STUDIA MATHEMATICA 200 (1) (2010)

Special symmetries of Banach spaces isomorphic to Hilbert spaces

by

Jarno Talponen (Aalto)

Abstract. We characterize Hilbert spaces among Banach spaces in terms of transitivity with respect to nicely behaved subgroups of the isometry group. For example, the following result is typical: If X is a real Banach space isomorphic to a Hilbert space and convex-transitive with respect to the isometric finite-dimensional perturbations of the identity, then X is already isometric to a Hilbert space.

1. Introduction. The expression "special symmetries" in the title refers to suitable subgroups of $\mathcal{G}(\mathrm{X}) = \{T\colon \mathrm{X} \to \mathrm{X} : T$ an isometric automorphism$\}$ where X is a real Banach space. We denote the closed unit ball of X by $\mathbf{B}_{\mathrm{X}}$ and the unit sphere by $\mathbf{S}_{\mathrm{X}}$. The *orbit* of $x \in \mathbf{S}_{\mathrm{X}}$ with respect to a family $\mathcal{F} \subset L(\mathrm{X})$ is given by $\mathcal{F}(x) = \{T(x) : T \in \mathcal{F}\}$. An inner product $(\cdot\,|\,\cdot)\colon \mathrm{X}\times\mathrm{X} \to \mathbb{R}$ is said to be *invariant* with respect to $\mathcal{F}$ if $(T(x)\,|\,T(y)) = (x\,|\,y)$ for each $x, y \in \mathrm{X}$, $T \in \mathcal{F}$. The concept of an invariant inner product is an important tool applied frequently in this article. We say that X is *transitive*, *almost transitive* or *convex-transitive* with respect to $\mathcal{F}$ if $\mathcal{F}(x) = \mathbf{S}_{\mathrm{X}}$, $\overline{\mathcal{F}(x)} = \mathbf{S}_{\mathrm{X}}$ or $\overline{\mathrm{conv}}(\mathcal{F}(x)) = \mathbf{B}_{\mathrm{X}}$, respectively, for all $x \in \mathbf{S}_{\mathrm{X}}$. If $\mathcal{F} = \mathcal{G}(\mathrm{X})$ above, then we will omit mentioning it. This article can be regarded as a part of the field generated around the well-known open *Banach–Mazur rotation problem*, which asks whether each transitive separable Banach space is isometrically a Hilbert space. See [3] for an exposition of the topic.

In [5] F. Cabello Sánchez studied the subgroup

$$\mathcal{G}_F = \{T \in \mathcal{G}(\mathrm{X}) : \mathrm{Rank}(T - \mathrm{Id}) < \infty\}$$

consisting of the finite-dimensional perturbations of the identity. There a classical result appearing in [1, 11] is applied, namely, that each finite-dimen-

2010 *Mathematics Subject Classification*: Primary 46C15, 43A35; Secondary 46B08, 43A07.

Key words and phrases: amenable, finite-dimensional isometric perturbations of the identity, isometry group, Mazur's problem, rotation.

DOI: 10.4064/sm200-1-2

sional Banach space admits an invariant inner product. This motivated the work in [5], where an elegant proof was presented for the following result:

THEOREM 1.1. *If the norm of* X *is transitive with respect to* $\mathcal{G}_F$, *then* X *is isometric to a Hilbert space.*

Cabello raised the question whether this result can be extended to the almost transitive setting. It turns out that the answer is affirmative under the additional assumption that X is isomorphic to a Hilbert space:

THEOREM 1.2. *Let* X *be a Banach space isomorphic to a Hilbert space. Then* X *is convex-transitive with respect to* $\mathcal{G}_F$ *if and only if* X *is isometric to a Hilbert space.*

This paper is also motivated by the following problems posed in [4, 5]:

- Is an almost transitive Banach space isometric to a Hilbert space if it is isomorphic to one?
- Find ideals $J \subset L(\mathrm{X})$ (with $F \subset J$) for which Theorem 1.1 remains true if the condition $T - \mathrm{Id} \in F$ is replaced by $T - \mathrm{Id} \in J$ (here F is the ideal of finite-rank operators).

Questions of this type are treated in what follows, and we will also show that the existence of an invariant inner product on X follows from the existence of an invariant inner product for each finitely generated subgroup of $\mathcal{G}(\mathrm{X})$ (see Theorem 2.2).

1.1. Preliminaries. We refer to [3], [7], [9], [10], [13] and [14] for some background information. Recall that a norm $\|\cdot\|$ on X is *maximal* if $\mathcal{G}_{(\mathrm{X},\|\cdot\|)} \subset \mathcal{G}_{(\mathrm{X},|||\cdot|||)}$ for an equivalent norm $|||\cdot|||$ implies that $\mathcal{G}_{(\mathrm{X},\|\cdot\|)} = \mathcal{G}_{(\mathrm{X},|||\cdot|||)}$. If X is convex-transitive, then the norm of X is maximal (see [6]). We denote by $\mathrm{Aut}(\mathrm{X})$ the group of isomorphisms $T\colon \mathrm{X} \to \mathrm{X}$.

Given a topological group G we denote by $\mathrm{UCB}(G)$ the space of uniformly continuous bounded functions on G. Here we consider the uniform structure Φ_G of G as being generated by a basis of entourages of the diagonal having the form

$$W = \{(g,h) \in G \times G : gh^{-1}, g^{-1}h \in V\}, \tag{1.1}$$

where V runs over a neighbourhood basis of e in G. The space $\mathrm{UCB}(G)$ is endowed with the $\|\cdot\|_\infty$-norm.

For convenience we isolate the following condition: Suppose that there is a positive functional $F \in \mathrm{UCB}(G)^*$ with $\|F\| = 1$ such that

$$F(f(\cdot g)) = F(f(\cdot)) \quad \text{for all } f \in \mathrm{UCB}(G),\ g \in G. \tag{1.2}$$

This type of condition can be viewed as a weaker version of amenability of G (see [12]). We note that the rotation group of L^p with the strong operator topology is extremely amenable for $1 \le p < \infty$ (see [9]).

Recall that the product topology of X^X inherited by $L(X)$ is called the *strong operator topology* (*SOT*).

We often consider subgroups $\mathcal{G} \subset \mathcal{G}(X)$ which enjoy the following property:

(∗) Given $n \in \mathbb{N}$, $T_1, \dots, T_n \in \mathcal{G}$ and a finite-codimensional subspace $Z \subset X$ there exists a finite-codimensional subspace $Y \subset Z$ such that $T_1(Y) = \cdots = T_n(Y) = Y$.

Clearly $\mathcal{G}_F$ is an example of a subgroup of $\mathcal{G}(X)$ satisfying (∗).

It is easy to see that if H is a Hilbert space, then $\mathcal{G}_F \subset \mathcal{G}(H)$ is dense in $\mathcal{G}(H)$ in the topology of uniform convergence on compact sets. On the other hand, given a Banach space X the group $\mathcal{G}(X)$ is SOT-closed in Aut(X).

2. Results

THEOREM 2.1. *Let* X *be a maximally normed Banach space which is isomorphic to a Hilbert space. Suppose that* $\mathcal{G}(X)$ *endowed with the strong operator topology is amenable in the sense of condition* (1.2). *Then* X *is isometrically isomorphic to a Hilbert space.*

Proof. We may assume without loss of generality that $(X, \|\cdot\|)$ and $(X, |\cdot|)$ are isomorphic via the identical mapping, where $|\cdot|$ is a norm induced by an inner product $(\cdot\,|\,\cdot)$ on X. We denote by $\mathcal{G}(X) = \mathcal{G}_{(X,\|\cdot\|)}$ and $\mathcal{G}_{(X,|\cdot|)}$ the corresponding rotation groups, and these are regarded with the strong operator topology. Recall that $\Phi_{\mathcal{G}(X)}$ is the natural uniformity given by the group $(\mathcal{G}(X), \mathrm{SOT})$ applied to (1.1).

Observe that $T \mapsto (Tx\,|\,Ty)$ defines a $\Phi_{\mathcal{G}(X)}$-uniformly continuous map $\mathcal{G}(X) \to \mathbb{R}$ for each $x, y \in X$. Indeed, this map is obtained by composing the $\Phi_{\mathcal{G}(X)}$-$\|\cdot\|_{X\oplus_2 X}$ uniformly continuous map $\mathcal{G}(X) \to X \oplus_2 X$, $T \mapsto (Tx, Ty)$, and the map $(Tx, Ty) \mapsto (Tx\,|\,Ty)$, which is $\|\cdot\|_{X\oplus_2 X}$-uniformly continuous as $\|\cdot\| \sim |\cdot|$. To check that $T \mapsto (Tx, Ty)$ is uniformly continuous, first consider a standard entourage

$$E = \{(x_1, y_1, x_2, y_2) \in X \oplus_2 X \times X \oplus_2 X : \|(x_1, y_1) - (x_2, y_2)\|_{X\oplus_2 X} < \epsilon\}$$

for some $\epsilon > 0$. The preimage of this is

$$\begin{aligned}
&\{(R, S) \in \mathcal{G}(X) \times \mathcal{G}(X) : \|(Rx, Ry) - (Sx, Sy)\|_{X\oplus_2 X} < \epsilon\} \\
&\qquad \supset \{(R, S) \in \mathcal{G}(X) \times \mathcal{G}(X) : \|Tx - Sx\|, \|Ty - Sy\| < \epsilon/2\} \\
&\qquad = \{(R, S) \in \mathcal{G}(X) \times \mathcal{G}(X) : \|x - T^{-1}Sx\|, \|y - T^{-1}Sy\| < \epsilon/2\}.
\end{aligned}$$

Hence it suffices to pick $V = \{R \in \mathcal{G}(X) : \|x - Rx\|, \|y - Ry\| < \epsilon/2\}$ in (1.1) to find an entourage of $\Phi_{\mathcal{G}(X)}$ in the preimage of E. We deduce that $T \mapsto (Tx, Ty)$ is $\Phi_{\mathcal{G}(X)}$-uniformly continuous.

According to the assumptions there is $F \in \mathrm{UCB}(\mathcal{G}(\mathrm{X}))^*$ with $\|F\| = 1$ such that $F(f(\cdot g)) = F(f(\cdot))$ for $f \in \mathrm{UCB}(\mathcal{G}(\mathrm{X}))$ and $g \in \mathcal{G}(\mathrm{X})$. For each $x, y \in \mathrm{X}$ we put

$$[x \,|\, y] = F(\{(g(x) \,|\, g(y))\}_{g\in\mathcal{G}(\mathrm{X})}).$$

This definition is sensible, since $g \mapsto (g(x) \,|\, g(y))$ defines an element in $\mathrm{UCB}(\mathcal{G}(\mathrm{X}))$ for each $x, y \in \mathrm{X}$. We claim that $[\cdot \,|\, \cdot]$ defines an inner product on X such that $|||x||| \doteq \sqrt{[x \,|\, x]}$ is equivalent to $\|\cdot\|$. Indeed, first note that $[\cdot \,|\, \cdot]\colon (\mathrm{X}, \|\cdot\|) \oplus_2 (\mathrm{X}, \|\cdot\|) \to \mathbb{R}$ is defined and bounded, since $(\cdot \,|\, \cdot)\colon (\mathrm{X}, \|\cdot\|) \oplus_2 (\mathrm{X}, \|\cdot\|) \to \mathbb{R}$ is bounded and $\|F\| = 1$. By using the bilinearity of $(\cdot \,|\, \cdot)$ and the linearity of F we see that $[\cdot \,|\, \cdot]$ is bilinear. Let $C \geq 1$ be such that $C^{-2}\|\cdot\|^2 \leq |\cdot|^2 \leq C^2\|\cdot\|^2$. Since F is positive and norm-1, we get

$$\begin{aligned} C^{-2}\|x\|^2 = \inf_g C^{-2}\|g(x)\|^2 &\leq F(\{(g(x) \,|\, g(x))\}_{g\in\mathcal{G}(\mathrm{X})}) \\ &\leq \sup_g C^2\|g(x)\| = C^2\|x\|, \end{aligned}$$

where $x \in \mathrm{X}$ and the supremum and infimum are taken over $\mathcal{G}(\mathrm{X})$. This means that $[\cdot \,|\, \cdot]$ is an inner product on X such that $|||\cdot|||$ is equivalent to $\|\cdot\|$.

Observe that

$$[h(x) \,|\, h(y)] = F(\{(gh(x) \,|\, gh(y))\}_{g\in\mathcal{G}(\mathrm{X})}) = F(\{(g(x) \,|\, g(y))\}_{g\in\mathcal{G}(\mathrm{X})}) = [x \,|\, y]$$

for each $h \in \mathcal{G}(\mathrm{X})$. The maximality of the norm of $(\mathrm{X}, \|\cdot\|)$ implies that $\mathcal{G}_{(\mathrm{X},\|\cdot\|)} = \mathcal{G}_{(\mathrm{X},|||\cdot|||)}$. The proof is completed by a standard argument using the fact that $(\mathrm{X}, |||\cdot|||)$ is transitive. ∎

Suppose that X is a Banach space with two equivalent norms $\|\cdot\|$ and $|||\cdot|||$ such that the group $\mathcal{G}$ generated by $\mathcal{G}_{(\mathrm{X},\|\cdot\|)} \cup \mathcal{G}_{(\mathrm{X},|||\cdot|||)}$ is operator norm bounded. Then there is one more equivalent norm $||||\cdot||||$ on X given by $||||x|||| = \sup_{g\in\mathcal{G}} \|g(x)\|$ and this is $\mathcal{G}$-invariant. Consequently, if the norms $\|\cdot\|$ and $|||\cdot|||$ are additionally maximal (resp. convex-transitive), then $\mathcal{G}_{(\mathrm{X},\|\cdot\|)} = \mathcal{G}_{(\mathrm{X},|||\cdot|||)}$ (resp. $\|\cdot\| = c|||\cdot|||$ for some constant $c > 0$).

The argument employed in the proof of [5, Lemma 2] can be modified to obtain the following dichotomy regarding the existence of invariant inner products.

Theorem 2.2. *Let* X *be a Banach space and* $C \geq 1$. *Suppose that for each* $n \in \mathbb{N}$ *and* $T_1, \ldots, T_n \in \mathcal{G}(\mathrm{X})$ *there exists an inner product* $(\cdot \,|\, \cdot)_*\colon \mathrm{X} \times \mathrm{X} \to \mathbb{R}$ *invariant under the rotations* $T_1, \ldots, T_n$ *such that* $C^{-2}\|x\|^2 \leq (x|x)_* \leq C^2\|x\|^2$ *for each* $x \in \mathrm{X}$. *Then there is already an inner product* $(\cdot \,|\, \cdot)_{\mathrm{X}}\colon \mathrm{X} \times \mathrm{X} \to \mathbb{R}$ *which is invariant under* $\mathcal{G}(\mathrm{X})$ *and satisfies* $C^{-2}\|x\|^2 \leq (x \,|\, x)_{\mathrm{X}} \leq C^2\|x\|^2$ *for* $x \in \mathrm{X}$.

Proof. We may assume without loss of generality that $\mathcal{G}(\mathrm{X})$ is not finitely generated. Let $\mathcal{N}$ be the net of finitely generated subgroups of $\mathcal{G}(\mathrm{X})$ ordered by inclusion. By the assumptions we may assign to each $\gamma \in \mathcal{N}$ an inner product $(\cdot\,|\,\cdot)_\gamma \colon \mathrm{X} \times \mathrm{X} \to \mathbb{R}$ invariant under γ and satisfying

$$C^{-1}\|x\|^2 \le (x\,|\,x)_\gamma \le C\|x\|^2 \quad \text{for } x \in \mathrm{X}.$$

Observe that the sets $\{\gamma \in \mathcal{N} : \delta \subset \gamma\}$, where $\delta \in \mathcal{N}$, form a filter base of a filter $\mathcal{F}$ on $\mathcal{N}$. Let us extend $\mathcal{F}$ to an ultrafilter $\mathcal{U}$ on $\mathcal{N}$. Note that $\mathcal{U}$ is non-principal, since for each $\eta \in \mathcal{N}$ there is $\delta \in \mathcal{N}$ with $\eta \subsetneq \delta$ such that $\eta \notin \{\gamma \in \mathcal{N} : \delta \subset \gamma\} \in \mathcal{U}$.

Define $B \colon \mathrm{X} \times \mathrm{X} \to \mathbb{R}^{\mathcal{N}}$ by setting $B(x,y) = \{(x\,|\,y)_\gamma\}_{\gamma\in\mathcal{N}}$ for $x, y \in \mathrm{X}$. We will consider $\mathbb{R}^{\mathcal{N}}$ equipped with the usual pointwise linear structure. Then B becomes a symmetric and bilinear map. Moreover, $B(x,x) \ge 0$ pointwise for $x \in \mathrm{X}$. Put $\vec{B} \colon \mathrm{X}\times\mathrm{X} \to \mathbb{R}$, $\vec{B}(x,y) = \lim_{\mathcal{U}} B(x,y)$ for $x, y \in \mathrm{X}$. Indeed, the above limit exists and is finite for all $x, y \in \mathrm{X}$, since

$$(x|y)_\gamma \le \sqrt{(x\,|\,x)_\gamma (y|y)_\gamma} \le C^2\|x\|\,\|y\| \quad \text{for all } \gamma \in \mathcal{N},\ x, y \in \mathrm{X}.$$

Moreover, similarly we get $C^{-2}\|x\|^2 \le \vec{B}(x,x) \le C^2\|x\|^2$ for all $x \in \mathrm{X}$. It follows that $\vec{B}$ is an inner product on X.

Observe that for all $T \in \mathcal{G}(\mathrm{X})$ and $x, y \in \mathrm{X}$ we have

$$\{\gamma \in \mathcal{N} : (Tx\,|\,Ty)_\gamma = (x|y)_\gamma\} \supset \{\gamma \in \mathcal{N} : T \in \gamma\} \in \mathcal{F} \subset \mathcal{U}.$$

Hence $\vec{B}(Tx, Ty) = \vec{B}(x,y)$ for $T \in \mathcal{G}(\mathrm{X})$ and $x, y \in \mathrm{X}$. Consequently, $\vec{B}$ is the required inner product. ∎

It is not known if an almost transitive Banach space isomorphic to a Hilbert space is in fact isometric to a Hilbert space (see [4]). The following consequence of Theorem 2.2 provides a partial answer to this problem.

Corollary 2.3. *Let* X *be a maximally normed Banach space,* H *a Hilbert space and* $C \ge 1$. *Suppose that for any* $n \in \mathbb{N}$ *and* $T_1, \ldots, T_n \in \mathcal{G}(\mathrm{X})$ *there exists an isomorphism* $\phi \colon \mathrm{X} \to \mathrm{H}$ *such that* $\max(\|\phi\|, \|\phi^{-1}\|) \le C$ *and* $\|\phi(x)\| = \|\phi(T_i x)\|$ *for all* $x \in \mathrm{X}$ *and* $i \in \{1, \ldots, n\}$. *Then* X *is already isometric to* H.

Proof. By putting $(x\,|\,y)_* = (\phi(x)\,|\,\phi(y))_{\mathrm{H}}$ for each $T_1, \ldots, T_n$ we obtain the assumptions of Theorem 2.2. Let $(\cdot\,|\,\cdot)_{\mathrm{X}} \colon \mathrm{X} \times \mathrm{X} \to \mathbb{R}$ be the resulting inner product. Then X endowed with the norm $|||x||| \doteq \sqrt{(x\,|\,x)_{\mathrm{X}}}$ is transitive being a Hilbert space. Since X is maximally normed, we get $\mathcal{G}_{(\mathrm{X},\|\cdot\|)} = \mathcal{G}_{(\mathrm{X},|||\cdot|||)}$. Thus X is transitive. It follows that $\|\cdot\| = c|||\cdot|||$ for some $c > 0$, and hence X is a Hilbert space. ∎

Theorem 2.4. *Let* $(\mathrm{X}, \|\cdot\|)$ *be a Banach space,* $(\mathrm{H}, (\cdot\,|\,\cdot)_{\mathrm{H}})$ *an inner product space,* $\mathcal{G} \subset \mathcal{G}(\mathrm{X})$ *a subgroup satisfying* $(*)$, *and* $S \colon \mathrm{X} \to \mathrm{H}$ *an isomorphism. Then there exists an inner product* $(\cdot\,|\,\cdot)_{\mathrm{X}}$ *on* X *such that*

(1) $\|S^{-1}\|^{-2}\|x\|^2 \leq (x\,|\,x)_{\mathrm{X}} \leq \|S\|^2\|x\|^2$ *for* $x \in \mathrm{X}$.
(2) $(Tx\,|\,Ty)_{\mathrm{X}} = (x\,|\,y)_{\mathrm{X}}$ *for* $x, y \in \mathrm{X}$ *and* $T \in \overline{\mathcal{G}}^{\mathrm{SOT}} \subset L(\mathrm{X})$.

Proof. It suffices to find $(\cdot\,|\,\cdot)_{\mathrm{X}}$ which satisfies conclusions (1) and (2) for merely $T \in \mathcal{G}$. Indeed, given $T \in \overline{\mathcal{G}}^{\mathrm{SOT}}$ and $x, y \in \mathrm{X}$ there is a sequence $(T_n) \subset \mathcal{G}$ such that $T_n(x) \to T(x)$ and $T_n(y) \to T(y)$ as $n \to \infty$. This yields

$$(T(x)\,|\,T(y))_{\mathrm{X}} - (x\,|\,y)_{\mathrm{X}} = \lim_{n\to\infty}((T_n(x)\,|\,T_n(y))_{\mathrm{X}} - (x\,|\,y)_{\mathrm{X}}) = 0$$

by using the $\mathcal{G}$-invariance and the $\|\cdot\|$-continuity of $(\cdot\,|\,\cdot)_{\mathrm{X}}$.

Let $\mathcal{M}$ be the set of all pairs (E, G) where $E \subset \mathrm{X}$ is a finite-codimensional subspace and $G \subset \mathcal{G}$ is a finitely generated subgroup such that $T(E) = E$ for $T \in G$.

From the definition of $\mathcal{G}$ we know that $\bigcup_{(E,G)\in\mathcal{M}} G = \mathcal{G}$ and $\bigcap_{(E,G)\in\mathcal{M}} E = \{0\}$. We equip $\mathcal{M}$ with the partial order $\leq$ defined as follows: $(E_1, G_1) \leq (E_2, G_2)$ if $E_1 \supset E_2$ and $G_1 \subset G_2$. So, $(\mathcal{M}, \leq)$ is a directed set.

Suppose that $\mathrm{Y} \subset \mathrm{H}$ is a subspace of a Hilbert space and H/Y is the corresponding quotient space. Then there exists a natural inner product on H/Y, namely

$$(\widehat{x}^{\mathrm{Y}}\,|\,\widehat{y}^{\mathrm{Y}})_{\mathrm{H}/\mathrm{Y}} = (x - P_{\mathrm{Y}}x\,|\,y - P_{\mathrm{Y}}y)_{\mathrm{H}}, \qquad x, y \in \mathrm{H},$$

where $\widehat{x}^{Y} = x + \mathrm{Y}$, $\widehat{y}^{\mathrm{Y}} = y + \mathrm{Y}$ and $P_{\mathrm{Y}}\colon \mathrm{X} \to \mathrm{Y}$ is the orthogonal projection onto Y.

Given $(E, G) \in \mathcal{M}$ we find that $T(E) = E$ for $T \in G$ and hence the mapping $\widehat{T}_E\colon \mathrm{X}/E \to \mathrm{X}/E$ given by $\widehat{T}_E(\widehat{x}^E) = T(x + E)$ defines a rotation on X/E for $T \in G$. Indeed, $\|\widehat{x}^E\|_{\mathrm{X}/E} = \mathrm{dist}(x, E)$ and $\mathrm{dist}(T(x), E) = \mathrm{dist}(x, E)$, as $T(E) = E$. Now, since X/E is finite-dimensional, the rotation group $\mathcal{G}_{\mathrm{X}/E}$ is compact in the operator norm topology.

For each $(E, G) \in \mathcal{M}$ we define a map $\widehat{S}_E\colon \mathrm{X}/E \to \mathrm{H}/S(E)$ by $\widehat{S}_E(\widehat{x}^E) = S(x + E)$. It is easy to see that

$$\begin{aligned}&\|S^{-1}\|^{-2}\|\widehat{x}^E\|^2_{\mathrm{X}/E} \leq (\widehat{S}_E(\widehat{x}^E)|\widehat{S}_E(\widehat{x}^E))_{\mathrm{H}/S(E)},\\ &(\widehat{S}_E(\widehat{x}^E)\,|\,\widehat{S}_E(\widehat{y}^E))_{\mathrm{H}/S(E)} \leq \|S\|^2\|\widehat{x}^E\|_{\mathrm{X}/E}\|\widehat{y}^E\|_{\mathrm{X}/E}\end{aligned} \tag{2.1}$$

for $x, y \in \mathrm{X}$. Consider $\mathbb{R}^{\mathcal{M}}$ with the pointwise linear structure. Define a map $B\colon \mathrm{X} \times \mathrm{X} \to \mathbb{R}^{\mathcal{M}}$ by

$$B(x, y)(E, G) = \int_{\mathcal{G}_{\mathrm{X}/E}} (\widehat{S}_E(\tau\widehat{x}^E)\,|\,\widehat{S}_E(\tau\widehat{y}^E))_{\mathrm{H}/S(E)}\,d\tau.$$

Above $\int_{\mathcal{G}_{\mathrm{X}/E}}$ is the invariant Haar integral over the compact group $\mathcal{G}_{\mathrm{X}/E}$. The invariance of the integral yields $B(Tx, Ty)(E, G) = B(x, y)(E, G)$ for $x, y \in \mathrm{X}$, $(E, G) \in \mathcal{M}$ and $T \in G$. By using (2.1) and the basic properties of the integral we obtain

$$\text{(2.2)}\qquad \begin{aligned} &\|S^{-1}\|^{-2}\|\widehat{x}^E\|^2_{\mathrm{X}/E} \le B(x,x)(E,G),\\ &B(x,y)(E,G) \le \|S\|^2\|\widehat{x}^E\|_{\mathrm{X}/E}\|\widehat{y}^E\|_{\mathrm{X}/E} \end{aligned}$$

for $x, y \in \mathrm{X}$ and $(E,G) \in \mathcal{M}$.

The family $\{\{\gamma \in \mathcal{M} : \gamma \ge \eta\}\}_{\eta\in\mathcal{M}}$ is a filter base on $\mathcal{M}$. Let $\mathcal{U}$ be a non-principal ultrafilter extending $\{\{\gamma \in \mathcal{M} : \gamma \ge \eta\}\}_{\eta\in\mathcal{M}}$. Put $(x\,|\,y)_{\mathrm{X}} = \lim_{\mathcal{U}} B(x,y)$ for $x, y \in \mathrm{X}$. It is easy to see that $(\cdot\,|\,\cdot)_{\mathrm{X}}$ is a bilinear mapping.

According to (2.2) we obtain $(x\,|\,y)_{\mathrm{X}} \le \|S\|^2\|x\|_{\mathrm{X}}\|y\|_{\mathrm{X}}$. Next, we aim to verify that $\|S^{-1}\|^{-2}\|x\|^2_{\mathrm{X}} \le (x\,|\,x)_{\mathrm{X}}$. Towards this, we will check that $\sup_{(E,G)\in\mathcal{M}} \|\widehat{x}^E\|_{\mathrm{X}/E} = \|x\|_{\mathrm{X}}$. Fix $x \in \mathbf{S}_{\mathrm{X}}$. Assume to the contrary that $\sup_{(E,G)\in\mathcal{M}} \|\widehat{x}^E\|_{\mathrm{X}/E} = c < 1$. Note that X is reflexive, being isomorphic to H. Thus the ball $x + c\mathbf{B}_{\mathrm{X}}$ is weakly compact. Putting

$$\{\{y \in E : \|x-y\| \le C\}\}_{(E,G)\in\mathcal{M}}$$

defines a net of non-empty closed convex subsets of $x + c\mathbf{B}_{\mathrm{X}}$. This net has a cluster point $z \in x + c\mathbf{B}_{\mathrm{X}}$ according to the weak compactness of $x + c\mathbf{B}_{\mathrm{X}}$. This means that $z \in \bigcap_{(E,G)\in\mathcal{M}} E$, which provides a contradiction, since $z \ne 0$. Consequently, (2.2) yields

$$\|S^{-1}\|^{-2}\ \|x\|^2_{\mathrm{X}} = \|S^{-1}\|^{-2}\ \lim_{\mathcal{U}} \|\widehat{x}^E\|^2_{\mathrm{X}/E} \le \lim_{\mathcal{U}} B(x,x) = (x\,|\,x)_{\mathrm{X}}.$$

Finally, we claim that $(Tx\,|\,Ty)_{\mathrm{X}} = (x\,|\,y)_{\mathrm{X}}$ for $x, y \in \mathrm{X}$ and $T \in \mathcal{G}$. Indeed, pick $T \in \mathcal{G}$ and $x, y \in \mathrm{X}$. Then

$$\begin{aligned} \{(E,G) \in \mathcal{M} : B(T(x),T(y))(E,G) = B(x,y)(E,G)\}&\\ \supset \{(E,G) \in \mathcal{M} : T \in G\} \in \mathcal{U},& \end{aligned}$$

so that $\lim_{\mathcal{U}}(B(Tx,Ty) - B(x,y)) = 0$. ∎

Corollary 2.5. *Let* X *be a maximally normed space* X *isomorphic to a Hilbert space. Suppose that there is a subgroup* $\mathcal{G} \subset \mathcal{G}(\mathrm{X})$ *which satisfies* $(*)$ *and* $\mathcal{G}(\mathrm{X}) \subset \overline{\mathcal{G}}^{\mathrm{SOT}}$. *Then* X *is isometrically a Hilbert space.* ∎

In Theorem 2.4 the isomorphism S was exploited in order to give bounds for the resulting inner product $(\cdot\,|\,\cdot)_{\mathrm{X}}$. In [5] a different approach was taken: the analogous construction was suitably normalized by using a special point x_0. By suitably combining the arguments in [5] and in the proof of Theorem 2.4 we obtain the following result.

Theorem 2.6. *Let* X *be a Banach space transitive with respect to a subgroup* $\mathcal{G} \subset \mathcal{G}(\mathrm{X})$ *which satisfies* $(*)$. *Then* X *is isometric to a Hilbert space.* ∎

Theorem 1.2 is an immediate consequence of the following result. This result implies that X must in particular be almost transitive, and we note that there exists an alternative route to this fact, since spaces both convex-transitive and superreflexive are additionally almost transitive (see e.g. [8]).

THEOREM 2.7. *Let* X *be a Banach space isomorphic to a Hilbert space and suppose* $\mathcal{G} \subset \mathcal{G}(\mathrm{X})$ *is a subgroup which satisfies* $(*)$ *and* $\mathcal{G}_F \subset \mathcal{G}$. *Then* X *is convex-transitive with respect to* $\overline{\mathcal{G}}^{\mathrm{SOT}} \subset L(\mathrm{X})$ *if and only if* X *is isometric to a Hilbert space.*

Proof. First note that a Hilbert space is transitive, in particular convex-transitive, and that $\mathcal{G}_F \subset \mathcal{G}(\mathrm{H})$ is SOT-dense in $\mathcal{G}(\mathrm{H})$, so that the "if" direction is clear.

Since X is isomorphic to a Hilbert space, we may apply Theorem 2.4 to obtain a $\overline{\mathcal{G}}^{\mathrm{SOT}}$-invariant inner product $(\cdot\,|\,\cdot)_{\mathrm{X}}$ on X such that $|||x|||^2 = (x\,|\,x)_{\mathrm{X}}$ defines a norm equivalent with $\|\cdot\|_{\mathrm{X}}$. Clearly $|||\cdot|||$ is $\overline{\mathcal{G}}^{\mathrm{SOT}}$-invariant as well. By rescaling $|||\cdot|||$ we may assume without loss of generality that $\|\cdot\|_{\mathrm{X}} \le |||\cdot|||$ and $\sup_{y\in \mathbf{S}_{(\mathrm{X},|||\cdot|||)}} \|y\|_{\mathrm{X}} = 1$. Put $C = \{x \in \mathrm{X} : |||x||| \le 1\}$.

Fix $x \in \mathbf{S}_{(\mathrm{X},\|\cdot\|_{\mathrm{X}})}$ and $\epsilon > 0$. Let $y \in \mathbf{S}_{(\mathrm{X},|||\cdot|||)}$ be such that $\|y\|_{\mathrm{X}} > 1-\epsilon/2$. Since $(\mathrm{X}, \|\cdot\|_{\mathrm{X}})$ is convex-transitive with respect to $\overline{\mathcal{G}}^{\mathrm{SOT}}$, we see that $(1-\epsilon/2)x \in \overline{\mathrm{conv}}^{\|\cdot\|_{\mathrm{X}}}(\{T(y) : T \in \overline{\mathcal{G}}^{\mathrm{SOT}}\})$. Since the norms $|||\cdot|||$ and $\|\cdot\|_{\mathrm{X}}$ are equivalent we deduce that there is a convex combination $\sum a_n T_n(y) \in \mathrm{conv}(\{T(y) : T \in \mathcal{G}_F\})$ such that $|||(1-\epsilon/2)x - \sum a_n T_n(y)||| < \epsilon/2$. By noting that $|||\sum a_n T_n(y)||| \le \sum a_n |||T_n(y)|||$ we obtain $\sup_{T\in\overline{\mathcal{G}}^{\mathrm{SOT}}} |||T(y)||| \ge |||x||| - \epsilon$. Hence $|||y||| \ge |||x||| - \epsilon$ by using the $\overline{\mathcal{G}}^{\mathrm{SOT}}$-invariance of $|||\cdot|||$. Since ϵ was arbitrary and $|||x||| \ge 1$, we deduce that $|||x||| = 1$, and it follows that $\|\cdot\|_{\mathrm{X}} = |||\cdot|||$. ∎

Finally, we will take a different approach and characterize the Hilbert spaces in terms of the subgroup of rotations that fix a given 1-dimensional subspace, rather than a finite-codimensional subspace.

PROPOSITION 2.8. *Let* X *be an almost transitive Banach space. Suppose that there exists* $z_0 \in \mathbf{S}_{\mathrm{X}}$ *such that for any* $\epsilon > 0$ *and* $x, y \in \mathbf{S}_{\mathrm{X}}$ *with* $\mathrm{dist}(x, [z_0]) = \mathrm{dist}(y, [z_0]) = 1$, *there is* $T \in \mathcal{G}(\mathrm{X})$ *with* $\|T(z_0) - z_0\| < \epsilon$ *and* $\|T(x) - y\| < \epsilon$. *Then* X *is isometric to an inner product space.*

Proof. It is well-known (see e.g. Corollary 2.42 and Diagram I in [3, p. 22], or Proposition 9.6.1 and discussion in [13]) that almost transitive finite-dimensional spaces are isometric to Hilbert spaces. Hence we may concentrate on the case $\dim(\mathrm{X}) \ge 3$. Let $A, B \subset \mathrm{X}$ be 2-dimensional subspaces such that $z_0 \in A$. Recall the classical result that a Banach space is isometric to a Hilbert space if and only if any couple of 2-dimensional subspaces are mutually isometric (see [2]). Thus, in order to establish the claim, it suffices to verify that the subspaces A and B are isometric.

Fix $0 < \epsilon < 1$, $x \in \mathbf{S}_{\mathrm{X}} \cap A$ such that $\mathrm{dist}(x, [z_0]) = 1$ and $w \in \mathbf{S}_{\mathrm{X}} \cap B$. Let $f \in \mathbf{S}_{\mathrm{X}^*}$ be such that $f(w) = 1$.

Since X is almost transitive, there is $T_1 \in \mathcal{G}(\mathrm{X})$ such that $\|T_1(w) - z_0\| < \epsilon/4$. Define a linear operator $S\colon \mathrm{X} \to \mathrm{X}$ by $S(v) = T_1(v) + f(v)(z_0 - T_1(w))$ for $v \in \mathrm{X}$ and note that $S(w) = z_0$. Observe that S is an isomorphism, since $\|T_1 - S_1\| < \epsilon/4$. Pick $y \in \mathbf{S}_{\mathrm{X}} \cap S(B)$ such that $\mathrm{dist}(y, [z_0]) = 1$. According to the assumptions there is $T_2 \in \mathcal{G}(\mathrm{X})$ such that

$$\max(\|T_2(z_0) - z_0\|, \|T_2(y) - x\|) < \epsilon/4.$$

Let $g, h \in 2\mathbf{B}_{\mathrm{X}^*}$ be such that $g(z_0) = h(y) = 1$, $y \in \mathrm{Ker}(g)$ and $z_0 \in \mathrm{Ker}(h)$. Define a linear operator $U\colon \mathrm{X} \to \mathrm{X}$ by

$$U(v) = T_2(v) + g(v)(z_0 - T_2(z_0)) + h(v)(x - T_2(y)) \quad \text{for } v \in \mathrm{X}.$$

Note that $U(z_0) = z_0$ and $U(y) = x$. Moreover, $\|T_2 - U\| < \epsilon$, so that U is an isomorphism. Observe that $U \circ S$ maps B linearly onto A. We conclude that A and B are almost isometric, since ϵ was arbitrary. Hence, being finite-dimensional spaces, A and B are isometric. ■

References

[1] H. Auerbach, *Sur les groupes linéaires bornés I*, Studia Math. 4 (1934), 113–127.

[2] H. Auerbach, S. Mazur et S. Ulam, *Sur une propriété caractéristique de l'ellipsoïde*, Monatsh. Math. Phys. 42 (1935), 45–48.

[3] J. Becerra Guerrero and A. Rodríguez-Palacios, *Transitivity of the norm on Banach spaces*, Extracta Math. 17 (2002), 1–58.

[4] F. Cabello Sánchez, *Regards sur le problème des rotations de Mazur*, ibid. 12 (1997), 97–116.

[5] —, *A theorem on isotropic spaces*, Studia Math. 133 (1999), 257–260.

[6] E. R. Cowie, *A note on uniquely maximal Banach spaces*, Proc. Edinburgh Math. Soc. 26 (1983), 85–87.

[7] M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucía, J. Pelant and V. Zizler, *Functional Analysis and Infinite-Dimensional Geometry*, CMS Books Math. 8, Springer, New York, 2001.

[8] C. Finet, *Uniform convexity properties of norms on a super-reflexive Banach space*, Israel J. Math. 53 (1986), 81–92.

[9] T. Giordano and V. Pestov, *Some extremely amenable groups related to operator algebras and ergodic theory*, J. Inst. Math. Jussieu 6 (2007), 279–315.

[10] S. Heinrich, *Ultraproducts in Banach space theory*, J. Reine Angew. Math. 313 (1980), 72–104.

[11] S. Mazur, *Quelques propriétés caractéristiques des espaces euclidiens*, C. R. Acad. Sci. Paris 207 (1938), 761–764.

[12] N. W. Rickert, *Amenable groups and groups with the fixed point property*, Trans. Amer. Math. Soc. 127 (1967), 221–232.

[13] S. Rolewicz, *Metric Linear Spaces*, 2nd ed., PWN, Warszawa, and Reidel, Dordrecht, 1985.
[14] B. Sims, *"Ultra"-techniques in Banach space theory*, Queen's Papers Pure Appl. Math. 60, Queen's Univ., Kingston, ON, 1982.

Jarno Talponen
Institute of Mathematics
Aalto University
P.O. Box 1100
FI-00076 Aalto, Finland
E-mail: talponen@cc.hut.fi

Received September 8, 2009
Revised version May 10, 2010 (6691)