Λ_n) if

$$\lim_{n\to\infty} \sup_{h\in K} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n h} \|[a, \tau_g(b)]\|\, dg = 0 \quad \text{for all } a, b \in A.$$

Whenever we write (A, ω, τ, K), we mean a C^*-dynamical system. In our final results in Section 5 we will make the further assumption that $K = G$, and this will be indicated by simply writing (A, ω, τ, G).

When working with such systems, we can use the GNS representation to represent τ on H by the formula

$$U_g \iota(a) := \iota(\tau_g(a))$$

and then uniquely extending $U_g : \iota(A) \to \iota(A)$ to H for every $g \in G$. This gives us a *representation* $U : K \to B(H) : g \mapsto U_g$ of the semigroup K as contractions, i.e. $U_g U_h = U_{gh}$ and $\|U_g\| \le 1$ for all $g, h \in K$ (in fact, U_g is unitary, but this is not essential), and similarly for the whole G instead of just K. We will consider this U to be the *GNS representation* of τ, and in the presence of a GNS representation we will use this notation in the rest of the paper. Note that $U_g \Omega = \iota(\tau_g(1)) = \iota(1) = \Omega$.

Remarks. The terminology in Definition 2.3(2) is not quite standard, but we will use it consistently in this paper. For simplicity, consider the case $K = G = \mathbb{Z}$ and the Følner sequence in $\mathbb{Z}$ given by $\Lambda_N = \{1, \dots, N\}$. The term "asymptotic abelian" (see for example [7]) is often used to describe the condition

$$\lim_{|n|\to\infty} \|[a, \tau_n(b)]\| = 0$$

which for the purposes of these remarks we will refer to as "strong asymptotic abelianness". Note that this condition implies asymptotic abelianness and uniform asymptotic abelianness in our sense above with respect to (Λ_N). In fact, if ω is a so-called factor state, then strong asymptotic abelianness also implies what is known as "strong mixing", namely

$$\lim_{|n|\to\infty} |\omega(a\tau_n(b)) - \omega(a)\omega(b)| = 0$$

(see [4, Example 4.3.24] for details), and hence weak mixing with respect to (Λ_N). We can also mention that from the results and discussions in [5, Section 5.3.2] regarding infinite temperature KMS states (i.e. at inverse temperature $\beta = 0$), it follows that if a unital C^*-algebra A has at least one trace, then it also has a "factor trace" (i.e. a factor state which is tracial). More concretely, the shift automorphism τ on the C^*-algebra $M = \bigotimes_{j\in\mathbb{Z}} B(\mathbb{C}^2)$ is strongly asymptotically abelian and leaves the trace ω of M (which happens to be a factor trace [14, Section 11.4]) invariant, hence $(M, \omega, \tau, \mathbb{Z})$ is a strongly asymptotic abelian and strongly mixing C^*-dynamical system.

3. Background. We now discuss a number of results which for the most part are known, but we formulate and adapt them in a way that will suit our needs in Section 5. In particular, these results are not stated in their most general forms, for instance the algebra A need only be a unital $*$-algebra rather than a C^*-algebra. Some more notation is also introduced.

Definition 3.1. Let S be an abelian semigroup of linear contractions on a Hilbert space H. A vector $x \in H\backslash\{0\}$ is called an *eigenvector with unimodular eigenvalues* (or "unimodular eigenvector" for short) of S if there exists a function $\lambda : S \to \mathbb{C}$ such that $|\lambda(U)| = 1$ and $Ux = \lambda(U)x$ for every $U \in S$. The closure of the span (finite linear combinations) of the unimodular eigenvectors of S will be denoted by H_0, and its orthogonal complement in H by H_v, hence $H = H_0 \oplus H_v$. The elements of H_0 are called *reversible*, while the elements of H_v are called *flight vectors*.

An important characterization of H_v in our setting (presented in Section 2) is the following:

Proposition 3.2. *Consider a representation $U : K \to B(H)$ of the abelian semigroup K as contractions on any Hilbert space H, giving the semigroup $S = \{U_g : g \in K\}$ in Definition* 3.1, *such that $K \to \mathbb{C} : g \mapsto \langle x, U_g y\rangle$ is $(\Sigma|_K)$-measurable for all $x, y \in H$. Then for $y \in H$ we have the following: $y \in H_v$ if and only if*

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |\langle x, U_g y\rangle|\, dg = 0 \quad \text{for all } x \in H.$$

The regularity of μ is not needed for this result.

This result follows as a special case of the results in [15, Section 2.4], but with some modifications to the proof of [15, Theorem 2.4.7] to compensate for the fact that we are using the form $\lim_{n\to\infty} \mu(\Lambda_n)^{-1} \int_{\Lambda_n} (\cdot)\, d\mu$ rather than an abstract invariant mean as [15] does (the fact that (Λ_n) is Følner and μ is invariant, plays an important role here). An important example is where U is a GNS representation of a $*$-dynamical system, and in such cases we will use the notation H_0 and H_v without further explanation. As a corollary we have the following related characterization of H_v:

COROLLARY 3.3. *Assume that $KK = K$. Consider any Følner sequences $(\Lambda_{1,n})$ and $(\Lambda_{2,n})$ in K, and a representation $U : K \to B(H)$ of K as contractions on any Hilbert space H such that $K \times K \to \mathbb{C} : (g,h) \mapsto \langle x, U_{gh} y\rangle$ is $(\Sigma \times \Sigma)|_{K\times K}$-measurable for all $x, y \in H$. Then for $y \in H$ we have the following*: $y \in H_v$ *if and only if*

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})} \int\limits_{\Lambda_{1,n}} \int\limits_{\Lambda_{2,n}} |\langle x, U_{gh} y\rangle| \, dg \, dh = 0 \tag{3.1}$$

for all $x \in H$.

Proof. We will apply Proposition 3.2 to $K' := K \times K \in \Sigma \times \Sigma$ as a semigroup in the "ambient" group $(G', \Sigma', \mu') := (G \times G, \Sigma \times \Sigma, \mu \times \mu)$. It is easily established that $\Lambda'_n := \Lambda_{2,n} \times \Lambda_{1,n}$ gives a Følner sequence in K'. Because of second countability, Σ' is the Borel σ-algebra of G'. As mentioned in Proposition 3.2, we need not worry about regularity of μ', but that μ' is invariant can be shown as follows: First prove that $\{V \in \Sigma' : V(g,h), V^c(g,h) \in \Sigma'\}$ is a σ-algebra and hence equal to Σ' for all $g, h \in G$, where $V^c := G' \setminus V$. From this we obtain $\Sigma'(g,h) \subset \Sigma'$, and then the right invariance of μ' follows by using the definition of a product measure.

Note that since K is abelian, $U' : K' \to B(H)$ defined by $U'_{(g,h)} := U_{gh}$ is a representation of K' as contractions. Since $KK = K$, the semigroups $S := \{U_g : g \in K\}$ and $S' := \{U'_{(g,h)} : (g,h) \in K'\}$ have the same reversible vectors H_0, and hence the same flight vectors H_v. But then by Proposition 3.2 a $y \in H$ is in H_v if and only

$$\lim_{n\to\infty} \frac{1}{\mu'(\Lambda'_n)} \int\limits_{\Lambda'_n} |\langle x, U'_{(g,h)} y\rangle| \, d(g,h) = 0 \quad \text{for all } x \in H,$$

which is exactly (3.1) by Fubini's Theorem. ∎

We have the following mean ergodic theorem: Let H be a Hilbert space and $U : K \to B(H) : g \mapsto U_g$ a representation of K as contractions such that $K \ni g \mapsto \langle x, U_g y\rangle$ is $(\Sigma|_K)$-measurable for all $x, y \in H$. Take P to be the projection of H onto $V := \{x \in H : U_g x = x \text{ for all } g \in K\}$. Then

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int\limits_{\Lambda_n} U_g x \, dg = Px \quad \text{for all } x \in H,$$

where integrals over sets $\Lambda \in \Sigma|_K$ with $\mu(\Lambda) < \infty$, of bounded Hilbert space valued functions f, with $\langle f, x\rangle$ measurable for every x in a dense linear subspace of H, are defined via the Riesz representation theorem, i.e. $\langle \int_\Lambda f \, d\mu, x\rangle := \int_\Lambda \langle f(g), x\rangle \, dg$. This integral has several simple properties, for example if $F : K \to \mathbb{R}$ is measurable and $\|f\| \le F$, then $|\langle \int_\Lambda f \, d\mu, x\rangle| \le \int_\Lambda |\langle f, x\rangle| \, d\mu \le (\int_\Lambda F \, d\mu)\|x\|$, hence $\|\int_\Lambda f \, d\mu\| \le \int_\Lambda F \, d\mu$.

Note that if (A, ω, τ, K) is ergodic with respect to some Følner sequence, then from the mean ergodic theorem it follows that it is ergodic with respect to every Følner sequence in K, since ergodicity is equivalent to the projection P in any GNS representation being $\Omega \otimes \Omega = \Omega\langle\Omega, \cdot\rangle$, i.e. ergodicity is independent of the Følner net being used, and then we can simply say that the system is *ergodic*. By the following result we have a similar situation for weak mixing:

Proposition 3.4. *In the GNS representation of* (A, ω, τ, K) *we have* $\mathbb{C}\Omega \subset H_0$. *Furthermore*, (A, ω, τ, K) *is weakly mixing with respect to* (Λ_n) *if and only if*

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |\langle x, U_g y\rangle - \langle x, \Omega\rangle\langle\Omega, y\rangle| \, dg = 0 \quad \textit{for all } x, y \in H, \tag{3.2}$$

which in turn holds if and only if $\dim H_0 = 1$. *In particular*, *if* (A, ω, τ, K) *is weakly mixing with respect to some Følner sequence in* K, *then it is weakly mixing with respect to every Følner sequence in* K.

Proof. The first equivalence: By setting $x = \iota(a^*)$ and $y = \iota(b)$, weak mixing with respect to (Λ_n) follows immediately from (3.2).

Conversely, consider any $x, y \in H$; then there are sequences (a_m) and (b_m) in A such that $\iota(a_m) \to x$ and $\iota(b_m) \to y$. Hence $K \to \mathbb{C} : g \mapsto \langle x, U_g y\rangle = \lim_{m\to\infty} \omega(a_m^* \tau_g(b_m))$ is Σ-measurable. Now we follow a standard argument from measure-theoretic ergodic theory (see for example [18, Theorem 1.23]): Consider any $\varepsilon > 0$, and an m for which $\|\iota(a_m) - x\| < \varepsilon$ and $\|\iota(b_m) - y\| < \varepsilon$. From Definition 2.4(2) it follows that there is an N such that for every $n > N$,

$$\frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |\langle x, U_g y\rangle - \langle x, \Omega\rangle\langle\Omega, y\rangle| \, dg \le (\|x\| + \|\iota(b_m)\| + \|\iota(a_m)\| + \|y\| + 1)\varepsilon$$

while $\|\iota(a_m)\| \le \|x\| + \varepsilon$ and $\|\iota(b_m)\| \le \|y\| + \varepsilon$. (Note that the properties of a Følner sequence have not been used yet.)

The second equivalence: Note that in general $\mathbb{C}\Omega \subset H_0$, since $U_g\Omega = \Omega$. As mentioned, $K \to \mathbb{C} : g \mapsto \langle x, U_g y\rangle$ is $(\Sigma|_K)$-measurable, so in particular for any $y \in H$ orthogonal to Ω, (3.2) tells us that

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} |\langle x, U_g y\rangle| \, dg = 0 \quad \text{for all } x \in H.$$

Hence $y \in H_v$ by Proposition 3.2, so $H_0^\perp \subset (\mathbb{C}\Omega)^\perp \subset H_v$; but $H_0^\perp = H_v$ by Definition 3.1, hence $H_0 = \mathbb{C}\Omega$. Conversely, if $H_0 = \mathbb{C}\Omega$, then for any $x, y \in H$ write $x = x_0 + x_v$ and $y = y_0 + y_v$ with $x_0, y_0 \in H_0$ and $x_v, y_v \in H_v$.

Then it follows from Proposition 3.2 that

$$\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int\limits_{\Lambda_n}|\langle x,U_g y\rangle-\langle x,\Omega\rangle\langle\Omega,y\rangle|\,dg$$
$$=\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int\limits_{\Lambda_n}|\langle x,U_g y_v\rangle+\langle x_0,y_0\rangle-\langle x_0,(\Omega\otimes\Omega)y_0\rangle|\,dg=0$$

as required. ∎

Hence when (A,ω,τ,K) is a C^*-dynamical system which is weakly mixing with respect to some Følner sequence in K, we will simply call (A,ω,τ,K) *weakly mixing.*

4. Preliminary limits. The goal of this section is to prove Proposition 4.3, which can be viewed as a collection of very simple nonconventional ergodic averages and is one of the tools used in Section 5. In Lemma 4.1 we still do not need A to be C^*-algebra, but from Lemma 4.2 onward we start using the properties of C^*-algebras. Keep in mind that we are still using the definitions and notations of Section 2.

LEMMA 4.1. *Let (A,ω,τ,K) be ergodic. Then*

$$\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n^{-1})}\int\limits_{\Lambda_n^{-1}}\omega(a\tau_g(b))\,dg=\omega(a)\omega(b)\qquad\text{for all } a,b\in A,$$

and furthermore (Λ_n^{-1}) is a Følner sequence in K^{-1}. Hence (A,ω,τ,K^{-1}) is an ergodic C^*-dynamical system.

Proof. Let $\mathcal{I}:G\to G:g\mapsto g^{-1}$. Since μ is a regular Haar measure, we have $\mu\circ\mathcal{I}=\mu$. Note that

$$\omega(a\tau_{(\cdot)}(b))|_{K^{-1}}=\omega(\tau_{\mathcal{I}(\cdot)}(a)b)|_{K^{-1}}=\overline{\omega(b^*\tau_{(\cdot)}(a^*))|_K}\circ\mathcal{I}|_{K^{-1}}$$

is $(\Sigma|_{K^{-1}})$-measurable by definition of a C^*-dynamical system and the fact that $\mathcal{I}$ is continuous and therefore measurable, hence (A,ω,τ,K^{-1}) is a C^*-dynamical system. Now we perform a simple calculation:

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n^{-1})}\int\limits_{\Lambda_n^{-1}}\omega(a\tau_g(b))\,dg&=\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int\limits_{\mathcal{I}(\Lambda_n)}\omega(\tau_{\mathcal{I}(g)}(a)b)\,d\mu(g)\\&=\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int\limits_{\Lambda_n}\omega(\tau_g(a)b)\,d(\mu\circ\mathcal{I})(g)\\&=\overline{\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int\limits_{\Lambda_n}\omega(b^*\tau_g(a^*))\,dg}=\overline{\omega(b^*)\omega(a^*)}.\end{aligned}$$

Lastly, it is straightforward to establish directly from the definition that (Λ_n^{-1}) is a Følner sequence in K^{-1}. ∎

LEMMA 4.2. *Let (A,ω,τ,K) be ergodic, with*

$$K \times K \to \mathbb{C} : (g,h) \mapsto \omega(a\tau_g(b)\tau_{h^{-1}g}(c))$$

$(\Sigma|_K \times \Sigma|_K)$-measurable for all $a,b,c \in A$. Then for any Følner sequences $(\Lambda_{1,n})$ and $(\Lambda_{2,n})$ in K, and any GNS representation of (A,ω), we have

$$\lim_{n\to\infty} \sup_{g_1,g_2\in K} \|M_{n,g_1,g_2}(\iota(\tau_g(b)\tau_{h^{-1}g}(c))) - \omega(b)\omega(c)\Omega\| = 0 \tag{4.1}$$

for all $b,c \in A$, where we write

$$M_{n,g_1,g_2}(f(g,h)) \equiv \frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})} \int\limits_{\Lambda_{1,n}g_1} \int\limits_{\Lambda_{2,n}g_2} f(g,h)\,dg\,dh$$

(*in particular, the symbol g on the left hand side indicates the integration variable over $\Lambda_{2,n}g_2$, and h the integration variable over $\Lambda_{1,n}g_1$*).

Proof. $M_{n,g_1,g_2}(\iota(\tau_g(b)\tau_{h^{-1}g}(c)))$ exists for n large enough by Fubini's Theorem, since $(g,h) \mapsto \omega(a\tau_g(b)\tau_{h^{-1}g}(c)) = \langle\iota(a^*), \iota(\tau_g(b)\tau_{h^{-1}g}(c))\rangle$ provides the required measurability.

Setting $c' := c - \omega(c)$ we have $\omega(c') = 0$, and assuming (4.1) holds for c' instead of c, and keeping in mind the invariance of μ, we have

$$\begin{aligned}
&\sup_{g_1,g_2\in K} \|M_{n,g_1,g_2}(\iota(\tau_g(b)\tau_{h^{-1}g}(c))) - \omega(b)\omega(c)\Omega\| \\
&\qquad\le \sup_{g_1,g_2\in K} \|M_{n,g_1,g_2}(\iota(\tau_g(b)\tau_{h^{-1}g}(c')))\| \\
&\qquad\quad + |\omega(c)| \sup_{g_2\in K} \bigg\| \frac{1}{\mu(\Lambda_{2,n})} \int\limits_{\Lambda_{2,n}g_2} \iota(\tau_g(b))\,dg - \omega(b)\Omega \bigg\| \to 0
\end{aligned}$$

in the n limit by the mean ergodic theorem, since (A,ω,τ,K) is ergodic and K is abelian, so that

$$\begin{aligned}
&\sup_{g_2\in K} \bigg\| \frac{1}{\mu(\Lambda_{2,n})} \int\limits_{\Lambda_{2,n}g_2} \iota(\tau_g(b))\,dg - \omega(b)\Omega \bigg\| \\
&\qquad = \sup_{g_2\in K} \bigg\| \frac{1}{\mu(\Lambda_{2,n})} \int\limits_{\Lambda_{2,n}} \iota(\tau_{gg_2}(b))dg - \omega(b)\Omega \bigg\| \\
&\qquad \le \sup_{g_2\in K} \|U_{g_2}\| \bigg\| \frac{1}{\mu(\Lambda_{2,n})} \int\limits_{\Lambda_{2,n}} U_g\iota(b)\,dg - \omega(b)\Omega \bigg\| \\
&\qquad \le \bigg\| \frac{1}{\mu(\Lambda_{2,n})} \int\limits_{\Lambda_{2,n}} U_g\iota(b)\,dg - \omega(b)\Omega \bigg\| \\
&\qquad \to \|(\Omega \otimes \Omega)\iota(b) - \omega(b)\Omega\| = 0.
\end{aligned}$$

Hence we can assume $\omega(c)=0$. From our hypothesis, $h\mapsto\omega(a\tau_{h^{-1}}(c))=\omega(a\tau_e(1)\tau_{h^{-1}e}(c))$ is $(\Sigma|_K)$-measurable. Also, since A is a C^*-algebra and π a $*$-homomorphism, we have $\pi(a)\in B(H)$ and $\|\pi(a)\|\le\|a\|$ for all $a\in A$. Therefore, since G is abelian,

$$\begin{aligned}
&\sup_{g_1,g_2\in K}\|M_{n,g_1,g_2}(\iota(\tau_g(b)\tau_{h^{-1}g}(c)))\| \\
&\quad=\sup_{g_1,g_2\in K}\bigg\|\frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})}\int\limits_{\Lambda_{2,n}g_2}\Big[\pi(\tau_g(b))U_g\int\limits_{\Lambda_{1,n}g_1}\iota(\tau_{h^{-1}}(c))\,dh\Big]\,dg\bigg\| \\
&\quad\le\sup_{g_1,g_2\in K}\frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})}\,\mu(\Lambda_{2,n}g_2)\|b\|\bigg\|\int\limits_{\Lambda_{1,n}g_1}\iota(\tau_{h^{-1}}(c))\,dh\bigg\| \\
&\quad=\|b\|\sup_{g_1\in K}\frac{1}{\mu(\Lambda_{1,n})}\bigg\|\int\limits_{\Lambda_{1,n}}\iota(\tau_{(hg_1)^{-1}}(c))\,dh\bigg\| \\
&\quad=\|b\|\sup_{g_1\in K}\frac{1}{\mu(\Lambda_{1,n})}\bigg\|U_{g_1^{-1}}\int\limits_{\Lambda_{1,n}}\iota(\tau_{h^{-1}}(c))\,dh\bigg\| \\
&\quad\le\|b\|\frac{1}{\mu(\Lambda_{1,n}^{-1})}\bigg\|\int\limits_{\Lambda_{1,n}^{-1}}U_h\iota(c)\,dh\bigg\|\to 0
\end{aligned}$$

by Lemma 4.1 and the mean ergodic theorem. ■

PROPOSITION 4.3. *Let (A,ω,τ,K) be ergodic. Consider any Følner sequences $(\Lambda_{1,n})$ and $(\Lambda_{2,n})$ in K, and use the notation M_{n,g_1,g_2} as in Lemma 4.2.*

(1) *If*

$$K\times K\to\mathbb{C}:(g,h)\mapsto\omega(\tau_h(a)\tau_{gh}(b)\tau_g(c))$$

is $(\Sigma|_K\times\Sigma|_K)$-measurable for all $a,b,c\in A$, then

$$\lim_{n\to\infty}\sup_{g_1,g_2\in K}|M_{n,g_1,g_2}(\omega(\tau_g(a)\tau_{gh}(b)\tau_h(c)))-\omega(a)\omega(b)\omega(c)|=0$$

for all $a,b,c\in A$.

(2) *If (A,ω,τ,K) is asymptotically abelian with respect to $(\Lambda_{1\alpha})$, ω is tracial, and*

$$K\times K\to\mathbb{C}:(g,h)\mapsto\omega(\tau_h(a_1)\tau_g(a_2)\tau_{gh}(a_3)\tau_g(a_4)\tau_h(a_5))$$

is $(\Sigma|_K\times\Sigma|_K)$-measurable for all $a_1,\dots,a_5\in A$, then

$$\lim_{n\to\infty}M_{n,e,e}(\omega(\tau_h(a_1)\tau_g(a_2)\tau_{gh}(a_3)\tau_g(a_4)\tau_h(a_5)))=\omega(a_5a_1)\omega(a_3)\omega(a_2a_4)$$

for all $a_1,\dots,a_5\in A$.

(3) *If (A,ω,τ,K) is uniformly asymptotically abelian with respect to $(\Lambda_{1,n})$, ω is tracial, and*

$$K\times K\to\mathbb{C}:(g,h)\mapsto\omega(\tau_h(a_1)\tau_g(a_2)\tau_{gh}(a_3)\tau_g(a_4)\tau_h(a_5))$$

is $(\Sigma|_K \times \Sigma|_K)$*-measurable for all* $a_1, \ldots, a_5 \in A$*, then*

$$\lim_{n\to\infty} \sup_{g_1,g_2\in K} |M_{n,g_1,g_2}(\omega(\tau_h(a_1)\tau_g(a_2)\tau_{gh}(a_3)\tau_g(a_4)\tau_h(a_5))) - \omega(a_5a_1)\omega(a_3)\omega(a_2a_4)| = 0$$

for all $a_1, \ldots, a_5 \in A$*.*

Proof. (1) Note that $\omega(\tau_h(a)\tau_{gh}(b)\tau_g(c)) = \omega(a\tau_g(b)\tau_{h^{-1}g}(c))$ and therefore the latter has the measurability required in Lemma 4.2, hence

$$\begin{aligned}\sup_{g_1,g_2\in K} &|M_{n,g_1,g_2}(\omega(\tau_h(a)\tau_{gh}(b)\tau_g(c))) - \omega(a)\omega(b)\omega(c)| \\ &= \sup_{g_1,g_2\in K} |\langle \iota(a^*), M_{n,g_1,g_2}(\iota(\tau_g(b)\tau_{h^{-1}g}(c))) - \omega(b)\omega(c)\Omega\rangle| \\ &\le \|\iota(a^*)\| \sup_{g_1,g_2\in K} \|M_{n,g_1,g_2}(\iota(\tau_g(b)\tau_{h^{-1}g}(c))) - \omega(b)\omega(c)\Omega\| \to 0.\end{aligned}$$

(3) Note that $(g,h) \mapsto \omega(\tau_h(a)\tau_g(1)\tau_{gh}(b)\tau_g(c)\tau_h(1)) = \omega(a\tau_g(b)\tau_{h^{-1}g}(c))$ is $(\Sigma|_K \times \Sigma|_K)$-measurable by hypothesis, hence we can apply Lemma 4.2 to obtain

$$\begin{aligned}&\sup_{g_1,g_2\in K} |M_{n,g_1,g_2}(\omega(\tau_h(a_1)\tau_g(a_2)\tau_{gh}(a_3)\tau_g(a_4)\tau_h(a_5))) - \omega(a_5a_1)\omega(a_3)\omega(a_2a_4)| \\ &\quad = \sup_{g_1,g_2\in K} |\langle \iota((a_5a_1)^*), \\ &\qquad\qquad M_{n,g_1,g_2}(\iota(\tau_{h^{-1}g}(a_2)\tau_g(a_3)\tau_{h^{-1}g}(a_4))) - \omega(a_3)\omega(a_2a_4)\Omega\rangle| \\ &\quad \le \|\iota((a_5a_1)^*)\| \\ &\qquad \times \sup_{g_1,g_2\in K} \|M_{n,g_1,g_2}(\iota(\tau_{h^{-1}g}(a_2)\tau_g(a_3)\tau_{h^{-1}g}(a_4) - \tau_g(a_3)\tau_{h^{-1}g}(a_2a_4)))\| \\ &\qquad + \|\iota((a_5a_1)^*)\| \\ &\qquad \times \sup_{g_1,g_2\in K} \|M_{n,g_1,g_2}(\iota(\tau_g(a_3)\tau_{h^{-1}g}(a_2a_4))) - \omega(a_3)\omega(a_2a_4)\Omega\| \\ &\quad \to 0,\end{aligned}$$

since

$$\begin{aligned}&\|\iota(\tau_{h^{-1}g}(a_2)\tau_g(a_3)\tau_{h^{-1}g}(a_4) - \tau_g(a_3)\tau_{h^{-1}g}(a_2a_4))\| \\ &\qquad = \|\iota(\tau_{h^{-1}g}([a_2,\tau_h(a_3)])\tau_{h^{-1}g}(a_4))\| \le \|[a_2,\tau_h(a_3)]\|\,\|a_4\|\end{aligned}$$

and then by uniform asymptotic abelianness with respect to $(\Lambda_{1,n})$,

$$\begin{aligned}&\sup_{g_1,g_2\in K} \|M_{n,g_1,g_2}(\iota(\tau_{h^{-1}g}(a_2)\tau_g(a_3)\tau_{h^{-1}g}(a_4) - \tau_g(a_3)\tau_{h^{-1}g}(a_2a_4)))\| \\ &\qquad \le \|a_4\| \sup_{g_1\in K} \frac{1}{\mu(\Lambda_{1,n})} \int_{\Lambda_{1,n}g_1} \|[a_2,\tau_h(a_3)]\|\,dh \to 0.\end{aligned}$$

(2) As for (3), but without the sup's. ∎

5. Main results. Now we will need a bit more structure in our Følner sequences, namely we will consider Følner sequences (Λ_n) such that $(\Lambda_n^{-1}\Lambda_n)$ is also Følner (in such cases $\Lambda_n^{-1}\Lambda_n$ need only be measurable for n large enough). A simple example of this is $\Lambda_n := \{g \in \mathbb{R}^q : \|g\| < n\}$, in the group $\mathbb{R}^q$, since $\Lambda_n^{-1}\Lambda_n = \Lambda_{2n}$, which also implies that (Λ_n) satisfies the Tempel'man condition. Furthermore, in this example (Λ_n) is uniformly Følner in $\mathbb{R}^q$. The same is true if we replace Λ_n by its closure. Another simple example with these properties (except for $\Lambda_n^{-1}\Lambda_n = \Lambda_{2n}$) is $\Lambda_n := \{0, 1, \ldots, n\}$ in $\mathbb{Z}$, as well as $\Lambda_n \times \cdots \times \Lambda_n$ in $\mathbb{Z}^q$. In each of the results in this section we will explicitly state which of these properties we are using. Again we remind the reader that throughout this section we are using the definitions and notations set up in Section 2, in particular the group G is abelian, and hence so is the semigroup K.

LEMMA 5.1. *Assume the existence of a uniform Følner sequence (Λ_n) in K, let H be a Hilbert space, and let $f : G \to H$ be a bounded function with $\langle f(\cdot), x\rangle$ and $\langle f(\cdot), f(\cdot)\rangle : G \times G \to \mathbb{C}$ Borel measurable for all $x \in H$. Assume that*

$$\gamma_h := \lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int\limits_{\Lambda_n} \langle f(g), f(gh)\rangle \, dg$$

exists for all $h \in G$. Also assume that we have the following limit (*the given iterated integral automatically exists for m large enough*, *by the other assumptions*):

$$\lim_{m\to\infty} \frac{1}{\mu(\Lambda_m)^2} \int\limits_{\Lambda_m} \int\limits_{\Lambda_m} \gamma_{h_1^{-1}h_2} \, dh_1 \, dh_2 = 0. \tag{5.1}$$

(1) *Then it follows that*

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_n)} \int\limits_{\Lambda_n} f \, d\mu = 0.$$

(*Note that the regularity of μ is not required here.*)

(2) *Assume furthermore that μ is the counting measure* (*i.e. $\mu(\Lambda)$ is the number of elements in the set $\Lambda \in \Sigma$, which means that Λ_n is a finite nonempty set for n large enough*) *and that*

$$\lim_{n\to\infty} \sup_{g_1\in K} \left| \frac{1}{\mu(\Lambda_n)} \int\limits_{\Lambda_n g_1} \langle f(g), f(gh)\rangle \, dg - \gamma_h \right| = 0 \quad \textit{for all } h \in G.$$

Then

$$\lim_{n\to\infty} \sup_{g_1\in K} \left\| \frac{1}{\mu(\Lambda_n)} \int\limits_{\Lambda_n g_1} f \, d\mu \right\| = 0.$$

(*The integrals in* (2) *are in fact finite sums*, *but for consistent notation we use integral signs.*)

Proof. Part (1) was proved in [3] following the basic structure of a proof of a special case given in [11], without assuming K is abelian. We now give an outline of this proof, but at appropriate points we also show the minor modifications needed to prove (2). We present it in several steps, with more details of each step to be found in [3].

(a) For m large enough, one obtains

$$\lim_{n\to\infty}\left\|\frac{1}{\mu(\Lambda_n)}\int\limits_{\Lambda_n} f\,d\mu-\frac{1}{\mu(\Lambda_n)}\frac{1}{\mu(\Lambda_m)}\int\limits_{\Lambda_n}\int\limits_{\Lambda_m} f(gh)\,dh\,dg\right\|=0,$$

and because K is abelian we find that

$$\lim_{n\to\infty}\sup_{g_1\in K}\left\|\frac{1}{\mu(\Lambda_n)}\int\limits_{\Lambda_n g_1} f\,d\mu-\frac{1}{\mu(\Lambda_n)}\frac{1}{\mu(\Lambda_m)}\int\limits_{\Lambda_n g_1}\int\limits_{\Lambda_m} f(gh)\,dh\,dg\right\|=0,$$

since

$$\begin{aligned}&\left\|\frac{1}{\mu(\Lambda_n)}\int\limits_{\Lambda_n g_1} f\,d\mu-\frac{1}{\mu(\Lambda_n)}\frac{1}{\mu(\Lambda_m)}\int\limits_{\Lambda_n g_1}\int\limits_{\Lambda_m} f(gh)\,dh\,dg\right\|\\ &\quad\le\frac{b_1}{\mu(\Lambda_n)}\sup_{h\in\Lambda_m}\mu((\Lambda_n g_1)\triangle(\Lambda_n g_1 h))=\frac{b_1}{\mu(\Lambda_n)}\sup_{h\in\Lambda_m}\mu(\Lambda_n\triangle(\Lambda_n h))\end{aligned}$$

where b_1 is an upper bound for $\|f(K)\|$.

(b) For any $\Lambda_1,\Lambda_2\in\Sigma|_K$ with $\mu(\Lambda_1),\mu(\Lambda_2)<\infty$ we have

$$\left\|\int\limits_{\Lambda_2}\int\limits_{\Lambda_1} f(gh)\,dh\,dg\right\|^2\le\mu(\Lambda_2)\int\limits_{\Lambda_1}\int\limits_{\Lambda_1}\int\limits_{\Lambda_2}\langle f(gh_1),f(gh_2)\rangle\,dg\,dh_1\,dh_2,$$

and in particular these iterated integrals exist.

(c) We also have

$$\lim_{n\to\infty}\frac{1}{\mu(\Lambda_n)}\int\limits_{\Lambda_n}\langle f(gh_1),f(gh_2)\rangle\,dg=\gamma_{h_1^{-1}h_2}\tag{5.2}$$

for all $h_1\in K$ and $h_2\in G$. However, if

$$\lim_{n\to\infty}\sup_{g_1\in K}\left|\frac{1}{\mu(\Lambda_n)}\int\limits_{\Lambda_n g_1}\langle f(g),f(gh)\rangle\,dg-\gamma_h\right|=0$$

for all $h\in G$, then since K is abelian, we in fact obtain

$$\lim_{n\to\infty}\sup_{g_1\in K}\left|\frac{1}{\mu(\Lambda_n)}\int\limits_{\Lambda_n g_1}\langle f(gh_1),f(gh_2)\rangle\,dg-\gamma_{h_1^{-1}h_2}\right|=0\tag{5.3}$$

for all $h_1 \in K$ and $h_2 \in G$, as follows:

$$\left|\frac{1}{\mu(\varLambda_n)} \int\limits_{\varLambda_n g_1} \langle f(gh_1), f(gh_2)\rangle \, dg - \gamma_{h_1^{-1}h_2}\right|$$
$$\le \frac{\mu((\varLambda_n g_1) \bigtriangleup (\varLambda_n g_1 h_1))}{\mu(\varLambda_n)} b_2 + \left|\frac{1}{\mu(\varLambda_n)} \int\limits_{\varLambda_n g_1} \langle f(g), f(gh_1^{-1}h_2)\rangle \, dg - \gamma_{h_1^{-1}h_2}\right|$$

where b_2 is an upper bound for $(g, h_1, h_2) \mapsto |\langle f(g), f(gh_1^{-1}h_2)\rangle|$. But $\mu((\varLambda_n g_1) \bigtriangleup (\varLambda_n g_1 h_1)) = \mu(\varLambda_n \bigtriangleup (\varLambda_n h_1))$, since K is abelian. (Note that the uniformity of the Følner sequence was not required here.)

(d) Next, from (5.2) and Lebesgue's Dominated Convergence Theorem, it follows that

$$\lim_{n\to\infty} \frac{1}{\mu(\varLambda_n)} \int\limits_{\varLambda}\int\limits_{\varLambda}\int\limits_{\varLambda_n} \langle f(gh_1), f(gh_2)\rangle \, dg\, dh_1\, dh_2 = \int\limits_{\varLambda}\int\limits_{\varLambda} \gamma_{h_1^{-1}h_2}\, dh_1\, dh_2$$

for $\varLambda \in \Sigma|_K$ with $\mu(\varLambda) < \infty$.

In the case of (2), and assuming (5.3) in the case of a sequence, we have instead

$$\lim_{n\to\infty} \sup_{g_1\in K} \left|\frac{1}{\mu(\varLambda_n)} \int\limits_{\varLambda}\int\limits_{\varLambda}\int\limits_{\varLambda_n g_1} \langle f(gh_1), f(gh_2)\rangle \, dg\, dh_1\, dh_2 - \int\limits_{\varLambda}\int\limits_{\varLambda} \gamma_{h_1^{-1}h_2}\, dh_1\, dh_2\right| = 0,$$

which is proven as follows: Since here $\varLambda \times \varLambda$ is a finite set and μ is the counting measure, integrability over $\varLambda \times \varLambda$ is no problem, and then

$$\lim_{n\to\infty} \sup_{g_1\in K} \left|\frac{1}{\mu(\varLambda_n)} \int\limits_{\varLambda}\int\limits_{\varLambda}\int\limits_{\varLambda_n g_1} \langle f(gh_1), f(gh_2)\rangle \, dg\, dh_1\, dh_2 - \int\limits_{\varLambda}\int\limits_{\varLambda} \gamma_{h_1^{-1}h_2}\, dh_1\, dh_2\right|$$
$$\le \int\limits_{\varLambda\times\varLambda} \lim_{n\to\infty} \sup_{g_1\in K} \left|\frac{1}{\mu(\varLambda_n)} \int\limits_{\varLambda_n g_1} \langle f(gh_1), f(gh_2)\rangle \, dg\, dh_1\, dh_2 - \gamma_{h_1^{-1}h_2}\right| d(h_1, h_2).$$

(e) Lastly, combining (a), (b) and (d) with (5.1), we obtain the required results. ∎

As a corollary, we have a "two-parameter" van der Corput lemma which will be used in the proof of Corollary 5.4:

Corollary 5.2. *Assume the existence of uniform Følner sequences* $(\varLambda_{1,n})$ *and* $(\varLambda_{2,n})$ *in* K, *let* H *be a Hilbert space, and let* $f : G^2 \to H$ *be a bounded function with* $\langle f(\cdot), x\rangle$ *and* $\langle f(\cdot), f(\cdot)\rangle : G^2 \times G^2 \to \mathbb{C}$ *Borel measurable for all* $x \in H$. *Assume that*

$$\gamma_{(g',h')} := \lim_{n\to\infty} \frac{1}{\mu(\varLambda_{1,n})\mu(\varLambda_{2,n})} \int\limits_{\varLambda_{1,n}} \int\limits_{\varLambda_{2,n}} \langle f(g,h), f(gg', hh')\rangle \, dg\, dh$$

exists for all $g', h' \in G$, that $G \times G \to \mathbb{C} : (g', h') \mapsto \gamma_{(g',h')}$ is Borel measurable, and that

$$\lim_{m\to\infty} \frac{1}{\mu(\Lambda_{1,m})\mu(\Lambda_{2,m})} \int_{\Lambda_{1,m}^{-1}\Lambda_{1,m}} \int_{\Lambda_{2,m}^{-1}\Lambda_{2,m}} |\gamma_{(g,h)}|\, dg\, dh = 0 \tag{5.4}$$

where we also assume that $\Lambda_{1,m}^{-1}\Lambda_{1,m}$ and $\Lambda_{2,m}^{-1}\Lambda_{2,m}$ are Borel for m large enough.

(1) *Then it follows that*

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})} \int_{\Lambda_{1,n}} \int_{\Lambda_{2,n}} f(g,h)\, dg\, dh = 0.$$

(2) *Assume furthermore that μ is the counting measure* (*in particular G is countable*), *and that*

$$\lim_{m\to\infty} \sup_{g_1,g_2\in K} \left| \frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})} \int_{\Lambda_{1,n}g_1} \int_{\Lambda_{2,n}g_2} \langle f(g,h), f(gg', hh')\rangle\, dg\, dh - \gamma_{(g',h')} \right| = 0$$

for all $g', h' \in G$. Then

$$\lim_{n\to\infty} \sup_{g_1,g_2\in K} \left\| \frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})} \int_{\Lambda_{1,n}g_1} \int_{\Lambda_{2,n}g_2} f(g,h)\, dg\, dh \right\| = 0.$$

Proof. We use a product as in Corollary 3.3's proof. It is easily shown than $\Lambda'_n := \Lambda_{2,n} \times \Lambda_{1,n}$ gives a uniform Følner sequence in $K' := K \times K$ viewed as a semigroup in the ambient group $(G', \Sigma', \mu') := (G \times G, \Sigma\times\Sigma, \mu\times\mu)$. Since G is second countable, the product σ-algebra Σ' is in fact the Borel σ-algebra of G'. Note that $(\Lambda'_m)^{-1}\Lambda'_m = (\Lambda_{2,m}^{-1}\Lambda_{2,m}) \times (\Lambda_{1,m}^{-1}\Lambda_{1,m})$, hence by Fubini's Theorem and [3, Lemma 2.8] we have

$$\left| \frac{1}{\mu'(\Lambda'_m)^2} \int_{\Lambda'_m} \int_{\Lambda'_m} \gamma_{(h_1,h_2)^{-1}(h_3,h_4)}\, d(h_1,h_2)\, d(h_3,h_4) \right| \le \frac{1}{\mu(\Lambda_{1,m})\mu(\Lambda_{2,m})} \int_{\Lambda_{1,m}^{-1}\Lambda_{1,m}} \int_{\Lambda_{2,m}^{-1}\Lambda_{2,m}} |\gamma_{(g,h)}|\, dg\, dh$$

for m large enough, hence (5.4) implies (5.1) for the product group G'. Now simply apply Lemma 5.1 and Fubini's Theorem. ∎

From now on we specialize to the case $G = K$.

Lemma 5.3. *Let (A, ω, τ, G) be ergodic with ω tracial, and such that*

$$G \times G \to \mathbb{C} : (g,h) \mapsto \omega(\tau_h(a_1)\tau_g(a_2)\tau_{gh}(a_3)\tau_g(a_4)\tau_h(a_5)) \tag{5.5}$$

is $\Sigma\times\Sigma$-measurable for all $a_1,\dots,a_5\in A$. Assume the existence of Følner sequences $(\Lambda_{1,n})$ and $(\Lambda_{2,n})$ in G such that $(\Lambda_{1,n}^{-1}\Lambda_{1,n})$ and $(\Lambda_{2,n}^{-1}\Lambda_{2,n})$ are also Følner in G. Consider the GNS representation of (A,ω) and any $a,b,c\in A$ with at least one of $\iota(a)$, $\iota(b)$ or $\iota(c^*)$ in H_v. Set $x_{g,h}:=\iota(\tau_{gh}(a)\tau_g(b)\tau_h(c))$. Write

$$\gamma_{g',h'}\equiv\lim_{n\to\infty}\frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})}\int\limits_{\Lambda_{1,n}}\int\limits_{\Lambda_{2,n}}\langle x_{g,h},x_{gg',hh'}\rangle\,dg\,dh$$

for all $g',h'\in G$.

(1) *If (A,ω,τ,G) is asymptotically abelian with respect to $(\Lambda_{1,n})$, then*

$$\gamma_{g',h'}=\omega(a^*\tau_{g'h'}(a))\omega(b^*\tau_{g'}(b))\omega(\tau_{h'}(c)c^*),\tag{5.6}$$

giving a $\Sigma\times\Sigma$-measurable mapping $G\times G\to\mathbb{C}:(g,h)\mapsto\gamma_{g,h}$ such that

$$\lim_{n\to\infty}\frac{1}{\mu(\Lambda_{1,n}^{-1}\Lambda_{1,n})\mu(\Lambda_{2,n}^{-1}\Lambda_{2,n})}\int\limits_{\Lambda_{1,n}^{-1}\Lambda_{1,n}}\int\limits_{\Lambda_{2,n}^{-1}\Lambda_{2,n}}|\gamma_{g,h}|\,dg\,dh=0.$$

(2) *If (A,ω,τ,G) is uniformly asymptotically abelian with respect to $(\Lambda_{1,n})$, then*

$$\lim_{n\to\infty}\sup_{g_1,g_2\in G}\left|\frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})}\int\limits_{\Lambda_{1,n}g_1}\int\limits_{\Lambda_{2,n}g_2}\langle x_{g,h},x_{gg',hh'}\rangle\,dg\,dh-\gamma_{g',h'}\right|=0$$

for all $g',h'\in G$.

Proof. (1) Since

$$\langle x_{g,h},x_{gg',hh'}\rangle=\omega(\tau_h(c^*)\tau_g(b^*)\tau_{gh}[a^*\tau_{g'h'}(a)]\tau_g[\tau_{g'}(b)]\tau_h[\tau_{h'}(c)]),$$

(5.6) follows from Proposition 4.3(2). Note that $[(g,h)\mapsto\omega(a^*\tau_{gh}(a))]=[\omega(a^*\tau_{(\cdot)}(a))\circ((g,h)\mapsto gh)]$ is $\Sigma\times\Sigma$-measurable, since $(g,h)\mapsto gh$ is measurable and (A,ω,τ,G) is a C^*-dynamical system. Similarly for $(g,h)\mapsto\omega(b^*\tau_g(b))$ and $(g,h)\mapsto\omega(\tau_h(c)c^*)=\overline{\omega(c\tau_h(c^*))}$. So $G\times G\to\mathbb{C}:(g,h)\mapsto\gamma_{g,h}$ is $\Sigma\times\Sigma$-measurable. Hence

$$J_n\equiv\frac{1}{\mu(\Lambda_{1,n}^{-1}\Lambda_{1,n})\mu(\Lambda_{2,n}^{-1}\Lambda_{2,n})}\int\limits_{\Lambda_{1,n}^{-1}\Lambda_{1,n}}\int\limits_{\Lambda_{2,n}^{-1}\Lambda_{2,n}}|\gamma_{g,h}|\,dg\,dh$$

exists by Fubini's Theorem for n large enough, and clearly $J_n\geq 0$. We consider the three cases $\iota(a)\in H_v$, $\iota(b)\in H_v$ and $\iota(c^*)\in H_v$ separately:

For $\iota(a) \in H_v$ we have

$$J_n \le \frac{\|b\|^2\|c\|^2}{\mu(\Lambda_{1,n}^{-1}\Lambda_{1,n})\mu(\Lambda_{2,n}^{-1}\Lambda_{2,n})} \int\limits_{\Lambda_{1,n}^{-1}\Lambda_{1,n}} \int\limits_{\Lambda_{2,n}^{-1}\Lambda_{2,n}} |\omega(a^*\tau_{gh}(a))|\, dg\, dh$$
$$= \frac{\|b\|^2\|c\|^2}{\mu(\Lambda_{1,n}^{-1}\Lambda_{1,n})\mu(\Lambda_{2,n}^{-1}\Lambda_{2,n})} \int\limits_{\Lambda_{1,n}^{-1}\Lambda_{1,n}} \int\limits_{\Lambda_{2,n}^{-1}\Lambda_{2,n}} |\langle \iota(a), U_{gh}\iota(a)\rangle|\, dg\, dh$$
$$\to 0$$

in the n limit, according to Corollary 3.3, since $(\Lambda_{1,n}^{-1}\Lambda_{1,n})$ and $(\Lambda_{2,n}^{-1}\Lambda_{2,n})$ are Følner in G, and note that $(g,h) \mapsto \langle \iota(a_1), U_{gh}\iota(a_2)\rangle = \omega(a_1^*\tau_{gh}(a_2))$ is $\Sigma\times\Sigma$-measurable by the same argument as above, hence $(g,h)\mapsto \langle x, U_{gh}y\rangle$ is $\Sigma\times\Sigma$-measurable for all $x,y\in H$ by considering sequences in $\iota(A)$ converging to x and y (as in Proposition 3.4's proof).

In much the same way for $\iota(b)\in H_v$ we have

$$J_n \le \frac{\|a\|^2\|c\|^2}{\mu(\Lambda_{1,n}^{-1}\Lambda_{1,n})\mu(\Lambda_{2,n}^{-1}\Lambda_{2,n})} \int\limits_{\Lambda_{1,n}^{-1}\Lambda_{1,n}} \int\limits_{\Lambda_{2,n}^{-1}\Lambda_{2,n}} |\omega(b^*\tau_g(b))|\, dg\, dh$$
$$= \frac{\|a\|^2\|c\|^2}{\mu(\Lambda_{2,n}^{-1}\Lambda_{2,n})} \int\limits_{\Lambda_{2,n}^{-1}\Lambda_{2,n}} |\langle \iota(b), U_g\iota(b)\rangle|\, dg \to 0$$

in the n limit according to Proposition 3.2. The case $\iota(c^*)\in H_v$ is similar to $\iota(b)\in H_v$, but using $\omega(\tau_h(c)c^*) = \overline{\omega(c\tau_h(c^*))}$.

(2) This follows directly from Proposition 4.3(3), the formula for $\langle x_{g,h}, x_{gg',hh'}\rangle$ given in (1)'s proof, and (5.6), since G is a group and hence uniform asymptotic abelianness implies asymptotic abelianness with respect to $(\Lambda_{1,n})$. ∎

COROLLARY 5.4. *Assume the situation in Lemma* 5.3, *but suppose instead of* (5.5) *that*

$$G^2\times G^2 \to \mathbb{C} : (g,h,j,k) \mapsto \omega(\tau_h(a_1)\tau_g(a_2)\tau_{gh}(a_3)\tau_{jk}(a_4)\tau_j(a_5)\tau_k(a_6))$$

is $\Sigma^2\times\Sigma^2$-measurable for all $a_1,\dots,a_6\in A$. Assume furthermore the existence of uniform Følner sequences $(\Lambda_{1,n})$ and $(\Lambda_{2,n})$ in G satisfying the Tempel'man condition and such that $(\Lambda_{1,n}^{-1}\Lambda_{1,n})$ and $(\Lambda_{2,n}^{-1}\Lambda_{2,n})$ are also Følner in G.

(1) *If (A,ω,τ,G) is asymptotically abelian with respect to (Λ_{1n}) then*

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})} \int\limits_{\Lambda_{1,n}} \int\limits_{\Lambda_{2,n}} x_{g,h}\, dg\, dh = 0.$$

(2) *If μ is the counting measure and (A,ω,τ,G) is uniformly asymptotically abelian with respect to $(\Lambda_{1,n})$, then*

$$\lim_{n\to\infty}\sup_{g_1,g_2\in G}\left\|\frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})}\int_{\Lambda_{1,n}g_1}\int_{\Lambda_{2,n}g_2} x_{g,h}\,dg\,dh\right\| = 0.$$

Proof. The mapping (5.5) can be expressed as

$$(g,h)\mapsto(g,h,g,h)\mapsto\omega(\tau_h(a_1)\tau_g(a_2)\tau_{gh}(a_3)\tau_{gh}(1)\tau_g(a_4)\tau_h(a_5)),$$

which by the hypothesis is $\Sigma\times\Sigma$-measurable as required in Lemma 5.3, since in any topological space $y\mapsto(y,y)$ is continuous in the product topology. Now simply apply Corollary 5.2 and the Tempel'man condition to $f(g,h)=x_{g,h}$, noting that $\|f(g,h)\|\le\|\tau_{gh}(a)\tau_g(b)\tau_h(c)\|\le\|a\|\,\|b\|\,\|c\|$, so f is bounded, and

$$(g,h)\mapsto\langle f(g,h),\iota(d)\rangle=\omega(\tau_h(c^*)\tau_g(b^*)\tau_{gh}(a^*)\tau_{je}(1)\tau_j(1)\tau_e(d))$$

is Σ^2-measurable, hence so is $(g,h)\mapsto\langle f(g,h),x\rangle$ for all $x\in H$, since $\iota(A)$ is dense in H, while

$$(g,h,j,k)\mapsto\langle f(g,h),f(j,k)\rangle=\omega(\tau_h(c^*)\tau_g(b^*)\tau_{gh}(a^*)\tau_{jk}(a)\tau_j(b)\tau_k(c))$$

is $\Sigma^2\times\Sigma^2$-measurable. ■

PROPOSITION 5.5. *Assume the situation in Corollary* 5.4, *and also that (A,ω,τ,G) is weakly mixing, but now set*

$$x_{g,h}(a,b,c):=\iota(\tau_{gh}(a)\tau_g(b)\tau_h(c))$$

and

$$L(a,b,c):=\omega(a)\omega(b)\omega(c)$$

for all $a,b,c\in A$.

(1) *If (A,ω,τ,G) is asymptotically abelian with respect to $(\Lambda_{1,n})$, then*

$$\lim_{n\to\infty}\frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})}\int_{\Lambda_{1,n}}\int_{\Lambda_{2,n}} x_{g,h}(a,b,c)\,dg\,dh=L(a,b,c)\Omega$$

for all $a,b,c\in A$.

(2) *If μ is the counting measure, and (A,ω,τ,G) is uniformly asymptotically abelian with respect to $(\Lambda_{1,n})$, then*

$$\lim_{n\to\infty}\sup_{g_1,g_2\in G}\left\|\frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})}\sum_{h\in\Lambda_{1,n}g_1}\sum_{g\in\Lambda_{2,n}g_2} x_{g,h}(a,b,c)-L(a,b,c)\Omega\right\|=0$$

for all $a,b,c\in A$.

Proof. Write $a_0:=\omega(a)1$ and $a_v:=a-a_0$. Then by Proposition 3.4, $\iota(a_0)=\omega(a)\Omega\in H_0$ and $\iota(a_v)\in H_v$, since $\langle\Omega,\iota(a_v)\rangle=\omega(1^*a_v)=0$. Similarly for b and c, so in particular $x_{g,h}(a_0,b_0,c_0)=L(a,b,c)\Omega$. Furthermore, $\iota(c_0^*)=\overline{\omega(c)}\Omega\in H_0$ and $\langle\iota(c_v^*),\Omega\rangle=\omega(c_v1)=0$ so $\iota(c_v^*)\in H_v$.

(1) By Corollary 5.4(1) we then have

$$\begin{aligned}\lim_{n\to\infty} &\frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})} \int_{\Lambda_{1,n}} \int_{\Lambda_{2,n}} x_{g,h}(a,b,c)\,dg\,dh \\ &= \lim_{n\to\infty} \frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})} \int_{\Lambda_{1,n}} \int_{\Lambda_{2,n}} x_{g,h}(a_0,b_0,c_0)\,dg\,dh \\ &= L(a,b,c) \lim_{n\to\infty} \frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})} \int_{\Lambda_{1,n}} \int_{\Lambda_{2,n}} \Omega\,dg\,dh = L(a,b,c)\Omega.\end{aligned}$$

(2) Similarly by Corollary 5.4(2), but switching to summation notation and also using the triangle inequality, we have

$$\begin{aligned}&\lim_{n\to\infty} \sup_{g_1,g_2\in G} \left\| \frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})} \sum_{h\in\Lambda_{1,n}g_1} \sum_{g\in\Lambda_{2,n}g_2} x_{g,h}(a,b,c) - L(a,b,c)\Omega \right\| \\ &\le \lim_{n\to\infty} \sup_{g_1,g_2\in G} \left\| \frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})} \sum_{h\in\Lambda_{1,n}g_1} \sum_{g\in\Lambda_{2,n}g_2} x_{g,h}(a_0,b_0,c_0) - L(a,b,c)\Omega \right\| \\ &= 0. \ \blacksquare\end{aligned}$$

This proposition in itself is interesting. If we translate all the requirements to a measure-theoretic system, (1) reduces to L^2 convergence of the following nonconventional ergodic average:

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})} \int_{\Lambda_{1,n}} \int_{\Lambda_{2,n}} (f_1\circ T_{gh})(f_2\circ T_g)(f_3\circ T_h)\,dg\,dh \\ = \left(\int f_1\,d\nu\right)\left(\int f_2\,d\nu\right)\left(\int f_3\,d\nu\right)\end{aligned}$$

for $f_1, f_2, f_3 \in L^\infty(\nu)$ where ν is a probability measure on some measurable space and with T_g an invertible measure preserving transformation of this probability space, keeping in mind that the GNS representation is now simply given by the set inclusion $\iota : L^\infty(\nu) \to L^2(\nu)$. Similarly for (2).

Now we arrive at the main result of this paper:

THEOREM 5.6. *Let (A,ω,τ,G) be a weakly mixing C^*-dynamical system with ω tracial, and such that*

$$G^2\times G^2 \to \mathbb{C} : (g,h,j,k) \mapsto \omega(\tau_h(a_1)\tau_g(a_2)\tau_{gh}(a_3)\tau_{jk}(a_4)\tau_j(a_5)\tau_k(a_6))$$

is $\Sigma^2\times\Sigma^2$-measurable for all $a_1,\dots,a_6 \in A$. Assume the existence of uniform Følner sequences $(\Lambda_{1,n})$ and $(\Lambda_{2,n})$ in G satisfying the Tempel'man condition and such that $(\Lambda_{1,n}^{-1}\Lambda_{1,n})$ and $(\Lambda_{2,n}^{-1}\Lambda_{2,n})$ are also Følner in G.

(1) *If* (A, ω, τ, G) *is asymptotically abelian with respect to* $(\Lambda_{1,n})$, *then*

$$\lim_{n\to\infty} \frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})} \int_{\Lambda_{1,n}} \int_{\Lambda_{2,n}} \omega(\tau_{gh}(a)\tau_g(b)\tau_h(c)d)\, dg\, dh = \omega(a)\omega(b)\omega(c)\omega(d)$$

for all $a, b, c, d \in A$.

(2) *If* μ *is the counting measure* (*in particular*, G *is countable*), *and* (A, ω, τ, G) *is uniformly asymptotically abelian with respect to* $(\Lambda_{1,n})$, *then*

$$\lim_{n\to\infty} \sup_{g_1,g_2\in G} \bigg| \frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})} \sum_{h\in\Lambda_{1,n}g_1} \sum_{g\in\Lambda_{2,n}g_2} \omega(\tau_{gh}(a)\tau_g(b)\tau_h(c)d) - \omega(a)\omega(b)\omega(c)\omega(d) \bigg| = 0$$

for all $a, b, c, d \in A$.

Proof. This follows easily from Proposition 5.5, namely for (1) we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})} \int_{\Lambda_{1,n}} \int_{\Lambda_{2,n}} \omega(\tau_{gh}(a)\tau_g(b)\tau_h(c)d)\, dg\, dh \\
&\qquad = \Big\langle \iota(d^*), \lim_{n\to\infty} \frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})} \int_{\Lambda_{1,n}} \int_{\Lambda_{2,n}} x_{g,h}(a,b,c)\, dg\, dh \Big\rangle \\
&\qquad = \langle \iota(d^*), L(a,b,c)\Omega \rangle = \omega(d)\omega(a)\omega(b)\omega(c),
\end{aligned}$$

and similarly for (2), using

$$\begin{aligned}
&\bigg| \frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})} \sum_{h\in\Lambda_{1,n}g_1} \sum_{g\in\Lambda_{2,n}g_2} \omega(\tau_{gh}(a)\tau_g(b)\tau_h(c)d) - \omega(a)\omega(b)\omega(c)\omega(d) \bigg| \\
&\qquad = \bigg| \Big\langle \iota(d^*), \frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})} \sum_{h\in\Lambda_{1,n}g_1} \sum_{g\in\Lambda_{2,n}g_2} x_{g,h}(a,b,c) - L(a,b,c)\Omega \Big\rangle \bigg| \\
&\qquad \le \|\iota(d^*)\| \bigg\| \frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})} \sum_{h\in\Lambda_{1,n}g_1} \sum_{g\in\Lambda_{2,n}g_2} x_{g,h}(a,b,c) - L(a,b,c)\Omega \bigg\|
\end{aligned}$$

and Proposition 5.5's notation. ∎

Note that the measurability of the $G^2 \times G^2 \to \mathbb{C}$ function in this theorem can be ensured by for example making the standard assumption that $g \mapsto \tau_g(a)$ is continuous. From Theorem 5.6 we can derive the following recurrence result:

COROLLARY 5.7. *Consider the situation in Theorem* 5.6 *and let* $\varepsilon > 0$ *be given. Consider any* $a, b, c, d \in A$.

(1) *If* (A, ω, τ, G) *is asymptotically abelian with respect to* $(\Lambda_{1,n})$, *then there is an* $n_0 \in \mathbb{N}$ *such that for every* $n > n_0$,

$$|\omega(\tau_{gh}(a)\tau_g(b)\tau_h(c)d)| > |\omega(a)\omega(b)\omega(c)\omega(d)| - \varepsilon$$

for some $g \in \Lambda_{2,n}$ *and some* $h \in \Lambda_{1,n}$.

(2) *If* μ *is the counting measure, and* (A, ω, τ, G) *is uniformly asymptotically abelian with respect to* $(\Lambda_{1,n})$, *then there is an* $n \in \mathbb{N}$ *such that for every* $g_1, g_2 \in G$ *we have*

$$|\omega(\tau_{gh}(a)\tau_g(b)\tau_h(c)d)| > |\omega(a)\omega(b)\omega(c)\omega(d)| - \varepsilon \tag{5.7}$$

for some $g \in \Lambda_{2,n}g_2$ *and some* $h \in \Lambda_{1,n}g_1$, *i.e. the set of* (g, h)*'s for which* (5.7) *holds is relatively dense in* $G \times G$.

Proof. We only prove (2), since (1)'s proof is similar. By Theorem 5.6(2) there exists an n such that

$$\frac{1}{\mu(\Lambda_{1,n})\mu(\Lambda_{2,n})} \sum_{h \in \Lambda_{1,n}g_1} \sum_{g \in \Lambda_{2,n}g_2} |\omega(\tau_{gh}(a)\tau_g(b)\tau_h(c)d)| > |\omega(a)\omega(b)\omega(c)\omega(d)| - \varepsilon$$

for all $g_1, g_2 \in G$, from which the result follows. Also keep in mind that relative denseness of a set E in $G \times G$ is often defined in the following equivalent way: $FE = G \times G$ for some finite set F in $G \times G$, in this case $F = (\Lambda_{2,n} \times \Lambda_{1,n})^{-1}$. ∎

In the case of a countable group, Corollary 5.7(2) therefore says that we "regularly" have recurrence. In the more general situation, Corollary 5.7(1) is not quite as strong; however, keep in mind that the intervals $\Lambda_1 = [0, 1]$, $\Lambda_2 \in [1, 3]$, $\Lambda_3 \in [3, 6], \ldots$ give a uniform Følner sequence with the required properties in $G = \mathbb{R}$, so in this case Corollary 5.7(1) says that from a certain interval onward, we do get recurrence in each interval, but with the intervals steadily growing in size. Similarly in $G = \mathbb{R}^q$, where for example we can use a sequence of balls as at the beginning of this section, but shifted so that they do not overlap.

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