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Fractional Laplacian with singular drift

by

TOMASZ JAKUBOWSKI (Wrocław)

Abstract. For $\alpha \in (1,2)$ we consider the equation $\partial_t u = \Delta^{\alpha/2} u + b \cdot \nabla u$, where b is a time-independent, divergence-free singular vector field of the Morrey class $M_1^{1-\alpha}$. We show that if the Morrey norm $\|b\|_{M_1^{1-\alpha}}$ is sufficiently small, then the fundamental solution is globally in time comparable with the density of the isotropic stable process.

1. Introduction. Let $d \geq 1$ be a natural number and let $\alpha \in (1,2)$. We denote by $p(t,x)$ the density function of the isotropic α-stable Lévy process in $\mathbb{R}^d$, i.e.

$$p(t,x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-t|\xi|^\alpha} \, d\xi, \qquad t>0,\ x \in \mathbb{R}^d. \tag{1.1}$$

For $\phi \in C_c^\infty(\mathbb{R}^d)$ we define the operator

$$\Delta^{\alpha/2}\phi(x) = \mathcal{A}_{d,\alpha} \lim_{\varepsilon \to 0^+} \int_{|y|>\varepsilon} \frac{\phi(x+y)-\phi(x)}{|y|^{d+\alpha}} \, dy,$$

where $\mathcal{A}_{d,\alpha} > 0$ is a constant depending only on α and d. The operator $\Delta^{\alpha/2}$ is the infinitesimal generator of the isotropic α-stable process with the transition density $p(t,x,y) = p(t,y-x)$,

$$\Delta^{\alpha/2}\phi(x) = \lim_{t \to 0} \frac{1}{t} \int_{\mathbb{R}^d} p(t,x,y)(\phi(y)-\phi(x)) \, dy.$$

Let $b(x) = (b_1(x), \ldots, b_d(x))$ be a vector field (not depending on time) satisfying the following conditions: there is a (finite) number C such that

$$\sup_{t>0} \sup_{x \in \mathbb{R}^d} t^{-d+\alpha-1} \int_{B(x,t)} |b(y)| \, dy \leq C \tag{1.2}$$

and

$$\operatorname{div} b = 0 \quad \text{in the sense of distributions.} \tag{1.3}$$

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Here $B(x,t)$ denotes the ball centered at x with radius t. The condition (1.2) means that b belongs to the Morrey space $M_1^{1-\alpha}$, and the smallest possible constant C in (1.2) is denoted by $\|b\|_{M_1^{1-\alpha}}$. Using the estimate (2.1) below and adapting the proof of [5, Lemma 11] we see that the condition (1.2) is equivalent to

$$\sup_{t>0}\sup_{x\in\mathbb{R}^d} t^{1-1/\alpha}\int_{\mathbb{R}^d} p(t,x,y)|b(y)|\,dy \le C_b \tag{1.4}$$

for another (finite) constant C_b. In fact, for every $C_b>0$, there exists $\delta>0$ such that (1.4) holds provided $\|b\|_{M_1^{1-\alpha}}<\delta$. The condition (1.2) is simpler and easier to verify than (1.4), but (1.4) is more convenient in proofs. In this paper we will study the equation

$$\partial_t u - \Delta^{\alpha/2}u - b\cdot\nabla u = 0, \qquad x\in\mathbb{R}^d,\ t>0. \tag{1.5}$$

We should note that (1.2) is less restrictive than the usual Kato condition (see (1.8) below). For example, our results apply to $d=2$ and

$$b_0(y) = c(y_2|y|^{-\alpha}, -y_1|y|^{-\alpha})$$

for small $c>0$ (see also Example 1). We note that $\operatorname{div} b_0 = 0$ in the sense of distributions and $|b_0(y)| = c|y|^{1-\alpha}$, hence, $b_0\notin\mathcal{K}_d^{\alpha-1}$. The main result of the paper is the following.

THEOREM 1. *There is a constant $\eta=\eta(\alpha,d)>0$ such that if $\|b\|_{M_1^{1-\alpha}} \le \eta$ then there is a function $\tilde p(t,x,y)$ such that for $\phi\in C_c^\infty(\mathbb{R},\mathbb{R}^d)$, $s\in\mathbb{R}$ and $x\in\mathbb{R}^d$,*

$$\int_s^\infty\int_{\mathbb{R}^d} \tilde p(u-s,x,z)[\partial_u\phi(u,z)+\Delta_z^{\alpha/2}\phi(u,z)+b(z)\cdot\nabla_z\phi(u,z)]\,dz\,du = -\phi(s,x), \tag{1.6}$$

and there is a constant $0<K<\infty$ depending only on d,α,η such that

$$K^{-1}p(t,x,y)\le \tilde p(t,x,y)\le Kp(t,x,y), \qquad t>0,\ x,y\in\mathbb{R}^d. \tag{1.7}$$

According to (1.6), $\tilde p$ is the integral kernel of the left inverse of $-(\partial_t + \Delta_z^{\alpha/2} + b\cdot\nabla_z)$. Put differently, the function $f\colon (u,x)\mapsto \tilde p(u-s,x,z)$ solves $(\partial_t - \Delta_x^{\alpha/2} - b\cdot\nabla_x)f = \delta_{(s,z)}$ in the sense of distributions, because $\operatorname{div} b = 0$. Thus, $\tilde p$ is the fundamental solution of (1.5). As a consequence, we also find that $\tilde p$ is the integral kernel of the Markov semigroup with (weak) generator $\Delta^{\alpha/2}+b\cdot\nabla$ (see Corollary 12 below).

Equations similar to (1.5) were widely studied for the Laplacian and more general elliptic operators (see, e.g., [2], [27], [28], [18]). At first, authors considered drifts b satisfying Kato-type conditions similar to (1.8) below, including drift functions b depending on time. Under such assumptions, Gaussian bounds for the resulting fundamental solution hold locally in time (i.e. with constants deteriorating for large time). To obtain estimates uniform in time, an additional assumption on the divergence of b is necessary. For example, in [21] Osada proved that the fundamental solution of $\partial_t u = Au + b\cdot\nabla u$ has upper and lower Gaussian bounds, where A is a uniformly elliptic operator in divergence form, b is the derivative of a bounded function and $\operatorname{div} b = 0$. Similar results were later obtained in [19] for the Laplacian and singular drifts b (exceeding the Kato class) under some smallness assumption on divergence. Recent results on Hölder continuity of solutions may be found in [23].

Additive perturbations of the fractional Laplacian were intensively studied in recent years (see, e.g., [13], [14], [5], [17], [4], [8], [6], [1], [11], [24], [25]). In particular, the equation (1.5) was considered in [5] for b in the Kato class $\mathcal{K}_d^{\alpha-1}$ without further assumptions on divergence (see also [16] and [17] for further developments). Recall that $b \in \mathcal{K}_d^{\alpha-1}$ if

$$\lim_{t\to 0} \sup_{x\in\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} s^{-1/\alpha} p(s,x,y)|b(y)|\, dy\, ds = 0. \tag{1.8}$$

The main result of [5] was the local in time comparability of $\tilde{p}$ and p, where the function $\tilde{p}$ was constructed as the perturbation series $\tilde{p} = \sum_{n=0}^{\infty} p_n$, with

$$p_0(t,x,y) = p(t,x,y), \tag{1.9}$$

$$p_n(t,x,y) = \int_0^t \Big[\int_{\mathbb{R}^d} p_{n-1}(t-s,x,z) b(z)\cdot\nabla_z p(s,z,y)\, dz\Big] ds. \tag{1.10}$$

The Kato condition on b ensures smallness of p_1 with respect to p for small time, and by iterating the result, we deduce that the perturbation series converges for small time. For large time one can either use the Chapman–Kolmogorov equations ([4]) or a more direct method developed in [17] for time dependent gradient perturbations. A similar approach was used to study the Green function of $\Delta^{\alpha/2} + b(x)\cdot\nabla_x$ (see [4], where these estimates are further extended to arbitrary bounded smooth sets), and to estimate general Schrödinger perturbations of transition densities ([15], [3]).

Gradient perturbations of the fractional Laplacian with divergence-free drift were recently studied by many authors (see e.g. [7], [9], [12]). The condition $\operatorname{div} b = 0$ arises naturally from the quasi-geostrophic equation, where the drift is of the form $b(x) = (b_1(x), b_2(x)) = (-\partial_{x_2}\Psi, \partial_{x_1}\Psi)$ for some Ψ (see e.g. [22], [10]). Interestingly, this additional assumption on the divergence of b also allows for considering more singular drifts in Morrey class. The present paper may be considered as a contribution to this theory.

Here is a summary of our approach. As in [13] we will employ the perturbation series, but in the present case the conditions on b only ensure the finiteness, rather than smallness, of p_1 (see Lemma 5). In fact, the integral in (1.10) may fail to be absolutely convergent as a double integral, and has to be interpreted as an iterated integral. To clarify, each iterated integral converges absolutely, but the convergence of the second iterated integral depends on subtle cancellations in the first iterated integral. This makes our proofs much more complicated and delicate. In particular, it is not obvious a priori that the functions p_n are well defined. To study p_n, we represent them as integrals of auxiliary functions $P_n(t,x,y,\underline{s},\underline{z})$ (see (3.1)), which we integrate over $(\mathbb{R}^d)^n$ and the n-dimensional simplex $S_n(0,t)$ (see (4.6)). We simultaneously consider the functions

$$|p|_n(t,x,y) = \int\limits_{S_n(0,t)} \Big| \int\limits_{(\mathbb{R}^d)^n} P_n(t,x,y,\underline{s},\underline{z})\, d\underline{z} \Big|\, d\underline{s},$$

majorants of p_n (see (3.3) and (4.6)). We use induction to estimate $|p|_n$ for $n \geq 2$ and to this end we split the integral over the simplex $S_n(0,t)$ into suitable $n+1$ parts. As a consequence, Motzkin numbers appear in the estimates of p_n (see [15] for another connection of the perturbation series with combinatorics). In order to ensure the convergence of the perturbation series, a good bound for p_1 is needed, which turns out to be a consequence of the smallness assumption $\|b\|_{M_1^{1-\alpha}} < \eta$ in Theorem 1. We expect the conclusion of Theorem 1 to hold if $\|b\|_{M_1^{1-\alpha}}$ is merely finite, but such an extension calls for different methods.

One of the tools used in this paper is the so called 3P theorem (see [5], [16], [17]). It allows us to suitably split a ratio of three functions p, when estimating p_n. Since for $\alpha = 2$ (Gaussian case) the 3P theorem does not hold, our method cannot be applied to perturbations of the classical Laplacian.

The paper is organized as follows. In Section 2 we collect basic properties of the transition density $p(t,x,y)$. In Section 3 we define and estimate the functions p_n. In Section 4 we prove Theorem 1.

All the functions considered are Borel measurable. When we write $f(x) \approx g(x)$, we mean that there is a number $0 < C < \infty$ independent of x, i.e. a *constant*, such that for every x we have $C^{-1}f(x) \leq g(x) \leq Cf(x)$. As usual we write $a \wedge b = \min(a,b)$ and $a \vee b = \max(a,b)$. The notation $C = C(a,b,\ldots,c)$ means that C is a constant which depends *only* on $a,b,\ldots,c$.

2. Preliminaries. Throughout the paper $d \geq 1$, and unless stated otherwise, $\alpha \in (1,2)$. In Lemmas 1, 2, 3 we recall well-known results about the density $p(t,x,y)$ of the isotropic d-dimensional α-stable process (see [5] for details).

LEMMA 1. *There exists a constant C such that, for $t \in (0,\infty)$ and $x \in \mathbb{R}^d$,*

$$C^{-1}\Big[t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}\Big] \leq p(t,x) \leq C\Big[t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}\Big]. \tag{2.1}$$

LEMMA 2 (3P). *There exists a constant C such that, for $s, t \in (0,\infty)$ and $x, y \in \mathbb{R}^d$,*

$$p(t,x,z)p(s,z,y) \le Cp(t+s,x,y)[p(t,x,z)+p(s,z,y)].$$

Let $p^{(m)}$ be the α-stable density in dimension m.

LEMMA 3. *For all $t > 0$ and $x \in \mathbb{R}^d$,*

$$\nabla_x p^{(d)}(t,x) = -2\pi x p^{(d+2)}(t,\tilde{x}), \tag{2.2}$$

where $\tilde{x} \in \mathbb{R}^{d+2}$ is such that $|\tilde{x}| = |x|$.

Applying (2.1) to (2.2), we get

$$|\nabla_x p(t,x)| \le Ct^{-1/\alpha} p(t,x), \qquad t \in (0,\infty),\ x \in \mathbb{R}^d. \tag{2.3}$$

We also note that by Lemma 2 and (1.4),

$$\begin{aligned} \int_{\mathbb{R}^d} p(t-s,x,z)|b(z)|p(s,z,y)\,dz \\ &\le cp(t,x,y) \int_{\mathbb{R}^d} (p(t-s,x,z)+p(s,z,y))|b(z)|\,dz \\ &\le cC_b[(t-s)^{1/\alpha-1} + s^{1/\alpha-1}]p(t,x,y). \end{aligned} \tag{2.4}$$

Our aim is to prove that the functions p_n defined in (1.9) and (1.10) satisfy $|p_n(t,x,y)| \le C_n p(t,x,y)$, where C_n are constants with $\sum_{n=0}^{\infty} C_n < \infty$. This requires appropriate assumptions on b. We will ensure smallness of

$$\int_0^t \Big| \int_{\mathbb{R}^d} p(t-s,x,z)b(z)\cdot\nabla_z p(s,z,y)\,dz \Big|\,ds. \tag{2.5}$$

To estimate the integral above, we will use the following lemma.

LEMMA 4. *For all $s, t \ge 0$ and $x, y \in \mathbb{R}^d$, we have*

$$\int_{\mathbb{R}^d} p(t,x,z)b(z)\cdot\nabla_z p(s,z,y)\,dz = -\int_{\mathbb{R}^d} \nabla_z p(t,x,z)\cdot b(z)p(s,z,y)\,dz. \tag{2.6}$$

Proof. Let $g \in C_c^\infty(\mathbb{R}^d)$ be such that

$$g(z) = \begin{cases} 1 & \text{for } |z| \le 1, \\ 0 & \text{for } |z| \ge 2. \end{cases}$$

Then the function $f_n(z) := g(z/n)p(t,x,z)p(s,z,y)$ is in $C_c^\infty(\mathbb{R}^d)$ and we have $\nabla_z f_n(z) \to \nabla_z(p(t,x,z)p(s,z,y))$ as $n \to \infty$. Furthermore,

$$|\nabla_z f_n(z)| \le c(|\nabla_z(p(t,x,z)p(s,z,y))| + p(t,x,z)p(s,z,y))$$

for some constant $c > 0$. By (2.3) and (2.4),

$$\int_{\mathbb{R}^d} [|\nabla_z(p(t,x,z)p(s,z,y))| + p(t,x,z)p(s,z,y)]|b(z)|\,dz < \infty.$$

Therefore, by (1.3) and Lebesgue's theorem,

$$0 = \lim_{n\to\infty} \int_{\mathbb{R}^d} \nabla_z f_n(z) \cdot b(z)\, dz = \int_{\mathbb{R}^d} \nabla_z (p(t,x,z)p(s,z,y)) \cdot b(z)\, dz,$$

which ends the proof. ■

Similarly, condition (1.3) implies that for all $s, t > 0$ and $\xi, y \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} [b(\xi) \cdot \nabla_\xi p(t,\xi,z)] b(z) \cdot \nabla_z p(s,z,y)\, dz = - \int_{\mathbb{R}^d} \nabla_z [b(\xi) \cdot \nabla_\xi p(t,\xi,z)] \cdot b(z) p(s,z,y)\, dz. \tag{2.7}$$

In the following lemma we will use (2.6) to show that the function p_1 introduced in (1.10) is well defined. In a similar way we will apply (2.7) to estimate p_n, $n \geq 2$.

LEMMA 5. *There exists a constant C such that for all $t>0$ and $x, y \in \mathbb{R}^d$,*

$$\int_0^t \Big| \int_{\mathbb{R}^d} p(t-s,x,z) b(z) \cdot \nabla_z p(s,z,y)\, dz \Big|\, ds \leq C p(t,x,y). \tag{2.8}$$

Proof. By Lemma 4, (2.3) and (2.4), we obtain

$$\begin{aligned}
&\int_0^t \Big| \int_{\mathbb{R}^d} p(t-s,x,z) b(z) \cdot \nabla_z p(s,z,y)\, dz \Big|\, ds \\
&\quad = \int_0^{t/2} \Big| \int_{\mathbb{R}^d} p(t-s,x,z) b(z) \cdot \nabla_z p(s,z,y)\, dz \Big|\, ds \\
&\qquad + \int_{t/2}^t \Big| \int_{\mathbb{R}^d} p(t-s,x,z) b(z) \cdot \nabla_z p(s,z,y)\, dz \Big|\, ds \\
&\quad = \int_0^{t/2} \Big| \int_{\mathbb{R}^d} \nabla_z p(t-s,x,z) \cdot b(z) p(s,z,y)\, dz \Big|\, ds \\
&\qquad + \int_{t/2}^t \Big| \int_{\mathbb{R}^d} p(t-s,x,z) b(z) \cdot \nabla_z p(s,z,y)\, dz \Big|\, ds \leq c p(t,x,y). \quad ■
\end{aligned}$$

In the following example we will see that

$$\int_0^t \int_{\mathbb{R}^d} |p(t-s,x,z) b(z) \cdot \nabla_z p(s,z,y)|\, dz\, ds = \infty$$

for some functions $b \in M_1^{1-\alpha}$. This should be compared with (2.8).

EXAMPLE 1. Let $d = 2$, $\alpha \in (1, 2)$ and $b_1(y) = (|y_2|^{1-\alpha}, |y_1|^{1-\alpha})$. Clearly, $\operatorname{div} b_1 = 0$, and

$$\int_{B(x,t)} |b_1(y)|\,dy \le \int_{B(0,t)} (|y_2|^{1-\alpha} + |y_1|^{1-\alpha})\,dy_2\,dy_1 = \frac{4\sqrt{\pi}\Gamma(1-\alpha/2)}{(3-\alpha)\Gamma((3-\alpha)/2)} t^{3-\alpha}.$$

Hence,

$$\|b\|_{M_1^{1-\alpha}} \le \frac{4\sqrt{\pi}\Gamma(1-\alpha/2)}{(3-\alpha)\Gamma((3-\alpha)/2)}.$$

Furthermore, taking $x = y = 0$ and applying (2.2) and (2.1), we get

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^d} &|p(t-s,z)b_1(z)\cdot\nabla_z p(s,z)|\,dz\,ds \\ &= 2\pi \int_0^t \int_{\mathbb{R}^d} p(t-s,z)|b_1(z)\cdot z| p^{(d+2)}(s,z)\,dz\,ds \\ &\ge c \int_0^{t/2} \int_{B(0,s^{1/\alpha})} (t-s)^{-2/\alpha}|b_1(z)\cdot z| s^{-4/\alpha}\,dz\,ds \\ &\ge c \int_0^{t/2} \int_{B(0,1)} s^{-1}(t-s)^{-2/\alpha}|b_1(w)\cdot w|\,dw\,ds = \infty.\end{aligned}$$

We note that Lemma 5 extends to $\alpha = 1$, but the following lemma does not, and this is why we generally assume $\alpha \in (1, 2)$ in this paper. Lemma 6 will allow us to estimate the functions p_n for $n \ge 2$.

LEMMA 6. *There exists a constant C such that for all $t>0$ and $x, y \in \mathbb{R}^d$,*

$$\begin{aligned}(2.9)\qquad &\int_0^{t/2} \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dw\,d\xi\,dr\,du \\ &p(u,x,\xi)\big|\nabla_w\big(b(\xi)\cdot\nabla_\xi p(r-u,\xi,w)\big)\cdot b(w)\big|p(t-r,w,y) < Cp(t,x,y).\end{aligned}$$

Proof. First, we show that there is a constant c_1 such that for all $t > 0$ and $z, w \in \mathbb{R}^d$,

$$|b(z)\cdot\nabla_z(b(w)\cdot\nabla_w p(t,z,w))| \le c_1|b(z)|\,|b(w)|p(t,z,w)t^{-2/\alpha}. \tag{2.10}$$

From (2.2) we get

$$\nabla_w p^{(d)}(t,z,w) = 2\pi(z-w)p^{(d+2)}(t,\tilde z,\tilde w),$$
$$\nabla_z p^{(d+2)}(t,\tilde z,\tilde w) = -2\pi(z-w)p^{(d+4)}(t,\hat z,\hat w),$$

where $\tilde{z} = (z, 0, 0) \in \mathbb{R}^{d+2}$ and $\hat{z} = (\tilde{z}, 0, 0) \in \mathbb{R}^{d+4}$ (and $\tilde{w}$, $\hat{w}$ are defined accordingly). Therefore,

$$
\begin{aligned}
&b(z) \cdot \nabla_z(b(w) \cdot \nabla_w p(t, z, w)) = 2\pi b(z) \cdot \nabla_z[b(w) \cdot (z - w)p^{(d+2)}(t, \tilde{z}, \tilde{w})] \\
&= 2\pi b(z) \cdot [b(w)p^{(d+2)}(t, \tilde{z}, \tilde{w}) - 2\pi(b(w) \cdot (z - w))(z - w)p^{(d+4)}(t, \hat{z}, \hat{w})] \\
&= 2\pi b(z) \cdot b(w)p^{(d+2)}(t, \tilde{z}, \tilde{w}) - 4\pi^2(b(w) \cdot (z-w))b(z) \cdot (z-w)p^{(d+4)}(t, \hat{z}, \hat{w}).
\end{aligned}
$$

Applying (2.1), we obtain (2.10).

Now, by (2.10) and (2.4),

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p(u, x, \xi)|\nabla_w(b(\xi) \cdot \nabla_\xi p(r - u, \xi, w)) \cdot b(w)|p(t - r, w, y)\, dw\, d\xi \\
&\le c_1 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p(u, x, \xi)|b(\xi)|(r - u)^{-2/\alpha} p(r - u, \xi, w)|b(w)|p(t - r, w, y)\, dw\, d\xi \\
&\le C_b c_2 \int_{\mathbb{R}^d} |b(\xi)|(r-u)^{-2/\alpha} p(u, x, \xi)p(t-u, \xi, y)((t-r)^{1/\alpha-1} + (r-u)^{1/\alpha-1})\, d\xi \\
&\le C_b^2 c_2 p(t, x, y)(r-u)^{-2/\alpha}((t-r)^{1/\alpha-1} + (r-u)^{1/\alpha-1})((t-u)^{1/\alpha-1} + u^{1/\alpha-1}).
\end{aligned}
$$

Now, we only need to show that for all $t > 0$,

$$
\int_0^{t/2}\int_{t/2}^{t} (r-u)^{-2/\alpha}((t-r)^{1/\alpha-1} + (r-u)^{1/\alpha-1})((t-u)^{1/\alpha-1} + u^{1/\alpha-1})\, dr\, du < c_3,
$$

for some constant c_3. By homogeneity, we may consider only $t = 1$. We note that $a^p + b^p \le 2^{1-p}(a + b)^p$ for $a, b \ge 0$ and $0 < p < 1$. Consequently, $a^{-p} + b^{-p} \le 2^{1-p}(a+b)^p(ab)^{-p}$. Hence, it suffices to show convergence of the integral

$$
\int_0^{1/2}\int_{1/2}^{1} (r - u)^{-1/\alpha-1}(1 - r)^{1/\alpha-1}u^{1/\alpha-1}\, dr\, du.
$$

Splitting the second integral into integrals on the intervals $(1/2, 3/4)$ and $(3/4, 1)$, we get

$$
\int_0^{1/2}\int_{1/2}^{1} (r - u)^{-1/\alpha-1}(1 - r)^{1/\alpha-1}u^{1/\alpha-1}\, dr\, du \le c \int_0^{1/2} (1/2 - u)^{-1/\alpha}u^{1/\alpha-1}du < \infty. \blacksquare
$$

3. Perturbation series. In this section we study the functions $|p|_n$, which are majorants of p_n (see (1.10)). For $a < b$ and $n \ge 1$, we denote

$$
S_n(a, b) = \{(s_1, \ldots, s_n) \in \mathbb{R}^n : a \le s_1 \le \cdots \le s_n \le b\}.
$$

For $t > 0$ and $x, y \in \mathbb{R}^d$, let

$$(3.1)\quad P_n(t,x,y,\underline{s},\underline{z}) = p(s_1,x,z_1)b(z_1)\cdot\nabla_{z_1}p(s_2-s_1,z_1,z_2)\dots b(z_n)\cdot\nabla_{z_n}p(t-s_n,z_n,y),$$

where $\underline{s} = (s_1,\dots,s_n) \in S_n(0,t)$ and $\underline{z} = (z_1,\dots,z_n) \in (\mathbb{R}^d)^n$.

Definition 7. *For any $t > 0$ and $x, y \in \mathbb{R}^d$, we define*

$$(3.2)\qquad |p|_0(t,x,y) = p(t,x,y),$$

$$(3.3)\qquad |p|_n(t,x,y) = \int_{S_n(0,t)} \Big| \int_{(\mathbb{R}^d)^n} P_n(t,x,y,\underline{s},\underline{z})\,d\underline{z}\Big|\,d\underline{s}.$$

We note that for $t > 0$, $x, y \in \mathbb{R}^d$ and $\underline{s}$ in the interior of $S_n(0,t)$, by Lemma 2, (1.4) and (2.3),

$$(3.4)\qquad \int_{(\mathbb{R}^d)^n} |P_n(t,x,y,\underline{s},\underline{z})|\,d\underline{z} < \infty.$$

Hence, the functions $|p|_n$ are well-defined (possibly infinite). The integral

$$\int_{S_n(0,t)}\int_{(\mathbb{R}^d)^n} |P_n(t,x,y,\underline{s},\underline{z})|\,d\underline{z}\,d\underline{s}$$

may be divergent because singularities of the gradient of the functions p in (3.1) may not be integrable in the whole simplex $S_n(0,t)$ (see Example 1). To estimate (3.3), we will use the decomposition

$$(3.5)\quad S_n(0,t) = S_n(0,t/2) \cup \Big(\bigcup_{k=1}^{n-1} S_{n-k}(0,t/2)\times S_k(t/2,t)\Big) \cup S_n(t/2,t),$$

along with Lemma 4 and (2.7) to move these singularities outside the region of integration.

Lemma 8. *If $1 \le k \le n-1$, $t > 0$, and $x, y \in \mathbb{R}^d$, then*

$$(3.6)\qquad \int_{S_{n-k}(0,t/2)\times S_k(t/2,t)} \Big| \int_{(\mathbb{R}^d)^n} P_n(t,x,y,\underline{s},\underline{z})\,d\underline{z}\Big|\,d\underline{s}$$
$$\le \int_0^{t/2}\int_{t/2}^{t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} dw\,d\xi$$
$$|p|_{n-k-1}(u,x,\xi)|\nabla_w(b(\xi)\cdot\nabla_\xi p(r-u,\xi,w))\cdot b(w)||p|_{k-1}(t-r,w,y)\,dr\,du.$$

Proof. By (3.4) and Fubini's theorem, we may change the order of integration in integrals over $(\mathbb{R}^d)^n$. We note that

$$(3.7)\qquad |p|_m(t-r,x,y) = \int_{S_m(r,t)} \Big| \int_{(\mathbb{R}^d)^m} P_m(t-r,x,y,\underline{s},\underline{z})\,d\underline{z}\Big|\,d\underline{s}.$$

Changing the order of integration:

$$\int_{S_m(a,b)} f(\underline{s})\,ds_m\,ds_{m-1}\ldots ds_1 = \int_{(a,b)\times S_{m-1}(a,s_m)} f(\underline{s})\,ds_{m-1}\ldots ds_1\,ds_m,$$

and using (2.7) and Fubini's theorem, we get

$$\begin{aligned}
&\int_{S_{n-k}(0,t/2)\times S_k(t/2,t)} \Big| \int_{(\mathbb{R}^d)^n} P_n(t,x,y,\underline{s},\underline{\xi})\,d\underline{\xi}\Big|\,d\underline{s} \\
&= \int_0^{t/2} \int_{S_{n-k-1}(0,r_{n-k})} \int_{t/2}^{t} \int_{S_{k-1}(u_k,t)} d\underline{u}\;du_k\,d\underline{r}\,dr_{n-k} \\
&\quad \Big| \int_{(\mathbb{R}^d)^n} P_{n-k-1}(r_{n-k},x,z_{n-k},\underline{r},\underline{z})b(z_{n-k})\cdot\nabla_{z_{n-k}}p(u_1-r_{n-k},z_{n-k},w_1) \\
&\qquad\qquad b(w_1)\cdot\nabla_{w_1}P_{k-1}(t-u_1,w_1,y,\underline{u},\underline{w})\,d\underline{w}\,dw_1\,d\underline{z}\,dz_{n-k}\Big| \\
&= \int_0^{t/2} \int_{S_{n-k-1}(0,r_{n-k})} \int_{t/2}^{t} \int_{S_{k-1}(u_k,t)} d\underline{u}\;du_k\,d\underline{r}\,dr_{n-k} \\
&\qquad \Big| \int_{(\mathbb{R}^d)^n} P_{n-k-1}(r_{n-k},x,z_{n-k},\underline{r},\underline{z})P_{k-1}(t-u_1,w_1,y,\underline{u},\underline{w}) \\
&\qquad b(w_1)\cdot\nabla_{w_1}[b(z_{n-k})\cdot\nabla_{z_{n-k}}p(u_1-r_{n-k},z_{n-k},w_1)]\,d\underline{w}\,dw_1\,d\underline{z}\,dz_{n-k}\Big|.
\end{aligned}$$

Here $\underline{r}=(r_1,\ldots,r_{n-k-1})$, $\underline{u}=(u_1,\ldots,u_{k-1})$, $\underline{z}=(z_1,\ldots,z_{n-k-1})$ and $\underline{w}=(w_2,\ldots,w_k)$. Splitting the integral over $(\mathbb{R}^d)^n$ into integrals over $(\mathbb{R}^d)^{n-k-1}$, $(\mathbb{R}^d)^{k-1}$ and $(\mathbb{R}^d)^2$, and applying (3.7), we get (3.6). ∎

Lemma 9. *For $n\ge 2$, $t>0$ and $x,y\in\mathbb{R}^d$,*

$$\begin{aligned}
|p|_n(t,x,y) \le{}& \int_0^{t/2}\int_{\mathbb{R}^d} |p|_{n-1}(u,x,z)|b(z)\cdot\nabla_z p(t-u,z,y)|\,dz\,du \\
&+ \int_{t/2}^{t}\int_{\mathbb{R}^d} |\nabla_z p(u,x,z)\cdot b(z)||p|_{n-1}(t-u,z,y)\,dz\,du \\
&+ \sum_{k=0}^{n-2}\int_0^{t/2}\int_{t/2}^{t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} dw\,dz\,dr\,du\,|p|_k(u,x,z) \\
&\times |b(z)\cdot\nabla_z(b(w)\cdot\nabla_w p(r-u,z,w))||p|_{n-2-k}(t-r,w,y).
\end{aligned}$$

Proof. By (3.5), we get

$$|p|_n(t,x,y) = \int\limits_{S_n(0,t/2)} \Big| \int\limits_{(\mathbb{R}^d)^n} P_n(t,x,y,\underline{s},\underline{\xi})\, d\underline{\xi} \Big|\, d\underline{s} \tag{3.8}$$

$$+ \int\limits_{S_n(t/2,t)} \Big| \int\limits_{(\mathbb{R}^d)^n} P_n(t,x,y,\underline{s},\underline{\xi})\, d\underline{\xi} \Big|\, d\underline{s} \tag{3.9}$$

$$+ \int\limits_{\bigcup_{k=1}^{n-1} S_{n-k}(0,t/2)\times S_k(t/2,t)} \Big| \int\limits_{(\mathbb{R}^d)^n} P_n(t,x,y,\underline{s},\underline{\xi})\, d\underline{\xi} \Big|\, d\underline{s} \tag{3.10}$$

The integral in (3.8) is estimated as follows:

$$\int\limits_{S_n(0,t/2)} \Big| \int\limits_{(\mathbb{R}^d)^n} P_n(t,x,y,\underline{s},\underline{\xi})\, d\underline{\xi} \Big|\, d\underline{s}$$

$$= \int\limits_0^{t/2} \int\limits_{S_{n-1}(0,s_n)} \Big| \int\limits_{(\mathbb{R}^d)^n} P_{n-1}(s_n,x,\xi_n,\underline{s}^*,\underline{\xi}^*) b(\xi_n)\cdot \nabla_{\xi_n} p(t-s_n,\xi_n,y)\, d\underline{\xi} \Big|\, d\underline{s}$$

$$\le \int\limits_0^{t/2} \int\limits_{S_{n-1}(0,s_n)} \int\limits_{\mathbb{R}^d} d\xi_n\, d\underline{s}^*\, ds_n$$

$$\times |b(\xi_n)\nabla_{\xi_n} p(t-s_n,\xi_n,y)| \Big| \int\limits_{(\mathbb{R}^d)^{n-1}} P_{n-1}(s_n,x,\xi_n,\underline{s}^*,\underline{\xi}^*)\, d\underline{\xi}^* \Big|$$

$$\le \int\limits_0^{t/2} \int\limits_{\mathbb{R}^d} |p|_{n-1}(s_n,x,\xi_n) |b(\xi_n)\cdot \nabla_{\xi_n} p(t-s_n,\xi_n,y)|\, d\xi_n\, ds_n,$$

where $\underline{s}^* = (s_1,\ldots,s_{n-1})$ and $\underline{\xi}^* = (\xi_1,\ldots,\xi_{n-1})$. Applying Lemma 4 and using a similar method, we estimate (3.9). Next, we split (3.10) into $n-1$ integrals over the sets $S_{n-k}(0,t/2)\times S_k(t/2,t)$ and apply Lemma 8 to each integral. ∎

By using the lemmas from Sections 2, 3 and induction, we will show that all functions $|p|_n$ are finite, and consequently, the functions p_n are well defined. Detailed estimates will be given in the next section.

4. Proof of Theorem 1. Before we pass to the proofs of the main theorem we briefly introduce the Motzkin numbers. In combinatorics the Motzkin number M_n represents the number of different ways of drawing non-intersecting chords on a circle between n points ([20]). The recurrence relation

$$M_n = M_{n-1} + \sum_{k=0}^{n-2} M_k M_{n-2-k}, \qquad M_0 = M_1 = 1, \tag{4.1}$$

leads to the generating function (see [26])

$$M(x)=\sum_{n=0}^{\infty} M_n x^n = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}. \tag{4.2}$$

We may now prove the main estimates of this paper.

LEMMA 10. *There is a constant C such that for all $t>0$, $x,y\in\mathbb{R}^d$ and $n\ge 1$,*

$$|p|_n(t,x,y)\le M_n C^n p(t,x,y). \tag{4.3}$$

Proof. Let c_1 be the constant such that (see the proof of Lemma 5)

$$\begin{aligned}\int_0^{t/2}\int_{\mathbb{R}^d} p(s,x,z)|b(z)\cdot\nabla_z p(t-s,z,y)|\,dz\,ds\\ +\int_{t/2}^{t}\int_{\mathbb{R}^d} |\nabla_z p(s,x,z)\cdot b(z)|p(t-s,z,y)\,dz\,ds \le c_1 p(t,x,y).\end{aligned} \tag{4.4}$$

Let $C=c_1\vee\sqrt{c_2}$, where c_2 is the constant from Lemma 6.

We use induction. For $n=1$ we apply Lemma 5. Suppose (4.3) holds for $n=1,\ldots,k-1$. By Lemma 9, we get

$$\begin{aligned}|p|_k(t,x,y)&\le C^{k-1}M_{k-1}\int_0^{t/2}\int_{\mathbb{R}^d} p(u,x,z)|b(z)\cdot\nabla_z p(t-u,z,y)|\,dz\,du\\
&\quad + C^{k-1}M_{k-1}\int_{t/2}^{t}\int_{\mathbb{R}^d}|\nabla_z p(u,x,z)\cdot b(z)|p(t-u,z,y)\,dz\,du\\
&\quad +\sum_{j=0}^{k-2} C^j M_j C^{k-2-j}M_{k-2-j}\int_0^{t/2}\int_{t/2}^{t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} dw\,dz\,dr\,du\\
&\qquad\qquad p(u,x,z)|b(z)\cdot\nabla_z(b(w)\cdot\nabla_w p(r-u,z,w))|p(t-r,w,y).\end{aligned}$$

Now by (4.4), Lemma 6 and (4.1), we obtain

$$\begin{aligned}|p|_k(t,x,y)&\le\Big(C^{k-1}c_1M_{k-1}+\sum_{j=0}^{k-2}C^jM_jC^{k-2-j}M_{k-2-j}c_2\Big)p(t,x,y)\\
&\le C^k\Big(M_{k-1}+\sum_{j=0}^{k-2}M_jM_{k-2-j}\Big)p(t,x,y)=C^kM_kp(t,x,y). \quad\blacksquare\end{aligned}$$

COROLLARY 11. *The functions p_n, $n \in \mathbb{N}$, may be defined by* (1.9) *and* (1.10) *and there exists a constant C such that for all $t > 0$, $x, y \in \mathbb{R}^d$ and $n \geq 1$,*

$$|p_n(t,x,y)| \leq M_n C^n p(t,x,y). \tag{4.5}$$

We note that for any $C > 0$, (4.5) holds provided $\|b\|_{M_1^{1-\alpha}}$ is sufficiently small (see remark below (1.4)).

Proof of Corollary 11. We simultaneously prove the above estimates of p_n and the fact that they are well defined. To this end we will first show that for $n \geq 1$,

$$p_n(t,x,y) = \int\limits_{S_n(0,t)} \Big[\int\limits_{(\mathbb{R}^d)^n} P_n(t,x,y,\underline{s},\underline{z})\, d\underline{z} \Big]\, d\underline{s}, \qquad t > 0,\ x,y \in \mathbb{R}^d. \tag{4.6}$$

According to our discussion in the Introduction, the right hand side of (4.6) should be considered an iterated integral. The inner integral is absolutely convergent (see (3.4)). By Lemma 10,

$$|p|_n(t,x,y) = \int\limits_{S_n(0,t)} \Big| \int\limits_{(\mathbb{R}^d)^n} P_n(t,x,y,\underline{s},\underline{z})\, d\underline{z} \Big|\, d\underline{s} < \infty,$$

so the right-hand side of (4.6) is well-defined.

To prove (4.6) we use induction. For $n = 1$, (4.6) matches the definition of p_1. Suppose (4.6) holds for $n \in \mathbb{N}$. By Lemmas 10, 2 and 3, for $u \in (0,t)$,

$$\begin{aligned}\int\limits_{\mathbb{R}^d} \int\limits_{S_n(0,t-u)} \Big| \int\limits_{(\mathbb{R}^d)^n} & P_n(t-u,x,\xi,\underline{s},\underline{z})\, d\underline{z} \Big|\, |b(\xi)|\, |\nabla_\xi p(u,\xi,y)|\, d\underline{s}\, d\xi \\ &\leq c \int\limits_{\mathbb{R}^d} p(t-u,x,\xi) |b(\xi)| u^{-1/\alpha} p(u,\xi,y)|\, d\xi \\ &\leq c((t-u)^{1/\alpha-1} + u^{1/\alpha-1}) u^{-1/\alpha} p(t,x,y) < \infty.\end{aligned} \tag{4.7}$$

Therefore, by Fubini's theorem with respect to $d\underline{s}\, d\xi$, and by (4.7),

$$\begin{aligned} p_{n+1}(t,x,y) &= \int_0^t \int\limits_{\mathbb{R}^d} p_n(t-u,x,\xi) b(\xi) \cdot \nabla_\xi p(u,\xi,y)\, d\xi\, du \\ &= \int_0^t \int\limits_{\mathbb{R}^d} \int\limits_{S_n(0,t-u)} \int\limits_{(\mathbb{R}^d)^n} P_n(t-u,x,\xi,\underline{s},\underline{z})\, d\underline{z}\, d\underline{s} b(\xi) \cdot \nabla_\xi p(u,\xi,y)\, d\xi\, du \\ &= \int_0^t \int\limits_{S_n(0,t-u)} \int\limits_{\mathbb{R}^d} \Big[\int\limits_{(\mathbb{R}^d)^n} P_n(t-u,x,\xi,\underline{s},\underline{z}) b(\xi) \cdot \nabla_\xi p(u,\xi,y)\, d\underline{z} \Big]\, d\xi\, d\underline{s}\, du. \end{aligned}$$

In particular p_{n+1} is well-defined, and $|p_{n+1}| < |p|_{n+1} < \infty$. ∎

Proof of Theorem 1. Let us fix $C > 0$ such that $M(C) < 2$ (see (4.2)). There exists $\eta = \eta(d,\alpha)$ such that (4.5) holds, provided $\|b\|_{M_1^{1-\alpha}} < \eta$, which we will assume in what follows. Then $\tilde{p}(t,x,y) = \sum_{n=0}^{\infty} p_n(t,x,y)$ satisfies

$$\tilde{p}(t,x,y) \le M(C)p(t,x,y),$$

and

$$\begin{aligned}\tilde{p}(t,x,y) &\ge p(t,x,y) - \sum_{n=1}^{\infty} |p_n(t,x,y)| \\ &= 2p(t,x,y) - \sum_{n=0}^{\infty} |p_n(t,x,y)| \ge (2 - M(C))p(t,x,y).\end{aligned}$$

We next prove that for $\phi \in C_c^\infty(\mathbb{R},\mathbb{R}^d)$, $s \in \mathbb{R}$ and $x \in \mathbb{R}^d$,

$$\int_s^\infty \int_{\mathbb{R}^d} \tilde{p}(u-s,x,z)(\partial_u \phi(u,z) + \Delta_z^{\alpha/2}\phi(u,z) + b(z)\cdot\nabla_z \phi(u,z))\,dz\,du = -\phi(s,x).$$

By the definition of $\tilde{p}$, we get

$$\begin{aligned}\tilde{p}(t,x,y) &= p(t,x,y) + \sum_{n=1}^{\infty} p_n(t,x,y) \\ &= p(t,x,y) + \sum_{n=1}^{\infty} \int_0^t \int_{\mathbb{R}^d} p_{n-1}(t-s,x,z)b(z)\cdot\nabla_z p(s,z,y)\,dz\,ds \\ &= p(t,x,y) + \int_0^t \int_{\mathbb{R}^d} \tilde{p}(t-s,x,z)b(z)\cdot\nabla_z p(s,z,y)\,dz\,ds.\end{aligned} \tag{4.8}$$

Here the application of Fubini's theorem is justified as in the proof of (4.6). The rest of the proof is the same as in [17, Theorem 1]. ■

COROLLARY 12. *The function $\tilde{p}$ satisfies the Chapman–Kolmogorov equation*

$$\int_{\mathbb{R}^d} \tilde{p}(s,x,z)\tilde{p}(t,z,y)\,dz = \tilde{p}(t+s,x,y), \qquad s,t>0,\ x,y \in \mathbb{R}^d, \tag{4.9}$$

and the family of operators $\tilde{P}_t$, defined by

$$\tilde{P}_t f(x) = \int_{\mathbb{R}^d} \tilde{p}(t,x,y)f(y)\,dy,$$

forms a Markov semigroup with (weak) generator $\Delta^{\alpha/2} + b(x)\cdot\nabla_x$.

Proof. For the proof of (4.9) see [17, Lemmas 15, 16]. By (4.8), (1.7) and Fubini's theorem,

$$\int_{\mathbb{R}^d} \tilde{p}(t,x,y)\,dy = \int_{\mathbb{R}^d} \Big(p(t,x,y) + \int_0^t \int_{\mathbb{R}^d} \tilde{p}(t-s,x,z)b(z)\cdot \nabla_z p(s,z,y)\,dz\,ds\Big)\,dy$$
$$= 1 + \int_0^t \int_{\mathbb{R}^d} \tilde{p}(t-s,x,z)b(z)\cdot \nabla_z\Big(\int_{\mathbb{R}^d} p(s,z,y)\,dy\Big)\,dz\,ds = 1.$$

Now, let $f,g \in C_c^\infty(\mathbb{R}^d)$. We will show that

$$\lim_{t\to 0} \int_{\mathbb{R}^d} \frac{\tilde{P}_t f(x) - f(x)}{t}\, g(x)\,dx = \int_{\mathbb{R}^d} (\Delta^{\alpha/2} f(x) + b(x)\cdot \nabla f(x)) g(x)\,dx. \tag{4.10}$$

By (4.8),

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{\tilde{P}_t f(x) - f(x)}{t}\, g(x)\,dx \\
&\quad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{p(t,x,y)(f(y)-f(x))}{t}\, g(x)\,dy\,dx \\
&\qquad + \frac{1}{t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d} \tilde{p}(t-s,x,z)b(z)\cdot\nabla_z p(s,z,y) f(y) g(x)\,dz\,ds\,dy\,dx \\
&\quad = I_1(t) + I_2(t).
\end{aligned}$$

The first summand converges to $\int_{\mathbb{R}^d} \Delta^{\alpha/2} f(x) g(x) dx$. By a careful use of Fubini's theorem,

$$I_2(t) = \frac{1}{t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^t \tilde{p}(t-s,x,z)p(s,z,y)b(z)\cdot \nabla_y f(y) g(x)\,dz\,ds\,dy\,dx.$$

If we denote by $p^{(b)}$ the function p perturbed by b then $\tilde{p}(t,x,y) = p^{(b)}(t,x,y)$ $= p^{(-b)}(t,y,x)$. Hence, $\int_{\mathbb{R}^d} \tilde{p}(t,x,y)dx = 1$ for $t>0$ and $y \in \mathbb{R}^d$. Therefore by (1.7),

$$\begin{aligned}
\Big| I_2(t) - \int_{\mathbb{R}^d} b(z)\cdot\nabla f(z)\, g(z)\,dz\Big| &\le c \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^t \frac{p(t-s,x,z)p(s,z,y)}{t} |b(z)| \\
&\quad \times |\nabla_y f(y) g(x) - \nabla_z f(z) g(z)|\,dz\,ds\,dy\,dx.
\end{aligned}$$

To prove that the last expression converges to 0 as $t\to 0$ we may follow the proof of [5, Theorem 1] with some slight modifications concerning the conditions on b. ∎

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Tomasz Jakubowski
Institute of Mathematics and Computer Science
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland
E-mail: Tomasz.Jakubowski@pwr.wroc.pl

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