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Norm convergence of some power series of operators in L^p with applications in ergodic theory

by

Christophe Cuny (Nouméa)

Abstract. Let X be a closed subspace of $L^p(\mu)$, where μ is an arbitrary measure and $1 < p < \infty$. Let U be an invertible operator on X such that $\sup_{n\in\mathbb{Z}} \|U^n\| < \infty$. Motivated by applications in ergodic theory, we obtain (optimal) conditions for the convergence of series like $\sum_{n\geq 1} (U^n f)/n^{1-\alpha}$, $0 \leq \alpha < 1$, in terms of $\|f + \cdots + U^{n-1}f\|_p$, generalizing results for unitary (or normal) operators in $L^2(\mu)$. The proofs make use of the spectral integration initiated by Berkson and Gillespie and, more particularly, of results from a paper by Berkson–Bourgain–Gillespie.

1. Introduction. Let $(\mathcal{M}, \mu)$ be an arbitrary measure space. Fix $1 < p < \infty$ and let X be a closed subspace of $L^p(\mu)$. Let U be an invertible operator on X, power bounded in the following sense: $\sup_{n\in\mathbb{Z}} \|U^n\| < \infty$. We will call such an operator *doubly power bounded*.

It is known (see Berkson–Gillespie [2] and the references therein) that such an operator admits a spectral decomposition consisting of projections acting in X. We will mostly refer to the paper of Berkson–Bourgain–Gillespie [1] for the properties we need.

One of our purposes is to obtain conditions on $f \in X$ that enable one to assign a meaning to singular integrals of the type $\int_{[0,\pi]} (1 - e^{it})^{-\alpha}\, dE(t)f$, $0 < \alpha < 1$, or $\int_{[0,\pi]} \log(1 - e^{it})\, dE(t)f$, where $\{E(t)\}_{t\in[0,2\pi]}$ is a family of projections to be defined later.

This question is of theoretical interest. In the case where U is a unitary operator, the functional calculus arising from spectral theory is much richer than in our situation and it is quite easy to achieve the above mentioned goal (see Gaposhkin [10], [11], or Cuny [6]). It is also shown in [11] that the

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previous question is related to the convergence of the series

$$\sum_{n\geq 1} \frac{U^n f}{n^{1-\alpha}}. \tag{1}$$

Our main interest is to show that it is still the case in the more general situation that we consider now.

As in Gaposhkin [11] (see also [6]), we want to find conditions for the convergence of (1), expressed in terms of $\{\|S_n(f)\|_p\}$, where $S_n(f) = f + \cdots + U^n f$. Hence, when we are concerned with the convergence of (1), the spectral theory will happen to be just a tool, not involved in the conditions.

In particular we will obtain

THEOREM 1.1. *Let $1 < p < \infty$ and $0 < \alpha < 1$. Let $f \in X$ be such that*

$$\sum_{n\geq 1} \frac{1}{n}\left(\frac{\|f + \cdots + U^{n-1}f\|_p}{n^{1-\alpha}}\right)^{\min(p,2)} < \infty. \tag{2}$$

Then the limit $\lim_{u,v\to 0^+} \int_{]u,2\pi-v]} (1-e^{it})^{-\alpha}\, dE(t)f$ exists in $L^p(\mu)$ and $\sum_{n\geq 1} (U^n f)/n^{1-\alpha}$ converges in $L^p(\mu)$.

THEOREM 1.2. *Let $1 < p < \infty$. Let $f \in X$ be such that*

$$\sum_{n\geq 2} \frac{1}{n\log n}\left(\frac{\|f + \cdots + U^{n-1}f\|_p \log n}{n}\right)^{\min(p,2)} < \infty. \tag{3}$$

Then the limit $\lim_{u,v\to 0^+} \int_{]u,2\pi-v]} \log(1-e^{it})\, dE(t)f$ exists in $L^p(\mu)$ and $\sum_{n\geq 1} (U^n f)/n$ converges in $L^p(\mu)$.

REMARK. If U is the isometry induced by an invertible transformation preserving μ, then the series $\sum_{n\geq 1} (U^n f)/n$ even converges μ-a.e. by Theorem 1.2 of [4].

When $p = 2$ and U is a unitary operator we recover the previously known conditions (see [6], or [11] for related results). In this case, it is even proved that (2) (for $p = 2$) is equivalent to the convergence of $\sum_{n\geq 1} (U^n f)/n^{1-\alpha}$.

For $p \neq 2$, we lose equivalence, but we will show that conditions (2) and (3) are optimal in the class of invertible power-bounded operators on some L^p. Finally, we give applications of Theorems 1.1 and 1.2 to pointwise convergence theorems for certain averages of U arising in ergodic theory.

2. The spectral decomposition and main properties. In this section we recall the basic properties of the spectral integration developed by Berkson–Gillespie [2], and we state the transferred theorems from multiplier theory, from [1], that are needed.

Let Y be a Banach space and denote by $\mathcal{B}(Y)$ the Banach algebra of bounded operators on Y. An idempotent element of $\mathcal{B}(Y)$ will be called a *projection*.

Definition. *A spectral family of projections* in the Banach space Y is a uniformly bounded, projection-valued function $F(\cdot) : \mathbb{R} \to \mathcal{B}(Y)$ which is right continuous on $\mathbb{R}$ in the strong operator topology (SOT), has at each $s \in \mathbb{R}$ a SOT left-hand limit (denoted $F(s^-)$), and satisfies:

(i) $F(s)F(t) = F(\min(s,t))$ for all $s, t \in \mathbb{R}$;
(ii) $\lim_{s\to-\infty} F(s) = 0$ (SOT);
(iii) $\lim_{s\to\infty} F(s) = I$ (SOT),

where I denotes the identity operator on Y.

If there is a compact interval $[\alpha, \beta]$ such that $F(\beta) = I$ (hence by (i), $F(s) = I$ for every $s \geq \beta$) and $F(s) = 0$ for every $s < \alpha$, then $F(\cdot)$ is said to be *concentrated on* $[\alpha, \beta]$.

Let $F(\cdot)$ be a spectral family of projections of Y, concentrated on a compact interval $J := [\alpha, \beta]$. Let $BV(J)$ be the Banach algebra of complex functions g having bounded variation on J, with norm $\|g\|_J$ defined by

$$\|g\|_J = |g(\beta)| + \operatorname{var}(g, J),$$

where $\operatorname{var}(g, J)$ denotes the variation of g on J.

Given a partition $\mathcal{P} = (\alpha = \lambda_0, \lambda_1, \ldots, \lambda_n = \beta)$ of J, write

$$\mathcal{S}(g, \mathcal{P}) = g(\alpha)F(\alpha) + \sum_{j=1}^{n} g(\lambda_j)(F(\lambda_j) - F(\lambda_{j-1})).$$

Then $\mathcal{S}(g, \mathcal{P})$ converges SOT as $\mathcal{P}$ runs through the partitions of J directed by refinement. The strong limit of $\mathcal{S}(g, \mathcal{P})$ is denoted by $\int_J g\, dF$ (it was denoted by $\int_J^{\oplus} g\, dF$ in [2]). The mapping $g \mapsto \int_J g\, dF$ is an identity preserving algebra homomorphism of $BV(J)$ into $\mathcal{B}(Y)$ such that

$$\Big\| \int_J g\, dF \Big\| \leq \|g\|_J \sup_{\lambda\in\mathbb{R}} \|F(\lambda)\| \quad \text{for every } g \in BV(J).$$

Let $1 < p < \infty$ and $(\mathcal{M}, \mu)$ be an arbitrary measure space. Let X be a closed subspace of $L^p(\mu)$. Let U be an invertible operator on X, power-bounded in the following sense:

$$c := \sup_{n\in\mathbb{Z}} \|U^n\| < \infty.$$

It follows from [2, Theorem (4.8)] that there is a unique spectral family of projections on X, denoted by $E(\cdot)$, concentrated on $[0, 2\pi]$, such that $E(2\pi^-) = I$ and $U = \int_{[0,2\pi]} e^{it}\, dE(t)$. Since $L^p(\mu)$ is reflexive, $L^p(\mu) =$

$\mathrm{Ker}(I-U)\oplus\overline{(I-U)L^p(\mu)}$. Then $E(0)$ is the corresponding projection onto $\mathrm{Ker}(I-U)$. Therefore, $E(0)f=0$ if and only if $\|f+\cdots+U^{n-1}f\|_p/n\to 0$.

Moreover there exists a constant C_p, depending only on p, such that

$$\sup_{t\in[0,2\pi)}\|E(t)\|\le c^2C_p. \tag{4}$$

Denote by $\mathbb{T}$ the set of unimodular complex numbers. We will identify the set $BV(\mathbb{T})$ of complex functions with bounded variation on $\mathbb{T}$ with a subalgebra of $BV([0,2\pi])$. For every function φ in $BV(\mathbb{T})$ we define its normalization $\tilde\varphi$ by

$$\tilde\varphi(t):=\frac{1}{2}\big(\lim_{s\to t^+}\varphi(e^{is})+\lim_{s\to t^-}\varphi(e^{is})\big)\qquad\forall t\in\mathbb{R}.$$

For every $\varphi\in BV(\mathbb{T})$, define

$$T_\varphi:=\int_{[0,2\pi]}\tilde\varphi(t)\,dE(t).$$

Denote by $M_p(\mathbb{T})$ the space of $\ell^p(\mathbb{Z})$-multipliers, that is, of bounded functions φ on $\mathbb{T}$ such that the convolution with $\{\hat\varphi(-n)\}_{n\in\mathbb{Z}}$ defines a bounded operator of $\ell^p(\mathbb{Z})$.

Recall that by the Stechkin Theorem (see e.g. [9]), $BV(\mathbb{T})$ is contained in $M_p(\mathbb{T})$ and $\|\varphi\|_{M_p(\mathbb{T})}\le C_p(\varphi(1)+\mathrm{var}(\varphi,\mathbb{T}))$. Then we have

Theorem 2.1 (Berkson–Gillespie, [2, Theorems (3.10)(ii) and (4.14)]). *For every $\varphi\in BV(\mathbb{T})$, we have*

$$\|T_\varphi\|\le c^2\|\varphi\|_{M_p(\mathbb{T})}. \tag{5}$$

We now define the dyadic decomposition of $\mathbb{T}$. For $j\in\mathbb{N}$, define $t_{-j}=\pi/2^j$, $t_j=2\pi-\pi/2^j$. Then, for every $j\in\mathbb{Z}$, define $\omega_j=e^{it_j}$, $\varGamma_j=\{e^{it}: t_j<t<t_{j+1}\}$ and $\varLambda_j$ to be the closure of $\varGamma_j$.

The Strong Marcinkiewicz Multiplier Theorem (see e.g. [9]) asserts that a bounded function on $\mathbb{T}$ with bounded variation on each $\varLambda_j$ uniformly bounded with respect to j is in $M_p(\mathbb{T})$:

Theorem 2.2. *Let $\varphi:\mathbb{T}\to\mathbb{C}$ be bounded and such that*

$$\sup_{j\in\mathbb{Z}}\mathrm{var}(\varphi,\varLambda_j)<\infty.$$

Then $\varphi\in M_p(\mathbb{T})$ and

$$\|\varphi\|_{M_p(\mathbb{T})}\le C_p\big(\sup_{z\in\mathbb{T}}|\varphi(z)|+\sup_{j\in\mathbb{Z}}\mathrm{var}(\varphi,\varLambda_j)\big).$$

Given $\varphi\in BV(\mathbb{T})$ the operator T_φ is meaningful as defined before. By (5), the Strong Marcinkiewicz Multiplier Theorem gives a much better control of $\|T_\varphi\|$ than what we would obtain by the Stechkin Theorem.

Denote by Σ_d the dyadic sigma-algebra, that is, the sigma-algebra generated by $\{\omega_j\}_{j\in\mathbb{Z}}$, $\{\Gamma_j\}_{j\in\mathbb{Z}}$ and $\{1\}$. The following was proved by Berkson–Bourgain–Gillespie [1].

THEOREM 2.3. *There exists a strongly countably additive spectral measure $\mathcal{E}$ on Σ_d, acting in X, such that*

$$\mathcal{E}(\Gamma_j) = E(t_{j+1}^-) - E(t_j) \qquad \forall j \in \mathbb{Z};$$
$$\mathcal{E}(\{1\}) = E(0); \qquad \mathcal{E}(\{\omega_j\}) = E(t_j) - E(t_j^-);$$
$$\sup_{\sigma\in\Sigma_d} \|\mathcal{E}(\sigma)\| \le c^2 C_p.$$

Moreover, $\mathcal{E}$ has the following property, to which we will refer as an analogue of the Littlewood–Paley Theorem, or simply *Littlewood–Paley.*

THEOREM 2.4. *There exists a positive constant $C_p > 0$, depending only on p, such that for every $f \in X$ and any mutually disjoint $\{\sigma_j\}_{j\ge1} \subset \Sigma_d$ with $\mathbb{T} = \bigcup_{j\ge1} \sigma_j$, we have*

$$c^{-2}C_p^{-1}\|f\|_p \le \Big\| \Big(\sum_{j\ge1} |\mathcal{E}(\sigma_j)f|^2 \Big)^{1/2} \Big\|_p \le c^2 C_p \|f\|_p.$$

If $x \in [0, 2\pi]$, $E(x^\pm)$ will mean that we are either looking at $E(x^+) = E(x)$ or at $E(x^-)$. We will also need the following transferred Riesz property (see [1, Theorem (3.15)]).

THEOREM 2.5. *There exists a positive constant C_p such that for any sequences $\{a_j\}_{j\ge1}, \{b_j\}_{j\ge1} \subset [0, 2\pi]$ and $\{g_j\}_{j\ge1} \subset X$, we have*

$$\Big\| \Big(\sum_{j\ge1} |(E(b_j^\pm) - E(a_j^\pm))g_j|^2 \Big)^{1/2} \Big\|_p \le c^2 C_p \Big\| \Big(\sum_{j\ge1} |g_j|^2 \Big)^{1/2} \Big\|_p.$$

By the triangle inequality, it is enough to show that

$$\Big\| \Big(\sum_{j\ge1} |E(b_j^\pm)g_j|^2 \Big)^{1/2} \Big\|_p \le c^2 C_p \Big\| \Big(\sum_{j\ge1} |g_j|^2 \Big)^{1/2} \Big\|_p,$$

which may be proved exactly as Theorem (3.15) of [1].

3. Conditions to define $\int_{[0,2\pi]} \psi(t)\, dE(t)f$ for certain unbounded functions ψ. Denote by $\mathcal{L}$ the set of positive functions $\varphi \in C^1(]0, \pi])$ such that

(i) φ is non-increasing,
(ii) there exists $K > 1$ such that $\varphi(t/2) \le K\varphi(t)$ for every $t \in]0, \pi]$.

Every function in $\mathcal{L}$ has bounded variation on any closed interval of $]0, \pi]$. We are interested in functions $\varphi \in \mathcal{L}$ such that $\lim_{t\to0^+} \varphi(t) = \infty$, the convergence to infinity being controlled by (ii).

For every $\varphi \in \mathcal{L}$, define

$$\mathcal{L}_\varphi := \{\psi \in C^1(]0,\pi]) : \exists C > 0 \text{ such that } |\psi| \le C\varphi,\ |\psi'| \le -C\varphi'\}.$$

For convenience, we define a sequence of arcs by $\varPi_n := \{e^{it} : t \in]t_{n-1}, t_n]\}$ for $n \in \mathbb{Z}$.

Proposition 3.1. *Let $\varphi \in \mathcal{L}$. Let $f \in X$ be such that*

$$\Big\|\Big(\sum_{n\ge 0} \varphi^2(\pi/2^n)|\mathcal{E}(\varPi_{-n})f|^2\Big)^{1/2}\Big\|_p < \infty. \tag{6}$$

Then, for every $\psi \in \mathcal{L}_\varphi$, $\lim_{u\to 0^+} \int_{]u,\pi]} \psi(t)\, dE(t)f$ exists in $L^p(\mu)$.

Proof. We show that the sequence $\{\int_{]\pi/2^n,\pi]} \psi(t)\, dE(t)f\}$ (which is well-defined, since ψ has bounded variation on any closed interval in $]0,\pi]$) is a Cauchy sequence; then the result will follow, since, using (4), for $u \in [\pi/2^{k+1}, \pi/2^k]$, we have

$$\begin{aligned}\Big\|\int\limits_{]u,\pi/2^k]} \psi(t)\, dE(t)f\Big\|_p &= \Big\|[E(\pi/2^k) - E(u)] \int\limits_{]\pi/2^{k+1},\pi/2^k]} \psi(t)\, dE(t)f\Big\|_p \\ &\le c^2 C_p \Big\|\int\limits_{]\pi/2^{k+1},\pi/2^k]} \psi(t)\, dE(t)f\Big\|_p.\end{aligned}$$

Let $n > 1$. We now define two functions ψ_n and ϕ_n on $\mathbb{T}$ as follows:

$$\psi_n : \mathbb{T} \to \mathbb{C},$$
$$e^{it} \mapsto \begin{cases} \psi(t) & \text{if } t \in]\pi/2^{k+1}, \pi/2^k[\text{ for some } 0 \le k \le n-1, \\ 0 & \text{otherwise,} \end{cases}$$
$$\phi_n : \mathbb{T} \to \mathbb{C},$$
$$e^{it} \mapsto \begin{cases} \varphi(\pi/2^k) & \text{if } t \in]\pi/2^{k+1}, \pi/2^k[\text{ for some } 0 \le k \le n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Then the function ψ_n/ϕ_n is well defined (with $0/0$ interpreted as 0), bounded on $\mathbb{T}$ by KC (use the fact that $\varphi \in \mathcal{L}$ and $|\psi| \le C\varphi$), and has bounded variation on any closed interval in $]0, 2\pi]$. Moreover, for every $0 \le k \le n-1$ and every $t \in]\pi/2^{k+1}, \pi/2^k[$,

$$\frac{d}{dt}\frac{\psi_n(e^{it})}{\phi_n(e^{it})} = \frac{\psi'(t)}{\varphi(\pi/2^k)}.$$

Hence

$$\Big|\frac{d}{dt}\frac{\psi_n(e^{it})}{\phi_n(e^{it})}\Big| \le \frac{-C\varphi'(t)}{\varphi(\pi/2^k)},$$

and

$$\operatorname{var}\Big(\frac{\psi_n}{\phi_n}, \varLambda_k\Big) \le C\frac{\varphi(\pi/2^{k+1}) - \varphi(\pi/2^k)}{\varphi(\pi/2^k)} \le C(K-1),$$

since $\varphi \in \mathcal{L}$. In particular,

$$\sup_{n\in\mathbb{N}} \sup_{j\in\mathbb{Z}} \operatorname{var}\left(\frac{\psi_n}{\phi_n}, \Lambda_j\right) < \infty,$$

and, by Theorem 2.1 and the Strong Marcinkiewicz Multiplier Theorem,

$$\sup_{n\in\mathbb{N}} \|T_{\psi_n/\phi_n}\| < \infty. \tag{7}$$

We obtain, for every $1 \le m < n$,

$$\begin{aligned}
\Big\| \int_{]\pi/2^n,\pi/2^m]} \psi(t)\, dE(t) f \Big\|_p &\le \Big\| \int_{]\pi/2^n,\pi/2^m]} \psi_n(e^{it})\, dE(t) f \Big\|_p \\
&\quad + \Big\| \int_{]\pi/2^n,\pi/2^m]} (\psi(t) - \psi_n(e^{it}))\, dE(t) f \Big\|_p \\
&\le \Big\| T_{\psi_n/\varphi_n} \int_{[0,2\pi]} \varphi_n(e^{it}) \mathbf{1}_{]\pi/2^n,\pi/2^m]}\, dE(t) f \Big\|_p \\
&\quad + C \Big\| \Big(\sum_{k=m}^{n} |\psi|^2(\pi/2^k) |\mathcal{E}(\{\omega_{-k}\}) f|^2 \Big)^{1/2} \Big\|_p \\
&\le \|T_{\psi_n/\phi_n}\| c^2 C_p \Big\| \Big(\sum_{k=m}^{n-1} \varphi^2(\pi/2^k) |\mathcal{E}(\Gamma_{-k}) f|^2 \Big)^{1/2} \Big\|_p \\
&\quad + C \Big\| \Big(\sum_{k=m}^{n} \varphi^2(\pi/2^k) |\mathcal{E}(\Pi_{-k}) f|^2 \Big)^{1/2} \Big\|_p,
\end{aligned}$$

where we have used the analogue of the Littlewood–Paley Theorem and the transferred Riesz property. Then the result follows from (7), our assumption and the Riesz property again (for the first term). ∎

Remark. It is not hard to see, by a similar proof, that the existence of $\lim_{u\to 0^+} \int_{]u,\pi]} \varphi(t)\, dE(t) f$ in $L^p(\mu)$ implies condition (6).

Of course we have a proposition similar to Proposition 3.1 for functions having a singularity at 2π.

Proposition 3.2. *Let $\varphi \in \mathcal{L}$. Let ψ be a complex function in $C^1([\pi, 2\pi[)$ such that $\psi(2\pi - \cdot) \in \mathcal{L}_\varphi$. Let $f \in X$ be such that*

$$\Big\| \Big(\sum_{n\ge 0} \varphi^2(\pi/2^n) |\mathcal{E}(\Pi_n) f|^2 \Big)^{1/2} \Big\|_p < \infty. \tag{8}$$

Then the limit $\lim_{u\to 0^+} \int_{[\pi,2\pi-u[} \psi(t)\, dE(t) f$ exists in $L^p(\mu)$.

We want to show that conditions (6) and (8) are implied by a condition expressed in terms of $\|f + \cdots + U^{n-1} f\|_p$.

We need some definitions. For every operator T on $L^p(\mu)$, and every $f \in L^p(\mu)$, define

$$A_n(T,f) = \frac{f + \cdots + T^{n-1}f}{n}.$$

For simplicity, we will write $A_n(U,f) = A_n(f)$.

Define also the following square function:

$$Q_n(T,f) = \Big(\sum_{k\geq n} |A_{2^k}(T,f) - A_{2^{k+1}}(T,f)|^2\Big)^{1/2}.$$

Again, we write $Q_n(U,f) = Q_n(f)$.

It follows from the proof of Theorem 5.4 (see 5.7) of [1] that there exists a constant C_p, depending only on p, such that, for every invertible, doubly power bounded operator T, and every $f \in L^p(\mu)$,

$$\|Q(T,f)\|_p \leq c^2 C_p \|f\|_p, \quad \text{where} \quad c = \sup_{n\in\mathbb{Z}} \|T^n\| < \infty. \tag{9}$$

This result was also obtained in [13, Theorem 2.3] in the case where T is induced by a probability preserving transformation.

Now, notice that, for every $n, k \geq 1$,

$$A_{2^{n+k}}(T,f) = A_{2^k}(T^{2^n}, A_{2^n}(T,f)),$$

hence

$$\begin{aligned}
Q_n(f) = Q_n(U,f) &= \Big(\sum_{k\geq 0} |A_{2^{k+n}}(f) - A_{2^{k+n+1}}(f)|^2\Big)^{1/2} \\
&= \Big(\sum_{k\geq 0} |A_{2^k}(U^{2^n}, A_{2^n}(f)) - A_{2^k+1}(U^{2^n}, A_{2^n}(f))|^2\Big)^{1/2} \\
&= Q_0(U^{2^n}, A_{2^n}(f)).
\end{aligned}$$

In particular, by (9),

$$\|Q_n(f)\|_p \leq c^2 C_p \|A_{2^n}(f)\|_p. \tag{10}$$

With those notations, we can state our next result.

Theorem 3.3. *Let $\{u_n\}_{n\geq 0}$ be a sequence of positive real numbers. Let $\{n_k\}_{k\geq 0}$ be a non-decreasing sequence of positive integers with $n_0 = 0$. Let $f \in L^p(\mu)$. Then*

$$\begin{aligned}
&\Big\|\Big(\sum_{n\in\mathbb{Z}} u_{|n|} |\mathcal{E}(\Pi_n) f|^2\Big)^{1/2}\Big\|_p \\
&\qquad\qquad \leq c^2 C_p \Big\|\Big(\sum_{n\geq 0} u_n |A_{2^n}(f) - A_{2^{n+1}}(f)|^2\Big)^{1/2}\Big\|_p,
\end{aligned} \tag{11}$$

$$\Big\|\Big(\sum_{k\ge 0} u_{n_{k+1}} Q_{n_k}^2(f)\Big)^{1/2}\Big\|_p \le c^2 C_p \sum_{k\ge 0} \big(\sqrt{u_{n_{k+1}}}\|A_{2^{n_k}}(f)\|_p\big)^{\min(p,2)}, \tag{12}$$

where $c = \sup_{n\in\mathbb{Z}} \|U^n\|$ and C_p is a universal constant depending only on $p > 1$. Moreover, if $\{u_n\}$ is non-decreasing, then

$$\Big\|\Big(\sum_{n\ge 0} u_n |A_{2^n}(f) - A_{2^{n+1}}(f)|^2\Big)^{1/2}\Big\|_p \le \Big\|\Big(\sum_{k\ge 0} u_{n_{k+1}} Q_{n_k}^2(f)\Big)^{1/2}\Big\|_p. \tag{13}$$

Proof. Let us prove (13). As $\{u_n\}$ is (in this case) non-decreasing, we have

$$\sum_{n\ge 0} u_n |A_{2^n}(f) - A_{2^{n+1}}(f)|^2 \le \sum_{k\ge 0} u_{n_{k+1}} \sum_{l=n_k}^{n_{k+1}-1} |A_{2^l}(f) - A_{2^{l+1}}(f)|^2,$$

hence the result.

Let us prove (12). Assume that $p \in]1,2]$. Then $\|\cdot\|_{\ell^2} \le \|\cdot\|_{\ell^p}$. Hence,

$$\Big(\sum_{k\ge 0} (\sqrt{u}_{n_{k+1}} Q_{n_k}(f))^2\Big)^{1/2} \le \Big(\sum_{k\ge 0} (\sqrt{u}_{n_{k+1}} Q_{n_k}(f)\Big)^p\Big)^{1/p},$$

and, by (10),

$$\begin{aligned}\Big\|\Big(\sum_{k\ge 0} (\sqrt{u}_{n_{k+1}} Q_{n_k}(f))^2\Big)^{1/2}\Big\|_p^p &\le \sum_{k\ge 0} u_{n_{k+1}}^{p/2} \|Q_{n_k}(f)\|_p^p \\ &\le c^{2p} C_p^p \sum_{k\ge 0} u_{n_{k+1}}^{p/2} \|A_{2^{n_k}}(f)\|_p^p.\end{aligned}$$

Assume now that $p \ge 2$. We have, using (10) and the triangle inequality in $L^{p/2}(\mu)$,

$$\begin{aligned}\Big\|\Big(\sum_{k\ge 0} u_{n_{k+1}} Q_{n_k}^2(f)\Big)^{1/2}\Big\|_p^2 &= \Big\|\sum_{k\ge 0} u_{n_{k+1}} Q_{n_k}^2(f)\Big\|_{p/2} \\ &\le \sum_{k\ge 0} u_{n_{k+1}} \|Q_{n_k}(f)\|_p^2 \\ &\le c^4 C_p^2 \sum_{k\ge 0} u_{n_{k+1}} \|A_{2^{n_k}}(f)\|_p^2.\end{aligned}$$

To prove (11), we show that

$$\Big\|\Big(\sum_{n\ge 0} u_n (|\mathcal{E}(\Pi_{-n})f|^2\Big)^{1/2}\Big\|_p \le c^2 C_p \Big\|\Big(\sum_{n\ge 0} u_n |A_{2^n}(f) - A_{2^{n+1}}(f)|^2\Big)^{1/2}\Big\|_p,$$

the proof for the second sum being the same.

For every $m \geq 1$, $t \in]0, 2\pi[$ and $k \in \mathbb{N}$, define $\sigma_m(t) = 1 + \cdots + e^{i(m-1)t}$ and

$$\begin{aligned}\gamma_k(t) &= \frac{\sigma_{2^k}(t)}{2^k} - \frac{\sigma_{2^{k+1}}(t)}{2^{k+1}} \\ &= \frac{\sigma_{2^k}(t)}{2^{k+1}} - e^{i2^k t}\frac{\sigma_{2^k}(t)}{2^{k+1}} = \frac{(1-e^{i2^k t})^2}{2^{k+1}(1-e^{it})}.\end{aligned} \tag{14}$$

We have $|A_{2^k}(f) - A_{2^{k+1}}(f)|^2 = |\int_{[0,2\pi]} \gamma_k(t)\, dE(t) f|^2$ and

$$\gamma_k(\pi/2^k)\mathcal{E}(\{\omega_k\})f = \mathcal{E}(\{\omega_k\})\Big(\int_{[0,2\pi]} \gamma_k(t)\, dE(t) f\Big).$$

Then, using the fact that

$$|\gamma_k(\pi/2^k)| = \left|\frac{4}{i2^{k+1}\sin(\pi/2^{k+1})}\right| \sim 4/\pi \quad \text{as } k \to \infty$$

and the transferred Riesz property, we obtain

$$\begin{aligned}&\Big\|\Big(\sum_{k\geq 0} u_k|\mathcal{E}(\{\omega_{-k}\})f|^2\Big)^{1/2}\Big\|_p \\ &\qquad \leq C\Big\|\Big(\sum_{k\geq 1} u_k|\mathcal{E}(\{\omega_{-k}\})(A_{2^k}(f) - A_{2^{k+1}}(f))|^2\Big)^{1/2}\Big\|_p \\ &\qquad \leq c^2 C_p\Big\|\Big(\sum_{k\geq 0} u_k|(A_{2^k}(f) - A_{2^{k+1}}(f))|^2\Big)^{1/2}\Big\|_p < \infty.\end{aligned}$$

Hence, it remains to prove that

$$\begin{aligned}&\Big\|\Big(\sum_{n\geq 0} u_n(|\mathcal{E}(\Gamma_{-n})f|^2\Big)^{1/2}\Big\|_p \\ &\qquad \leq c^2 C_p\Big\|\Big(\sum_{n\geq 0} u_n|(A_{2^n}(f) - A_{2^{n+1}}(f))|^2\Big)^{1/2}\Big\|_p.\end{aligned}$$

Let $n \geq 1$. We define two functions ψ_n and ϕ_n on $\mathbb{T}$ as follows:

$$\begin{aligned}&\psi_n : \mathbb{T} \to \mathbb{C}, \\ &\qquad e^{it} \mapsto \begin{cases}\sqrt{u_k} & \text{if } t \in \Gamma_{-k} \text{ for some } 0 \leq k \leq n, \\ 0 & \text{otherwise,}\end{cases} \\ &\phi_n : \mathbb{T} \to \mathbb{C}, \\ &\qquad e^{it} \mapsto \begin{cases}\sqrt{u_k}\gamma_k(t) & \text{if } t \in \Gamma_{-k} \text{ for some } 0 \leq k \leq n, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

The functions ψ_n and ϕ_n are in $BV(\mathbb{T})$.

By construction, using the analogue of the Littlewood–Paley Theorem, we have, for every $n \geq 1$,

$$\Big\| \Big(\sum_{k=0}^{n} u_k |\mathcal{E}(\Gamma_{-k}) f|^2 \Big)^{1/2} \Big\|_p \leq c^2 C_p \Big\| \int_{[0,2\pi]} \psi_n(e^{it})\, dE(t) f \Big\|_p. \tag{15}$$

Hence we are in a position to use the method employed in the proof of Proposition 3.1 with the present choice of ψ_n and ϕ_n.

Note that the function ψ_n/ϕ_n is well defined ($0/0$ interpreted as 0), with bounded variation on $\mathbb{T}$. For every $t \in]\pi/2^{k+1}, \pi/2^k[$,

$$\left| \frac{\psi_n(e^{it})}{\phi_n(e^{it})} \right| = \left| \frac{1}{\gamma_k(t)} \right| = \frac{2^{k+1}|1-e^{it}|}{|1-e^{2^k it}|^2} \leq \pi.$$

On the other hand, for every $k \geq 1$ and every $t \in]\pi/2^{k+1}, \pi/2^k[$, we have

$$\left| \frac{d}{dt} \frac{\psi_n(e^{it})}{\phi_n(e^{it})} \right| = \left| \frac{2^{k+1} i e^{it}}{(1-e^{i2^k t})^2} - 2\frac{2^{2k+1} i e^{i2^k t}(1-e^{it})}{(1-e^{i2^k t})^3} \right| \leq (1+\sqrt{2})\pi 2^k.$$

Hence

$$\operatorname{var}\Big(\frac{\psi_n}{\phi_n}, \Lambda_k \Big) \leq \int_{\pi/2^{k+1}}^{\pi/2^k} (1+\sqrt{2})\pi 2^k \, dt = (1+\sqrt{2})\pi^2/2$$

and $\sup_{n\geq 1} \sup_{k\in\mathbb{Z}} \operatorname{var}(\psi_n/\phi_n, \Lambda_k) < \infty$. Then, by Theorem 2.1 and the Strong Marcinkiewicz Multiplier Theorem, there exists $K > 0$ such that $\|T_{\psi_n/\phi_n}\| \leq K$ for every $n \geq 1$.

Using the analogues of the Littlewood–Paley Theorem and of the Riesz property, we deduce

$$\begin{aligned}
\Big\| \int_{[0,2\pi]} \psi_n(e^{it})\, dE(t) f \Big\|_p &= \Big\| T_{\psi_n/\phi_n} \Big(\int_{[0,2\pi]} \varphi_n(t)\, dE(t) f \Big) \Big\|_p \\
&\leq \|T_{\psi_n/\phi_n}\| c^2 C_p \Big\| \Big(\sum_{k=0}^{n} u_k \Big| \int_{]\pi/2^{k+1}, \pi/2^k[} \gamma_k(t)\, dE(t) f \Big|^2 \Big)^{1/2} \Big\|_p \\
&\leq c^2 C_p \|T_{\psi_n/\varphi_n}\| \Big\| \Big(\sum_{k=0}^{n} u_k \Big| \int_{[0,2\pi]} \gamma_k(t)\, dE(t) f \Big|^2 \Big)^{1/2} \Big\|_p \\
&= c^2 C_p \|T_{\psi_n/\phi_n}\| \Big\| \Big(\sum_{k=0}^{n} u_k |A_{2^k}(f) - A_{2^{k+1}}(f)|^2 \Big)^{1/2} \Big\|_p \\
&\leq c^2 C_p K \Big\| \Big(\sum_{k\geq 0} u_k |A_{2^k}(f) - A_{2^{k+1}}(f)|^2 \Big)^{1/2} \Big\|_p < \infty.
\end{aligned}$$

Letting n go to infinity in (15), we obtain the desired result. ∎

In applications, we will take $n_k = k$ or $n_k = 2^k$. It is convenient to show that for suitable $\{u_n\}$ the right-hand side of (12) may be replaced by a series involving the whole sequence $\{\|A_n(f)\|_p\}$.

We say that a positive function b is in the *Zygmund class* if for every $\delta > 0$, $x \mapsto x^\delta b(x)$ (respectively $x \mapsto x^{-\delta} b(x)$) is increasing (resp. decreasing) at infinity.

LEMMA 3.4. *Let T be an operator on a Banach space Y such that $\sup_{n\ge 1} \|T^n\|_Y < \infty$. Let b be a function in the Zygmund class, $\gamma > 1$ and $r \ge 1$. For every $f \in Y$, the following are equivalent:*

(i) $\sum_{n\ge1} b(n)\|f + \cdots + T^{n-1}f\|_Y^r / n^\gamma < \infty$.
(ii) $\sum_{n\ge1} b(2^n)\|f + \cdots + T^{2^n-1}f\|_Y^r / 2^{n(\gamma-1)} < \infty$.

If either (i) *or* (ii) *is satisfied then $b(n)\|f + \cdots + T^{n-1}f\|_Y^r / n^{\gamma-1} \to 0$ as $n \to \infty$. If $\gamma = r + 1$ and $b = \log^\delta$ for some $\delta \in \mathbb{R} - \{-1\}$, we even have $\|f + \cdots + T^{n-1}f\|_Y^r (\log n)^{\delta+1}/n^r \to 0$ and* (i) *and* (ii) *are equivalent to*

(iii) $\sum_{n\ge1} 2^{n(\delta+1)}(\|f + \cdots + T^{2^{2^n}-1}f\|_Y / 2^{2^n})^r < \infty$.

For the proof, see Appendix A.

COROLLARY 3.5. *Let $1 < p < \infty$, $0 < \alpha < 1$, and $f \in X$.*

(i) *If* (2) *holds, then* $\|(\sum_{n\in\mathbb{Z}} 2^{2|n|\alpha} |\mathcal{E}(\Pi_n)f|^2)^{1/2}\|_p < \infty$.
(ii) *If* (3) *holds, then* $\|(\sum_{n\in\mathbb{Z}} n^2 |\mathcal{E}(\Pi_n)f|^2)^{1/2}\|_p < \infty$.

In particular we can apply Propositions 3.1 *and* 3.2 *either with $\varphi(t) = t^{-\alpha}$ and $\psi(t) = (1 - e^{it})^{-\alpha}$ or with $\varphi(t) = \log(2\pi/t)$ and $\psi(t) = \log(1 - e^{it})$.*

Proof. Assume (2). By Lemma 3.4 with $b \equiv 1$, $r = \min(p, 2)$, $\gamma = 1 + (1-\alpha)\min(p, 2)$, we obtain $\sum_n (2^{n\alpha}\|A_{2^n}\|_p)^{\min(p,2)} < \infty$. Then apply Theorem 3.3 with $n_k = k$, $u_n = 2^{2n\alpha}$ to obtain (i).

Assume (3). By Lemma 3.4 with $b \log^{\min(p,2)-1}$, $r = \min(p, 2)$, $\gamma = r + 1$, we obtain $\sum_n (2^n \|A_{2^{2^n}}\|_p)^{\min(p,2)} < \infty$. Then apply Theorem 3.3 with $n_k = 2^k$, $u_n = n^2$ to obtain (ii). ∎

4. Conditions for the norm convergence of power series of U. In this section, we want to obtain conditions for the norm convergence of general power series including $\sum_{n\ge1} (U^n f)/n^{1-\alpha}$, for $0 \le \alpha < 1$. It is proved in [8] that the convergence of the latter is equivalent to f being in the range of $(I - U)^\alpha$, where this operator is well defined, using the power series expansion of $(1 - z)^\alpha$ (see [8]).

As shown in [8], [19], [4], [6] or [5], the convergence of power series in U allows one to obtain pointwise ergodic theorems with rate and has applications in probability theory.

The spectral representation of U will enable us to relate the convergence of general power series to the studies in the previous section.

Given a sequence $\{a_n\}$ we want to find conditions on $f \in L^p(\mu)$ such that $\sum_{n\ge1} a_n U^n f$ converges in $L^p(\mu)$. Writing $S_n(f) := f + \cdots + U^{n-1}f$, we have

$$\begin{aligned}(16)\qquad \sum_{k=1}^{n} a_k U^k f &= \sum_{k=1}^{n} a_k(S_{k+1}(f) - S_k(f))\\ &= \sum_{k=2}^{n} S_k(f)(a_{k-1} - a_k) + a_n S_{n+1}(f) - a_1 S_1(f).\end{aligned}$$

Hence to obtain the desired convergence it suffices to show that the series $\sum_{n\ge1} S_n(f)(a_{n-1} - a_n)$ converges in $L^p(\mu)$ and $\lim_{n\to\infty} \|a_n S_{n+1}(f)\|_p = 0$.

We will now study the convergence of the series on the right-hand side of (16). Actually, for regular sequences $\{a_n\}$, one automatically obtains $\lim_{n\to\infty} \|a_n S_{n+1}(f)\|_p = 0$ (as one can see in the examples of the next lemma). Also the conditions for the convergence of $\sum_{n\ge1} S_n(f)(a_{n-1} - a_n)$ that we will obtain (such as (2) or (3)) imply that $\lim_{n\to\infty} \|a_n S_{n+1}(f)\|_p = 0$, using Lemma 3.4.

LEMMA 4.1. *Let T be an operator on the Banach space Y with $\sup_{n\ge1} \|T^n\| < \infty$. Let $f \in Y$ and $\beta \in]0,1]$. Assume that one of the following is satisfied:*

(i) $\sum_{n\ge1} (T^n f)/n^\beta$ *converges in Y,*
(ii) $\sum_{n\ge1} S_n(f)/n^{1+\beta}$ *converges Y.*

Then $\|S_n(f)\|_Y/n^\beta \to 0$ as $n \to \infty$. In particular, (i) *and* (ii) *are equivalent.*

REMARK. The lemma shows that the condition $\sum_{n\ge1} \|S_n(f)\|_Y/n^{1+\beta} < \infty$ is sufficient for the convergence in Y of $\sum_{n\ge1} (T^n f)/n^\beta$.

For the proof of Lemma 4.1, see Appendix B.

We go back to the study of the convergence of series of the type $\sum_{n\ge1} \alpha_n S_n(f)$.

Denote by $\mathcal{K}$ the set of positive functions $\varphi \in C^1(]0,\pi])$ satisfying the following set of conditions:

(A1) φ and $-\varphi'$ are non-increasing,
(A2) $t \mapsto t\varphi(t)$ and $t \mapsto -t^2\varphi'(t)$ are non-decreasing,
(A3) $\lim_{t\to0^+} t\varphi(t) = 0$.

In particular, the functions $t \mapsto \log(2\pi/t)$, and $t \mapsto 1/t^\alpha$ for $0 < \alpha < 1$, belong to $\mathcal{K}$.

Notice that $\mathcal{K} \subset \mathcal{L}$ by (A2), and by (A1) and (A2),

$$(17)\qquad 0 \le -t\varphi'(t) \le \varphi(t) \quad \forall t \in (0,\pi].$$

Then, by (A3), we obtain

$$\lim_{t\to 0^+} t^2\varphi'(t) = 0. \tag{18}$$

Recall that for every $t \in \mathbb{R}$ and every $n \geq 1$, $\sigma_n(t) = 1 + \cdots + e^{i(n-1)t}$.

Given a function $\varphi \in \mathcal{K}$, we denote by $\mathcal{K}_\varphi$ the set of sequences $\{\alpha_n\} \subset \mathbb{C}$ such that there exists a constant $C > 0$ for which, for every $p > n \geq 1$ and every $t \in [-\pi, \pi] \setminus \{0\}$,

$$\Big|\sum_{k=1}^{n} \alpha_k\sigma_k(t)\Big| \leq C\varphi(\pi/n), \qquad \Big|\sum_{k=1}^{n} \alpha_k\sigma_k'(t)\Big| \leq -C\varphi'(\pi/n), \tag{19}$$

$$\Big|\sum_{k=n}^{p} \alpha_k\sigma_k(t)\Big| \leq C\frac{\varphi(\pi/n)}{n|t|}, \qquad \Big|\sum_{k=n}^{p} \alpha_k\sigma_k'(t)\Big| \leq -C\frac{\varphi'(\pi/n)}{n^2t^2}. \tag{20}$$

For instance, the previous conditions will be fulfilled in the following situation.

Proposition 4.2. *Let $\varphi \in \mathcal{K}$. Let $\{\alpha_n\} \subset \mathbb{R}^+$. Assume that $\{n\alpha_n\}$ is non-increasing, and that there exists $K > 0$ such that for every $n \geq 1$,*

$$\sum_{k=1}^{n} k^2\alpha_k \leq -K\varphi'(\pi/n), \tag{21}$$

$$\sum_{k\geq n} \alpha_k \leq -K\frac{\varphi'(\pi/n)}{n^2}. \tag{22}$$

Then $\{\alpha_n\} \in \mathcal{K}_\varphi$.

For the proof, see Appendix C.

For instance, Proposition 4.2 applies with $\alpha_n = 1/n^{2-\alpha}$ for $0 \leq \alpha < 1$, and $\varphi(t) = \log(2\pi/t)$ if $\alpha = 0$ and $\varphi(t) = t^{-\alpha}$ otherwise.

Proposition 4.3. *Let $\varphi \in \mathcal{K}$ and $\{\alpha_n\} \in \mathcal{K}_\varphi$. Then $\sum_{n\geq 1} \alpha_n\sigma_n(t)$ converges uniformly on any compact subset of $\mathbb{R}\backslash 2\pi\mathbb{Z}$ to a function W with bounded variation on any compact subset of $\mathbb{R}\backslash 2\pi\mathbb{Z}$. Moreover the restrictions to $]0, \pi]$ of W and $W(2\pi - \cdot)$ belong to $\mathcal{L}_\varphi$.*

Proof. Since $\varphi \in \mathcal{K}$, the first condition of (20) implies that the series $\sum_{n\geq 0} \alpha_n\sigma_n$ converges uniformly on every compact subset of $]0, 2\pi[$ to a function W defined and continuous on $]0, 2\pi[$. Then, by the second condition of (20), W has bounded variation on any closed interval in $]0, 2\pi[$ (actually $W \in C^1(]0, \pi])$) and (20) still holds when $p \to \infty$.

Let $t \in [-\pi, \pi]$, $t \neq 0$. Using the monotonicity properties of $\varphi \in \mathcal{K}$, (19) and (20), we have

$$|W(t)| = \Big|\sum_{n\geq 1}\alpha_n\sigma_n(t)\Big| \leq \Big|\sum_{n=1}^{[\pi/|t|]}\alpha_n\sigma_n(t)\Big| + \Big|\sum_{n\geq[\pi/|t|]+1}\alpha_n\sigma_n(t)\Big|$$
$$\leq 2C\varphi(|t|),$$
$$|W'(t)| = \Big|\sum_{n\geq 1}\alpha_n\sigma_n'(t)\Big| \leq \Big|\sum_{n=1}^{[\pi/|t|]}\alpha_n\sigma_n'(t)\Big| + \Big|\sum_{n\geq[\pi/|t|]+1}\alpha_n\sigma_n'(t)\Big|$$
$$\leq -\tilde{C}\varphi'(|t|). \blacksquare$$

PROPOSITION 4.4. *Let $\varphi \in \mathcal{K}$ and $\{\alpha_n\}_{n\in\mathbb{N}} \in \mathcal{K}_\varphi$. Let $W = \sum_{n\geq 1}\alpha_n\sigma_n$ be as in Proposition* 4.3. *Let $f \in X$ be such that $E(0)f = 0$ and*

$$\Big(\sum_{n\in\mathbb{Z}}\varphi^2(\pi/2^{|n|})|\mathcal{E}(\Pi_n)f|^2\Big)^{1/2} \in L^p(\mu). \tag{23}$$

Then there exists $\{g_m\} \subset L^p(\mu)$, with $\lim_{m\to\infty}\|g_m\|_p = 0$, such that, for every $m \geq 1$, taking $n = [\log_2 m]$,

$$\sum_{k=1}^{m}\alpha_k S_k(f) = g_m + \int\limits_{]\pi/2^n, 2\pi-\pi/2^n]} W(t)\,dE(t)f.$$

In particular, by Proposition 3.1, *$\sum_{n\geq 1}\alpha_n S_n(f)$ converges in $L^p(\mu)$.*

REMARKS. The proposition may be seen as a version of Lemma 5 of [11] (see also Theorem 4 of the same paper) where the case of unitary operators is considered. It can be checked that the conditions in [11] are actually the same as ours in this case. In [11], even the μ-a.e. convergence of $\{g_m\}$ is obtained. The study of the μ-a.e. convergence will be done in the forthcoming work [7].

Proof of Proposition 4.4. Let $m \geq 1$ and $n = [\log_2 m]$. Define $W_m(t) = \sum_{k=0}^{m}\alpha_k\sigma_k(t)$ and write

$$\sum_{k=1}^{m}\alpha_k S_k(f) = \int\limits_{[0,2\pi]} W_m(t)\,dE(t)f$$
$$= \int\limits_{]0,\pi/2^n]} W_m(t)\,dE(t)f + \int\limits_{]2\pi-\pi/2^n, 2\pi]} W_m(t)\,dE(t)f$$
$$+ \int\limits_{]\pi/2^n, 2\pi-\pi/2^n]} (W_m(t) - W(t))\,dE(t)f + \int\limits_{]\pi/2^n, 2\pi-\pi/2^n]} W(t)\,dE(t)f.$$

The proposition will follow from the next lemmata, whose proofs are given in Appendix D.

LEMMA 4.5. *Let $\varphi \in \mathcal{K}$ and $\{\alpha_n\}_{n\in\mathbb{N}} \in \mathcal{K}_\varphi$. There exists $L > 0$ such that for every $f \in X$ and every $m \geq 1$, we have (with $n = [\log_2 m]$)*

$$\Big\| \int_{]\pi/2^n, 2\pi-\pi/2^n]} (W_m(t) - W(t))\, dE(t) f \Big\|_p \tag{24}$$
$$\leq L \frac{\varphi(\pi/2^n)}{2^n} \Big\| \Big(\sum_{k=1-n}^{n-1} 2^{2|k|} |\mathcal{E}(\Pi_{-k}) f|^2 \Big)^{1/2} \Big\|_p .$$

Lemma 4.6. *Let $\varphi \in \mathcal{K}$ and $\{\alpha_n\}_{n\in\mathbb{N}} \in \mathcal{K}_\varphi$. There exists $L > 0$ such that for every $f \in X$ and every $m \geq 1$, we have (with $n = [\log_2 m]$)*

$$\Big\| \int_{]0,\pi/2^n]} W_m(t)\, dE(t) f \Big\|_p \leq L\varphi(\pi/2^n) \Big\| \Big(\sum_{k\geq n} |\mathcal{E}(\Pi_{-k}) f|^2 \Big)^{1/2} \Big\|_p , \tag{25}$$

$$\Big\| \int_{]2\pi-\pi/2^n, 2\pi]} W_m(t)\, dE(t) f \Big\|_p \leq L\varphi(\pi/2^n) \Big\| \Big(\sum_{k\geq n} |\mathcal{E}(\Pi_{k}) f|^2 \Big)^{1/2} \Big\|_p . \tag{26}$$

Since $(\varphi(\pi/2^n))$ is non-decreasing, (23) implies that the right-hand sides of (25) and (26) converge to zero as $m \to \infty$, by the monotone convergence theorem.

It remains to prove that (23) implies the convergence to zero of the right-hand side of (24). We need the following version of Kronecker's Lemma (whose proof, based on Abel summation, is left to the reader).

Lemma 4.7 (Kronecker's Lemma). *Let $\{a_n\}$ be a sequence of real numbers decreasing to zero, and $\{b_n\}$ be a sequence of non-negative real numbers such that $\sum_{n\geq 1} a_n b_n < \infty$. Then $a_n \sum_{k=1}^n b_k \to 0$ and $\sup_{n\geq 1} a_n \sum_{k=1}^n b_k \leq \sum_{n\geq 1} a_n b_n$.*

Assume (23). Apply Kronecker's Lemma with $a_n = \varphi(\pi/2^n)/2^n$ and $b_n = |\mathcal{E}(\Pi_{\pm n}) f|^2(x))$ for μ-a.e. $x \in \mathcal{M}$ and Lebesgue's Dominated Convergence Theorem to see that the right-hand side of (24) converges to zero. ∎

Now we are in a position to prove Theorems 1.1 and 1.2.

Proof of Theorems 1.1 and 1.2. Let $0 < \alpha < 1$ and $f \in X$ be such that (2) holds. It follows from Corollary 3.5 that the limit in Theorem 1.1 exists and that condition (23) holds.

On the other hand, by (2) and Lemma 3.4, $\|S_n(f)\|_p / n^{1-\alpha} \to 0$. Hence $E(0) f = 0$ (see Section 2).

Hence, we can apply Proposition 4.4 to the cases mentioned after Proposition 4.2 and obtain the convergence in $L^p(\mu)$ of $\sum_{n\geq 1} S_n(f)/n^{1+\alpha}$. Then we conclude by means of Lemma 4.1.

Theorem 1.2 can be proved exactly the same way with suitable modifications. ∎

5. Optimality of the conditions. We now discuss the sufficient conditions obtained in Theorems 1.1 and 1.2 for the convergence of series in $L^p(\mu)$.

By Lemma 4.1, for every operator T on a Banach space Y such that $\sup_{n\ge 1}\|T^n\|<\infty$, if $f\in Y$ satisfies

$$\sum_{n\ge 1}\frac{\|f+\cdots+T^{n-1}f\|_Y}{n^{2-\alpha}}<\infty \tag{27}$$

for some $\alpha\in[0,1[$ then $\sum_{n\ge 1}(T^nf)/n^{1-\alpha}$ converges in $Y=L^p(\mu)$.

Condition (27) always implies the conditions of Theorems 1.1 and 1.2. Indeed, by Corollary 3.4, if (27) holds then

$$\lim_{n\to\infty}\frac{\|f+\cdots+T^{n-1}f\|_Y}{n^{1-\alpha}}=0 \quad \text{if } \alpha\in\,]0,1[,$$
$$\lim_{n\to\infty}\frac{\|f+\cdots+T^{n-1}f\|_Y}{n}\log n=0 \quad \text{if } \alpha=0.$$

Using (27) again, we see that the conditions of Theorems 1.1 and 1.2 hold for the corresponding α, with $Y=L^p$.

Now, from a practical point of view, one can see that Theorems 1.1 and 1.2 yield a quantitative improvement of condition (27): if one has an estimate of the type $\|f+\cdots+U^{n-1}f\|_p=O(n^\gamma(\log n)^\delta)$, there is a gain in the power of the logarithm.

Before proving the optimality of our conditions, let us discuss the specific case $p=2$.

When $p=2$, Theorems 1.1 and 1.2 have been proved in [6] (see also [11] and the references therein) for U either an isometry or a (sub)normal contraction of L^2.

Here we consider invertible power bounded operators. By a result of Nagy [16], if U is an invertible operator on L^2 (or on a Hilbert space) such that $\sup_{n\in\mathbb{Z}}\|U^n\|<\infty$, then U is similar to a unitary operator. Hence our results do not bring any novelty in the case $p=2$.

We will distinguish the cases $p\le 2$ and $p\ge 2$.

5.1. The case $p\in\,]1,2]$. Let ν be a finite measure on the Borel sets of $[-\pi,\pi]$. Define an operator V on $L^1(\nu)$ by $Vf(t)=e^{it}f(t)$. Then V is an invertible isometry on each $L^p(\nu)$, $p\ge 1$.

In this paper, we are only concerned with norm convergence. It is well known that if U is a unitary operator on $L^2(\mu)$ then, for every $f\in L^2(\mu)$, there exists a positive finite measure ν_f on $[-\pi,\pi]$ (the spectral measure of f relative to U) such that for all $\alpha_1,\dots,\alpha_m\in\mathbb{C}$ and $n_1,\dots n_m\in\mathbb{Z}$,

$$\Big\|\sum_{k=1}^m\alpha_kU^{n_k}f\Big\|_2^2=\int_{-\pi}^{\pi}\Big|\sum_{k=1}^m\alpha_ke^{in_kt}\Big|^2\,d\nu_f(t).$$

In particular, in order to prove Theorems 1.1 and 1.2 for f, it suffices to prove them for the function $t\mapsto 1$ and V acting on $L^2([-\pi,\pi],\nu_f)$.

For $p \neq 2$, one cannot deduce results for general operators from the study of V (on some $L^p([-\pi,\pi],\nu)$, but we will see that when $p \in]1,2]$ condition (2) of Theorem 1.1 is optimal.

Proposition 5.1. *Let V be the operator on $L^1([-\pi,\pi],\nu)$ defined above. Let $p \geq 1$ and $\alpha \in]0,1[$. For every $f \in L^p(\nu)$ the following are equivalent:*

(i) $\displaystyle\sum_{n\geq 1} \frac{\|f+\cdots+V^{n-1}f\|^p_{L^p(\nu)}}{n^{1+(1-\alpha)p}} < \infty$,

(ii) $\displaystyle\int_{-\pi}^{\pi} \frac{|f(t)|^p}{|t|^{\alpha p}}\,\nu(dt) < \infty$,

(iii) $\displaystyle\sum_{n\geq 1} \frac{V^n f}{n^{1-\alpha}}$ *converges in* $L^p(\nu)$.

Proof. We may and do assume that $\nu(\{0\}) = 0$. Let $t \in [-\pi,\pi] \setminus \{0\}$. We have

$$\sum_{n\geq 1} \frac{\|f+\cdots+V^{n-1}f\|_p^p}{n^{1+(1-\alpha)p}}$$
$$= \int_{-\pi}^{\pi} \sum_{1\leq n\leq 1/|t|} \frac{|\sigma_n(t)|^p}{n^{1+(1-\alpha)p}} |f(t)|^p\,\nu(dt) + \int_{-\pi}^{\pi} \sum_{n>1/|t|} \frac{|\sigma_n(t)|^p}{n^{1+(1-\alpha)p}} |f(t)|^p\,\nu(dt)$$
$$\leq C \int_{-\pi}^{\pi} \frac{|f(t)|^p}{|t|^{\alpha p}}\,\nu(dt),$$

since $|\sigma_n(t)| \leq C\min(n, 1/|t|)$. Similarly, using the estimate, $|\sigma_n(t)| \geq Cn$ for $1 \leq n \leq 1/|t|$, we have

$$\sum_{n\geq 1} \frac{\|f+\cdots+V^{n-1}f\|_p^p}{n^{1+(1-\alpha)p}} \geq \int_0^{2\pi} \frac{|f(t)|^p}{|t|^{\alpha p}}\,\nu(dt),$$

and (i)$\Leftrightarrow$(ii).

To see that (ii)$\Leftrightarrow$(iii), just notice that

$$\sup_{n\geq 1} \left|\sum_{k=1}^{n} \frac{e^{ikt}}{k^{1-\alpha}}\right| \leq \frac{C}{|t|^\alpha} \quad \text{and} \quad \left|\sum_{n\geq 1} \frac{e^{int}}{n^{1-\alpha}}\right| \sim \frac{C_\alpha}{|t|^\alpha}, \quad t \to 0, \tag{28}$$

and use Fatou's Lemma for (iii)$\Rightarrow$(ii) and Lebesgue's Dominated Convergence Theorem for (ii)$\Rightarrow$(iii). ∎

It follows from Proposition 5.1 that condition (2), for $1 < p < 2$, in Theorem 1.1 cannot be improved in the context of doubly power bounded operators on L^p spaces. One can see (looking at irrational rotations as below) that the equivalence of (i) and (iii) does not hold, in general, for other operators than V.

We also have the following.

PROPOSITION 5.2. *Let V be the operator on $L^1([-\pi,\pi],\nu)$ defined above. Let $p \ge 1$. For every $f \in L^p(\nu)$ the following are equivalent:*

(i) $\displaystyle\sum_{n\ge 1} \frac{\|f+\dots+V^{n-1}f\|^p_{L^p(\nu)}}{n^{1+p}}(\log n)^{p-1} < \infty$,

(ii) $\displaystyle\int_{-\pi}^{\pi} (\log(1/|t|))^p |f(t)|^p\,\nu(dt) < \infty$,

(iii) $\displaystyle\sum_{n\ge 1} \frac{V^n f}{n}$ *converges in $L^p(\nu)$.*

This proposition can be proved just as Proposition 5.1.

To conclude with the case $p \in]1,2]$, we make the following remark. Assume U acts on each L^p $(1 \le p \le 2)$, for instance, take U induced by an invertible measure preserving transformation. Then condition (2) of Theorem 1.1 (for $1 < p < 2$) really "looks like" what one would obtain by interpolating condition (2) for $p = 2$ and condition (27) with $Y = L^1$. However, we have not succeeded in doing it.

5.2. The case $p \in]2,\infty[$. Let $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Q}$ and denote by R_θ the rotation by θ, i.e. $R_\theta g = g(\cdot+\theta)$ for every function g on $[0,2\pi)$. Then R_θ induces an invertible (positive) isometry on any $L^p([0,2\pi),\lambda)$, $p \ge 1$, where λ is the normalised Lebesgue measure on $[0,2\pi)$.

For $\rho > 1$, denote by $\mathbf{L}_\rho$ the set of non-decreasing sequences $\{n_k\} \subset \mathbb{N}$ for which $\inf_{k\ge 1} n_{k+1}/n_k \ge \rho$. Fix $\mathbf{n} = \{n_k\} \in \mathbf{L}_\rho$ and define the following subspace of $L^2([0,2\pi),\lambda)$:

$$\kappa_{\mathbf{n}} = \Big\{ g \in L^2([0,2\pi),\lambda) : \exists\{c_k\} \in \ell^2(\mathbb{N}),\ g(x) = \sum_{k\ge 1} c_k e^{in_k x} \Big\}.$$

Clearly, for all $\rho > 1$ and $\mathbf{n} \in \mathbf{L}_\rho$, $\kappa_{\mathbf{n}}$ is R_θ-invariant for every $\theta \in \mathbb{R}$.

It follows from the (classical) Littlewood–Paley Theorem (see e.g. Theorem 2.1, p. 225, Vol. II of [20]) that for every $p > 1$, there exists $C_p = C_p(\rho) > 0$ such that for every $g \in \kappa_{\mathbf{n}}$, we have

$$\frac{1}{C_p}\|g\|_p \le \|g\|_2 \le C_p\|g\|_p. \tag{29}$$

In particular, $\kappa_{\mathbf{n}} \subset \bigcap_{p>1} L^p([0,2\pi),\lambda)$.

We have the following

PROPOSITION 5.3. *Let $\rho > 1$, $\mathbf{n} \in \mathbf{L}_\rho$ and $f \in \kappa_{\mathbf{n}}$. Let $p > 1$, $\theta \in \mathbb{R}\setminus 2\pi\mathbb{Q}$ and $\alpha \in]0,1[$. The following are equivalent:*

(i) $\sum_{n\geq 1}(R_\theta^n f)/n^{1-\alpha}$ *converges in* $L^p([0,2\pi),\lambda)$.
(ii) $\sum_{n\geq 1}\|f+\cdots+R_\theta^{n-1}f\|_p^2/n^{1+2(1-\alpha)}<\infty$.

In particular, condition (2) *of Theorem* 1.1 *is optimal for* $p>2$.

Remark. Similarly, for $f\in\kappa_{\mathbf{n}}$, the convergence of $\sum_{n\geq 1}(R_\theta^n f)/n$ in $L^p([0,2\pi),\lambda)$ is equivalent to $\sum_{n\geq 1}\|f+\cdots+R_\theta^n f\|_p^2/n^3\log n<\infty$.

Proof of Proposition 5.3. By (29), (i) and (ii) are respectively equivalent to the same condition for $p=2$. Now the assertion is true for $p=2$ (even for all $f\in L^2([0,2\pi),\lambda)$) either by Proposition 5.1 or by Proposition 2.3 of [6]. ∎

To see that condition (2) of Theorem 1.1 is optimal, it suffices to show that there exists $\mathbf{n}\in\mathbf{L}_\rho$ for some $\rho>1$, and $f\in\kappa_{\mathbf{n}}$ such that (i) is satisfied.

Given $\theta\in\mathbb{R}\setminus 2\pi\mathbb{Q}$, define $n_0=1$ and for $k\geq 0$, $n_{k+1}=\inf\{n>\rho n_k : n\theta\in 2\pi\mathbb{Z}+[\pi/2,3\pi/2]\}$. The sequence $\{n_k\}$ is well defined since $2\pi\mathbb{Z}+\{n\theta\}_{n\geq 1}$ is dense in $\mathbb{R}$ by the assumption on θ.

Define $f(x)=\sum_{k\geq 1}e^{in_kx}/k$. For every $s>r\geq 1$, we have

$$\begin{aligned}\Big\|\sum_{m=r}^{s}\frac{R_\theta^m f}{m^{1-\alpha}}\Big\|_2^2&=\sum_{k\geq 1}\frac{1}{k^2}\Big|\sum_{m=r}^{s}\frac{e^{imn_k\theta}}{m^{1-\alpha}}\Big|^2\\&=\sum_{k\geq 1}\frac{1}{k^2}\Big|\sum_{m=r}^{s}\frac{\sigma_m(n_k\theta)-\sigma_{m-1}(n_k\theta)}{m^{1-\alpha}}\Big|^2\\&\leq\frac{4C}{\pi^2r^{2-2\alpha}}\sum_{k\geq 1}\frac{1}{k^2}\xrightarrow[r\to\infty]{}0,\end{aligned}$$

where we have used Abel summation by parts. ∎

Proposition 5.4. *Let* $\theta\in\mathbb{R}\setminus 2\pi\mathbb{Z}$ *and* $\alpha\in\,]0,1[$. *Let* $\rho>1$ *and* $\mathbf{n}\in\mathbf{L}_\rho$. *If* $p>2$ (*resp. if* $1<p<2$), *then the condition*

$$\sum_{n\geq 1}\frac{\|\sum_{k=0}^{n-1}R_\theta^k f\|_p^p}{n^{1+(1-\alpha)p}}<\infty \tag{30}$$

on $f\in\kappa_{\mathbf{n}}$ *is not sufficient (resp. not necessary), in general, for the convergence in* $L^p([0,2\pi),\lambda)$ *of* $\sum_{n\geq 1}(R_\theta^n f)/n^{1-\alpha}$.

Proof. Set $n_0=1$ and $n_{k+1}=\inf\{n\geq\rho n_k : n\theta\in 2\pi\mathbb{Z}+[1/(k+1),1/k]\}$. Let $\{d_k\}_{k\geq 1}\in\ell^2$ and define $f(x)=\sum_{k\geq 1}d_ke^{in_kx}$. Using (28), it is not hard to see that the convergence of $\sum_{n\geq 1}(R_\theta^n f)/n^{1-\alpha}$ in $L^2([0,2\pi),\lambda)$ (hence in $L^p([0,2\pi),\lambda)$) is equivalent to $\sum_{k\geq 1}|d_k|^2/|1-e^{in_k\theta}|^{2\alpha}<\infty$, which, by our choice of $\{n_k\}$, is equivalent to $\sum_{k\geq 1}|d_k|^2k^{2\alpha}<\infty$.

On the other hand, using (29), we have

$$\sum_{n\geq 1}\frac{\|\sum_{k=0}^{n-1}R_\theta^k f\|_p^p}{n^{1+(1-\alpha)p}}\leq C_p\sum_{n\geq 1}\frac{1}{n^{1+(1-\alpha)p}}\Big(\sum_{k\geq 1}|d_k|^2|\gamma_n(n_k\theta)|^2\Big)^{p/2}.$$

Recall that $|\gamma_n(u)|\leq C\min(1/u,n)$ for $u\in\,]0,\pi]$, and $|\gamma_n(u)|\geq Cn$ for $u\in\,]0,2/(\pi n)]$. By our choice of $\{n_k\}$, taking $d_k=1/(k^{1/2+\alpha}(\log k)^\delta)$, we obtain

$$\frac{1}{C}\sum_{n\geq 1}\frac{1}{n(\log n)^{p\delta}}\leq\sum_{n\geq 1}\frac{\|\sum_{k=0}^{n-1}R_\theta^k f\|_p^p}{n^{1+(1-\alpha)p}}\leq C\sum_{n\geq 1}\frac{1}{n(\log n)^{p\delta}}.$$

With our choice of $\{d_k\}$, the convergence of the series $\sum_{n\geq 1}(R_\theta^n f)/n^{1-\alpha}$ in $L^2([0,2\pi),\lambda)$ is equivalent to $\sum_{k\geq 1}1/(k(\log k)^{2\delta})<\infty$. Hence the result clearly follows. ∎

6. Applications in ergodic theory. We now give some applications in ergodic theory. We start with conditions to obtain a rate in the pointwise ergodic theorem. Let $(\Omega,\mathcal{F},m)$ be a σ-finite measure space. Given a Dunford–Schwartz operator on $(\Omega,\mathcal{F},m)$ (i.e. an operator which is a contraction on each $L^p(m)$), Derriennic and Lin [8] obtained rates in the pointwise ergodic theorem for functions $f\in(I-T)^\alpha L^p(m)$ with $1<p<\infty$ and $0<\alpha<1$. The operator $(I-T)^\alpha$ was defined there thanks to the power series expansion of $(1-z)^\alpha$.

It was also shown in [8] that $f\in(I-T)^\alpha L^p(m)$ if and only if the series $\sum_{n\geq 1}(T^n f)/n^{1-\alpha}$ converges in $L^p(m)$. In particular, when T is induced by an (invertible) measure preserving transformation, one can apply the previous section.

Recall the result of Derriennic–Lin.

Theorem 6.1. *Let T be a Dunford–Schwartz operator on the measure space $(\Omega,\mathcal{F},m)$. Let $p>1$ and $\alpha\in\,]0,1[$. Let $f\in(I-T)^\alpha L^p(m)$. If $q:=p/(p-1)$ then the following hold.*

(i) *If $\alpha>1-1/p$, then $n^{-1/p}\sum_{k=0}^{n-1}T^k f\to 0$ m-a.e.*
(ii) *If $\alpha=1-1/p$, then $n^{-1/p}(\log n)^{-1/q}\sum_{k=0}^{n-1}T^k f\to 0$ m-a.e.*
(iii) *If $\alpha<1-1/p$, then $n^{-1+\alpha}\sum_{k=0}^{n-1}T^k f\to 0$ m-a.e.*

It is shown in [6] that the rates in (i)–(iii) are best possible for functions in $(I-T)^\alpha L^p(m)$.

Applying the previous results, we obtain

Theorem 6.2. *Let $(\Omega,\mathcal{F},m,\theta)$ be an invertible ergodic dynamical system (the measure m is θ-invariant). Let T be the Dunford–Schwartz operator* *induced by θ. Let $\alpha \in]0,1[$, $p \in]1,2]$ and $f \in L^p(m)$ be such that*

$$\sum_{n\geq 1} \frac{\|f + \cdots + f\circ\theta^{n-1}\|_p^p}{n^{1+(1-\alpha)p}} < \infty$$

(e.g. $\sup_{n\geq 1} \|f + \cdots + f\circ\theta^{n-1}\|_p (\log n)^{1/p+\eta}/n^{1-\alpha}$ for some $\eta > 0$). Then $f \in (I-T)^\alpha L^p(m)$ and Theorem 6.1 applies.

Remark. For $p > 2$, we have a similar result. We have assumed θ is invertible, to be able to apply Theorem 1.1. If θ is not invertible, but acting on a Lebesgue space, one may use the natural extension (see e.g. Rokhlin [17] or Maharam [14]).

We give a theorem which is a combination of results of Cohen and Lin [3] and of Weber [18] obtained for general (not necessarily invertible) power bounded operators on some fixed L^p.

Theorem 6.3. *Let T be a power bounded operator on the measure space $(\Omega, \mathcal{F}, m)$. Let $\alpha \in]0,1[$ and $p > 1$. Let $f \in L^p(m)$ be such that there exists $\eta > 0$ with $\sup_{n\geq 1} \|f + \cdots + f\circ\theta^{n-1}\|_p (\log n)^{1/p+\eta}/n^{1-\alpha} < \infty$. Then, for every $\varepsilon > 0$:*

(i) *If $\alpha > 1 - 1/p$, then*

$$\frac{1}{n^{1/p}(\log n)^{1/p+\varepsilon}} \sum_{k=0}^{n-1} T^k f \to 0 \quad m\text{-a.e.}$$

(ii) *If $\alpha = 1 - 1/p$, then*

$$\frac{1}{n^{1/p}(\log n)^{1+\varepsilon-\eta}} \sum_{k=0}^{n-1} T^k f \to 0 \quad m\text{-a.e.}$$

(iii) *If $\alpha < 1 - 1/p$, then*

$$\frac{1}{n^{1-\alpha}(\log n)^{\varepsilon-\eta}} \sum_{k=0}^{n-1} T^k f \to 0 \quad m\text{-a.e.}$$

Remarks. 1. (i) and (ii) of Theorem 6.3 follow from Theorem 1.3 of [18] applied with $M_n = n$, $L(x) = x(\log x)^{1+\varepsilon}$, $\Psi(x) = x^{p(1-\alpha)}/(\log x)^{1+\eta}$ and $\varphi(n)$ given by the corresponding denominator in (i) or (ii).

2. The rate in (iii) follows from Theorem 2.13 of Cohen–Lin [3] applied with $\varphi(n)$ given by the denominator in (iii), $q = p(1-\alpha)$ and $d^q(0,n) = n^{p(1-\alpha)}/(\log n)^{1/p+\eta}$.

To compare Theorem 6.2 with the results of Weber and Cohen–Lin, one should let η go to zero in Theorem 6.3. The rates in cases (i) and (ii) are better in Theorem 6.2 and essentially the same in case (iii).

When $p = 2$, the use of the spectral theory for unitary operators enabled us in [6] and [5] to consider more general power series (as proposed by

Zhao–Woodroofe) than the one giving $(I-T)^\alpha$, and thus, to obtain more precise rates.

There are technical difficulties (that should not be hard to overcome) to extend the results of [6] and [5] to the case $p \neq 2$. Those difficulties are essentially due to the fact that the spectral integration of Berkson–Gillespie works for functions with bounded variation (hence we "need" a control of the differential of the functions) while in the unitary case one may integrate a much wider class of functions.

We now give another application in ergodic theory, using a result of [4].

THEOREM 6.4. *Let $(\Omega, \mathcal{F}, m)$ be a σ-finite measure space and θ be an invertible $\mathcal{F}$-measurable transformation preserving m. Let $p > 1$ and $f \in L^p(m)$ be such that*

$$\sum_{n\geq 2} \frac{1}{n\log n}\bigg(\frac{\|f+\cdots+f\circ\theta^{n-1}\|_p^2 \log n}{n}\bigg)^{\min(p,2)} < \infty.$$

Then $\sum_{n\geq 1} f\circ\theta^n/n$ converges in $L^p(m)$ and m-a.e. Moreover,

$$\sup_{n\geq 1}\bigg|\sum_{k=1}^{n}\frac{f\circ\theta^k}{k}\bigg| \in L^p(m).$$

Proof. By Theorem 1.2, $\sum_{n\geq 1} f\circ\theta^n/n$ converges in $L^p(m)$. The a.e. convergence and the integrability of the maximal function then follow from Theorem 2.1 of [4]. ∎

A. Proof of Lemma 3.4. It is well-known (and easy to check) that $N(g) := \sup_{n\geq 0}\|T^n g\|_Y$ defines a norm on Y equivalent to the norm $\|\cdot\|_Y$, such that T becomes a contraction for N. In particular, the sequence $\{N(f+\cdots+T^{n-1}f)\}$ is subadditive. Then Lemma 3.4 is a corollary of the following lemma which generalizes Lemma 2.7 of [15].

LEMMA A.1. *Let $\{w_n\}$ be a subadditive sequence of positive numbers. Let b be slowly varying and $\gamma > 1$ and $r \geq 1$. The following are equivalent:*

(i) $\sum_{n\geq 1} b(n) w_n^r/n^\gamma < \infty$.
(ii) $\sum_{n\geq 1} b(2^n) w_{2^n}^r/2^{n(\gamma-1)} < \infty$.

If either (i) *or* (ii) *is satisfied then $b(n)w_n^r/n^{\gamma-1} \to 0$ as $n\to\infty$. If $\gamma = r+1$ and $b = \log^\delta$ for some $\delta \in \mathbb{R}$, then* (i) *and* (ii) *are equivalent to*

(iii) $\sum_{n\geq 1} 2^{n(\delta+1)}(w_{2^{2^n}}/2^{2^n})^r < \infty$,

and we have $w_n^r(\log n)^{\delta+1}/n^r \to 0$ as $n\to\infty$ if $\delta \neq -1$, and $w_n^r(\log\log n)/n^r \to 0$ as $n\to\infty$ if $\delta = -1$.

Proof. (i)⇒(ii). For every $k \in \{4^{n-1}, \ldots, 4^n/2-1\}$, we have $w_{4^n}^r \leq 2^{r-1}(w_k^r + w_{4^n-k}^r)$. Since $\{b(m)/m^\gamma\}$ is non-increasing (for m large), one obtains

$$\frac{4^n}{2}\,\frac{b(4^n)w_{4^n}^r}{4^{\gamma n}} \le 2^{r-1}\sum_{k=4^{n-1}}^{4^n/2-1}\Big(\frac{b(k)w_k^r}{k^\gamma}+\frac{b(4^n-k)w_{4^n-k}}{(4^n-k)^\gamma}\Big).$$

Hence $\sum_{n\ge 1} b(2^{2n})w_{2^{2n}}^r/2^{2(\gamma-1)n} < \infty$ and the result follows as $w_{2^{2n+1}} \le 2w_{2^{2n}}$.

(ii)⇒(i). Let $\delta>0$ be such that $\gamma-1-\delta r>0$. Let $k\ge 0$ and $2^k\le n\le 2^{k+1}-1$. Then, with $1/r+1/r'=1$, we have

$$w_n^r \le \Big(\sum_{l=0}^k w_{2^l}\Big)^r \le \Big(\sum_{l=0}^k 2^{l\delta r'}\Big)^{r/r'}\sum_{l=0}^k \frac{w_{2^l}^r}{2^{l\delta r}} \le C2^{k\delta r}\sum_{l=0}^k \frac{w_{2^l}^r}{2^{l\delta r}}.$$

Hence

$$\begin{aligned}\sum_{n\ge 1}\frac{b(n)w_n^r}{n^\gamma} &= \sum_{k\ge 0}\sum_{n=2^k}^{2^{k+1}-1}\frac{b(n)w_n^r}{n^\gamma} \le C\sum_{k\ge 0}2^{k\delta r}\sum_{l=0}^k\frac{w_{2^l}^r}{2^{l\delta r}}\sum_{n=2^k}^{2^{k+1}-1}\frac{b(n)}{n^\gamma}\\ &\le C'\sum_{l\ge 0}\frac{w_{2^l}^r}{2^{l\delta r}}\sum_{k\ge l}2^{k\delta r}\frac{b(2^k)}{2^{k(\gamma-1)}} \le C''\sum_{l\ge 0}\frac{b(2^l)w_{2^l}^r}{2^{(\gamma-1)l}} < \infty.\end{aligned}$$

Assume that (ii) is satisfied. Then, clearly, $b(2^n)w_{2^n}^r/2^{n(\gamma-1)} \to 0$ as $n\to\infty$. Using the fact that $w_m \le \sum_{k=0}^{\log_2 m} w_{2^k}$, one can see that $b(n)w_n^r/n^{\gamma-1} \to 0$ as $n\to\infty$.

Assume that $b=\log^\delta$ and $\gamma=r+1$. Noticing that $\{w_{2^n}/2^n\}$ is non-increasing, it is not difficult to see that (iii) is equivalent to (ii). Then, for every $m\ge 1$,

$$o(1) = \sum_{m/2\le n\le m}\Big(\frac{w_{2^n}}{2^n}\Big)^r (n\log 2)^\delta \ge \Big(\frac{w_{2^m}}{2^m}\Big)^r(\log 2)^\delta\sum_{m/2\le n\le m} n^\delta.$$

Hence, $(w_{2^n}^r/2^{nr})n^{\delta+1} \to 0$ as $n\to\infty$, when $\delta\ne -1$. If $\delta=-1$, use $\sum_{\sqrt{m}\le n\le m}$ instead of $\sum_{m/2\le n\le m}$. ∎

B. Proof of Lemma 4.1. Since $1/(k-1)^\beta - 1/k^\beta = \beta/k^{1+\beta} + O(1/k^{2+\beta})$ and since T is power bounded, if we use (16) the equivalence of (i) and (ii) will follow once we prove that $\|S_n(f)\|_Y/n^\beta \to 0$ as $n\to\infty$. This convergence follows from Kronecker's Lemma if we know that (i) is satisfied.

Hence, assume that (ii) is satisfied. By Cauchy's criterion,

$$\sum_{k=n}^{2n-1}\frac{S_k(f)}{k^{1+\beta}} = S_n(f)\sum_{k=n}^{2n-1}\frac{1}{k^{1+\beta}} + T^n\Big(\sum_{k=1}^{n-1}\frac{S_k(f)}{(n+k)^{1+\beta}}\Big) \xrightarrow{n\to\infty} 0.$$

Clearly, since $\sup_{n\ge 1}\|T^n\| < \infty$, it suffices to show that the sequence $\{\|\sum_{k=1}^{n-1} S_k(f)/(n+k)^{1+\beta}\|_Y\}$ converges to 0.

Define $R_n = \sum_{k\ge n} S_k(f)/k^{1+\beta}$. We have

$$\sum_{k=1}^{n-1}\frac{S_k(f)}{(n+k)^{1+\beta}} = \sum_{k=1}^{n-1}(R_k - R_{k+1})\frac{k^{1+\beta}}{(n+k)^{1+\beta}}$$
$$= \sum_{k=2}^{n-1} R_k\bigg(\frac{k^{1+\beta}}{(n+k)^{\beta}} - \frac{(k-1)^{1+\beta}}{(n+k-1)^{1+\beta}}\bigg) + \frac{R_1}{n^{1+\beta}} - \frac{R_n(n-1)^{1+\beta}}{(2n-1)^{1+\beta}}.$$

Let $\varepsilon > 0$. By assumption, there exists $n_0 > 0$ such that $\|R_k\|_Y < \varepsilon$ for every $k \ge n_0$. As $x \mapsto x/(n+x)$ is increasing to 1 on $]0,\infty[$, one obtains

$$\bigg\|\sum_{k=1}^{n-1}\frac{S_k(f)}{(n+k)^{1+\beta}}\bigg\|_Y \le \frac{n_0^{2+\beta}\sup_{1\le k\le n_0}\|R_k\|_Y}{n^{1+\beta}} + 2\varepsilon,$$

which proves the desired result. ■

C. Proof of Proposition 4.2. Recall that $|\sigma_n(t)| \le C\min(n, 1/|t|)$ for $n \ge 1$ and $t \in [-\pi,\pi]\setminus\{0\}$. Write $L_n = \sum_{k=1}^n k^2\alpha_k$ for $n \ge 1$ and $L_0 = 0$. Then, for every $n \ge 1$, using the fact that $t \mapsto -t^2\varphi'(t)$ is non-increasing, we have

$$\sum_{k=1}^{n}\alpha_k|\sigma_k(t)| \le \sum_{k=1}^{n}\frac{L_k - L_{k-1}}{k} = \sum_{k=1}^{n-1}\frac{L_k}{k(k+1)} + \frac{L_n}{n}$$
$$\le K\sum_{k=1}^{n-1}\frac{-\varphi'(\pi/k)}{k^2} - K\frac{\varphi'(\pi/n)}{n}$$
$$\le \frac{K}{\pi}\int_1^n \frac{d\varphi(\pi/x)}{dx}\,dx - K\frac{\varphi'(\pi/n)}{n} \le 2K\varphi(\pi/n),$$

where we have used (17) in the last step. That proves the first part of (19).

The second part of (19) follows immediately from (21) and the fact that $|\sigma_n'(t)| = |\sum_{k=0}^{n-1} ke^{ikt}| \le Cn^2$.

Similarly, using (22), we have

$$\sum_{k\ge n}\alpha_k|\sigma_k(t)| \le \frac{-C\varphi'(\pi/n)}{n^2|t|}, \tag{31}$$

and the first part of (20) follows, using (17).

The second part of (20) needs more care. Notice that

$$\sigma_k'(t) = \frac{ie^{it}\sigma_k(t) - ike^{ikt}}{1-e^{it}}$$

Hence

$$\sum_{k\ge n}\alpha_k\sigma_k'(t) = \frac{ie^{it}}{1-e^{it}}\sum_{k\ge n}\alpha_k\sigma_k(t) - \frac{i}{1-e^{it}}\sum_{k\ge n}k\alpha_k e^{ikt}.$$

The first term above was treated in (31).

For the second term, notice that, by (22) (using the fact that $\{n\alpha_n\}$, hence $\{\alpha_n\}$ is non-increasing),

$$\frac{n\alpha_n}{2} \le \sum_{n/2\le k\le n} \alpha_n \le \frac{-4K\varphi'(2\pi/n)}{n^2} \le \frac{-4K\varphi'(\pi/n)}{n^2},$$

since $-\varphi'$ is non-increasing. Then by Abel summation, we obtain

$$\Big|\sum_{k=n}^{p} k\alpha_k e^{ikt}\Big| \le \sum_{k=n}^{p}(k\alpha_k - (k+1)\alpha_{k+1})|\sigma_k(t)| \\ + n\alpha_n|\sigma_n(t)| + (p+1)\alpha_{p+1}|\sigma_p(t)| \le \frac{2Cn\alpha_n}{|t|},$$

which proves the second part of (20). ■

D. Proof of Lemmata 4.5 and 4.6

Proof of Lemma 4.5. We will prove that

$$\Big\|\int_{]\pi/2^n,\pi]} (R_m(t)-R(t))\,dE(t)f\Big\|_p \le L\frac{\varphi(\pi/2^n)}{2^n}\Big\|\Big(\sum_{k=0}^{n-1} 2^{2k}|\mathcal{E}(\varPi_{-k})f|^2\Big)^{1/2}\Big\|_p,$$

the proof for $\int_{]\pi,2\pi-\pi/2^n]}$ being similar. Recall that, for every $t\in\,]0,\pi[$,

$$|R_m(t)-R(t)| \le C\frac{\varphi(\pi/m)}{mt}, \tag{32}$$

$$|R_m'(t)-R'(t)| \le -C\frac{\varphi'(\pi/m)}{m^2t^2}. \tag{33}$$

Let $m\ge 1$ and $n=[\log_2 m]$. Define a function ψ_m on $\mathbb{T}$ by $\psi_m(e^{it}) = R_m(t)-R(t)$ if $t\in\,]\pi/2^{k+1},\pi/2^k[$ for some $k\in\{0,\dots,n-1\}$, and 0 otherwise. Then, using the analogue of Littlewood–Paley, we have

$$\begin{aligned}(34)\quad &\Big\|\int_{]\pi/2^n,\pi]} (R_m(t)-R(t))\,dE(t)f\Big\|_p \\ &\le \Big\|\int_{[0,2\pi]} \psi_m(e^{it})\,dE(t)f\Big\|_p \\ &\quad + \Big\|\int_{[0,2\pi]} (R_m(t)-R(t)-\psi_m(e^{it}))\mathbf{1}_{]\pi/2^n,\pi]}\,dE(t)f\Big\|_p \\ &\le \Big\|\int_{[0,2\pi]} \psi_m(e^{it})\,dE(t)f\Big\|_p + \frac{c^2C_pC\varphi(\pi/2^n)}{\pi 2^n}\Big\|\Big(\sum_{k=0}^{n-1} 2^{2k}|\mathcal{E}(\{\omega_{-k}\})f|^2\Big)^{1/2}\Big\|_p,\end{aligned}$$

where we have used (32) for the last inequality. The second term of (34) can be estimated by means of the Riesz property, hence it remains to deal with the first term.

Define a function ϕ_m by

$$\phi_m : \mathbb{T} \to \mathbb{R},$$
$$e^{it} \mapsto \begin{cases} 2^k \dfrac{\varphi(\pi/2^n)}{2^n} & \text{if } t \in \,]\pi/2^{k+1}, \pi/2^k[\, \text{ for some } k \in \{0, \ldots n-1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then the function ψ_m/ϕ_m is well defined (with $0/0$ interpreted as 0), supported by the arc $\{e^{it} : t \in \,]\pi/2^n, \pi[\}$. Moreover, since $R \in \mathcal{L}_\varphi$ and $\varphi \in \mathcal{K}$, we have $\psi_m/\phi_m \in BV(\mathbb{T})$ for every $m \geq 1$, and, as $\varphi \in \mathcal{K}$,

$$K := \sup_{m\geq 1} \left\| \frac{\psi_m}{\phi_m} \right\|_\infty + \sup_{m \geq 1} \sup_{j \in \mathbb{Z}} \operatorname{var}\left(\frac{\psi_m}{\phi_m}, \Lambda_j \right) < \infty.$$

By the Strong Marcinkiewicz Multiplier Theorem and the analogue of Littlewood–Paley, we obtain

$$\begin{aligned} \Big\| \int\limits_{[0,2\pi]} \psi_m(e^{it})\, dE(t) f \Big\|_p &= \Big\| T_{\psi_m/\tilde{\phi}_m} \Big(\int\limits_{[0,2\pi]} \phi_m(t)\, dE(t) f \Big) \Big\|_p \\ &\leq \|T_{\psi_m/\phi_m}\| \Big\| \Big(\sum_{k=0}^{n-1} 2^{2k} \frac{\varphi^2(\pi/2^n)}{2^{2n}} |\mathcal{E}(\Gamma_{-k}) f|^2 \Big)^{1/2} \Big\|_p \\ &\leq K \frac{\varphi(\pi/2^n)}{2^n} \Big\| \Big(\sum_{k=0}^{n-1} 2^{2k} |\mathcal{E}(\Pi_{-k}) f|^2 \Big)^{1/2} \Big\|_p, \end{aligned}$$

where we have used the analogue of the Riesz property. ■

Proof of Lemma 4.6. Let us show that

$$\Big\| \int\limits_{]0,\pi/2^n]} R_m(t)\, dE(t) f \Big\|_p \leq \varphi(\pi/2^n) \Big\| \Big(\sum_{k\geq n} |\mathcal{E}(\Pi_{-k}) f|^2 \Big)^{1/2} \Big\|_p.$$

The proof of (26) may be done the same way.

Define ψ_m and φ_m on $\mathbb{T}$ by $\psi_m(e^{it}) = W_m(t)$ and $\phi_m(e^{it}) = \varphi(\pi/2^n)$ if $t \in \,]0, \pi/2^n[$, and 0 otherwise. The function ψ_m/ϕ_m is well defined (with $0/0$ interpreted as 0), supported by the arc $\{e^{it} : t \in \,]0, \pi/2^n[\}$, and belongs to $BV(\mathbb{T})$. By (19), ψ_m/ϕ_m is bounded on $\mathbb{T}$, uniformly with respect to m.

Moreover, by (19) and (17), for every $t \in \,]0, \pi/2^n[$,

$$\left| \frac{d}{dt} \frac{\psi_m(e^{it})}{\varphi(\pi/2^n)} \right| \leq C 2^n.$$

Hence

$$\sup_{m \geq 1} \sup_{j \in \mathbb{Z}} \operatorname{var}\left(\frac{\psi_m}{\phi_m}, \Lambda_j \right) < \infty.$$

So, for every $n \geq 1$, T_{ψ_m/ϕ_m} is well defined and, by Theorem 2.1 and the

Strong Marcinkiewicz Multiplier Theorem,

$$L = \sup_{m\geq 1} \|T_{\psi_m/\phi_m}\| < \infty.$$

We obtain

$$\Big\| \int\limits_{]0,\pi/2^n]} R_m(t)\, dE(t) f\Big\|_p \leq \Big\| \int\limits_{[0,2\pi]} \psi_m(t)\, dE(t) f\Big\|_p + \varphi(\pi/2^n)\|\mathcal{E}(\{\omega_{-n}\})f\|_p,$$

where the second term is bounded by $c^2C_p\varphi(\pi/2^n)\|\mathcal{E}(\Pi_{-n})f\|_p$.

On the other hand, using the analogue of the Littlewood–Paley theorem and of the Riesz property,

$$\begin{aligned}\Big\| \int\limits_{[0,2\pi]} \psi_m(t)\, dE(t) f\Big\|_p &= \Big\| T_{\psi_m/\phi_m} \int\limits_{[0,2\pi]} \phi_m(t)\, dE(t) f\Big\|_p \\ &\leq L\varphi(\pi/2^n)\Big\| \sum_{k\geq n} |\mathcal{E}(\Pi_{-k})f|^2)^{1/2}\Big\|_p,\end{aligned}$$

which finishes the proof of Lemma 4.6. ∎

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Christophe Cuny
Équipe ERIM
University of New Caledonia
B.P. R4
98800 Nouméa, New Caledonia
E-mail: cuny@univ-nc.nc

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