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Spatial Besov regularity for stochastic partial differential equations on Lipschitz domains

by

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Abstract. We use the scale of Besov spaces $B^{\alpha}_{\tau,\tau}(\mathcal{O})$, $1/\tau = \alpha/d+1/p$, $\alpha > 0$, p fixed, to study the spatial regularity of solutions of linear parabolic stochastic partial differential equations on bounded Lipschitz domains $\mathcal{O} \subset \mathbb{R}$. The Besov smoothness determines the order of convergence that can be achieved by nonlinear approximation schemes. The proofs are based on a combination of weighted Sobolev estimates and characterizations of Besov spaces by wavelet expansions.

1. Introduction. In this paper, the spatial Besov regularity of solutions of linear stochastic evolution equations on bounded Lipschitz domains is studied. We combine regularity results by Kim [30, 31] on stochastic partial differential equations (SPDEs, for short) on Lipschitz domains in terms of weighted Sobolev spaces with methods used in Dahlke and DeVore [12], where the Besov regularity of (deterministic) elliptic equations on Lipschitz domains is investigated. Our considerations are motivated by the question whether adaptive and other nonlinear approximation methods for solutions of SPDEs on Lipschitz domains pay off in the sense that they yield better convergence rates than uniform methods. Thus referring to a numerical theme and combining concepts and methods from different areas and scientific communities, the article is addressed to readers of both worlds: stochastic analysis and numerical analysis. Therefore, we give a rather detailed account in the first part of the paper, emphasizing conceptual and notational clarity.

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Our setting is as follows. On a finite interval $[0,T] \subset [0,\infty)$ let $(w_t^\kappa)_{t\in[0,T]}$, $\kappa \in \mathbb{N} = \{1,2,\dots\}$, be independent, one-dimensional standard Brownian motions with respect to a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ of σ-algebras on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$. Throughout the paper we assume that $(\mathcal{F}_t)_{t\in[0,T]}$ is normal, i.e. the filtration satisfies the usual hypotheses (see, e.g., [17, Section 3.3]). Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded Lipschitz domain. We consider the model equation

$$du = \sum_{\mu,\nu=1}^{d} a^{\mu\nu} u_{x_\mu x_\nu}\,dt + \sum_{\kappa=1}^{\infty} g^\kappa\,dw_t^\kappa, \quad u(0,\cdot\,) = u_0, \tag{1.1}$$

for $t \in [0,T]$ and $x \in \mathcal{O}$. Here du is Itô's stochastic differential with respect to t, $(a^{\mu\nu})_{1\le\mu,\nu\le d} \in \mathbb{R}^{d\times d}$ is a strictly positive definite, symmetric matrix and the coefficients g^κ, $\kappa \in \mathbb{N}$, are random functions depending on t and x such that the mappings $\Omega\times[0,T] \ni (\omega,t) \mapsto g^\kappa(\omega,t,\cdot\,)$ are predictable processes with values in certain function spaces. For details see Section 2.3.

Equation (1.1) is understood in a weak or distributional sense, i.e. u is a solution of (1.1) if for all $\varphi \in C_0^\infty(\mathcal{O})$ the equality

$$\begin{aligned}\langle u(t,\cdot\,),\varphi\rangle &= \langle u_0,\varphi\rangle + \sum_{\mu,\nu=1}^{d}\int_0^t \langle a^{\mu\nu}u_{x_\mu x_\nu}(s,\cdot\,),\varphi\rangle\,ds + \sum_{\kappa=1}^{\infty}\int_0^t \langle g^\kappa(s,\cdot\,),\varphi\rangle\,dw_s^\kappa\end{aligned}$$

holds for all $t \in [0,T]$ $\mathbb{P}$-almost surely. Here and throughout the paper we write $\langle u,\varphi\rangle$ for the application of a distribution $u \in \mathcal{D}'(\mathcal{O})$ to a test function $\varphi \in C_0^\infty(\mathcal{O})$. The existence and uniqueness of solutions of (1.1), respectively (1.3) below, in certain classes $\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T)$ of stochastic processes has been shown in [30, 31]; see also the earlier papers by Krylov, Lototsky and Kim, e.g. [29, 32, 33, 37]. Roughly speaking, the classes $\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T)$ are L_p-spaces of functions on $\Omega\times[0,T]$ with values in weighted Sobolev spaces $H^\gamma_{p,\theta-p}(\mathcal{O})$ that can be regarded as generalizations of the classical Sobolev spaces with zero Dirichlet boundary condition. Again we refer to Section 2.3 for precise definitions.

Let us remark that in Examples 3.3, 3.4 and 3.5, illustrating our Besov regularity result in Section 3, the solution of (1.1) in the class $\mathfrak{H}^\gamma_{2,\theta}(\mathcal{O},T)$ coincides with the unique weak solution with zero Dirichlet boundary condition in the sense of Da Prato and Zabczyk [17], and hence can be represented by the well known stochastic variation-of-constants formula

$$u(t,\cdot\,) = e^{tA}u_0 + \int_0^t e^{(t-s)A}G(s)\,dW_s, \quad t \in [0,T]. \tag{1.2}$$

Here $(e^{tA})_{t\ge0}$ is the semigroup of contractions on $L_2(\mathcal{O})$ generated by the

partial differential operator $A = \sum_{\mu,\nu=1}^{d} a^{\mu\nu} \frac{\partial^2}{\partial x_\mu \partial x_\nu}$ with zero Dirichlet boundary condition considered as an unbounded operator on $L_2(\mathcal{O})$, $(G(t))_{t\in[0,T]}$ is an operator-valued process and $(W_t)_{t\in[0,T]}$ is a cylindrical Wiener process on $\ell_2(\mathbb{N})$ (see Remarks 2.13 and 2.14). Similarly, for $p > 2$ a solution $u \in \mathfrak{H}^{\gamma}_{p,\theta}(\mathcal{O}, T)$ to (1.1) may be interpreted within the approach to stochastic evolution equations in Banach spaces of Brzeźniak [3, 4] and van Neerven, Veraar and Weis [41, 42] (see Remark 3.7).

As already mentioned, our motivation to study the Besov regularity of SPDEs is the theme of nonlinear approximation of the solution processes. For deterministic settings, a detailed overview of nonlinear approximation and an exposition of the characterization of its efficiency in terms of the Besov smoothness of the target functions can be found in DeVore [19] (see also Cohen [7, Chapters 3 and 4]). Let us consider an example of approximation by wavelets in $L_p(\mathcal{O})$, the L_p-space of real-valued functions on $\mathcal{O}$, $p \in (1, \infty)$. To this end, let $\{\psi_\lambda : \lambda \in \nabla\}$ be a wavelet basis on $\mathcal{O}$ and let $f \in L_p(\mathcal{O})$ be a target function which we want to approximate by functions $f_N \in L_p(\mathcal{O})$ belonging to certain approximation spaces S_N, where N is the number of parameters used to describe the elements of S_N. We specify the index set of the wavelet basis by writing $\nabla = \bigcup_{j\geq j_0-1} \nabla_j$; the wavelets ψ_λ, $\lambda \in \nabla_j$, $j \geq j_0$, are those at scale levels $j \geq j_0$ respectively, and ψ_λ, $\lambda \in \nabla_{j_0-1}$, are the scaling functions at the coarsest level $j_0 \in \mathbb{Z}$. In the case of uniform wavelet approximation up to a highest scale level $j_0 - 1 + n$, $n \in \mathbb{N}$, the approximation spaces are

$$S_N = S_{N(n)} = \Big\{ \sum_{j=j_0-1}^{j_0-1+n} \sum_{\lambda\in\nabla_j} c_\lambda \psi_\lambda : c_\lambda \in \mathbb{R},\, \lambda \in \nabla_j,\, j \in \{j_0 - 1, \dots, j_0 - 1 + n\} \Big\},$$

where $N = N(n) = |\bigcup_{j=j_0-1}^{j_0-1+n} \nabla_j| \in \mathbb{N}$ is the cardinality of the set of all indices up to scale level $j_0 - 1 + n$. Let $e_N(f) = \inf_{f_N\in S_N} \|f - f_N\|_{L_p(\mathcal{O})}$ be the corresponding approximation error measured in $L_p(\mathcal{O})$. It is well known that—under certain technical assumptions on the wavelet basis—the decay rate of $e_N(f)$ is linked to the L_p-Sobolev smoothness of the target function. More precisely, there exists an upper bound $r \in \mathbb{N}$ depending on the wavelet basis such that, for all $s \in [0, r]$,

$$f \in W_p^s(\mathcal{O}) \;\Rightarrow\; e_N(f) \leq C \cdot N^{-s/d},\ N = N(n),\ n \in \mathbb{N},$$

for some constant $C > 0$ which does not depend on N. The fractional order Sobolev spaces $W_p^s(\mathcal{O})$ are defined in the next section. One can also show the converse:

$$\exists C > 0\ \forall n \in \mathbb{N} : e_N(f) \leq C \cdot N^{-s/d},\ N = N(n) \;\Rightarrow\; f \in W_p^{s'}(\mathcal{O}),\ s' < s.$$

If we consider instead best N-term approximation as a form of nonlinear approximation, the approximation spaces are

$$\Sigma_N = \Big\{ \sum_{\lambda\in\Lambda} c_\lambda \psi_\lambda : \Lambda \subset \nabla,\ |\Lambda| \le N,\ c_\lambda \in \mathbb{R},\ \lambda \in \Lambda \Big\},$$

$N \in \mathbb{N}$, and the decay rate of the error $\sigma_N(f) := \inf_{f_N\in\Sigma_N} \|f - f_N\|_{L_p(\mathcal{O})}$ is governed by the smoothness of f measured in certain $L_\tau(\mathcal{O})$-norms, $\tau < p$, which are weaker than the $L_p(\mathcal{O})$-norm: For all $\alpha \in [0, r]$,

$$f \in B^\alpha_{\tau,\tau}(\mathcal{O}),\ \frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p} \ \Rightarrow\ \sigma_N(f) \le C \cdot N^{-\alpha/d},\ N \in \mathbb{N},$$

$B^\alpha_{\tau,\tau}(\mathcal{O})$ being a Besov space as defined in Section 2.2. Therefore, if the target function f belongs to $B^\alpha_{\tau^*,\tau^*}(\mathcal{O})$, $1/\tau^* = \alpha/d + 1/p$, for some $\alpha \in [0, r]$, and if in addition $\beta := \sup\{s \in \mathbb{R} : f \in W^s_p(\mathcal{O})\} < \alpha$, then the convergence rate of uniform wavelet approximations is inferior to the convergence rate of the best N-term wavelet approximation. The latter can be considered as a benchmark for the convergence rate of adaptive numerical algorithms (see [8, 9, 11]). This situation is illustrated in Figure 1, where each point $(1/\tau, s)$ represents the smoothness spaces of functions with "s derivatives in $L_\tau(\mathcal{O})$". Note that the nonlinear approximation line $\{(1/\tau, s) \in [0,\infty)^2 : 1/\tau = s/d + 1/p\}$ is also the Sobolev embedding line. For bounded domains, all spaces left to this line as well as the spaces $B^s_{\tau,\tau}(\mathcal{O})$ on the line are continuously embedded in $L_p(\mathcal{O})$.

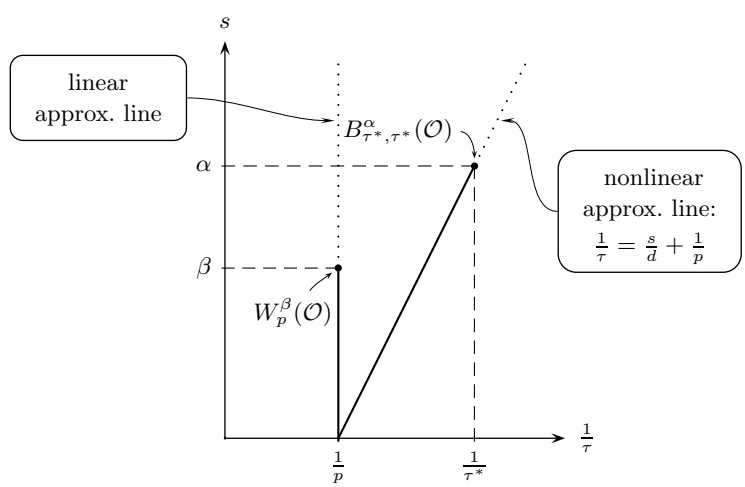

Fig. 1. Linear vs. nonlinear approximation illustrated in a DeVore–Triebel diagram

Let us return to equation (1.1) and assume that the solution $u = u(\omega, t, x)$, $(\omega, t, x) \in \Omega \times [0, T] \times \mathcal{O}$, vanishes on the boundary $\partial\mathcal{O}$, satisfying a zero Dirichlet boundary condition. It is clear that the smoothness of $x \mapsto u(\omega, t, x)$ depends on the smoothness of the mappings $x \mapsto g^\kappa(\omega, t, x)$, $\kappa \in \mathbb{N}$. However,

even if the spatial smoothness of the g^κ is high, the Sobolev smoothness of $x \mapsto u(\omega, t, x)$ can be additionally limited by singularities of the spatial derivatives of u at the boundary of $\mathcal{O}$, due to the zero Dirichlet boundary condition and the shape of the domain. Such corner singularities are typical examples for the fact that the spatial L_p-Sobolev regularity of u may be exceeded by the regularity in the scale of Besov spaces $B^\alpha_{\tau,\tau}(\mathcal{O})$, $1/\tau = \alpha/d + 1/p$. In this paper, we present a result on the spatial Besov regularity of the solution u to (1.1) which has the following structure: If

$$u \in L_p(\Omega \times [0,T], \mathcal{P}, \mathbb{P} \otimes \lambda;\, W^s_p(\mathcal{O}))$$

and if the functions g^κ, $\kappa \in \mathbb{N}$, are sufficiently regular, then

$$u \in L_\tau(\Omega \times [0,T], \mathcal{P}, \mathbb{P} \otimes \lambda;\, B^\alpha_{\tau,\tau}(\mathcal{O}))$$

for certain $\alpha > s$ and $1/\tau = \alpha/d + 1/p$. Here $\mathcal{P}$ is the predictable σ-algebra with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and λ denotes Lebesgue measure on $[0,T]$. This result is important for the theoretical foundation of adaptive numerical methods for the approximation of u. The proof is based on a wavelet expansion of an extension of $\mathcal{O} \ni x \mapsto u(\omega, t, x)$ to $\mathbb{R}^d$, which allows us to estimate the $B^\alpha_{\tau,\tau}(\mathcal{O})$-norm in terms of the wavelet coefficients. We apply a strategy similar to the one used in Dahlke and DeVore [12], where the Besov regularity of (deterministic) elliptic equations on Lipschitz domains is investigated with the help of an estimate of weighted Sobolev norms of harmonic functions. Our substitute for the latter is an estimate of weighted Sobolev norms of the solution of (1.1) provided by Kim [30, 31].

There exists an extensive literature on the Besov regularity of SPDEs. In general, however, the assumptions on the domain and the scale of parameters considered do not fit into our setting. To mention an example, the semigroup approach to SPDEs of Da Prato and Zabczyk [17], which is placed in a Hilbert space framework, has been generalized to M-type 2 Banach spaces by Brzeźniak [3, 4], for the purpose of gaining better Hölder regularity results. Roughly speaking, the operator A appearing in equation (1.2) is considered as the generator of a semigroup on $L_p(\mathcal{O})$ for some $p \geq 2$, and the stochastic integral in (1.2) is considered as a stochastic integral in an interpolation space X between $L_p(\mathcal{O})$ and $D(A) \subset L_p(\mathcal{O})$, the domain of A, realizing a zero Dirichlet boundary condition. If $\partial\mathcal{O}$ is sufficiently smooth, then $D(A) = W^2_p(\mathcal{O}) \cap \mathring{W}^1_p(\mathcal{O})$ and $X \subseteq B^s_{p,2}(\mathcal{O})$ for some $s \in [0,2]$. In this situation, the Sobolev embedding theorem leads to Hölder regularity results, and these results become better for large p. With the help of a theory of stochastic integration in wider classes of Banach spaces, this approach has been generalized in the works of van Neerven, Veraar and Weis (see, e.g., [40, 41, 42], compare also Brzeźniak and van Neerven [5]). In contrast to these works the problem considered here is of a

different nature. Firstly, we are explicitly interested in domains with non-smooth boundary. For polygonal non-convex domains, it is well known that $W_2^2(\mathcal{O}) \cap \mathring{W}_2^1(\mathcal{O}) \subsetneq D(A)$, where $D(A) := \{u \in \mathring{W}_2^1(\mathcal{O}) : Au \in L_2(\mathcal{O})\}$, $A = \Delta = \sum_{\mu=1}^d \partial^2/\partial x_\mu^2$ (see Grisvard [23, 24], and for more general Lipschitz domains see Jerison and Kenig [28]). Secondly, we are interested in the special scale $B^\alpha_{\tau,\tau}(\mathcal{O})$, $1/\tau = \alpha/d + 1/p$, $\tau > 0$, p fixed, including in particular spaces which are not Banach spaces but quasi-Banach spaces. The parameter τ decreases if α increases and $B^\alpha_{\tau,\tau}(\mathcal{O})$ fails to be a Banach space for $\tau < 1$. While our methods work in this setting, any direct approach requires (at least!) a fully-fledged theory of stochastic integration in quasi-Banach spaces which is not yet available. We refer to Remark 3.7 for a concrete comparison of our result with a related result by van Neerven, Veraar and Weis [42].

Let us emphasize that our result can be extended to more general linear equations of the type

$$\begin{cases} du = \displaystyle\sum_{\mu,\nu=1}^d (a^{\mu\nu} u_{x_\mu x_\nu} + b^\mu u_{x_\mu} + cu + f)\,dt \\ \qquad\quad + \displaystyle\sum_{\kappa=1}^\infty \Big(\sum_{\mu=1}^d \sigma^{\mu\kappa} u_{x_\mu} + \eta^\kappa u + g^\kappa \Big)\,dw_t^\kappa, \quad u(0,\cdot) = u_0, \end{cases} \tag{1.3}$$

including, in particular, the case of multiplicative noise. Here the coefficients $a^{\mu\nu}$, b^μ, c, $\sigma^{\mu\kappa}$, η^κ and the free terms f and g^κ are random functions depending on t and x. This extension is possible because one of our main tools, the weighted Sobolev norm estimate of Corollary 2.12, holds for equations of type (1.1) as well as for equations of type (1.3). Since this mainly adds notational complications, we will focus on equation (1.1) and refer to Appendix B for a short account of how to treat equations of type (1.3).

The paper is organized as follows: In Section 2 we collect the notation, definitions and preliminary results needed later on. Some general notation is introduced in Section 2.1. Section 2.2 provides the necessary facts on Besov spaces and wavelet decompositions. In Section 2.3 a short introduction to the general L_p-theory of SPDEs on Lipschitz domains due to Kim [30, 31] is given, including the definitions of the already mentioned spaces $H^\gamma_{p,\theta-p}(\mathcal{O})$, $\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T)$. Finally, in Section 3 the Besov regularity result (Theorem 3.1) is stated and proved, and some concrete examples of application are given.

2. Preliminaries

2.1. Some notation and conventions. In this and the next subsection, $\mathcal{O} \subseteq \mathbb{R}^d$ can be an arbitrary (not necessarily bounded) Lipschitz do-

main. A domain is called *Lipschitz* if each point on the boundary $\partial\mathcal{O}$ has a neighbourhood whose intersection with the boundary—after relabeling and reorienting the coordinate axes if necessary—is the graph of a Lipschitz function.

By $\mathcal{D}'(\mathcal{O})$ we denote the space of Schwartz distributions on $\mathcal{O}$. If not explicitly stated otherwise, all function spaces or spaces of distributions are meant to be spaces of real-valued functions or distributions. If $f \in \mathcal{D}'(\mathcal{O})$ and $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ is a multi-index, we write $D^\alpha f = \partial^{|\alpha|} f / \partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}$ for the corresponding derivative with respect to $x = (x_1, \dots, x_d) \in \mathcal{O}$, where $|\alpha| = \alpha_1 + \dots + \alpha_d$. As in (1.1) and (1.3) we also use the notation $f_{x_\mu x_\nu} = \partial^2 f / \partial x_\mu \partial x_\nu$, $f_{x_\mu} = \partial f / \partial x_\mu$. For $m \in \mathbb{N}_0$, $D^m f = \{D^\alpha f : |\alpha| = m\}$ is the set of all mth order derivatives of f which is identified with an $\mathbb{R}^{\binom{d+m-1}{m}}$-valued distribution. Given $p \in [1, \infty)$ and $m \in \mathbb{N}_0$, $W_p^m(\mathcal{O})$ denotes the classical Sobolev space consisting of all (equivalence classes of) measurable functions $f : \mathcal{O} \to \mathbb{R}$ such that $\|f\|_{W_p^m(\mathcal{O})} = \|f\|_{L_p(\mathcal{O})} + |f|_{W_p^m(\mathcal{O})} = (\int_{\mathcal{O}} |f(x)|^p \, dx)^{1/p} + \sum_{|\alpha|=m} (\int_{\mathcal{O}} |D^\alpha f(x)|^p \, dx)^{1/p}$ is finite. For $p \in (1, \infty)$ and $s \in (m, m+1)$, $m \in \mathbb{N}_0$, we define the fractional order Sobolev space $W_p^s(\mathcal{O})$ to be the Besov space $B_{p,p}^s(\mathcal{O})$ introduced in the next subsection. (This scale of fractional order Sobolev spaces can also be obtained by real interpolation of $W_p^n(\mathcal{O})$, $n \in \mathbb{N}_0$. One can show that $W_2^n(\mathcal{O}) = B_{2,2}^n(\mathcal{O})$ for all $n \in \mathbb{N}$ and $W_p^n(\mathcal{O}) \subset B_{p,p}^n(\mathcal{O})$ for all $n \in \mathbb{N}$, $p > 2$; see, e.g. Triebel [49, Remark 2.3.3/4 and Theorem 4.6.1(b)] together with Dispa [21].) Given any countable index set $\mathcal{J}$, the space of p-summable sequences indexed by $\mathcal{J}$ is denoted by $\ell_p = \ell_p(\mathcal{J})$ and $|\cdot|_{\ell_p}$ is the corresponding norm. Usually we have $\ell_p = \ell_p(\mathbb{N})$, but for instance we may also use the notation $|D^m f(x)|_{\ell_p}^p = \sum_{|\alpha|=m} |D^\alpha f(x)|^p$ for $f \in W_p^m(\mathcal{O})$.

Given a distribution $f \in \mathcal{D}'(\mathcal{O})$ and a smooth and compactly supported test function $\varphi \in C_0^\infty(\mathcal{O})$, we write $\langle f, \varphi \rangle$ for the application of f to φ. If H is a Hilbert space, then $\langle \cdot, \cdot \rangle_H$ denotes the inner product in H. Given another Hilbert space U, we denote by $L_{(\mathrm{HS})}(H, U)$ and $L_{(\mathrm{nuc})}(H, U)$ the spaces of Hilbert–Schmidt operators and nuclear operators from H to U respectively (see, e.g., Pietsch [45, Sections 6 and 15] or Da Prato and Zabczyk [17, Appendix C] for definitions). We also abbreviate $L_{(\mathrm{HS})}(H) = L_{(\mathrm{HS})}(H, H)$ and $L_{(\mathrm{nuc})}(H) = L_{(\mathrm{nuc})}(H, H)$. $\mathcal{M}_T^{2,c}(H, (\mathcal{F}_t))$ is the space of continuous, square-integrable, H-valued martingales with respect to the filtration $(\mathcal{F}_t)_{t \in [0,T]}$. For $\Omega \times [0, T]$ we use the shorthand notation Ω_T, and

$$\mathcal{P} = \sigma\big(\{(s, t] \times F_s : 0 \le s < t \le T,\, F_s \in \mathcal{F}_s\} \cup \{\{0\} \times F_0 : F_0 \in \mathcal{F}_0\}\big)$$

is the predictable σ-algebra. $\mathbb{P} \otimes \lambda$ is the product measure of the probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ and Lebesgue measure λ on $([0, T], \mathcal{B}([0, T]))$, where $\mathcal{B}([0, T])$ denotes the Borel σ-algebra on $[0, T]$. Given any measure space

$(A, \mathcal{A}, m)$, any (quasi-)normed space B with (quasi-)norm $\|\cdot\|_B$ and any summability index $p > 0$, we denote by $L_p(A, \mathcal{A}, m; B)$ the L_p-space of all strongly measurable functions $u : A \to B$ whose (quasi-)norm $\|u\|_{L_p(A,\mathcal{A},m;B)} := (\int_A \|u(z)\|_B^p \, m(dz))^{1/p}$ is finite.

All equalities of random variables or random (generalized) functions appearing in this paper are meant to be $\mathbb{P}$-almost sure equalities. Throughout the paper, C denotes a positive constant which may change its value from line to line.

2.2. Besov spaces and wavelet decompositions. In this section we give the definition of Besov spaces and describe their characterization in terms of wavelets. Our standard reference in this context is the monograph of Cohen [7].

For a function $f : \mathcal{O} \to \mathbb{R}$ and a natural number $n \in \mathbb{N}$ let

$$\Delta_h^n f(x) := \prod_{i=0}^{n} \mathbb{1}_{\mathcal{O}}(x + ih) \cdot \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} f(x + jh)$$

be the nth difference of f with step $h \in \mathbb{R}^d$. For $p \in (0, \infty)$ the modulus of smoothness is given by

$$\omega^n(t, f)_p := \sup_{|h|<t} \|\Delta_h^n f\|_{L_p(\mathcal{O})}, \qquad t > 0.$$

One approach to introduce Besov spaces is the following.

Definition 2.1. Let $s, p, q \in (0, \infty)$ and $n \in \mathbb{N}$ with $n > s$. Then $B^s_{p,q}(\mathcal{O})$ is the collection of all functions $f \in L_p(\mathcal{O})$ such that

$$|f|_{B^s_{p,q}(\mathcal{O})} := \left(\int_0^\infty [t^{-s} \omega^n(t, f)_p]^q \, \frac{dt}{t} \right)^{1/q} < \infty.$$

These classes are equipped with a (quasi-)norm by taking

$$\|f\|_{B^s_{p,q}(\mathcal{O})} := \|f\|_{L_p(\mathcal{O})} + |f|_{B^s_{p,q}(\mathcal{O})}.$$

Remark 2.2. For a more general definition of Besov spaces, including the cases where $p, q = \infty$ and $s < 0$, see, e.g., Triebel [50].

We want to describe $B^s_{p,q}(\mathbb{R}^d)$ by means of wavelet expansions. To this end let φ be a scaling function of tensor product type on $\mathbb{R}^d$ and let ψ_i, $i = 1, \ldots, 2^d - 1$, be corresponding multivariate mother wavelets such that, for a given $r \in \mathbb{N}$ and some $N > 0$, the following locality, smoothness and

vanishing moment conditions hold. For all $i = 1, \dots, 2^d - 1$,

$$\operatorname{supp} \varphi, \operatorname{supp} \psi_i \subset [-N, N]^d, \tag{2.1}$$

$$\varphi, \psi_i \in C^r(\mathbb{R}^d), \tag{2.2}$$

$$\int x^\alpha \psi_i(x)\, dx = 0 \quad \text{for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \le r. \tag{2.3}$$

We assume that

$$\{\varphi_k, \psi_{i,j,k} : (i,j,k) \in \{1, \dots, 2^d - 1\} \times \mathbb{N}_0 \times \mathbb{Z}^d\}$$

is a Riesz basis of $L_2(\mathbb{R}^d)$, where we use the standard abbreviations for dyadic shifts and dilations of the scaling function and the corresponding wavelets:

$$\varphi_k(x) := \varphi(x - k), \quad x \in \mathbb{R}^d, \tag{2.4}$$

for $k \in \mathbb{Z}^d$, and

$$\psi_{i,j,k}(x) := 2^{jd/2} \psi_i(2^j x - k), \quad x \in \mathbb{R}^d, \tag{2.5}$$

for $(i,j,k) \in \{1, \dots, 2^d - 1\} \times \mathbb{N}_0 \times \mathbb{Z}^d$. Further, we assume that there exists a dual Riesz basis satisfying the same requirements. More precisely, there exist functions $\widetilde{\varphi}$ and $\widetilde{\psi}_i$, $i = 1, \dots, 2^d - 1$, such that conditions (2.1)–(2.3) hold if φ and ψ are replaced by $\widetilde{\varphi}$ and $\widetilde{\psi}_i$, and the biorthogonality relations

$$\langle \widetilde{\varphi}_k, \psi_{i,j,k} \rangle = \langle \widetilde{\psi}_{i,j,k}, \varphi_k \rangle = 0, \quad \langle \widetilde{\varphi}_k, \varphi_\ell \rangle = \delta_{k,\ell}, \quad \langle \widetilde{\psi}_{i,j,k}, \psi_{u,v,\ell} \rangle = \delta_{i,u} \delta_{j,v} \delta_{k,\ell},$$

are fulfilled. Here we use analogous abbreviations to (2.4) and (2.5) for the dyadic shifts and dilations of $\widetilde{\varphi}$ and $\widetilde{\psi}_i$, and $\delta_{k,l}$ denotes the Kronecker symbol. We refer to Cohen [7, Chapter 2] for the construction of biorthogonal wavelet bases (see also Daubechies [18] and Cohen, Daubechies and Feauveau [10]). To keep notation simple, we will write

$$\psi_{i,j,k,p} := 2^{jd(1/p - 1/2)} \psi_{i,j,k} \quad \text{and} \quad \widetilde{\psi}_{i,j,k,p'} := 2^{jd(1/p' - 1/2)} \widetilde{\psi}_{i,j,k},$$

for the L_p-normalized wavelets and the correspondingly modified duals, with $p' := p/(p-1)$ if $p \in (0, \infty)$, $p \neq 1$, and $p' := \infty$, $1/p' := 0$ if $p = 1$.

The following theorem shows how Besov spaces can be described by decay properties of the wavelet coefficients, if the parameters fulfil certain conditions.

THEOREM 2.3. *Let $p, q \in (0, \infty)$ and $s > \max\{0, d(1/p - 1)\}$. Choose $r \in \mathbb{N}$ such that $r > s$ and construct a biorthogonal wavelet Riesz basis as described above. Then a locally integrable function $f : \mathbb{R}^d \to \mathbb{R}$ is in the Besov space $B^s_{p,q}(\mathbb{R}^d)$ if, and only if,*

$$f = \sum_{k \in \mathbb{Z}^d} \langle f, \widetilde{\varphi}_k \rangle \varphi_k + \sum_{i=1}^{2^d - 1} \sum_{j \in \mathbb{N}_0} \sum_{k \in \mathbb{Z}^d} \langle f, \widetilde{\psi}_{i,j,k,p'} \rangle \psi_{i,j,k,p} \tag{2.6}$$

(*convergence in* $\mathcal{D}'(\mathbb{R}^d)$) *with*

$$\Big(\sum_{k\in\mathbb{Z}^d}|\langle f,\widetilde{\varphi}_k\rangle|^p\Big)^{1/p}+\Big(\sum_{i=1}^{2^d-1}\sum_{j\in\mathbb{N}_0}2^{jsq}\Big(\sum_{k\in\mathbb{Z}^d}|\langle f,\widetilde{\psi}_{i,j,k,p'}\rangle|^p\Big)^{q/p}\Big)^{1/q}<\infty, \tag{2.7}$$

and (2.7) *is an equivalent* (*quasi-*)*norm for* $B^s_{p,q}(\mathbb{R}^d)$.

Remark 2.4. A proof of this theorem for the case $p \geq 1$ can be found in Meyer [38, §10 of Chapter 6]. For the general case see for example Kyriazis [35] or Cohen [7, Theorem 3.7.7]. Of course, if (2.7) holds then the infinite sum in (2.6) converges also in $B^s_{p,q}(\mathbb{R}^d)$. If $s > \max\{0, d(1/p-1)\}$ we have the embedding $B^s_{p,q}(\mathbb{R}^d) \subset L_u(\mathbb{R}^d)$ for some $u > 1$ (see, e.g., Cohen [7, Corollary 3.7.1]).

Let us now fix a value $p \in (1,\infty)$ and consider the scale of Besov spaces $B^s_{\tau,\tau}(\mathbb{R}^d)$, $1/\tau = s/d + 1/p$, $s > 0$. A simple computation gives the following result.

Corollary 2.5. *Let* $p \in (1,\infty)$, $s > 0$ *and* $\tau \in \mathbb{R}$ *be such that* $1/\tau = s/d+1/p$. *Choose* $r \in \mathbb{N}$ *such that* $r > s$ *and construct a biorthogonal wavelet Riesz basis as described above. Then a locally integrable function* $f : \mathbb{R}^d \to \mathbb{R}$ *is in the Besov space* $B^s_{\tau,\tau}(\mathbb{R}^d)$ *if, and only if,*

$$f=\sum_{k\in\mathbb{Z}^d}\langle f,\widetilde{\varphi}_k\rangle\varphi_k+\sum_{i=1}^{2^d-1}\sum_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{Z}^d}\langle f,\widetilde{\psi}_{i,j,k,p'}\rangle\psi_{i,j,k,p} \tag{2.8}$$

(*convergence in* $\mathcal{D}'(\mathbb{R}^d)$) *with*

$$\Big(\sum_{k\in\mathbb{Z}^d}|\langle f,\widetilde{\varphi}_k\rangle|^\tau\Big)^{1/\tau}+\Big(\sum_{i=1}^{2^d-1}\sum_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{Z}^d}|\langle f,\widetilde{\psi}_{i,j,k,p'}\rangle|^\tau\Big)^{1/\tau}<\infty, \tag{2.9}$$

and (2.9) *is an equivalent* (*quasi-*)*norm for* $B^s_{\tau,\tau}(\mathbb{R}^d)$.

2.3. SPDEs on Lipschitz domains and weighted Sobolev spaces. From now on, let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded Lipschitz domain.

We have already mentioned corner singularities as typical examples where the regularity of a function on $\mathcal{O} \subset \mathbb{R}^d$ in the Besov scale $B^\alpha_{\tau,\tau}(\mathcal{O})$, $1/\tau = \alpha/d + 1/p$, $\alpha > 0$, can exceed the regularity in the Sobolev scale $W^s_p(\mathcal{O})$, $s > 0$. This reflects the sparsity of the large wavelet coefficients of such a function (given a wavelet basis on the domain $\mathcal{O}$). A general way to deal with smoothness regardless of certain singularities at the boundary is to use weighted Sobolev spaces, where the weight function is a power of the distance to the boundary. The L_p-theory of SPDEs on Lipschitz domains by Kim [30, 31] is based on spaces of this type, namely the weighted Sobolev

spaces $H^\gamma_{p,\theta}(\mathcal{O})$, $p \in (1,\infty)$, θ, $\gamma \in \mathbb{R}$, introduced in Lototsky [37]. They are defined in terms of the Bessel-potential spaces

$$H^\gamma_p(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) : \|u\|_{H^\gamma_p(\mathbb{R}^d)} = \|(1-\Delta)^{\gamma/2}u\|_{L_p(\mathbb{R}^d)} < \infty\}.$$

Here, $\mathcal{S}'(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d)$ is the space of (real valued) tempered distributions and $(1-\Delta)^{\gamma/2} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is the pseudo-differential operator with symbol $\mathbb{R}^d \ni \xi \mapsto (1+|\xi|^2)^{\gamma/2}$, i.e. $(1-\Delta)^{\gamma/2}u = \mathcal{F}^{-1}((1+|\xi|^2)^{\gamma/2}\mathcal{F}u)$, where $\mathcal{F}$ denotes the Fourier transform on the (complex valued) tempered distributions.

For $x \in \mathcal{O}$ we write $\rho(x) := \operatorname{dist}(x, \partial\mathcal{O})$ for the distance between x and the boundary of the domain $\mathcal{O}$. Fix $c > 1$ and $k_0 > 0$, and for $n \in \mathbb{Z}$ consider the subsets $\mathcal{O}_n$ of $\mathcal{O}$ given by

$$\mathcal{O}_n := \{x \in \mathcal{O} : c^{-n-k_0} < \rho(x) < c^{-n+k_0}\}.$$

Let $\zeta_n \in C^\infty_0(\mathcal{O}_n)$, $n \in \mathbb{Z}$, be non-negative functions satisfying $\sum_{n\in\mathbb{Z}} \zeta_n(x) = 1$ and $|D^m\zeta_n(x)| \le C \cdot c^{mn}$ for all $n \in \mathbb{Z}$, $m \in \mathbb{N}_0$, $x \in \mathcal{O}$, and a constant $C > 0$ that does not depend on n, m or x. The functions ζ_n can be constructed by mollifying the indicator functions of the sets $\mathcal{O}_n$ (see, e.g., Hörmander [27, Section 1.4]). If $\mathcal{O}_n$ is empty we set $\zeta_n \equiv 0$. For $u \in \mathcal{D}'(\mathcal{O})$, $\zeta_n u$ is a distribution on $\mathcal{O}$ with compact support which can be extended by zero to $\mathbb{R}^d$. This extension is a tempered distribution, i.e. $\zeta_n u \in \mathcal{S}'(\mathbb{R}^d)$.

Definition 2.6. Let ζ_n, $n \in \mathbb{Z}$, be as above and $p \in (1,\infty)$, $\theta, \gamma \in \mathbb{R}$. Then

$$H^\gamma_{p,\theta}(\mathcal{O}) := \Big\{u \in \mathcal{D}'(\mathcal{O}) : \|u\|^p_{H^\gamma_{p,\theta}(\mathcal{O})} := \sum_{n\in\mathbb{Z}} c^{n\theta}\|\zeta_{-n}(c^n\,\cdot)u(c^n\,\cdot)\|^p_{H^\gamma_p(\mathbb{R}^d)} < \infty\Big\}.$$

According to Lototsky [37] this definition is independent of the specific choice of c, k_0 and ζ_n, $n \in \mathbb{Z}$, in the sense that one gets equivalent norms. If $\gamma = m \in \mathbb{N}_0$ then the spaces can be characterized as

$$H^0_{p,\theta}(\mathcal{O}) = L_{p,\theta}(\mathcal{O}) := L_p(\mathcal{O}, \rho(x)^{\theta-d}\,dx),$$
$$H^m_{p,\theta}(\mathcal{O}) = \{u : \rho^{|\alpha|}D^\alpha u \in L_{p,\theta}(\mathcal{O}) \text{ for all } \alpha \in \mathbb{N}^d_0 \text{ with } |\alpha| \le m\},$$

and one has the norm equivalence

(2.10)
$$C^{-1}\|u\|^p_{H^m_{p,\theta}(\mathcal{O})} \le \sum_{\alpha\in\mathbb{N}^d_0,\,|\alpha|\le m} \int_{\mathcal{O}} |\rho(x)^{|\alpha|}D^\alpha u(x)|^p \rho(x)^{\theta-d}\,dx \le C\|u\|^p_{H^m_{p,\theta}(\mathcal{O})}.$$

Analogous notation is used for $\ell_2 = \ell_2(\mathbb{N})$-valued functions $g = (g^\kappa)_{\kappa\in\mathbb{N}}$. For $p \in (1,\infty)$, $\theta, \gamma \in \mathbb{R}$ and ζ_n, $n \in \mathbb{Z}$, as above,

$$H^\gamma_p(\mathbb{R}^d;\ell_2) := \big\{g \in (\mathcal{S}'(\mathbb{R}^d))^{\mathbb{N}} : (1-\Delta)^{\gamma/2} g^\kappa \in L_p(\mathbb{R}^d) \text{ for all } \kappa \in \mathbb{N} \text{ and } \|g\|_{H^\gamma_p(\mathbb{R}^d;\ell_2)} := \big\| |((1-\Delta)^{\gamma/2} g^\kappa)_{\kappa\in\mathbb{N}}|_{\ell_2} \big\|_{L_p(\mathbb{R}^d)} < \infty\big\},$$

$$H^\gamma_{p,\theta}(\mathcal{O};\ell_2) := \Big\{g \in (\mathcal{D}'(\mathcal{O}))^{\mathbb{N}} : \|g\|^p_{H^\gamma_{p,\theta}(\mathcal{O};\ell_2)} := \sum_{n\in\mathbb{Z}} c^{n\theta} \|\zeta_{-n}(c^n\,\cdot) g(c^n\,\cdot)\|^p_{H^\gamma_p(\mathbb{R}^d;\ell_2)} < \infty\Big\}.$$

Remark 2.7. (a) One can consider the spaces $H^\gamma_{p,\theta}(\mathcal{O})$ as generalizations of the classical Sobolev spaces on $\mathcal{O}$ with zero boundary conditions. For $\gamma = m \in \mathbb{N}_0$ we have the identity

$$H^m_{p,d-mp}(\mathcal{O}) = \mathring{W}^m_p(\mathcal{O}),$$

and the norms in both spaces are equivalent (see Theorem 9.7 in Kufner [34]). Here $\mathring{W}^m_p(\mathcal{O})$ is the closure of $C_0^\infty(\mathcal{O})$ in the classical Sobolev space $W^m_p(\mathcal{O})$.

(b) Note that, in contrast to the spaces $W^s_p(\mathcal{O}) = B^s_{p,p}(\mathcal{O})$, $s \in (m, m+1)$, $m \in \mathbb{N}_0$, which can be regarded as *real* interpolation spaces of the classical Sobolev spaces $W^m_p(\mathcal{O})$, $m \in \mathbb{N}_0$ (see, e.g., Triebel [50, Section 1.11.8] and Dispa [21]), the spaces $H^\gamma_{p,\theta}(\mathcal{O})$, $\gamma \in (m, m+1)$, $m \in \mathbb{N}_0$, are *complex* interpolants of the respective integer smoothness spaces (cf. Lototsky [37, Proposition 2.4]).

We can now define spaces of stochastic processes and random functions in terms of the weighted Sobolev spaces introduced above.

Definition 2.8. For $\gamma, \theta \in \mathbb{R}$ and $p \in (1,\infty)$ we set

$$\mathbb{H}^\gamma_{p,\theta}(\mathcal{O},T) := L_p(\Omega_T, \mathcal{P}, \mathbb{P}\otimes\lambda;\, H^\gamma_{p,\theta}(\mathcal{O})),$$
$$\mathbb{H}^\gamma_{p,\theta}(\mathcal{O},T;\ell_2) := L_p(\Omega_T, \mathcal{P}, \mathbb{P}\otimes\lambda;\, H^\gamma_{p,\theta}(\mathcal{O};\ell_2)),$$
$$U^\gamma_{p,\theta}(\mathcal{O}) := L_p(\Omega, \mathcal{F}_0, \mathbb{P};\, H^{\gamma-2/p}_{p,\,\theta+2-p}(\mathcal{O})),$$

and for $p \in [2,\infty)$,

$$\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T) := \Big\{u \in \mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T) : u(0,\cdot) \in U^\gamma_{p,\theta}(\mathcal{O}) \text{ and } du = f\,dt + \sum_{\kappa=1}^\infty g^\kappa\,dw^\kappa_t \text{ for some } f \in \mathbb{H}^{\gamma-2}_{p,\theta+p}(\mathcal{O},T),\ g \in \mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)\Big\},$$

equipped with the norm

$$\|u\|_{\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T)} := \|u\|_{\mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T)} + \|f\|_{\mathbb{H}^{\gamma-2}_{p,\theta+p}(\mathcal{O},T)} + \|g\|_{\mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)} + \|u(0,\cdot)\|_{U^\gamma_{p,\theta}(\mathcal{O})}.$$

The equality $du = f\,dt + \sum_{\kappa=1}^{\infty} g^\kappa\,dw_t^\kappa$ above is shorthand for

$$\langle u(t,\cdot),\varphi\rangle = \langle u(0,\cdot),\varphi\rangle + \int_0^t \langle f(s,\cdot),\varphi\rangle\,ds + \sum_{\kappa=1}^{\infty}\int_0^t \langle g^\kappa(s,\cdot),\varphi\rangle\,dw_s^\kappa \tag{2.11}$$

for all $\varphi \in C_0^\infty(\mathcal{O})$, $t \in [0,T]$.

REMARK 2.9. (a) If $p \in [2,\infty)$, then the sum of stochastic integrals in (2.11) converges in the space $\mathcal{M}_T^{2,c}(\mathbb{R},(\mathcal{F}_t))$ of continuous, square-integrable, $\mathbb{R}$-valued martingales with respect to $(\mathcal{F}_t)_{t\in[0,T]}$. For the convenience of the reader we include a proof in Appendix A.

(b) Using the arguments of Krylov in [32, Remark 3.3], we get the uniqueness (up to indistinguishability) of the pair $(f,g) \in \mathbb{H}_{p,\theta+p}^{\gamma-2}(\mathcal{O},T) \times \mathbb{H}_{p,\theta}^{\gamma-1}(\mathcal{O},T;\ell_2)$ which satisfies (2.11). Consequently, the norm in $\mathfrak{H}_{p,\theta}^{\gamma}(\mathcal{O},T)$ is well defined.

DEFINITION 2.10. We call a predictable $\mathcal{D}'(\mathcal{O})$-valued stochastic process $u = (u(t,\cdot))_{t\in[0,T]}$ a *solution* to equation (1.1) if it is a solution to (2.11) where f is replaced by $\sum_{\mu,\nu=1}^d a^{\mu\nu}u_{x_\mu x_\nu}$ and $u(0,\cdot) = u_0$.

The next result is taken from Kim [30, 31].

THEOREM 2.11. *Let $\gamma \in \mathbb{R}$.*

(i) *For $p \in [2,\infty)$, there exists a constant $\kappa_0 \in (0,1)$, depending only on d, p, $(a^{\mu\nu})_{1\le\mu,\nu\le d}$ and $\mathcal{O}$, such that for any $\theta \in (d+p-2-\kappa_0, d+p-2+\kappa_0)$, $g \in \mathbb{H}_{p,\theta}^{\gamma-1}(\mathcal{O},T;\ell_2)$ and $u_0 \in U_{p,\theta}^{\gamma}(\mathcal{O})$, equation (1.1) has a unique solution u in the class $\mathfrak{H}_{p,\theta}^{\gamma}(\mathcal{O},T)$. For this solution*

$$\|u\|^p_{\mathfrak{H}_{p,\theta}^{\gamma}(\mathcal{O},T)} \le C(\|g\|^p_{\mathbb{H}_{p,\theta}^{\gamma-1}(\mathcal{O},T;\ell_2)} + \|u_0\|^p_{U_{p,\theta}^{\gamma}(\mathcal{O})}), \tag{2.12}$$

where the constant C depends only on d, p, γ, θ, $(a^{\mu\nu})_{1\le\mu,\nu\le d}$, T and $\mathcal{O}$.

(ii) *There exists $p_0 > 2$ such that the following statement holds: if $p \in [2,p_0)$, then there exists a constant $\kappa_1 \in (0,1)$, depending only on d, p, $(a^{\mu\nu})_{1\le\mu,\nu\le d}$ and $\mathcal{O}$, such that for any $\theta \in (d-\kappa_1, d+\kappa_1)$, $g \in \mathbb{H}_{p,\theta}^{\gamma-1}(\mathcal{O},T;\ell_2)$ and $u_0 \in U_{p,\theta}^{\gamma}(\mathcal{O})$, equation (1.1) has a unique solution u in the class $\mathfrak{H}_{p,\theta}^{\gamma}(\mathcal{O},T)$. For this solution, estimate (2.12) holds.*

We will need the following straightforward consequence of Theorem 2.11. Recall that if $m \in \mathbb{N}$ and $f \in \mathcal{D}'(\mathcal{O})$ is sufficiently regular, then $|D^m f|_{\ell_p}$ stands for $(\sum_{|\alpha|=m} |D^\alpha f|^p)^{1/p}$, the (pointwise) ℓ_p-norm of the vector of the mth order derivatives of f.

COROLLARY 2.12. *In the situation of Theorem* 2.11 *with* $\gamma = m \in \mathbb{N}$, *for every* $\tau \in [0,p]$,

$$\int_\Omega \int_0^T \|\rho^{m-\delta}|D^m u(\omega,t,\cdot)|_{\ell_p}\|^\tau_{L_p(\mathcal{O})}\,dt\,\mathbb{P}(d\omega) \leq C(\|g\|_{\mathbb{H}^{m-1}_{p,\theta}(\mathcal{O},T;\ell_2)} + \|u_0\|_{U^m_{p,\theta}(\mathcal{O})})^\tau,$$

where $\delta = 1 + (d-\theta)/p$.

Proof. Theorem 2.11 implies, in particular, that

$$\|u\|_{\mathbb{H}^m_{p,\theta-p}(\mathcal{O},T)} \leq C(\|g\|_{\mathbb{H}^{m-1}_{p,\theta}(\mathcal{O},T;\ell_2)} + \|u_0\|_{U^m_{p,\theta}(\mathcal{O})}),$$

and we have

$$\begin{aligned}\|u\|^p_{\mathbb{H}^m_{p,\theta-p}(\mathcal{O},T)} &= \int_\Omega \int_0^T \|u(\omega,t,\cdot)\|^p_{H^m_{p,\theta-p}(\mathcal{O},T)}\,dt\,\mathbb{P}(d\omega)\\ &\geq C\int_\Omega \int_0^T \sum_{k=0}^m \|\rho^{k+(\theta-p-d)/p}|D^k u(\omega,t,\cdot)|_{\ell_p}\|^p_{L_p(\mathcal{O})}\,dt\,\mathbb{P}(d\omega)\\ &\geq C\int_\Omega \int_0^T \|\rho^{m-\delta}|D^m u(\omega,t,\cdot)|_{\ell_p}\|^p_{L_p(\mathcal{O})}\,dt\,\mathbb{P}(d\omega)\end{aligned}$$

with $\delta = 1+(d-\theta)/p \in ((2-\kappa_0)/p, (p+\kappa_0)/p)$. Now let $\tau \in [0,p]$. Jensen's inequality for concave functions (see, e.g., Schilling [47, Theorem 12.14]) yields

$$\begin{aligned}\int_\Omega \int_0^T \|\rho^{m-\delta}|D^m u(\omega,t,\cdot)|_{\ell_p}\|^\tau_{L_p(\mathcal{O})}\,dt\,\mathbb{P}(d\omega) &\leq C(T)(\|g\|^p_{\mathbb{H}^{m-1}_{p,\theta}(\mathcal{O},T;\ell_2)} + \|u_0\|^p_{U^m_{p,\theta}(\mathcal{O})})^{\tau/p}\\ &\leq C(\|g\|_{\mathbb{H}^{m-1}_{p,\theta}(\mathcal{O},T;\ell_2)} + \|u_0\|_{U^m_{p,\theta}(\mathcal{O})})^\tau.\end{aligned}$$

In the last step we have used the fact that all norms on $\mathbb{R}^2$ are equivalent. ∎

REMARK 2.13. Consider the Hilbert space case $p = 2$ and assume $g \in \mathbb{H}^\gamma_{2,\theta}(\mathcal{O},T;\ell_2)$. The expression $\sum_{\kappa=1}^\infty \int_0^t g^\kappa(s,\cdot)\,dw^\kappa_s$ can be considered as an $H^\gamma_{2,\theta}(\mathcal{O})$-valued stochastic integral $\int_0^t G(s)\,dW_s$ with respect to a cylindrical Wiener process $(W_t)_{t\in[0,T]}$ on ℓ_2 whose coordinate processes are $(w^\kappa_t)_{t\in[0,T]}$, $\kappa \in \mathbb{N}$. (See, e.g., Da Prato and Zabczyk [17] or Peszat and Zabczyk [44] for stochastic integration with respect to cylindrical processes.) Here $(G(t))_{t\in[0,T]}$ is a stochastic process in the space $L_{(\mathrm{HS})}(\ell_2, H^\gamma_{2,\theta}(\mathcal{O}))$ of Hilbert–

Schmidt operators defined by

$$G(\omega,t):\ell_2\to H^\gamma_{2,\theta}(\mathcal{O}),\qquad (x^\kappa)_{\kappa\in\mathbb{N}}\mapsto\sum_{\kappa\in\mathbb{N}}g^\kappa(\omega,t,\cdot)x^\kappa,\quad (\omega,t)\in\Omega_T,$$

and it is an element of the space $L_2(\Omega_T;L_{(\mathrm{HS})}(\ell_2,H^\gamma_{2,\theta}(\mathcal{O})))$. Indeed, for fixed $(\omega,t)\in\Omega_T$ we have

$$\begin{aligned}\|G(\omega,t)\|^2_{L_{(\mathrm{HS})}(\ell_2,H^\gamma_{2,\theta}(\mathcal{O}))}&=\sum_{\kappa\in\mathbb{N}}\|g^\kappa(\omega,t,\cdot)\|^2_{H^\gamma_{2,\theta}(\mathcal{O})}\\&=\sum_{\kappa\in\mathbb{N}}\sum_{n\in\mathbb{Z}}c^{n\theta}\|\zeta_{-n}(c^n\,\cdot)g^\kappa(\omega,t,c^n\,\cdot)\|^2_{H^\gamma_2(\mathbb{R}^d)}\\&=\sum_{n\in\mathbb{Z}}c^{n\theta}\|\zeta_{-n}(c^n\,\cdot)g(\omega,t,c^n\,\cdot)\|^2_{H^\gamma_2(\mathbb{R}^d;\ell_2)}\end{aligned}$$

by Tonelli's theorem, so that

$$\|G\|_{L_2(\Omega_T;L_{(\mathrm{HS})}(\ell_2;H^\gamma_{2,\theta}(\mathcal{O})))}=\|g\|_{\mathbb{H}^\gamma_{2,\theta}(\mathcal{O},T;\ell_2)}.$$

As a consequence, (1.1) can be rewritten in the form

$$du=\sum_{\mu,\nu=1}^d a^{\mu\nu}u_{x_\mu x_\nu}\,dt+dM_t,\quad u(0,\cdot)=u_0,\tag{2.13}$$

where $(M_t)_{t\in[0,T]}\in\mathcal{M}^{2,c}_T(H^\gamma_{2,\theta}(\mathcal{O}),(\mathcal{F}_t))$ is the $H^\gamma_{2,\theta}(\mathcal{O})$-valued, square-integrable martingale given by

$$M_t:=\int_0^t G(s)\,dW_s,\quad t\in[0,T].$$

Remark 2.14. In Examples 3.3–3.5 below the solution u of (1.1) in $\mathfrak{H}^\gamma_{2,\theta}(\mathcal{O},T)$ as given by Theorem 2.11 coincides with the weak solution of (2.13) with zero Dirichlet boundary condition in the sense of Da Prato and Zabczyk [17].

In these examples we consider equation (2.13) driven by certain Wiener processes $(M_t)_{t\in[0,T]}$ in $L_2(\mathcal{O})$ with $u_0\in U^2_{2,2}(\mathcal{O})$, $d=2$ and the solution u is in the class $\mathfrak{H}^2_{2,2}(\mathcal{O},T)\subset\mathbb{H}^2_{2,0}(\mathcal{O},T)$. (Strictly speaking, in Example 3.5 $(M_t)_{t\in[0,T]}$ is not a Wiener process, but it is one conditioned on the family of random variables Y_λ, $\lambda\in\nabla$.) Thus, by Remark 2.7(a) we know that u is an element of $L_2(\Omega_T;\mathring{W}^1_2(\mathcal{O}))$. Let us now introduce the operator

$$(A,D(A)):=\Big(\sum_{\mu,\nu=1}^d a^{\mu\nu}\frac{\partial^2}{\partial x_\mu\partial x_\nu},\Big\{u\in\mathring{W}^1_2(\mathcal{O}):\sum_{\mu,\nu=1}^d a^{\mu\nu}u_{x_\mu x_\nu}\in L_2(\mathcal{O})\Big\}\Big)$$

and consider the equation

$$du=Au\,dt+dM_t,\quad u(0,\cdot)=u_0,\qquad t\in[0,T].\tag{2.14}$$

A weak solution of (2.14) in the sense of Da Prato and Zabczyk [17] is an $L_2(\mathcal{O})$-valued predictable process $u = (u(t,\cdot))_{t\in[0,T]}$ with $\mathbb{P}$-almost surely Bochner integrable trajectories $t \mapsto u(\omega,t,\cdot)$ satisfying

$$\langle u(t,\cdot),\zeta\rangle_{L_2(\mathcal{O})} = \langle u_0,\zeta\rangle_{L_2(\mathcal{O})} + \int_0^t \langle u(s,\cdot),A^*\zeta\rangle_{L_2(\mathcal{O})}\,ds + \langle M_t,\zeta\rangle_{L_2(\mathcal{O})} \tag{2.15}$$

for all $t \in [0,T]$ and $\zeta \in D(A^*)$. It is given by the variation of constants formula

$$u(t,\cdot) = e^{tA}u_0 + \int_0^t e^{(t-s)A}\,dM_s, \quad t \in [0,T],$$

where $(e^{tA})_{t\geq 0}$ is the contraction semigroup on $L_2(\mathcal{O})$ generated by A.

It is clear that the solution $u \in \mathfrak{H}^2_{2,2}(\mathcal{O},T)$ given by Theorem 2.11 satisfies

$$\begin{aligned}\langle u(t,\cdot),\varphi\rangle_{L_2(\mathcal{O})} &\\ = \langle u_0,\varphi\rangle_{L_2(\mathcal{O})} + \int_0^t \langle A^{1/2}u(s,\cdot),A^{1/2}\varphi\rangle_{L_2(\mathcal{O})}\,ds &+ \langle M_t,\varphi\rangle_{L_2(\mathcal{O})}\end{aligned} \tag{2.16}$$

for all $t \in [0,T]$ and $\varphi \in C_0^\infty(\mathcal{O})$. Note that the operator A is self-adjoint because the coefficients $a^{\mu\nu}$, $1 \leq \mu,\nu \leq d$, are constants. Since every $\zeta \in D(A^*) = D(A) \subset \mathring{W}^1_2(\mathcal{O})$ is the limit in $W^1_2(\mathcal{O})$ of a sequence $(\varphi_k)_{k\in\mathbb{N}} \subset C_0^\infty(\mathcal{O})$, one can let $k \to \infty$ for $\varphi = \varphi_k$ in (2.16) to obtain (2.15).

3. Besov regularity for SPDEs. In this section we state and prove our main result. We give some concrete examples to illustrate its applicability. The result is formulated in terms of the L_τ-spaces

$$L_\tau(\Omega_T;B^s_{\tau,\tau}(\mathcal{O})) = L_\tau(\Omega_T,\mathcal{P},\mathbb{P}\otimes\lambda;B^s_{\tau,\tau}(\mathcal{O})), \quad \tau \in (0,\infty),\ s \in (0,\infty),$$

and the spaces introduced in the preceding section.

THEOREM 3.1. *Let $g \in \mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)$ and $u_0 \in U^\gamma_{p,\theta}(\mathcal{O})$ for some $\gamma \in \mathbb{N}$, where θ and p satisfy one of the following conditions:*

(i) $p \in [2,\infty)$ *and* $\theta \in (d+p-2-\kappa_0, d+p-2+\kappa_0)$,

(ii) $p \in [2,p_0)$ *and* $\theta \in (d-\kappa_1, d+\kappa_1)$,

with κ_0, κ_1 and p_0 from Theorem 2.11. Let u be the unique solution in the class $\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T)$ of equation (1.1) and assume furthermore that

$$u \in L_p(\Omega_T;\,B^s_{p,p}(\mathcal{O})) \quad \text{for some } s \in \left(0,\gamma\wedge\left(1+\frac{d-\theta}{p}\right)\right]. \tag{3.1}$$

Then

$$u \in L_\tau(\Omega_T;B^\alpha_{\tau,\tau}(\mathcal{O})), \quad \frac{1}{\tau} = \frac{\alpha}{d}+\frac{1}{p}, \quad \text{for all } \alpha \in \left(0,\gamma\wedge\frac{sd}{d-1}\right),$$

and

$$\|u\|_{L_\tau(\Omega_T;B^\alpha_{\tau,\tau}(\mathcal{O}))} \le C(\|g\|_{\mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)} + \|u_0\|_{U^\gamma_{p,\theta}(\mathcal{O})} + \|u\|_{L_p(\Omega_T;B^s_{p,p}(\mathcal{O}))}). \tag{3.2}$$

The constant C depends only on d, p, γ, α, s, θ, $(a^{\mu\nu})_{1\le\mu,\nu\le d}$, T and $\mathcal{O}$.

REMARK 3.2. Since the constant $\kappa_1 = \kappa_1(d,p,(a^{\mu\nu}),\mathcal{O})$ is greater than zero, we can always choose $\theta = d$, provided $p \in [2,p_0)$ with $p_0 > 2$ from Theorem 2.11. In this case, we know that for each $\gamma \in \mathbb{N}$ we have a unique solution u in the class $\mathfrak{H}^\gamma_{p,d}(\mathcal{O},T)$ if the free term g and the initial condition u_0 are sufficiently regular. In particular, we get

$$u \in \mathbb{H}^\gamma_{p,d-p}(\mathcal{O},T) = L_p(\Omega_T;H^\gamma_{p,d-p}(\mathcal{O})) \subseteq L_p(\Omega_T;\mathring{W}^1_p(\mathcal{O})) \subseteq L_p(\Omega_T;B^1_{p,p}(\mathcal{O})).$$

Thus, the additional requirement (3.1) holds with $s = 1$. Since $\mathcal{O}$ is an arbitrary bounded Lipschitz domain, the regularity of the solution u in the scale $L_p(\Omega_T;W^s_p(\mathcal{O}))$, $s > 0$, is in general limited by some $s^* < 2$, i.e. there exists $s^* < 2$, such that $u \notin L_p(\Omega_T;W^s_p(\mathcal{O}))$ for all $s \ge s^*$ (see [36]).

However, if $\gamma \ge 2$ our result shows that we obtain higher regularity than $s = 1$ in the nonlinear approximation scale, namely

$$u \in L_\tau(\Omega_T;B^\alpha_{\tau,\tau}(\mathcal{O})), \quad \frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}, \quad \text{for all } \alpha < \frac{d}{d-1}.$$

Proof of Theorem 3.1. We fix α and τ as stated in the theorem and choose a wavelet Riesz basis

$$\{\varphi_k, \psi_{i,j,k} : (i,j,k) \in \{1,\dots,2^d-1\} \times \mathbb{N}_0 \times \mathbb{Z}^d\}$$

of $L_2(\mathbb{R}^d)$ which satisfies the assumptions from Section 2.2 with $r > \gamma$. Given $(j,k) \in \mathbb{N}_0 \times \mathbb{Z}^d$ let

$$Q_{j,k} := 2^{-j}k + 2^{-j}[-N,N]^d,$$

so that $\operatorname{supp}\psi_{i,j,k} \subset Q_{j,k}$ for all $i \in \{1,\dots,2^d-1\}$ and $\operatorname{supp}\varphi_k \subset Q_{0,k}$ for all $k \in \mathbb{Z}^d$. Remember that the supports of the corresponding dual basis meet the same requirements. For our purpose the set of all indices associated with those wavelets that may have common support with the domain $\mathcal{O}$ will play an important role and we denote them by

$$\Lambda := \{(i,j,k) \in \{1,\dots,2^d-1\} \times \mathbb{N}_0 \times \mathbb{Z}^d : Q_{j,k} \cap \mathcal{O} \ne \emptyset\}.$$

In particular, we will also use the following notation:

$$\Gamma := \{k \in \mathbb{Z}^d : Q_{0,k} \cap \mathcal{O} \ne \emptyset\}.$$

Due to the assumption $u \in L_p(\Omega_T;B^s_{p,p}(\mathcal{O}))$ we have $u(\omega,t,\cdot) \in B^s_{p,p}(\mathcal{O})$ for $\mathbb{P}\otimes\lambda$-almost every $(\omega,t) \in \Omega_T$. As $\mathcal{O}$ is a Lipschitz domain there exists a

bounded linear extension operator $\mathcal{E} : B^s_{p,p}(\mathcal{O}) \to B^s_{p,p}(\mathbb{R}^d)$, i.e. there exists a constant $C > 0$ such that for $\mathbb{P} \otimes \lambda$-almost every $(\omega, t) \in \Omega_T$,

$$\mathcal{E}u(\omega,t,\cdot)|_{\mathcal{O}} = u(\omega,t,\cdot) \quad \text{and} \quad \|\mathcal{E}u(\omega,t,\cdot)\|_{B^s_{p,p}(\mathbb{R}^d)} \leq C\|u(\omega,t,\cdot)\|_{B^s_{p,p}(\mathcal{O})}$$

(see, e.g., Rychkov [46]). In the following we will omit the $\mathcal{E}$ in our notation and write u instead of $\mathcal{E}u$.

Theorem 2.3 tells us that for almost all $(\omega, t) \in \Omega_T$ the following equality holds on the domain $\mathcal{O}$:

$$u(\omega,t,\cdot) = \sum_{k\in\Gamma} \langle u(\omega,t,\cdot), \widetilde{\varphi}_k\rangle \varphi_k + \sum_{(i,j,k)\in\Lambda} \langle u(\omega,t,\cdot), \widetilde{\psi}_{i,j,k,p'}\rangle \psi_{i,j,k,p},$$

where the sums converge unconditionally in $B^s_{p,p}(\mathbb{R}^d)$. Furthermore (cf. Corollary 2.5), for $\mathbb{P} \otimes \lambda$-almost all $(\omega, t) \in \Omega_T$,

$$\begin{aligned} (3.3) \qquad &\|u(\omega,t,\cdot)\|^\tau_{B^\alpha_{\tau,\tau}(\mathcal{O})} \\ &\qquad \leq C\Big[\sum_{k\in\Gamma} |\langle u(\omega,t,\cdot), \widetilde{\varphi}_k\rangle|^\tau + \sum_{(i,j,k)\in\Lambda} |\langle u(\omega,t,\cdot), \widetilde{\psi}_{i,j,k,p'}\rangle|^\tau\Big]. \end{aligned}$$

Hence, it is enough to prove that

$$(3.4) \qquad \int_\Omega \int_0^T \sum_{k\in\Gamma} |\langle u(\omega,t,\cdot), \widetilde{\varphi}_k\rangle|^\tau \, dt\, \mathbb{P}(d\omega) \leq C\|u\|^\tau_{L_p(\Omega_T; B^s_{p,p}(\mathcal{O}))}$$

and

$$\begin{aligned} (3.5) \qquad &\int_\Omega \int_0^T \sum_{(i,j,k)\in\Lambda} |\langle u(\omega,t,\cdot), \widetilde{\psi}_{i,j,k,p'}\rangle|^\tau \, dt\, \mathbb{P}(d\omega) \\ &\qquad \leq C(\|g\|_{\mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)} + \|u_0\|_{U^\gamma_{p,\theta}(\mathcal{O})} + \|u\|_{L_p(\Omega_T; B^s_{p,p}(\mathcal{O}))})^\tau. \end{aligned}$$

We start with (3.4). The index set Γ introduced above is finite because of the boundedness of $\mathcal{O}$, so that we can use Jensen's inequality to get, for $\mathbb{P} \otimes \lambda$-almost all $(\omega, t) \in \Omega_T$,

$$\begin{aligned} \sum_{k\in\Gamma} |\langle u(\omega,t,\cdot), \widetilde{\varphi}_k\rangle|^\tau &\leq C\Big[\Big(\sum_{k\in\Gamma} |\langle u(\omega,t,\cdot), \widetilde{\varphi}_k\rangle|^p\Big)^{1/p}\Big]^\tau \\ &\leq C\|u(\omega,t,\cdot)\|^\tau_{B^s_{p,p}(\mathcal{O})}. \end{aligned}$$

In the last step we used Theorem 2.3 and the boundedness of the extension operator. Integration with respect to $\mathbb{P} \otimes \lambda$ and another application of Jensen's inequality yield (3.4).

To prove (3.5), we introduce the following notation:

$$\begin{aligned}
\rho_{j,k} &:= \operatorname{dist}(Q_{j,k}, \partial\mathcal{O}) = \inf_{x\in Q_{j,k}} \rho(x),\\
\Lambda_j &:= \{(i,l,k) \in \Lambda : l = j\},\\
\Lambda_{j,m} &:= \{(i,j,k) \in \Lambda_j : m2^{-j} \le \rho_{j,k} < (m+1)2^{-j}\},\\
\Lambda_j^0 &:= \Lambda_j \setminus \Lambda_{j,0}, \qquad \Lambda^0 := \bigcup_{j\in\mathbb{N}_0} \Lambda_j^0,
\end{aligned}$$

where $j, m \in \mathbb{N}_0$ and $k \in \mathbb{Z}^d$. We split the left hand side of (3.5) into

$$\begin{aligned}
(3.6)\qquad &\int_\Omega \int_0^T \sum_{(i,j,k)\in\Lambda^0} |\langle u(\omega,t,\cdot), \widetilde{\psi}_{i,j,k,p'}\rangle|^\tau \, dt\, \mathbb{P}(d\omega)\\
&\qquad + \int_\Omega \int_0^T \sum_{(i,j,k)\in\Lambda\setminus\Lambda^0} |\langle u(\omega,t,\cdot), \widetilde{\psi}_{i,j,k,p'}\rangle|^\tau \, dt\, \mathbb{P}(d\omega) =: I + II
\end{aligned}$$

and estimate each term separately.

Let us begin with I. Fix $(i,j,k) \in \Lambda^0$ and $(\omega,t) \in \Omega_T$ such that

$$\int_{\mathcal{O}} \big|\rho(x)^{\gamma-s} |D^\gamma u(\omega,t,x)|_{\ell_p}\big|^p \, dx < \infty.$$

By Corollary 2.12 this holds for $\mathbb{P}\otimes\lambda$-almost all $(\omega,t) \in \Omega_T$. By a Whitney-type inequality, also known as the Deny–Lions lemma (see, e.g., DeVore and Sharpley [20, Theorem 3.4]), there exists a polynomial $P_{j,k}$ of total degree less than γ such that

$$\|u(\omega,t,\cdot) - P_{j,k}\|_{L_p(Q_{j,k})} \le C2^{-j\gamma} |u(\omega,t,\cdot)|_{W_p^\gamma(Q_{j,k})},$$

where the last norm is finite since $\rho_{j,k} = \operatorname{dist}(Q_{j,k}, \partial\mathcal{O}) > 0$. Since $\widetilde{\psi}_{i,j,k,p'}$ is orthogonal to every polynomial of total degree less than γ, one gets

$$\begin{aligned}
|\langle u(\omega,t,\cdot), \widetilde{\psi}_{i,j,k,p'}\rangle| &= |\langle u(\omega,t,\cdot) - P_{j,k}, \widetilde{\psi}_{i,j,k,p'}\rangle|\\
&\le \|u(\omega,t,\cdot) - P_{j,k}\|_{L_p(Q_{j,k})} \, \|\widetilde{\psi}_{i,j,k,p'}\|_{L_{p'}(Q_{j,k})}\\
&\le C2^{-j\gamma} |u(\omega,t,\cdot)|_{W_p^\gamma(Q_{j,k})}\\
&\le C2^{-j\gamma} \rho_{j,k}^{s-\gamma} \Big(\int_{Q_{j,k}} \big|\rho(x)^{\gamma-s} \, |D^\gamma u(\omega,t,x)|_{\ell_p}\big|^p \, dx \Big)^{1/p}\\
&=: C2^{-j\gamma} \rho_{j,k}^{s-\gamma} \mu_{j,k}(\omega,t).
\end{aligned}$$

Fix $j \in \mathbb{N}_0$. Summing over all indices $(i,j,k) \in \Lambda_j^0$ and applying Hölder's inequality with exponents $p/\tau > 1$ and $p/(p-\tau)$ one finds

$$
\begin{aligned}
(3.7)\qquad &\sum_{(i,j,k)\in\Lambda_j^0} |\langle u(\omega,t,\cdot),\widetilde{\psi}_{i,j,k,p'}\rangle|^\tau \\
&\qquad \leq C \sum_{(i,j,k)\in\Lambda_j^0} 2^{-j\gamma\tau}\rho_{j,k}^{(s-\gamma)\tau}\mu_{j,k}(\omega,t)^\tau \\
&\qquad \leq C\Big(\sum_{(i,j,k)\in\Lambda_j^0}\mu_{j,k}(\omega,t)^p\Big)^{\tau/p}\Big(\sum_{(i,j,k)\in\Lambda_j^0} 2^{\frac{-pj\gamma\tau}{p-\tau}}\rho_{j,k}^{\frac{(s-\gamma)p\tau}{p-\tau}}\Big)^{\frac{p-\tau}{p}}.
\end{aligned}
$$

Since any $x\in\mathcal{O}$ lies outside of all but at most a constant number $C>0$ of the cubes $Q_{j,k}$, $k\in\mathbb{Z}^d$, we get the following bound for the first factor on the right hand side:

$$
\begin{aligned}
(3.8)\qquad &\Big(\sum_{(i,j,k)\in\Lambda_j^0}\mu_{j,k}(\omega,t)^p\Big)^{\tau/p} \\
&\qquad = \Big(\sum_{(i,j,k)\in\Lambda_j^0}\int_{Q_{j,k}} \big|\rho(x)^{\gamma-s}\,|D^\gamma u(\omega,t,x)|_{\ell_p}\big|^p\,dx\Big)^{\tau/p} \\
&\qquad \leq C\|\rho^{\gamma-s}|D^\gamma u(\omega,t,\cdot)|_{\ell_p}\|^\tau_{L_p(\mathcal{O})}.
\end{aligned}
$$

In order to estimate the second factor in (3.7) we use the Lipschitz character of the domain $\mathcal{O}$, which implies that

$$
(3.9)\qquad |\Lambda_{j,m}| \leq C2^{j(d-1)} \quad \text{for all } j,m\in\mathbb{N}_0.
$$

The constant $C>0$ does not depend on j or m. Moreover, the boundedness of $\mathcal{O}$ yields $\Lambda_{j,m}=\emptyset$ for all $j,m\in\mathbb{N}_0$ with $m\geq C2^j$. Consequently,

$$
\begin{aligned}
(3.10)\qquad &\Big(\sum_{(i,j,k)\in\Lambda_j^0} 2^{\frac{-pj\gamma\tau}{p-\tau}}\rho_{j,k}^{\frac{(s-\gamma)p\tau}{p-\tau}}\Big)^{p-\tau/p} \\
&\qquad \leq \Big(\sum_{m=1}^{C2^j}\sum_{(i,j,k)\in\Lambda_{j,m}} 2^{\frac{-pj\gamma\tau}{p-\tau}}\rho_{j,k}^{\frac{(s-\gamma)p\tau}{p-\tau}}\Big)^{\frac{p-\tau}{p}} \\
&\qquad \leq C\Big(\sum_{m=1}^{C2^j} 2^{j(d-1)}\,2^{-j\frac{p\gamma\tau}{p-\tau}}(m\,2^{-j})^{\frac{(s-\gamma)p\tau}{p-\tau}}\Big)^{\frac{p-\tau}{p}} \\
&\qquad \leq C\big(2^{j(d-1-\frac{sp\tau}{p-\tau})}+2^{j(d-\frac{\gamma p\tau}{p-\tau})}\big)^{\frac{p-\tau}{p}}.
\end{aligned}
$$

Now, let us sum over all $j\in\mathbb{N}_0$ and integrate over Ω_T with respect to $\mathbb{P}\otimes\lambda$ on both sides of (3.7). By using (3.10), (3.8) and Corollary 2.12 we get

$$\int_\Omega \int_0^T \sum_{(i,j,k)\in\Lambda^0} |\langle u(\omega,t,\cdot),\widetilde{\psi}_{i,j,k,p'}\rangle|^\tau \, dt\,\mathbb{P}(d\omega)$$
$$\le C \sum_{j\in\mathbb{N}_0} (2^{j(d-1-\frac{sp\tau}{p-\tau})} + 2^{j(d-\frac{\gamma p\tau}{p-\tau})})^{\frac{p-\tau}{p}}$$
$$\cdot \int_\Omega \int_0^T \|\rho^{\gamma-s}\,|D^\gamma u(\omega,t,\cdot)|_{\ell_p}\|^\tau_{L_p(\mathcal{O})}\, dt\,\mathbb{P}(d\omega)$$
$$\le C\Big(\sum_{j\in\mathbb{N}_0} 2^{j(d-1-\frac{sp\tau}{p-\tau})\frac{p-\tau}{p}} + \sum_{j\in\mathbb{N}_0} 2^{j(d-\frac{\gamma p\tau}{p-\tau})\frac{p-\tau}{p}}\Big)$$
$$\cdot (\|g\|_{\mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)} + \|u_0\|_{U^\gamma_{p,\theta}(\mathcal{O})})^\tau.$$

One can see that the sums on the right hand side converge if, and only if, $\alpha \in (0, \gamma \wedge s\frac{d}{d-1})$. Finally,

$$\int_\Omega \int_0^T \sum_{(i,j,k)\in\Lambda^0} |\langle u(\omega,t,\cdot),\widetilde{\psi}_{i,j,k,p'}\rangle|^\tau \, dt\,\mathbb{P}(d\omega)$$
$$\le C(\|g\|_{\mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)} + \|u_0\|_{U^\gamma_{p,\theta}(\mathcal{O})})^\tau. \tag{3.11}$$

Now we estimate the term II in (3.6). First we fix $j \in \mathbb{N}_0$ and use Hölder's inequality and (3.9) to get

$$\sum_{(i,j,k)\in\Lambda_{j,0}} \big|\langle u(\omega,t,\cdot),\widetilde{\psi}_{i,j,k,p'}\rangle\big|^\tau$$
$$\le C\, 2^{j(d-1)\frac{p-\tau}{p}} \Big[\sum_{(i,j,k)\in\Lambda_{j,0}} |\langle u(\omega,t,\cdot),\widetilde{\psi}_{i,j,k,p'}\rangle|^p\Big]^{\tau/p}.$$

Summing over all $j \in \mathbb{N}_0$ and using Hölder's inequality again yields

$$\sum_{(i,j,k)\in\Lambda\setminus\Lambda^0} |\langle u(\omega,t,\cdot),\widetilde{\psi}_{i,j,k,p'}\rangle|^\tau$$
$$= \sum_{j\in\mathbb{N}_0} \Big[\sum_{(i,j,k)\in\Lambda_{j,0}} |\langle u(\omega,t,\cdot),\widetilde{\psi}_{i,j,k,p'}\rangle|^\tau\Big]$$
$$\le C \sum_{j\in\mathbb{N}_0} \Big[2^{j(d-1)\frac{p-\tau}{p}}\Big(\sum_{(i,j,k)\in\Lambda_{j,0}} |\langle u(\omega,t,\cdot),\widetilde{\psi}_{i,j,k,p'}\rangle|^p\Big)^{\tau/p}\Big]$$
$$\le C\Big[\sum_{j\in\mathbb{N}_0} 2^{j(\frac{(d-1)(p-\tau)}{p}-\tau p)\frac{p}{p-\tau}}\Big]^{\frac{p-\tau}{p}}$$
$$\cdot \Big[\sum_{j\in\mathbb{N}_0}\sum_{(i,j,k)\in\Lambda_{j,0}} 2^{jsp}|\langle u(\omega,t,\cdot),\widetilde{\psi}_{i,j,k,p'}\rangle|^p\Big]^{\tau/p}.$$

Using Theorem 2.3 and the boundedness of the extension operator, for $\mathbb{P}\otimes\lambda$-almost every $(\omega,t)\in\Omega_T$ one gets

$$\sum_{(i,j,k)\in\Lambda\setminus\Lambda^0}|\langle u(\omega,t,\cdot),\widetilde{\psi}_{i,j,k,p'}\rangle|^\tau \le C\|u(\omega,t,\cdot)\|^\tau_{B^s_{p,p}(\mathcal{O})}\Big(\sum_{j\in\mathbb{N}_0}2^{j(\frac{(d-1)(p-\tau)}{p}-\tau p)\frac{p}{p-\tau}}\Big)^{\frac{p-\tau}{p}}.$$

The series on the right hand side converges if and only if $\alpha\in(0,sd/(d-1))$. But this is part of our assumptions, so that for $\mathbb{P}\otimes\lambda$-almost every $(\omega,t)\in\Omega_T$,

$$\sum_{(i,j,k)\in\Lambda\setminus\Lambda^0}|\langle u(\omega,t,\cdot),\widetilde{\psi}_{i,j,k,p'}\rangle|^\tau \le C\|u(\omega,t,\cdot)\|^\tau_{B^s_{p,p}(\mathcal{O})}.$$

Let us integrate over Ω_T with respect to $\mathbb{P}\otimes\lambda$ and use Jensen's inequality to get

$$\begin{aligned}\int_\Omega\int_0^T\sum_{(i,j,k)\in\Lambda\setminus\Lambda^0}|\langle u(\omega,t,\cdot),\widetilde{\psi}_{i,j,k,p'}\rangle|^\tau\,dt\,\mathbb{P}(d\omega)&\le C\int_\Omega\int_0^T\|u(\omega,t,\cdot)\|^\tau_{B^s_{p,p}(\mathcal{O})}\,dt\,\mathbb{P}(d\omega)\\&\le C\Big(\int_\Omega\int_0^T\|u(\omega,t,\cdot)\|^p_{B^s_{p,p}(\mathcal{O})}\,dt\,\mathbb{P}(d\omega)\Big)^{\tau/p}.\end{aligned}$$

Because of (3.11) this proves (3.5). Now (3.4) and (3.3) finish the proof. ∎

Next, we give some examples for an application of Theorem 3.1. We are mainly interested in the Hilbert space case $p=2$ since it provides a natural setting for numerical discretization techniques like adaptive wavelet schemes.

Example 3.3. Let us first consider equation (1.1) in the form (2.13) where the driving process $(M_t)_{t\in[0,T]}$ is a Wiener process in $\mathring{W}^1_2(\mathcal{O})$ with covariance operator $Q\in L_{(\mathrm{nuc})}(\mathring{W}^1_2(\mathcal{O}))$. It can be represented as a stochastic integral process $(\int_0^t G(s)\,dW_s)_{t\in[0,T]}$ with respect to the cylindrical Wiener process $(W_t)_{t\in[0,T]}$ on ℓ_2 by defining the integrand process $(G(t))_{t\in[0,T]}$ in the space $L_{(\mathrm{HS})}(\ell_2,\mathring{W}^1_2(\mathcal{O}))$ of Hilbert–Schmidt operators as the constant deterministic process

$$G(\omega,t):\ell_2\to\mathring{W}^1_2(\mathcal{O}),\quad (x^\kappa)_{\kappa\in\mathbb{N}}\mapsto\sum_{\kappa\in\mathbb{N}}\sqrt{\lambda_\kappa}x^\kappa e_\kappa,\quad(\omega,t)\in\Omega_T,\tag{3.12}$$

where $(e_\kappa)_{\kappa\in\mathbb{N}}$ is an orthonormal basis of $\mathring{W}_2^1(\mathcal{O})$ consisting of eigenvectors of Q with positive eigenvalues $(\lambda_\kappa)_{\kappa\in\mathbb{N}}$.

This corresponds to defining $g = (g^\kappa)_{\kappa\in\mathbb{N}}$ in (1.1) by

$$g^\kappa(\omega,t,\cdot) := \sqrt{\lambda_\kappa} e_\kappa, \quad \kappa\in\mathbb{N},\ (\omega,t)\in\Omega_T. \tag{3.13}$$

It is easy to see that g is an element of $\mathbb{H}^1_{2,d}(\mathcal{O},T;\ell_2)$. By definition

$$\begin{aligned}\|g\|^2_{\mathbb{H}^1_{2,d}(\mathcal{O},T;\ell_2)} &= T^2\sum_{n\in\mathbb{Z}} c^{nd}\|\zeta_{-n}(c^n\cdot)(\sqrt{\lambda_\kappa}e_\kappa(c^n\cdot))_{\kappa\in\mathbb{N}}\|^2_{H^1_2(\mathbb{R}^d;\ell_2)}\\ &= T^2\sum_{\kappa\in\mathbb{N}}\lambda_\kappa\sum_{n\in\mathbb{Z}} c^{nd}\|\zeta_{-n}(c^n\cdot)e_\kappa(c^n\cdot)\|^2_{H^1_2(\mathbb{R}^d)}\\ &= T^2\sum_{\kappa\in\mathbb{N}}\lambda_\kappa\|e_\kappa\|^2_{H^1_{2,d}(\mathcal{O})}.\end{aligned} \tag{3.14}$$

Using the norm equivalence (2.10), one has

$$\begin{aligned}\|g\|^2_{\mathbb{H}^1_{2,d}(\mathcal{O},T;\ell_2)} &\le CT^2\sum_{\kappa\in\mathbb{N}}\lambda_\kappa\sum_{|\alpha|\le 1}\|\rho^{|\alpha|}D^\alpha e_\kappa\|^2_{L_2(\mathcal{O})}\\ &\le CT^2\sum_{\kappa\in\mathbb{N}}\lambda_\kappa\sum_{|\alpha|\le 1}\|D^\alpha e_\kappa\|^2_{L_2(\mathcal{O})} = CT^2\sum_{\kappa\in\mathbb{N}}\lambda_\kappa < \infty.\end{aligned}$$

Thus, in a 2-dimensional setting, Theorem 2.11 with $d=\theta=\gamma=2$ tells us that for every initial condition $u_0\in U^2_{2,2}(\mathcal{O}) = L_2(\Omega,\mathcal{F}_0,\mathbb{P};H^1_{2,2}(\mathcal{O}))$ equation (1.1) has a unique solution u in the class $\mathfrak{H}^2_{2,2}(\mathcal{O},T)\subset\mathbb{H}^2_{2,0}(\mathcal{O},T) = L_2(\Omega_T;H^2_{2,0}(\mathcal{O}))$. As a trivial consequence,

$$u\in L_2(\Omega_T;W^1_2(\mathcal{O})) = L_2(\Omega_T;B^1_{2,2}(\mathcal{O}))$$

because we have the equality

$$H^2_{2,0}(\mathcal{O}) = \big\{u\in\mathcal{D}'(\mathcal{O}) : \rho^{|\alpha|-1}D^\alpha u\in L_2(\mathcal{O}) \text{ for all } \alpha\in\mathbb{N}_0^2 \text{ with } |\alpha|\le 2\big\}.$$

(In fact, according to Remark 2.7 we even know that $u\in L_2(\Omega_T;\mathring{W}^1_2(\mathcal{O}))$.)

Note that in general it is not true that u belongs to $L_2(\Omega_T;W^s_2(\mathcal{O}))$ for all $s<2$. Since $\mathcal{O}$ is an arbitrary bounded Lipschitz domain, certain second derivatives might explode near the boundary and the norm $\|u(\omega,t,\cdot)\|_{W^2_2(\mathcal{O})}$ as well as $\|u(\omega,t,\cdot)\|_{W^s_2(\mathcal{O})}$, where $s\in(1,2)$, might not be finite. For $\mathcal{O}$ being a polygonal domain, an explicit upper bound for the regularity in the Sobolev scale $L_2(\Omega_T;W^s_2(\mathcal{O}))$, $s\ge 0$, has been derived in [36]. Adapting techniques used in Grisvard [23, 24] to our stochastic setting, it has been shown that $u\notin L_2(\Omega_T;W^s_2(\mathcal{O}))$ if $s>1+\pi/\gamma_{\max}$, where $\gamma_{\max}$ is the measure of the largest interior angle at a corner of ∂O.

However, in the situation under study, Theorem 3.1 with $s=1$ states that

$$u\in L_\tau(\Omega_T;B^\alpha_{\tau,\tau}(\mathcal{O})),\quad \frac{1}{\tau}=\frac{\alpha}{2}+\frac{1}{2},\quad \text{for all } \alpha<2.$$

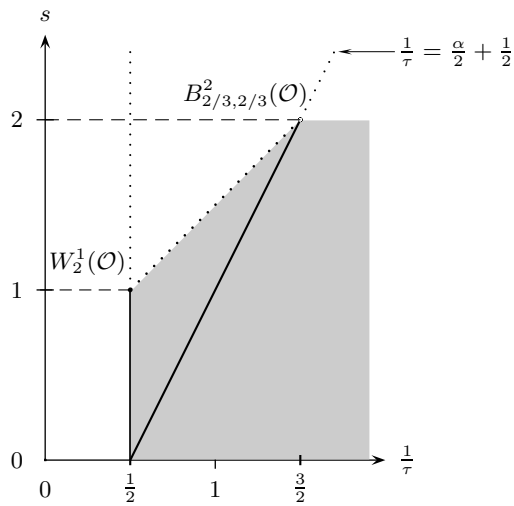

Fig. 2. Besov regularity in the scale $B^{\alpha}_{\tau,\tau}(\mathcal{O})$, $1/\tau = \alpha/2 + 1/2$, vs. Sobolev regularity of the solution, illustrated in a DeVore–Triebel diagram

This is illustrated in Figure 2, where each point $(1/\tau, s)$ represents the smoothness spaces of functions with "s derivatives in $L_\tau(\mathcal{O})$". Based on the knowledge that $u \in L_2(\Omega_T; W^1_2(\mathcal{O}))$ and $u \in L_\tau(\Omega_T; B^{\alpha}_{\tau,\tau}(\mathcal{O}))$ for all $\alpha < 2$, $1/\tau = \alpha/2 + 1/2$, interpolation and embedding theorems show that u also belongs to each of the spaces $L_\tau(\Omega_T; B^{s}_{\tau,\tau}(\mathcal{O}))$, $0 < \tau < 2$, $s < (1/2+1/\tau)\wedge 2$. This is indicated by the shaded area.

Example 3.4. In view of equality (3.14) it is clear that we can apply Theorems 2.11 and 3.1 in the same way as in Example 3.3, i.e. with $d = \theta = \gamma = 2$ and $s = 1$, if the driving process $(M_t)_{t\in[0,T]}$ in (2.13) is a Wiener process in $W^1_2(\mathcal{O})$ with covariance operator $Q \in L_{(\mathrm{nuc})}(W^1_2(\mathcal{O}))$, and even if it is a Wiener process in $H^1_{2,2}(\mathcal{O})$ with covariance operator $Q \in L_{(\mathrm{nuc})}(H^1_{2,2}(\mathcal{O}))$. In the first case $(M_t)_{t\in[0,T]}$ does not satisfy a zero Dirichlet boundary condition as in Example 3.3, and in the second case $(M_t)_{t\in[0,T]}$ behaves even more irregularly near the boundary in the sense that the first derivatives are allowed to blow up near $\partial\mathcal{O}$.

In these cases we choose $(e_\kappa)_{\kappa\in\mathbb{N}}$ in (3.12) and (3.13) to be an orthonormal basis of the space $W^1_2(\mathcal{O})$, respectively $H^1_{2,2}(\mathcal{O})$, consisting of eigenvectors of $Q \in L_{(\mathrm{nuc})}(W^1_2(\mathcal{O}))$, respectively $Q \in L_{(\mathrm{nuc})}(H^1_{2,2}(\mathcal{O}))$, with corresponding eigenvalues $(\lambda_\kappa)_{\kappa\in\mathbb{N}}$.

As in Example 3.3 the solution u lies in the space $L_2(\Omega_T; W^1_2(\mathcal{O})) = L_2(\Omega_T; B^1_{2,2}(\mathcal{O}))$ and, by Theorem 3.1, it also lies in $L_2(\Omega_T; B^{\alpha}_{\tau,\tau}(\mathcal{O}))$, $1/\tau = \alpha/2 + 1/2$, $\alpha < 2$ (see Figure 2).

EXAMPLE 3.5. Let the driving process $(M_t)_{t\in[0,T]}$ in (2.13) be a time-dependent version of the stochastic wavelet expansion introduced in Abramovich et al. [1] in the context of Bayesian nonparametric regression and generalized in Bochkina [2] and Cioica et al. [6]. This noise model is formulated in terms of a wavelet basis expansion on the domain $\mathcal{O} \subset \mathbb{R}^d$ with random coefficients of prescribed sparsity and thus tailor-made for applying adaptive techniques with regard to the numerical approximation of the corresponding SPDEs. Via the choice of certain parameters specifying the distributions of the wavelet coefficients it also allows for an explicit control of the spatial Besov regularity of $(M_t)_{t\in[0,T]}$. We first describe the general noise model and then deduce a further example for the application of Theorem 3.1.

Let $\{\psi_\lambda : \lambda \in \nabla\}$ be a multiscale Riesz basis for $L_2(\mathcal{O})$ consisting of scaling functions at a fixed scale level $j_0 \in \mathbb{Z}$ and of wavelets at level j_0 and all finer levels. As in the introduction, the notation we use here is different from that used in Section 2.2 because we do not consider a basis on the whole space $\mathbb{R}^d$ but on the bounded domain $\mathcal{O}$. Information like scale level, spatial location and type of the wavelets or scaling functions are encoded in the indices $\lambda \in \nabla$. We refer to Cohen [7, Sections 2.12, 2.13 and 3.9] and Dahmen and Schneider [14]–[16] for detailed descriptions of multiscale bases on bounded domains. Adopting the notation of Cohen we write $\nabla = \bigcup_{j\geq j_0-1} \nabla_j$, where for $j \geq j_0$ the set $\nabla_j \subset \nabla$ contains the indices of all wavelets ψ_λ at scale level j and where $\nabla_{j_0-1} \subset \nabla$ is the index set referring to the scaling functions at scale level j_0 which we denote by ψ_λ, $\lambda \in \nabla_{j_0-1}$, for notational simplicity. We make the following assumptions concerning our basis. Firstly, the cardinalities of the index sets ∇_j, $j \geq j_0 - 1$, satisfy

$$C^{-1}2^{jd} \leq |\nabla_j| \leq C2^{jd}, \quad j \geq j_0 - 1. \tag{3.15}$$

Secondly, we assume that the basis admits norm equivalences similar to those described in Theorem 2.3. There exists an $r \in \mathbb{N}$ (depending on the smoothness of the scaling functions ψ_λ, $\lambda \in \nabla_{j_0-1}$, and on the degree of polynomial exactness of their linear span) such that, given $p, q > 0$, $\max\{0, d(1/p-1)\} < s < r$, and a real valued distribution $f \in \mathcal{D}'(\mathcal{O})$, we have $f \in B^s_{p,q}(\mathcal{O})$ if and only if f can be represented as $f = \sum_{\lambda\in\nabla} c_\lambda \psi_\lambda$, $(c_\lambda)_{\lambda\in\nabla} \subset \mathbb{R}$ (convergence in $\mathcal{D}'(\mathcal{O})$), such that

$$\Big(\sum_{j=j_0-1}^{\infty} 2^{jq(s+d(1/2-1/p))} \Big(\sum_{\lambda\in\nabla_j} |c_\lambda|^p \Big)^{q/p} \Big)^{1/q} < \infty. \tag{3.16}$$

Furthermore, $\|f\|_{B^s_{p,q}(\mathcal{O})}$ is equivalent to the quasi-norm (3.16). Concrete constructions of bases satisfying these assumptions can be found in the literature mentioned above. Concerning the family $(w^\kappa_t)_{t\in[0,T]}$, $\kappa \in \mathbb{N}$, of independent standard Brownian motions in (1.1) respectively (2.13), we modify

our notation and write $(w_t^\lambda)_{t\in[0,T]}$, $\lambda\in\nabla$, instead. The description of the noise model involves parameters $a\geq 0$, $b\in[0,1]$, $c\in\mathbb{R}$, with $a+b>1$. For every $j\geq j_0-1$ we set $\sigma_j=(j-(j_0-2))^{cd/2}2^{-a(j-(j_0-1))d/2}$ and let Y_λ, $\lambda\in\nabla_j$, be Bernoulli distributed random variables on $(\Omega,\mathcal{F}_0,\mathbb{P})$ with parameter $p_j=2^{-b(j-(j_0-1))d}$ such that the random variables and processes Y_λ, $(w_t^\lambda)_{t\in[0,T]}$, $\lambda\in\nabla$, are stochastically independent. Now we are ready to define $(M_t)_{t\in[0,T]}$ by

$$M_t:=\sum_{j=j_0-1}^{\infty}\sum_{\lambda\in\nabla_j}\sigma_j Y_\lambda\psi_\lambda\cdot w_t^\lambda,\quad t\in[0,T]. \tag{3.17}$$

Using (3.16), (3.15) and $a+b>1$, it is easy to check that the infinite sum converges in $L_2(\Omega_T;L_2(\mathcal{O}))$ as well as in the space $\mathcal{M}_T^{2,c}(L_2(\mathcal{O}),(\mathcal{F}_t))$ of continuous, square-integrable, $L_2(\mathcal{O})$-valued martingales with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Moreover, by the choice of the parameters a, b and c one has an explicit control of the convergence of the infinite sum in (3.17) in the (quasi-)Banach spaces $L_{p_2}(\Omega_T;B^s_{p_1,q}(\mathcal{O}))$, $s<r$, $p_1,q>0$, $p_2\leq q$. (Compare Cioica et al. [6] which can easily be adapted to our setting.)

With regard to Theorems 2.11 and 3.1 let $d=p=\gamma=\theta=2$. Equation (2.13) with $(M_t)_{t\in[0,T]}$ defined as above corresponds to equation (1.1) if we set

$$g^\lambda(\omega,t,\cdot):=\sigma_jY_\lambda(\omega)\psi_\lambda(\cdot),\qquad \lambda\in\nabla_j,\ j\geq j_0-1,\ (\omega,t)\in\Omega_T,$$

and sum over all $\lambda\in\nabla$ instead of $\kappa\in\mathbb{N}$. In the following we write $\ell_2=\ell_2(\nabla)$. Since $a+b>1$ and $\|g\|_{\mathbb{H}^0_{2,2}(\mathcal{O},T;\ell_2)}=\sqrt{2/T}\|M\|_{L_2(\Omega_T;L_2(\mathcal{O}))}$ we have $g\in\mathbb{H}^0_{2,2}(\mathcal{O},T;\ell_2)$. Let us impose a bit more smoothness on g and assume that $a+b>2$. This is sufficient to ensure that $g\in\mathbb{H}^1_{2,2}(\mathcal{O},T;\ell_2)$: Using (2.10) one sees that the $\mathbb{H}^1_{2,2}(\mathcal{O},T;\ell_2)$-norm of $g=(g^\lambda)_{\lambda\in\nabla}$ satisfies

$$\begin{aligned}
\|g\|^2_{\mathbb{H}^1_{2,2}(\mathcal{O},T;\ell_2)}&=\mathbb{E}\int_0^T\sum_{n\in\mathbb{Z}}c^{n2}\|\zeta_{-n}(c^n\cdot)g(t,c^n\cdot)\|^2_{H^1_2(\mathbb{R}^d;\ell_2)}\,dt\\
&=\mathbb{E}\int_0^T\sum_{\lambda\in\nabla}\|g^\lambda(t,\cdot)\|^2_{H^1_{2,2}(\mathcal{O})}\,dt\\
&=T\mathbb{E}\sum_{j=j_0-1}^{\infty}\sum_{\lambda\in\nabla_j}\sigma_j^2Y_\lambda^2\|\psi_\lambda\|^2_{H^1_{2,2}(\mathcal{O})}\\
&\leq C\sum_{j=j_0-1}^{\infty}\sum_{\lambda\in\nabla_j}\sigma_j^2p_j\sum_{|\alpha|\leq 1}\|\rho^{|\alpha|}D^\alpha\psi_\lambda\|^2_{L_2(\mathcal{O})}\\
&\leq C\sum_{j=j_0-1}^{\infty}\sum_{\lambda\in\nabla_j}\sigma_j^2p_j\|\psi_\lambda\|^2_{W^1_2(\mathcal{O})}.
\end{aligned}$$

Since $W_2^1(\mathcal{O}) = B_{2,2}^1(\mathcal{O})$ with equivalent norms we can use the equivalence (3.16) with $f = \psi_\lambda$ to get

$$\begin{aligned}\|g\|^2_{\mathbb{H}^1_{2,2}(\mathcal{O},T;\ell_2)} &\le C \sum_{j=j_0-1}^{\infty} \sum_{\lambda\in\nabla_j} \sigma_j^2 p_j 2^{2j} \\ &= C \sum_{j=j_0-1}^{\infty} |\nabla_j|(j-(j_0-2))^{2c} 2^{-2a(j-(j_0-1))} 2^{-2b(j-(j_0-1))} 2^{2j} \\ &\le C \sum_{j=j_0-1}^{\infty} (j-(j_0-2))^{2c} 2^{-2j(a+b-2)}.\end{aligned}$$

In the last step we used (3.15) with $d = 2$. Thus $g \in \mathbb{H}^1_{2,2}(\mathcal{O},T;\ell_2)$. As in Example 3.3 we may apply Theorems 2.11 and 3.1 to conclude that for every initial condition $u_0 \in L_2(\Omega,\mathcal{F}_0,\mathbb{P};H^1_{2,2}(\mathcal{O}))$ there exists a unique solution of equation (1.1) in the class $\mathfrak{H}^2_{2,2}(\mathcal{O},T)$, for which it is in general not true that it belongs to $L_2(\Omega_T;W_2^s(\mathcal{O}))$ for all $s<2$, but it does belong to every space $L_2(\Omega_T;B^\alpha_{\tau,\tau}(\mathcal{O}))$ with $\alpha<2$ and $\tau = 2/(\alpha+1)$.

Remark 3.6. In practice, many adaptive wavelet-based algorithms are realized with the energy norm of the problem which is equivalent to a Sobolev norm. Let us denote by $\{\eta_\lambda : \lambda\in\nabla\}$ a wavelet Riesz basis of $W_2^s(\mathcal{O})$ for some $s>0$, which can be obtained by rescaling the wavelet basis $\{\psi_\lambda : \lambda\in\nabla\}$ of $L_2(\mathcal{O})$ (see, e.g., Cohen [7] or Dahmen [13]). For the best N-term approximation in this Sobolev norm, it is well known that

$$u \in B^\alpha_{\tau,\tau}(\mathcal{O}),\ \frac{1}{\tau} = \frac{\alpha-s}{d} + \frac{1}{2} \ \Rightarrow\ \sigma_{N,W_2^s(\mathcal{O})}(u) \le CN^{-(\alpha-s)/d},$$

where

$$\sigma_{N,W_2^s(\mathcal{O})}(u) := \inf\Big\{\|u-u_N\|_{W_2^s(\mathcal{O})} : u_N = \sum_{\lambda\in\Lambda} c_\lambda\eta_\lambda,\ \Lambda\subset\nabla,\ |\Lambda|\le N,\ c_\lambda\in\mathbb{R},\ \lambda\in\Lambda\Big\}.$$

Therefore, similar to the $L_2(\mathcal{O})$-setting, the approximation order of the best N-term wavelet scheme in $W_2^s(\mathcal{O})$ depends on the Besov regularity of the object one wants to approximate.

There exist adaptive wavelet-based algorithms which are guaranteed to converge and which indeed asymptotically realize the convergence rate of best N-term approximation with respect to the Sobolev norm. For example, Cohen, Dahmen and DeVore [8] designed such an adaptive numerical scheme for solving (deterministic) elliptic PDEs. First results for parabolic problems were obtained by Schwab and Stevenson [48].

Once again, the use of adaptive algorithms is justified if the rate of approximation that can be achieved is higher than in classical uniform schemes.

Let u_N, $N \in \mathbb{N}$, denote a uniform approximation scheme (e.g. a Galerkin approximation) of u. It is well-known that under certain natural conditions (see, e.g., Dahlke, Dahmen and DeVore [11], DeVore [19] or Hackbusch [25])

$$\|u - u_N\|_{W_2^s(\mathcal{O})} \leq C N^{-(\alpha-s)/d} \|u\|_{W_2^\alpha(\mathcal{O})}.$$

This means that, even in this case, adaptivity can pay off if the Besov smoothness of the solution is higher than its Sobolev regularity.

Let us discuss this relationship in more detail for the examples above. We consider approximation in $W_2^1(\mathcal{O})$. As already mentioned in Example 3.3, in general we cannot expect that the spatial Sobolev regularity of the solution is higher than $3/2$ (see [36]). Therefore, uniform schemes yield an approximation rate of $O(N^{-1/4})$.

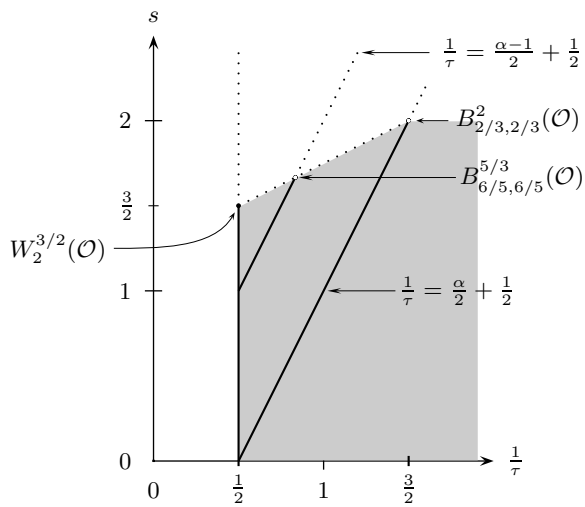

Fig. 3. Besov regularity in the scale $B^\alpha_{\tau,\tau}(\mathcal{O})$, $1/\tau = (\alpha-1)/2+1/2$, vs. Sobolev regularity of the solution illustrated in a DeVore–Triebel diagram

On the other hand, our main result shows that

$$u \in L_\tau(\Omega_T; B^\alpha_{\tau,\tau}(\mathcal{O})), \qquad \frac{1}{\tau} = \frac{\alpha}{2} + \frac{1}{2}, \qquad \text{for all } \alpha < 2.$$

Therefore, by interpolation and embedding of Besov spaces we can achieve that the solution is contained in all the spaces $L_\tau(\Omega_T; B^\alpha_{\tau,\tau}(\mathcal{O}))$ corresponding to the points in the trapezoid with vertices $(1/2, 0)$, $(1/2, 3/2)$, $(3/2, 2)$, $(3/2, 0)$ and to the points to the right of this trapezoid in the DeVore–Triebel diagram of Figure 3. As a consequence, by a short computation we get

$$u \in L_\tau(\Omega_T; B^\alpha_{\tau,\tau}(\mathcal{O})), \qquad \frac{1}{\tau} = \frac{\alpha-1}{2} + \frac{1}{2}, \qquad \text{for all } \alpha < \frac{5}{3}.$$

Thus, best N-term wavelet approximation provides order $O(N^{-1/3})$, so that again the use of adaptivity is completely justified.

REMARK 3.7. Examples 3.3–3.5 are stated in the Hilbert space setting $p = 2$ and in Remarks 2.13, 2.14 we pointed out how this setting relates to the semigroup approach to SPDEs by Da Prato and Zabczyk [17]. A drawback of the restriction to $p = 2$ is that it does not allow for optimal regularity results since for bounded domains the $L_2(\mathcal{O})$-norm is of course weaker than the $L_p(\mathcal{O})$-norm for $p > 2$. Let us, therefore, exemplarily consider Example 3.3 for $p \geq 2$ and make a comparison with closely related results for stochastic evolution equations in UMD Banach spaces by van Neerven, Veraar and Weis [42]. ("UMD" is an abbreviation for "unconditional martingale differences".)

Firstly, we sketch how our setting relates to the one in [42]. Let $p \geq 2$, $\gamma_0 \geq 0$, $\theta \in \mathbb{R}$ and $g \in \mathbb{H}^{\gamma_0}_{p,\theta}(\mathcal{O}, T; \ell_2)$. Similar to Remark 2.13, the expression $\sum_{\kappa=1}^{\infty} \int_0^t g^\kappa(s,\cdot)\, dw_s^\kappa$ can be considered as an $H^{\gamma_0}_{p,\theta}(\mathcal{O})$-valued stochastic integral $M_t = \int_0^t G(s)\, dW_s$ with respect to a cylindrical Wiener process $(W_t)_{t\in[0,T]}$ on ℓ_2. The integrand process $(G(t))_{t\in[0,T]}$, defined exactly as in Remark 2.13, takes values in the space $\gamma(\ell_2, H^{\gamma_0}_{p,\theta}(\mathcal{O}))$ of γ-radonifying operators (see van Neerven [39] for a comprehensive survey on this class of operators). Using the definition of $H^{\gamma_0}_{p,\theta}(\mathcal{O}; \ell_2)$, equality (2.3) in [42] with $X = \mathbb{R}$, Theorem 3.20 in [39] and the Kahane–Khintchine inequalities [39, Proposition 2.7], one sees that, for all $(\omega, t) \in \Omega \times [0, T]$,

$$C^{-1}\|G(\omega,t)\|_{\gamma(\ell_2, H^{\gamma_0}_{p,\theta}(\mathcal{O}))} \leq \|g(\omega,t,\cdot)\|_{H^{\gamma_0}_{p,\theta}(\mathcal{O};\ell_2)} \leq C\|G(\omega,t)\|_{\gamma(\ell_2, H^{\gamma_0}_{p,\theta}(\mathcal{O}))}. \tag{3.18}$$

Since $H^{\gamma_0}_{p,\theta}(\mathcal{O})$ is a UMD Banach of type 2, Corollary 3.10 and Proposition 4.3 in van Neerven, Veraar and Weis [40] yield

$$\mathbb{E} \sup_{t\in[0,T]} \Big\| \int_0^t G(s)\, dW_s \Big\|^p_{H^{\gamma_0}_{p,\theta}(\mathcal{O})} \leq C\|G\|^p_{L_p(\Omega_T;\gamma(\ell_2, H^{\gamma_0}_{p,\theta}(\mathcal{O})))} \leq C\|g\|^p_{\mathbb{H}^{\gamma_0}_{p,\theta}(\mathcal{O},T;\ell_2)}.$$

The UMD property of $H^{\gamma_0}_{p,\theta}(\mathcal{O})$ follows from the fact that $L_p(\mathcal{O})$ is a UMD space and that closed subspaces of UMD spaces are UMD. The type 2 property follows since $L_p(\mathcal{O})$ is type 2, from the norm equivalence (2.10) and complex interpolation. In this remark, let the operator $(A, D(A))$ be defined as in Remark 2.14 but with integrability parameter 2 replaced by p. In the following situation the solution u to (1.1) in the class $\mathfrak{H}^2_{p,d}(\mathcal{O}, T)$ coincides with the weak solution u to (2.14) as defined in Veraar [51, Definition 7.5.8] (which is in this case even a strong solution).

Secondly, we consider, analogously to Example 3.3, equation (2.14) where the driving process $M = (M_t)_{t\in[0,T]}$ is a Wiener process in the space $\mathring{W}^1_p(\mathcal{O}) = H^1_{p,d-p}(\mathcal{O})$ and $u_0 \in U^2_{p,d}(\mathcal{O})$. We also assume that $p \in [2, p_0)$, where p_0 has the same meaning as in Theorem 2.11. The process M has a represen-

tation as a stochastic integral process $(\int_0^t G(s)\,dW_s)_{t\in[0,T]}$ with respect to a cylindrical Wiener process $(W_t)_{t\in[0,T]}$ on ℓ_2, where $(G(t))_{t\in[0,T]}$ is a constant deterministic process in $\gamma(\ell_2, \mathring{W}^1_p(\mathcal{O}))$ with suitably chosen value, say $G \in \gamma(\ell_2, \mathring{W}^1_p(\mathcal{O}))$. The existence of G follows from the Karhunen–Loève theorem [39, Theorem 7.3] and the fact that the Cameron–Martin space of M is separable (see, e.g., Hairer [26]). This corresponds to defining $g = (g^\kappa)_{\kappa\in\mathbb{N}}$ in (1.1) by

$$g^\kappa(\omega,t,\cdot) := Gb_\kappa \in \mathring{W}^1_p(\mathcal{O}), \qquad \kappa\in\mathbb{N},\ (\omega,t)\in\Omega_T, \tag{3.19}$$

where $(b_\kappa)_{\kappa\in\mathbb{N}}$ is the natural orthonormal basis in ℓ_2. From (3.18) it follows that g, defined by (3.13), belongs to $\mathbb{H}^1_{p,d-p}(\mathcal{O},T;\ell_2) \subset \mathbb{H}^1_{p,d}(\mathcal{O},T;\ell_2)$. Hence we can apply Theorems 2.11 and 3.1 with $\gamma = 2$, $\theta = d$ and $s = 1$ and obtain a unique solution $u \in \mathfrak{H}^2_{p,d}(\mathcal{O},T)$ to (1.1), such that

$$u \in L_\tau(\Omega_T; B^\alpha_{\tau,\tau}(\mathcal{O})), \quad \frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}, \quad \text{for all } \alpha < \frac{d}{d-1}.$$

Let us compare this situation to the spatial regularity we get from the results in van Neerven, Veraar and Weis [42], where the semigroup approach to SPDEs is used. We have to check the abstract condition assumed in [42] that the operater $(-A, D(A))$ has bounded H^∞-calculus of angle $< \pi/2$. For $d \geq 3$, this follows from Duong and McIntosh [22] and the fact that $(A, D(A))$ is the generator of an analytic semigroup of contractions on $L_p(\mathcal{O})$ whose resolvent set contains $(0,\infty)$. The latter can be shown analogously to the proof of Theorem 3.6 in Pazy [43, Chapter 7] if one uses the Green formula for bounded Lipschitz domains according to [24, Theorem 1.5.1]. Thus, if one knows that $G \in \gamma(\ell_2, D((-A)^{1/2}))$ and if u_0 is sufficiently regular, one can apply Theorem 4.5 in [42] (with $H = \ell_2$) to deduce that the solution $u \in \mathfrak{H}^2_{p,d}(\mathcal{O},T)$ to (1.1) considered above is even a strong solution to (2.14) and

$$u \in L_p(\Omega_T; D(A)).$$

For C^2-domains $\mathcal{O}$ it is well known that $D((-A)^{1/2}) = \mathring{W}^1_p(\mathcal{O})$ (cf. [42, Section 9]), which fits the definition of G as an element of $\gamma(\ell_2, \mathring{W}^1_p(\mathcal{O}))$ above.

Applying the results in van Neerven, Veraar and Weis [41], similar comparisons with our result can be made for more irregular Wiener processes M in (2.14). Also, more general equations of type (1.3)—for which our result remains valid, see Appendix B—can be embedded in the setting of [41, 42].

Appendix A. Convergence of stochastic integrals. In this section we give a proof of the $\mathcal{M}^{2,c}_T(\mathbb{R},(\mathcal{F}_t))$-convergence of the sum of the stochastic integral processes $(\int_0^t \langle g^\kappa(s,\cdot),\varphi\rangle\,dw^\kappa_s)_{t\in[0,T]}$, $\kappa\in\mathbb{N}$, appearing in (2.11).

Let us assume that $g \in \mathbb{H}^\gamma_{p,\theta}(\mathcal{O},T;\ell_2)$ for some $p \in [2,\infty)$ and $\gamma,\theta \in \mathbb{R}$. We use an analogous strategy to [32, Remark 3.2]. Due to the independence of the Brownian motions $(w_t^\kappa)_{t\in[0,T]}$, $\kappa \in \mathbb{N}$, the covariation process $([w^\kappa, w^\ell]_t)_{t\in[0,T]}$ vanishes if $\kappa \neq \ell$, and by Itô's isometry we have

$$\begin{aligned}\mathbb{E}\Big|\sum_{\kappa=1}^\infty \int_0^T \langle g^\kappa(s,\cdot),\varphi\rangle\,dw_s^\kappa\Big|^2 &\\ &= \mathbb{E}\Big[\sum_{\kappa=1}^\infty \int_0^{(\cdot)} \langle g^\kappa(s,\cdot),\varphi\rangle\,dw_s^\kappa,\ \sum_{\kappa=1}^\infty \int_0^{(\cdot)} \langle g^\kappa(s,\cdot),\varphi\rangle\,dw_s^\kappa\Big]_T\\ &= \mathbb{E}\sum_{\kappa=1}^\infty \int_0^T |\langle g^\kappa(s,\cdot),\varphi\rangle|^2\,ds.\end{aligned}$$

We are going to show that the last term is at most a constant times $\|g\|^2_{\mathbb{H}^\gamma_{p,\theta}(\mathcal{O},T;\ell_2)}$, which is finite due to our assumption. Then the convergence of the integral processes in $\mathcal{M}^{2,c}_T(\mathbb{R},(\mathcal{F}_t))$ follows by Doob's maximal inequality for martingales.

For $u \in \mathcal{D}'(\mathcal{O})$ and $n \in \mathbb{Z}$ we use the notation $u_n := \zeta_{-n}(c^n\cdot)u(c^n\cdot) \in \mathcal{S}'(\mathbb{R}^d)$. Let us abbreviate $L_\tau(\mathbb{R}^d)$ by L_τ for all $\tau \geq 1$. Setting $p' = p/(p-1)$, we denote by $\langle\cdot,\cdot\rangle_{L_p\times L_{p'}} : L_p \times L_{p'} \to \mathbb{R}$ the dual form obtained by continuous extension of $\langle\varphi,\psi\rangle = \int\varphi(x)\psi(x)\,dx$, $\varphi,\psi \in C_0^\infty(\mathbb{R}^d)$. Now we estimate as follows:

$$\begin{aligned}\sum_{\kappa=1}^\infty \int_0^T |\langle g^\kappa(s,\cdot),\varphi\rangle|^2\,ds &= \sum_{\kappa=1}^\infty \int_0^T \Big|\sum_{n\in\mathbb{Z}} c^{nd}\langle g_n^\kappa(s,\cdot),\varphi_n\rangle\Big|^2\,ds\\ &= \sum_{\kappa=1}^\infty \int_0^T \Big|\sum_{n\in\mathbb{Z}} c^{nd}\langle(1-\Delta)^{\gamma/2}g_n^\kappa(s,\cdot),(1-\Delta)^{-\gamma/2}\varphi_n\rangle_{L_p\times L_{p'}}\Big|^2\,ds\\ &\leq \sum_{\kappa=1}^\infty \int_0^T \Big(\sum_{n\in\mathbb{Z}} c^{nd}\big\||(1-\Delta)^{\gamma/2}g_n^\kappa(s,\cdot)|\cdot|(1-\Delta)^{-\gamma/2}\varphi_n|^{1/2}\big\|_{L_2}\\ &\qquad\qquad\cdot\big\||(1-\Delta)^{-\gamma/2}\varphi_n|^{1/2}\big\|_{L_2}\Big)^2\,ds\\ &\leq \sum_{\kappa=1}^\infty \int_0^T \Big[\Big(\sum_{n\in\mathbb{Z}} c^{2nd}\big\||(1-\Delta)^{\gamma/2}g_n^\kappa(s,\cdot)|\cdot|(1-\Delta)^{-\gamma/2}\varphi_n|^{1/2}\big\|_{L_2}^2\Big)^{1/2}\\ &\qquad\qquad\cdot\Big(\sum_{n\in\mathbb{Z}}\|(1-\Delta)^{-\gamma/2}\varphi_n\|_{L_1}\Big)^{1/2}\Big]^2\,ds.\end{aligned}$$

Here we have used Hölder's inequality twice. Since φ has compact support in $\mathcal{O}$ and ζ_{-n} equals zero outside $\mathcal{O}_{-n}$, the functions φ_n vanish on $\mathbb{R}^d$ for all but

finitely many $n \in \mathbb{Z}$. As a consequence, the sum $\sum_{n\in\mathbb{Z}} \|(1-\Delta)^{-\gamma/2}\varphi_n\|_{L_1}$ has only finitely many non-zero terms. Therefore,

$$\begin{aligned}
&\sum_{\kappa=1}^{\infty}\int_0^T |\langle g^\kappa(s,\cdot),\varphi\rangle|^2\,ds\\
&\quad\le C\sum_{\kappa=1}^{\infty}\int_0^T\sum_{n\in\mathbb{Z}} c^{2nd}\big\||(1-\Delta)^{\gamma/2}g_n^\kappa(s,\cdot)|\cdot|(1-\Delta)^{-\gamma/2}\varphi_n|^{1/2}\big\|_{L_2}^2\,ds\\
&\quad= C\int_0^T\sum_{n\in\mathbb{Z}} c^{2nd}\Big\langle\sum_{\kappa=1}^{\infty}|(1-\Delta)^{\gamma/2}g_n^\kappa(s,\cdot)|^2,|(1-\Delta)^{-\gamma/2}\varphi_n|\Big\rangle_{L_{p/2}\times L_{p/(p-2)}}ds,
\end{aligned}$$

where the constant C depends on φ. In the last step we used the fact that

$$(1-\Delta)^{\gamma/2}g_n(s,\cdot)=((1-\Delta)^{\gamma/2}g_n^\kappa(s,\cdot))_{\kappa\in\mathbb{N}}\in L_p(\mathbb{R}^d;\ell_2)$$

$\mathbb{P}\otimes\lambda$-almost everywhere in Ω_T, which results from g being an element of $\mathbb{H}^\gamma_{p,\theta}(\mathcal{O};\ell_2)$. Applying again Hölder's inequality we obtain

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} c^{2nd}\Big\langle\sum_{\kappa=1}^{\infty}|(1-\Delta)^{\gamma/2}g_n^\kappa(s,\cdot)|^2,|(1-\Delta)^{-\gamma/2}\varphi_n|\Big\rangle_{L_{p/2}\times L_{p/(p-2)}}\\
&\le\sum_{n\in\mathbb{Z}} c^{2n\theta}\big\||(1-\Delta)^{\gamma/2}g_n(s,\cdot)|_{\ell_2}\big\|_{L_p}^{1/2}c^{2n(d-\theta)}\|(1-\Delta)^{-\gamma/2}\varphi_n\|_{L_{p/(p-2)}}\\
&\le C\Big(\sum_{n\in\mathbb{Z}} c^{n\theta}\big\||(1-\Delta)^{\gamma/2}g_n(s,\cdot)|_{\ell_2}\big\|_{L_p}^{p}\Big)^{2/p}\\
&\qquad\qquad\cdot\Big(\sum_{n\in\mathbb{Z}} c^{2n(d-\theta)p/(p-2)}\|(1-\Delta)^{-\gamma/2}\varphi_n\|_{L_{p/(p-2)}}^{p/(p-2)}\Big)^{(p-2)/p}\\
&\le C\Big(\sum_{n\in\mathbb{Z}} c^{n\theta}\big\||(1-\Delta)^{\gamma/2}g_n(s,\cdot)|_{\ell_2}\big\|_{L_p}^{p}\Big)^{2/p},
\end{aligned}$$

where we have used the fact that $p\le 2$ and that only finitely many of the φ_n, $n\in\mathbb{Z}$, are non-zero. All in all we have shown

$$\sum_{\kappa=1}^{\infty}\int_0^T|\langle g^\kappa(s,\cdot),\varphi\rangle|^2\,ds\le C\int_0^T\Big(\sum_{n\in\mathbb{Z}} c^{n\theta}\big\||(1-\Delta)^{\gamma/2}g_n(s,\cdot)|_{\ell_2}\big\|_{L_p}^{p}\Big)^{2/p}ds.$$

Finally, taking the expectation and applying Jensen's inequality yields

$$\mathbb{E}\sum_{\kappa=1}^{\infty}\int_0^T|\langle g^\kappa(s,\cdot),\varphi\rangle|^2\,ds\le C\|g\|^2_{\mathbb{H}^\gamma_{p,\theta}(\mathcal{O},T;\ell_2)},$$

and this finishes the proof.

Appendix B. General linear equations. In the introduction we have indicated that our main result can be extended to equation (1.3). The major reason is, that by a result of Kim [30, 31] an estimate similar to the one proved in Corollary 2.12 holds not only for the model equation (1.1) but for equations of the type (1.3), provided the coefficients $a^{\mu\nu}, b^\mu$, c, $\sigma^{\mu\kappa}$ and η^κ, the free terms f and g^κ and the initial value u_0 satisfy certain conditions. We can use this fact to extend our regularity result to such equations. In this section we point out more precisely how to do this.

For the convenience of the reader we begin by presenting the result from Kim [31, Theorem 2.12] (see also Remark 2.13 therein). Therefore, we need some additional notation. For $x, y \in \mathcal{O}$ we write $\rho(x,y) := \rho(x) \wedge \rho(y)$. For $\alpha \in \mathbb{R}$, $\delta \in (0,1]$ and $k \in \mathbb{N}_0$ we set

$$[f]_k^{(\alpha)} := \sup_{x\in\mathcal{O}} \rho(x)^{k+\alpha} |D^k f(x)|,$$

$$[f]_{k+\delta}^{(\alpha)} := \sup_{\substack{x,y\in\mathcal{O}\\ |\beta|=k}} \rho(x,y)^{k+\alpha} \frac{|D^\beta f(x) - D^\beta f(y)|}{|x-y|^\delta},$$

$$|f|_k^{(\alpha)} := \sum_{j=0}^{k} [f]_j^{(\alpha)} \quad \text{and} \quad |f|_{k+\delta}^{(\alpha)} := |f|_k^{(\alpha)} + [f]_{k+\delta}^{(\alpha)},$$

whenever it makes sense. We shall use the same notation for ℓ_2-valued functions (just replace the absolute values in the above definitions by the ℓ_2-norms). Furthermore, let us fix an arbitrary function

$$\mu_0 : [0,\infty) \to [0,\infty),$$

vanishing only on the set of non-negative integers, i.e. $\mu_0(j) = 0$ if, and only if, $j \in \mathbb{N}_0$. We set

$$t_+ := t + \mu_0(t).$$

Now we are able to present the assumptions on the coefficients of (1.3) (see Kim [30, Assumptions 2.5 and 2.6] as well as [31, Assumptions 2.10]).

[K1] For any fixed $x \in \mathcal{O}$, the coefficients

$$a^{\mu\nu}(\cdot,\cdot,x), b^\mu(\cdot,\cdot,x), c(\cdot,\cdot,x), \sigma^{\mu\kappa}(\cdot,\cdot,x), \eta^\kappa(\cdot,\cdot,x) : \Omega \times [0,T] \to \mathbb{R}$$

are predictable processes with respect to the given normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$.

[K2] (Stochastic parabolicity) There are constants $\delta_0, K > 0$ such that for all $(\omega,t,x) \in \Omega \times [0,T] \times \mathcal{O}$ and $\lambda \in \mathbb{R}^d$,

$$\delta_0 |\lambda|^2 \le \overline{a^{\mu\nu}}(\omega,t,x)\lambda_\mu\lambda_\nu \le K|\lambda|^2,$$

where $\overline{a^{\mu\nu}} := a^{\mu\nu} - \frac{1}{2}\langle \sigma^\mu, \sigma^\nu \rangle_{\ell_2}$ for $\mu,\nu \in \{1,\dots,d\}$.

[K3] For all $(\omega,t)\in\Omega\times[0,T]$,

$$|a^{\mu\nu}(\omega,t,\cdot)|^{(0)}_{|\gamma|_+}+|b^{\mu}(\omega,t,\cdot)|^{(1)}_{|\gamma|_+}+|c(\omega,t,\cdot)|^{(2)}_{|\gamma|_+}+|\sigma^{\mu}(\omega,t,\cdot)|^{(0)}_{|\gamma|_+}+|\eta(\omega,t,\cdot)|^{(1)}_{|\gamma+1|_+}\leq K.$$

[K4] The coefficients $a^{\mu\nu}$ and σ^{μ} are uniformly continuous in $x\in\mathcal{O}$, i.e. for any $\epsilon>0$ there is a $\delta=\delta(\epsilon)>0$ such that

$$|a^{\mu\nu}(\omega,t,x)-a^{\mu\nu}(\omega,t,y)|+|\sigma^{\mu}(\omega,t,x)-\sigma^{\mu}(\omega,t,y)|_{\ell_2}\leq\epsilon$$

for all $(\omega,t)\in\Omega\times[0,T]$ whenever $x,y\in\mathcal{O}$ with $|x-y|\leq\delta$.

[K5] The behaviour of the coefficients b^{μ}, c and ν can be controlled near the boundary of $\mathcal{O}$ in the following way:

$$\lim_{\substack{\rho(x)\to 0\\ x\in\mathcal{O}}}\sup_{\substack{\omega\in\Omega\\ t\in[0,T]}}\{\rho(x)|b^{\mu}(\omega,t,x)|+\rho(x)^2|c(\omega,t,x)|+\rho(x)|\eta(\omega,t,x)|_{\ell_2}\}=0.$$

Here is the main result of Kim [31].

THEOREM B.1. *Let $\gamma\in\mathbb{R}$.*

(i) *Let assumptions* [K1]–[K5] *be satisfied for some $p\in[2,\infty)$ with $K,\delta_0>0$. Then there exists a constant $\kappa_0=\kappa_0(d,p,\delta_0,K,\mathcal{O})\in(0,1)$ such that, if $\theta\in(d+p-2-\kappa_0,d+p-2+\kappa_0)$, then for any $f\in\mathbb{H}^{\gamma-2}_{p,\theta+p}(\mathcal{O},T)$, $g\in\mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)$ and $u_0\in U^{\gamma}_{p,\theta}(\mathcal{O})$, equation* (1.3) *with initial value u_0 admits a unique solution $u\in\mathfrak{H}^{\gamma}_{p,\theta}(\mathcal{O},T)$, i.e. there exists a $\mathcal{D}'(\mathcal{O})$-valued predictable process $u\in\mathbb{H}^{\gamma}_{p,\theta-p}(\mathcal{O},T)$, unique up to indistinguishability, such that for any $\varphi\in C_0^{\infty}(\mathcal{O})$ the equality*

$$\begin{aligned}\langle u(t,\cdot),\varphi\rangle&=\langle u(0,\cdot),\varphi\rangle\\&\quad+\int_0^t\langle a^{\mu\nu}(s,\cdot)u_{x_\mu x_\nu}(s,\cdot)+b^{\mu}(s,\cdot)u_{x_\mu}(s,\cdot)+c(s,\cdot)u(s,\cdot)+f(s,\cdot),\varphi\rangle\,ds\\&\quad+\sum_{\kappa=1}^{\infty}\int_0^t\langle\sigma^{\mu\kappa}(s,\cdot)u_{x_\mu}(s,\cdot)+\eta^{\kappa}(s,\cdot)u(s,\cdot)+g^{\kappa}(s,\cdot),\varphi\rangle\,dw_s^{\kappa}\end{aligned}$$

holds for all $t\in[0,T]$ with probability 1. *Moreover, for this solution we have*

$$\|u\|^p_{\mathfrak{H}^{\gamma}_{p,\theta}(\mathcal{O},T)}\leq C(\|f\|^p_{\mathbb{H}^{\gamma-2}_{p,\theta+p}(\mathcal{O},T)}+\|g\|^p_{\mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)}+\|u_0\|^p_{U^{\gamma}_{p,\theta}(\mathcal{O})}),\tag{B.1}$$

where C is a constant depending only on d, γ, p, θ, δ_0, K, T and $\mathcal{O}$.

(ii) *There exists $p_0>2$ such that the following statement holds: if assumptions* [K1]–[K5] *are satisfied for some $p\in[2,p_0)$ with $K,\delta_0>0$, then there exists a constant $\kappa_1=\kappa_1(d,p,\delta_0,K,\mathcal{O})\in(0,1)$ such that, if $\theta\in(d-\kappa_1,d+\kappa_1)$, then for any $f\in\mathbb{H}^{\gamma-2}_{p,\theta+p}(\mathcal{O},T)$, $g\in\mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)$*

and $u_0 \in U^\gamma_{p,\theta}(\mathcal{O})$, equation (1.3) *with initial value u_0 admits a unique solution $u \in \mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T)$. For this solution, estimate* (B.1) *holds.*

An immediate consequence of this theorem is the following estimate.

COROLLARY B.2. *In the situation of Theorem* B.1 *with $\gamma = m \in \mathbb{N}$, the following inequality holds for every $\tau \in [0,p]$*

$$\int_\Omega \int_0^T \|\rho^{m-\delta}|D^m u(\omega,t,\cdot)|_{\ell_p}\|^\tau_{L_p(\mathcal{O})}\,dt\,\mathbb{P}(d\omega) \le C(\|f\|_{\mathbb{H}^{m-2}_{p,\theta+p}(\mathcal{O},T)} + \|g\|_{\mathbb{H}^{m-1}_{p,\theta}(\mathcal{O},T;\ell_2)} + \|u_0\|_{U^m_{p,\theta}(\mathcal{O})})^\tau,$$

where $\delta = 1 + (d-\theta)/p$.

Proof. Just repeat the arguments of the proof of Corollary 2.12 and use estimate (B.1) instead of (2.12) at the beginning. ∎

Now we can present our main result in the generalized setting.

THEOREM B.3. *Let $\gamma \in \mathbb{N}$ and let assumptions* [K1]–[K5] *be satisfied with appropriate constants $K, \delta_0 > 0$. Moreover, let $f \in \mathbb{H}^{\gamma-2}_{p,\theta+p}(\mathcal{O},T)$, $g \in \mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)$ and $u_0 \in U^\gamma_{p,\theta}(\mathcal{O})$, where p and θ satisfy one of the following conditions:*

(i) $p \in [2,\infty)$ *and* $\theta \in (d+p-2-\kappa_0, d+p-2+\kappa_0)$,

(ii) $p \in [2,p_0)$ *and* $\theta \in (d-\kappa_1, d+\kappa_1)$,

with κ_0, κ_1 and p_0 from Theorem B.1*. Denote by u the unique solution of equation* (1.3) *in the class $\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T)$. Assume furthermore that*

$$u \in L_p(\Omega_T; B^s_{p,p}(\mathcal{O})) \quad \text{for some } s \in \left(0, \gamma \wedge \left(1 + \frac{d-\theta}{p}\right)\right].$$

Then

$$u \in L_\tau(\Omega_T; B^\alpha_{\tau,\tau}(\mathcal{O})), \quad \frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}, \quad \text{for all } \alpha \in \left(0, \gamma \wedge \frac{sd}{d-1}\right),$$

and

$$\|u\|_{L_\tau(\Omega_T;B^\alpha_{\tau,\tau}(\mathcal{O}))} \le C(\|f\|_{\mathbb{H}^{\gamma-2}_{p,\theta+p}(\mathcal{O},T)} + \|g\|_{\mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)} + \|u_0\|_{U^\gamma_{p,\theta}(\mathcal{O})} + \|u\|_{L_p(\Omega_T;B^s_{p,p}(\mathcal{O}))}).$$

Proof. We can argue as in the proof of Theorem 3.1. We just have to use Corollary B.2 where we used Corollary 2.12. ∎

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