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# Mixed-type reverse order law and its equivalents

by

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**Abstract.** We present new results related to various equivalents of the mixed-type reverse order law for the Moore–Penrose inverse for operators on Hilbert spaces. Recent finite-dimensional results of Tian are extended to Hilbert space operators.

**1. Introduction.** The reverse order law of the form $(AB)^\dagger = B^\dagger A^\dagger$ does not hold in general for the Moore–Penrose inverse. The classical equivalent condition ($A^*A$ commutes with $BB^\dagger$, and $BB^*$ commutes with $AA^\dagger$) is proved in [G] for complex matrices, in [B1], [B2] and [I] for closed-range bounded linear operators on Hilbert spaces, and in [KDjC] in rings with involutions. However, various weaker conditions than the reverse order law are also investigated, and a significant number of results have already been published (see [Dj1], [Dj2], [DjD], [DjR], [T1]–[T5], [WG], [W1], [W2]). In particular, the reverse order law of the form $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$ is investigated in [Hw].

In this paper we present a set of equivalents of the mixed type reverse order law $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$ for the ordinary and weighted Moore–Penrose inverses of bounded linear operators on Hilbert spaces. Some finite-dimensional results from [T4] are extended to infinite-dimensional settings. We use operator matrices, which naturally appear in the theory of closed-range bounded linear operators on Hilbert spaces. Hence, our methods of proof are essentially different from the method used in [T4].

Let $X, Y, Z$ be Hilbert spaces, and let $\mathcal{L}(X, Y)$ be the set of all bounded linear operators from $X$ to $Y$. For $A \in \mathcal{L}(X, Y)$ we use, respectively, $\mathcal{N}(A)$, $\mathcal{R}(A)$ and $A^*$ to denote the null space, the range space and the adjoint of $A$.

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The *Moore–Penrose inverse* of $A \in \mathcal{L}(X,Y)$ (if it exists) is the unique operator $A^\dagger \in \mathcal{L}(Y,X)$ satisfying the following:

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.$$

It is well-known that $A^\dagger$ exists if and only if $\mathcal{R}(A)$ is closed.

Let $M \in \mathcal{L}(Y)$ and $N \in \mathcal{L}(X)$ be positive and invertible operators. The *weighted Moore–Penrose inverse* of $A \in \mathcal{L}(X,Y)$ with respect to the weights $M$ and $N$ (if it exists) is the unique operator $A^\dagger_{M,N} \in \mathcal{L}(Y,X)$ satisfying the following:

$$AA^\dagger_{M,N}A = A, \quad A^\dagger_{M,N}AA^\dagger_{M,N} = A^\dagger_{M,N},$$
$$(MAA^\dagger_{M,N})^* = MAA^\dagger_{M,N}, \quad (NA^\dagger_{M,N}A)^* = NA^\dagger_{M,N}A.$$

Also, $A^\dagger_{M,N}$ exists if and only if $\mathcal{R}(A)$ is closed. If $M = I_Y$ and $N = I_X$, then $A^\dagger_{I_Y,I_X}$ is the standard Moore–Penrose inverse $A^\dagger$ of $A$.

We assume that the reader is familiar with the generalized invertibility and the Moore–Penrose inverse (see, for example, [BIG], [C], [H]).

We continue with several auxiliary results.

Lemma 1.1. *Let $A \in \mathcal{L}(X,Y)$ have a closed range. Then $A$ has the following matrix decomposition with respect to the orthogonal decompositions $X = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$ and $Y = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$:*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

*where $A_1$ is invertible. Moreover,*

$$A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}.$$

The proof is straightforward.

Lemma 1.2 ([DjD]). *Let $A \in \mathcal{L}(X,Y)$ have a closed range. Let $X_1$ and $X_2$ be closed and mutually orthogonal subspaces of $X$ such that $X = X_1 \oplus X_2$. Let $Y_1$ and $Y_2$ be closed and mutually orthogonal subspaces of $Y$ such that $Y = Y_1 \oplus Y_2$. Then the operator $A$ has the following matrix representations with respect to the orthogonal decompositions $X = X_1 \oplus X_2 = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$ and $Y = \mathcal{R}(A) \oplus \mathcal{N}(A^*) = Y_1 \oplus Y_2$ :*

(a)

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

*where $D = A_1A_1^* + A_2A_2^*$ maps $\mathcal{R}(A)$ into itself and $D > 0$ (meaning $D \geq 0$*

*invertible*). *Also,*

$$A^\dagger = \begin{bmatrix} A_1^* D^{-1} & 0 \\ A_2^* D^{-1} & 0 \end{bmatrix}.$$

(b)

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix},$$

*where* $D = A_1^* A_1 + A_2^* A_2$ *maps* $\mathcal{R}(A^*)$ *into itself and* $D > 0$ (*meaning* $D \geq 0$ *invertible*). *Also,*

$$A^\dagger = \begin{bmatrix} D^{-1} A_1^* & D^{-1} A_2^* \\ 0 & 0 \end{bmatrix}.$$

*Here* $A_i$ *denote different operators in different cases.*

The reader should notice the difference between the following notations. If $A, B \in \mathcal{L}(X)$, then $[A, B] = AB - BA$ denotes the commutator of $A$ and $B$. On the other hand, if $U \in \mathcal{L}(X, Z)$ and $V \in \mathcal{L}(Y, Z)$, then $[U \;\; V] : \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right] \to Z$ denotes the matrix form of the corresponding operator. In the following lemma, a lot of well-known and important facts and properties concerning the Moore–Penrose inverse are collected, especially those which we use in the proof of the main theorem.

Lemma 1.3 ([BIG], [DjR]). *Let* $A \in \mathcal{L}(X, Y)$ *be a closed range operator, and let* $M \in \mathcal{L}(Y)$ *and* $N \in \mathcal{L}(X)$ *be positive definite and invertible operators. Then:*

$$\begin{aligned}
&A^* = A^\dagger A A^* = A^* A A^\dagger; \\
&A^\dagger = A^* (AA^*)^\dagger = (A^* A)^\dagger A^*; \\
&\mathcal{R}(A) = \mathcal{R}(AA^\dagger) = \mathcal{R}(AA^*); \\
&\mathcal{R}(A^\dagger) = \mathcal{R}(A^*) = \mathcal{R}(A^\dagger A) = \mathcal{R}(A^* A); \\
&\mathcal{R}(I - A^\dagger A) = \mathcal{N}(A^\dagger A) = \mathcal{N}(A) = \mathcal{R}(A^*)^\perp; \\
&\mathcal{R}(I - AA^\dagger) = \mathcal{N}(AA^\dagger) = \mathcal{N}(A^\dagger) = \mathcal{N}(A^*) = \mathcal{R}(A)^\perp; \\
&\mathcal{R}(A^\dagger_{M,N}) = N^{-1} \mathcal{R}(A^*), \; \mathcal{N}(A^\dagger_{M,N}) = M^{-1} \mathcal{N}(A^*); \\
&A^\dagger_{M,N} = N^{-1/2} (M^{1/2} A N^{-1/2})^\dagger M^{1/2}.
\end{aligned}$$

The following result is well-known; it can be found in [C, p. 127] and in [I].

Lemma 1.4. *Let* $A \in \mathcal{L}(Y, Z)$ *and* $B \in \mathcal{L}(X, Y)$ *have closed ranges. Then* $AB$ *has a closed range if and only if* $A^\dagger A B B^\dagger$ *has a closed range.*

The following result is proved in [DjD, Lemma 2.1].

Lemma 1.5. *Let $X, Y$ be Hilbert spaces, let $C \in \mathcal{L}(X,Y)$ have a closed range, and let $D \in \mathcal{L}(Y)$ be Hermitian and invertible. Then $\mathcal{R}(DC) = \mathcal{R}(C)$ if and only if $[D, CC^\dagger] = 0$.*

We shall also use the following result from [DW], which can be easily extended from complex matrices to bounded linear Hilbert space operators.

Lemma 1.6. *Let $H_i$ $(i = \overline{1,4})$ be Hilbert spaces, and let $C \in \mathcal{L}(H_1, H_2)$, $X \in \mathcal{L}(H_2, H_3)$ and $B \in \mathcal{L}(H_3, H_4)$ be closed range operators. Then*

$$C(BXC)^\dagger B = X^\dagger$$

*if and only if*

$$\mathcal{R}(B^*BX) = \mathcal{R}(X) \quad \text{and} \quad \mathcal{N}(XCC^*) = \mathcal{N}(X).$$

Let $\mathcal{A}$ be a unital $C^*$-algebra with unit 1. Denote the set of all projections by $\mathcal{P}(\mathcal{A}) = \{p \in \mathcal{A} : p^2 = p = p^*\}$. In [L, Theorem 10.a] the following results are proved.

Lemma 1.7 ([L]). *Let $p, q \in \mathcal{P}(\mathcal{A})$. Then the following statements are equivalent:*

(a) *$pq$ is Moore–Penrose invertible;*
(b) *$qp$ is Moore–Penrose invertible;*
(c) *$(1-p)(1-q)$ is Moore–Penrose invertible;*
(d) *$(1-q)(1-p)$ is Moore–Penrose invertible.*

Lemma 1.8 ([L]). *Let $p, q \in \mathcal{P}(\mathcal{A})$. If $pq$ is Moore–Penrose invertible, then*

$$(qp)^\dagger = pq - p[(1-p)(1-q)]^\dagger q.$$

We shall use these results in the case of $\mathcal{A} = \mathcal{L}(X)$.

**2. Main results.** Many necessary and sufficient conditions for $(AB)^\dagger = B^\dagger A^\dagger$ to hold were given in the literature. In the paper of Tian [T3], one can find the following important relation: $(AB)^\dagger = B^\dagger A^\dagger$ iff $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$ and $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A$. Therefore, it is of interest to find conditions equivalent to $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$. The next theorem is our main result, and it represents a generalization of results from [T4] to the infinite-dimensional setting.

Theorem 2.1. *Let $X$, $Y$, $Z$ be Hilbert spaces, and let $A \in \mathcal{L}(Y, Z)$ and $B \in \mathcal{L}(X, Y)$ be operators such that $A$, $B$ and $AB$ have closed ranges. The following statements are equivalent:*

(a1) $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$;

(a2) $(AB)^\dagger = B^*(A^*ABB^*)^\dagger A^*$;

(a3) $(AB)^\dagger = B^\dagger A^\dagger - B^\dagger((I - BB^\dagger)(I - A^\dagger A))^\dagger A^\dagger$;

(b1) $((A^\dagger)^*B)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^*$;

(b2) $((A^\dagger)^*B)^\dagger = B^*((A^*A)^\dagger BB^*)^\dagger A^\dagger$;

(b3) $((A^\dagger)^*B)^\dagger = B^\dagger A^* - B^\dagger((I-BB^\dagger)(I-A^\dagger A))^\dagger A^*$;

(c1) $(A(B^\dagger)^*)^\dagger = B^*(A^\dagger ABB^\dagger)^\dagger A^\dagger$;

(c2) $(A(B^\dagger)^*)^\dagger = B^\dagger(A^*A(BB^*)^\dagger)^\dagger A^*$;

(c3) $(A(B^\dagger)^*)^\dagger = B^*A^\dagger - B^*((I-BB^\dagger)(I-A^\dagger A))^\dagger A^\dagger$;

(d1) $(B^\dagger A^\dagger)^\dagger = A(BB^\dagger A^\dagger A)^\dagger B$;

(d2) $(B^\dagger A^\dagger)^\dagger = (A^\dagger)^*((BB^*)^\dagger(A^*A)^\dagger)^\dagger(B^\dagger)^*$;

(d3) $(B^\dagger A^\dagger)^\dagger = AB - A((I-A^\dagger A)(I-BB^\dagger))^\dagger B$;

(e1) $(A^\dagger AB)^\dagger A^\dagger = B^\dagger(ABB^\dagger)^\dagger$;

(e2) $(A^\dagger AB)^\dagger A^* = B^\dagger((A^\dagger)^*BB^\dagger)^\dagger$;

(e3) $(A^\dagger A(B^\dagger)^*)^\dagger A^\dagger = B^*(ABB^\dagger)^\dagger$;

(e4) $(BB^\dagger A^\dagger)^\dagger B = A(B^\dagger A^\dagger A)^\dagger$;

(e5) $(A^*AB)^\dagger A^* = B^*(ABB^*)^\dagger$;

(e6) $((A^*A)^\dagger B)^\dagger A^\dagger = B^*((A^\dagger)^*BB^*)^\dagger$;

(e7) $(A^*A(B^\dagger)^*)^\dagger A^* = B^\dagger(A(BB^*)^\dagger)^\dagger$;

(e8) $B^\dagger((A^*)^\dagger(BB^*)^\dagger)^\dagger = ((A^*A)^\dagger(B^*)^\dagger)^\dagger A^\dagger$;

(e9) $(AA^*ABB^*B)^\dagger = B^\dagger(A^*ABB^*)^\dagger A^\dagger$;

(f1) $(A^\dagger AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger$ *and* $(ABB^\dagger)^\dagger = (A^\dagger ABB^\dagger)^\dagger A^\dagger$;

(f2) $(A^\dagger AB)^\dagger = B^*(A^\dagger ABB^*)^\dagger$ *and* $(ABB^\dagger)^\dagger = (A^*ABB^\dagger)^\dagger A^*$;

(f3) $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A - B^\dagger((I-BB^\dagger)(I-A^\dagger A))^\dagger A^\dagger A$ *and*
$(ABB^\dagger)^\dagger = BB^\dagger A^\dagger - BB^\dagger((I-BB^\dagger)(I-A^\dagger A))^\dagger A^\dagger$;

(g1) $\mathcal{R}((AB)^\dagger) = \mathcal{R}(B^\dagger(A^\dagger ABB^\dagger)A^\dagger)$ *and*
$\mathcal{R}(((AB)^\dagger)^*) = \mathcal{R}((B^\dagger(A^\dagger ABB^\dagger)A^\dagger)^*)$;

(g2) $\mathcal{R}((AB)^\dagger) = \mathcal{R}(B^\dagger A^\dagger)$ *and* $\mathcal{R}((B^*A^*)^\dagger) = \mathcal{R}((A^*)^\dagger(B^*)^\dagger)$;

(g3) $\mathcal{R}(AA^*AB) = \mathcal{R}(AB)$ *and* $\mathcal{R}(B^*B(AB)^*) = \mathcal{R}((AB)^*)$.

*Proof.* The existence of various terms appearing in the statements of the theorem follows mainly from Lemma 1.4, and from properties of kernels and ranges of operators (see Lemma 1.3). The existence of the Moore–Penrose inverse of the products like $(I-BB^\dagger)(I-A^\dagger A)$ follows from Lemma 1.7.

Using Lemma 1.1, we conclude that the operator $B$ has the matrix form

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix},$$

where $B_1$ is invertible. Then

$$B^\dagger = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix}.$$

From Lemma 1.2 it also follows that the operator $A$ has the matrix form

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $D = A_1A_1^* + A_2A_2^*$ is invertible and positive in $\mathcal{L}(\mathcal{R}(A))$. Then

$$A^\dagger = \begin{bmatrix} A_1^*D^{-1} & 0 \\ A_2^*D^{-1} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix}.$$

First we find an equivalent form for (a1). We have

$$S = A^\dagger ABB^\dagger = \begin{pmatrix} A_1^*D^{-1}A_1 & 0 \\ A_2^*D^{-1}A_1 & 0 \end{pmatrix},$$

and consequently

$$S^\dagger = (S^*S)^\dagger S^* = \begin{pmatrix} (A_1^*D^{-1}A_1)^\dagger A_1^*D^{-1}A_1 & (A_1^*D^{-1}A_1)^\dagger A_1^*D^{-1}A_2 \\ 0 & 0 \end{pmatrix}.$$

It follows that

$$B^\dagger S^\dagger A^\dagger = \begin{pmatrix} B_1^{-1}(A_1^*D^{-1}A_1)^\dagger A_1^*D^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore,

$$(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger B^\dagger$$

is equivalent to

$$(A_1B_1)^\dagger = B_1^{-1}(A_1^*D^{-1}A_1)^\dagger A_1^*D^{-1} = B_1^{-1}(D^{-1/2}A_1)^\dagger D^{-1/2}.$$

By checking the Penrose equations, the last formula holds if and only if

$$[B_1B_1^*, (D^{-1/2}A_1)^\dagger D^{-1/2}A_1] = 0 \text{ and } [D, D^{-1/2}A_1(D^{-1/2}A_1)^\dagger] = 0. \tag{2.1}$$

Hence, (a1) is equivalent to (2.1).

Let us now find some more statements equivalent to (a1). Using Lemma 1.5, we deduce that (2.1) is equivalent to

$$\mathcal{R}(DA_1) = \mathcal{R}(A_1) \quad \text{and} \quad \mathcal{R}(B_1B_1^*A_1^*) = \mathcal{R}(A_1^*),$$

and to

$$\mathcal{R}(DA_1) = \mathcal{R}(A_1) \quad \text{and} \quad \mathcal{N}(A_1B_1B_1^*) = \mathcal{N}(A_1).$$

Lemma 1.6 applied for $X = A_1B_1$, $C = B_1^{-1}$, $B = D^{-1/2}$ shows that the equality

$$(A_1B_1)^\dagger = B_1^{-1}(D^{-1/2}A_1)^\dagger D^{-1/2}$$

is equivalent to

$$\mathcal{R}(D^{-1}A_1B_1) = \mathcal{R}(A_1B_1) \quad \text{and} \quad \mathcal{N}(A_1B_1(B_1^*B_1)^{-1}) = \mathcal{N}(A_1B_1),$$

and to

$$\mathcal{R}(D^{-1}A_1B_1) = \mathcal{R}(A_1B_1) \quad \text{and} \quad \mathcal{R}((B_1^*B_1)^{-1}(A_1B_1)^*) = \mathcal{R}((A_1B_1)^*).$$

Now, we find a statement equivalent to (g3). The condition

$$\mathcal{R}(AA^*AB) = \mathcal{R}(AB) \quad \text{and} \quad \mathcal{R}(B^*B(AB)^*) = \mathcal{R}((AB)^*)$$

is equivalent to

$$\mathcal{R}(DA_1B_1) = \mathcal{R}(A_1B_1) \quad \text{and} \quad \mathcal{R}(B_1^*B_1(A_1B_1)^*) = \mathcal{R}((A_1B_1)^*),$$

which is equivalent to (2.1). Hence, (g3) is equivalent to (a1).

Analogously, the equivalencies (b1)⇔(g3), (c1)⇔(g3) and (d1)⇔(g3) can be proved.

Let us now prove, for example, (c2)⇔(g3). Using the above notation, and

$$T = A^*A(BB^*)^\dagger = \begin{pmatrix} A_1^*A_1(B_1B_1^*)^{-1} & 0 \\ A_2^*A_1(B_1B_1^*)^{-1} & 0 \end{pmatrix},$$

it is easy to see that

$$\begin{aligned} T^\dagger &= (T^*T)^\dagger T^* \\ &= \begin{pmatrix} (D^{1/2}A_1(B_1B_1^*)^{-1})^\dagger D^{-1/2}A_1 & (D^{1/2}A_1(B_1B_1^*)^{-1})^\dagger D^{-1/2}A_2 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Now,

$$(A(B^\dagger)^*)^\dagger = B^\dagger(A^*A(BB^*)^\dagger)^\dagger A^*$$

if and only if

$$(A_1(B_1^*)^{-1})^\dagger = B_1^{-1}(D^{1/2}A_1(B_1B_1^*)^{-1})^\dagger D^{1/2}.$$

Applying Lemma 1.6 for $X = A_1(B_1^*)^{-1}$, $C = B_1^{-1}$, $B = D^{1/2}$ shows that the last equality is equivalent to

$$\begin{aligned} &\mathcal{R}(DA_1(B_1^*)^{-1}) = \mathcal{R}(A_1(B_1^*)^{-1}) \quad \text{and} \\ &\mathcal{N}(A_1(B_1^*)^{-1}B_1^{-1}(B_1^*)^{-1}) = \mathcal{N}(A_1(B_1^*)^{-1}), \end{aligned}$$

i.e.

$$\mathcal{R}(DA_1B_1) = \mathcal{R}(A_1B_1) \quad \text{and} \quad \mathcal{R}(B_1^{-1}A_1^*) = \mathcal{R}((A_1B_1)^*),$$

so we have just proved that (c2) is equivalent to (g3).

Analogously, we prove the equivalencies (a2)⇔(g3), (b2)⇔(g3) and (d2)⇔(g3).

In proving equivalencies including e-statements, there are no other techniques besides those we have already used in the previous part of the proof.

The table of appropriate equivalent statements is given below as some kind of overview, and also for the sake of completeness:

(a1) $(A_1B_1)^\dagger = B_1^{-1}(D^{-1/2}A_1)^\dagger D^{-1/2}$;
(a2) $(A_1B_1)^\dagger = B_1^*(D^{1/2}A_1B_1B_1^*)^\dagger D^{1/2}$;
(b1) $(D^{-1}A_1B_1)^\dagger = B_1^{-1}(D^{-1/2}A_1)^\dagger D^{1/2}$;
(b2) $(D^{-1}A_1B_1)^\dagger = B_1^*(D^{-3/2}A_1B_1B_1^*)^\dagger D^{-1/2}$;
(c1) $(A_1(B_1^*)^{-1})^\dagger = B_1^*(D^{-1/2}A_1)^\dagger D^{-1/2}$;
(c2) $(A_1(B_1^*)^{-1})^\dagger = B_1^{-1}(D^{1/2}A_1(B_1B_1^*)^{-1})^\dagger D^{1/2}$;
(d1) $(B_1^{-1}A_1^*D^{-1})^\dagger = D^{1/2}(A_1^*D^{-1/2})^\dagger B_1$;
(d2) $(B_1^{-1}A_1^*D^{-1})^\dagger = D^{-1/2}((B_1B_1^*)^{-1}A_1^*D^{-3/2})^\dagger(B_1^*)^{-1}$;
(e1) $(D^{-1/2}A_1B_1)^\dagger D^{-1/2} = B_1^{-1}A_1^\dagger$;
(e2) $(D^{-1/2}A_1B_1)^\dagger D^{-1/2} = B_1^{-1}(D^{-1}A_1)^\dagger D^{-1}$;
(e3) $(D^{-1/2}A_1(B_1^*)^{-1})^\dagger = B_1^*A_1^\dagger D^{1/2}$;
(e4) $(B_1^{-1}A_1^*D^{-1/2})^\dagger = D^{-1/2}(A_1^*D^{-1})^\dagger B_1$;
(e5) $(D^{1/2}A_1B_1)^\dagger = B_1^*(A_1B_1B_1^*)^\dagger D^{-1/2}$;
(e6) $(D^{-1}A_1B_1B_1^*)^\dagger = (B_1^*)^{-1}(D^{-3/2}A_1B_1)^\dagger D^{-1/2}$;
(e7) $(D^{1/2}A_1(B_1^*)^{-1})^\dagger = B_1^{-1}(A_1(B_1B_1^*)^{-1})^\dagger D^{-1/2}$;
(e8) $(D^{-1}A_1(B_1B_1^*)^{-1})^\dagger = B_1(D^{-3/2}A_1(B_1^*)^{-1})^\dagger D^{-1/2}$;
(e9) $(DA_1B_1B_1^*B_1)^\dagger = B_1^{-1}(D^{1/2}A_1B_1B_1^*)^\dagger D^{-1/2}$.

Each of those statements is equivalent to:

$$\mathcal{R}(D^\alpha A_1B_1) = \mathcal{R}(A_1B_1) \quad \text{and} \quad \mathcal{N}(A_1B_1(B_1^*B_1)^\beta) = \mathcal{N}(A_1B_1),$$

for some $\alpha, \beta \in \{-1, 1\}$. More precisely, we have:

| $\alpha$ | $\beta$ | Statement |
|---|---|---|
| 1 | 1 | a2, d1, e3, e6 |
| 1 | −1 | b1, c2, e1, e8 |
| −1 | 1 | b2, c1, e4, e5 |
| −1 | −1 | a1, d2, e2, e7, e9 |

Using Lemma 1.5, we have

$$\begin{aligned}\mathcal{R}(D^\alpha A_1B_1) = \mathcal{R}(A_1B_1) &\Leftrightarrow [D^\alpha, A_1B_1(A_1B_1)^\dagger] = 0\\ &\Leftrightarrow [D, A_1B_1(A_1B_1)^\dagger] = 0,\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}(A_1B_1(B_1^*B_1)^\beta) = \mathcal{N}(A_1B_1) &\Leftrightarrow \mathcal{R}((B_1^*B_1)^\beta(A_1B_1)^*) = \mathcal{R}((A_1B_1)^*)\\ &\Leftrightarrow [(B_1^*B_1)^\beta, (A_1B_1)^*((A_1B_1)^*)^\dagger] = 0\\ &\Leftrightarrow [(B_1^*B_1)^\beta, (A_1B_1)^\dagger A_1B_1] = 0\\ &\Leftrightarrow [B_1^*B_1, (A_1B_1)^\dagger A_1B_1] = 0,\end{aligned}$$

which means that each statement mentioned in the table above is equivalent to (g3). Now, we prove the equivalencies (x3)⇔(x1), where $x \in \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{f}\}$.

First, we prove (a3)⇔(a1):

$$\text{(a3)} \;\Leftrightarrow\; (AB)^{\dagger} = B^{\dagger}A^{\dagger} - B^{\dagger}[(I - BB^{\dagger})(I - A^{\dagger}A)]^{\dagger}A^{\dagger}.$$

Using Lemma 1.8 for $P = BB^{\dagger}$ and $Q = A^{\dagger}A$, we have

$$(A^{\dagger}ABB^{\dagger})^{\dagger} = BB^{\dagger}A^{\dagger}A - BB^{\dagger}[(I - BB^{\dagger})(I - A^{\dagger}A)]^{\dagger}A^{\dagger}A. \tag{2.2}$$

If we premultiply this expression by $B^{\dagger}$ and postmultiply it by $A^{\dagger}$, we obtain

$$B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger} = B^{\dagger}A^{\dagger} - B^{\dagger}[(I - BB^{\dagger})(I - A^{\dagger}A)]^{\dagger}A^{\dagger} = (AB)^{\dagger},$$

as desired.

Analogously, we can prove that (b3)⇔(b1) and (c3)⇔(c1); the part (d3)⇔(d1) is very similar—the difference is in taking $Q = BB^{\dagger}$ and $P = A^{\dagger}A$.

Let us now prove (f3)⇔(f1):

$$\text{(f3.1)} \;\Leftrightarrow\; (A^{\dagger}AB)^{\dagger} = B^{\dagger}A^{\dagger}A - B^{\dagger}((I - BB^{\dagger})(I - A^{\dagger}A))^{\dagger}A^{\dagger}A.$$

If we premultiply (2.2) by $B^{\dagger}$, we have

$$B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger} = B^{\dagger}A^{\dagger}A - B^{\dagger}((I - BB^{\dagger})(I - A^{\dagger}A))^{\dagger}A^{\dagger}A = (A^{\dagger}AB)^{\dagger},$$

i.e. part (f1.1). Also,

$$\text{(f3.2)} \Leftrightarrow (ABB^{\dagger})^{\dagger} = BB^{\dagger}A^{\dagger} - BB^{\dagger}((I - BB^{\dagger})(I - A^{\dagger}A))^{\dagger}A^{\dagger}.$$

If we postmultiply (2.2) by $A^{\dagger}$, we have

$$(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger} = BB^{\dagger}A^{\dagger} - BB^{\dagger}((I - BB^{\dagger})(I - A^{\dagger}A))^{\dagger}A^{\dagger} = (ABB^{\dagger})^{\dagger},$$

i.e. part (f1.2). We have thus finished this part of the proof.

Let us now see what are the equivalents of statements (f1) and (f2). A simple computation shows that (f1) is equivalent to the conjunction of the following two statements:

$$(D^{-1/2}A_1B_1)^{\dagger}D^{-1/2}A_i = B_1^{-1}(D^{-1/2}A_1)^{\dagger}D^{-1/2}A_i, \qquad i = 1, 2; \tag{2.3}$$

$$A_1^{\dagger} = (D^{-1/2}A_1)^{\dagger}D^{-1/2}. \tag{2.4}$$

Suppose that (f1) holds; if we substitute (2.4) in (2.3), then postmultiply each of the modified equations (2.3) by $A_i^*$, and add them, we get

$$(D^{-1/2}A_1B_1)^{\dagger} = B_1^{-1}A_1^{\dagger}D^{1/2},$$

which holds if and only if

$$[D, A_1A_1^{\dagger}] = 0 \quad \text{and} \quad [B_1B_1^*, A_1^{\dagger}A_1] = 0,$$

which is, by Lemma 1.5, equivalent to

$$\mathcal{R}(DA_1) = \mathcal{R}(A_1) \quad \text{and} \quad \mathcal{R}(B_1B_1^*A_1^*) = \mathcal{R}(A_1^*),$$

i.e. we get statement (a1). It is not difficult to see that the reverse implication also holds.

An easy computation shows that (f2) is equivalent to the conjunction of the following two statements:

(2.5) $(D^{-1/2}A_1B_1)^\dagger D^{-1/2}A_i = B_1^*(D^{-1/2}A_1B_1B_1^*)^\dagger D^{-1/2}A_i, \quad i = 1, 2;$

(2.6) $A_1^\dagger = (D^{-1/2}A_1)^\dagger D^{-1/2}.$

Suppose that (f2) holds; if we postmultiply each equation of (2.5) by $A_i^*$, and add them, we obtain

$$(D^{-1/2}A_1B_1)^\dagger = B_1^*(D^{-1/2}A_1B_1B_1^*)^\dagger,$$

which holds, by Lemma 1.6, if and only if $\mathcal{N}(A_1B_1B_1^*B_1) = \mathcal{N}(A_1B_1)$. As in the previous part of the proof, (2.6) is equivalent to $\mathcal{R}(DA_1) = \mathcal{R}(A_1)$. So, (f2)⇒(a1). The reverse implication is easy.

Let us now find statements equivalent to (g1) and (g2).

First, (g1):

$$\begin{aligned}
&\mathcal{R}(B^\dagger(A^\dagger ABB^\dagger)A^\dagger) = \mathcal{R}((AB)^\dagger) = \mathcal{R}((AB)^*)\\
&\quad\Leftrightarrow\ \mathcal{R}(B_1^*A_1^*) = \mathcal{R}(B_1^{-1}(D^{-1/2}A_1)^\dagger D^{-1/2}) = \mathcal{R}(B_1^{-1}(D^{-1/2}A_1)^\dagger)\\
&\quad\Leftrightarrow\ B_1\mathcal{R}(B_1^*A_1^*) = \mathcal{R}(B_1B_1^*A_1^*) = \mathcal{R}((D^{-1/2}A_1)^\dagger) = \mathcal{R}((D^{-1/2}A_1)^*)\\
&\qquad = \mathcal{R}(A_1^*),
\end{aligned}$$

so we actually have

$$\mathcal{R}(B_1B_1^*A_1^*) = \mathcal{R}(A_1^*).$$

The second condition, $\mathcal{R}(((AB)^\dagger)^*) = \mathcal{R}((B^\dagger(A^\dagger ABB^\dagger)A^\dagger)^*)$, becomes:

$$\begin{aligned}
&\mathcal{N}(B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger) = \mathcal{N}((AB)^\dagger) = \mathcal{N}((AB)^*)\\
&\Leftrightarrow\ \mathcal{N}(A_1^*) = \mathcal{N}(B_1^*A_1^*) = \mathcal{N}(B_1^{-1}(D^{-1/2}A_1)^\dagger D^{-1/2}) = \mathcal{N}((D^{-1/2}A_1)^\dagger D^{-1/2})\\
&\Leftrightarrow\ \mathcal{R}(A_1) = \mathcal{R}(D^{-1/2}(A_1^*D^{-1/2})^\dagger)\\
&\Leftrightarrow\ D^{1/2}\mathcal{R}(A_1) = \mathcal{R}(D^{1/2}A_1) = \mathcal{R}((A_1^*D^{-1/2})^\dagger) = \mathcal{R}((A_1^*D^{-1/2})^*)\\
&\qquad = \mathcal{R}(D^{-1/2}A_1),
\end{aligned}$$

so we have

$$\mathcal{R}(DA_1) = \mathcal{R}(A_1).$$

Those two conditions are equivalent to (a1), so we have just proved (g1)⇔(a1).

Now, (g2):

$$\begin{aligned}
\mathcal{R}(B^\dagger A^\dagger) = \mathcal{R}((AB)^\dagger) = \mathcal{R}((AB)^*)&\\
&\Leftrightarrow \mathcal{R}(B_1^*A_1^*) = \mathcal{R}(B_1^*A_1^*D^{-1}) = \mathcal{R}(B_1^{-1}A_1^*)\\
&\Leftrightarrow B_1\mathcal{R}(B_1^*A_1^*) = \mathcal{R}(B_1B_1^*A_1^*) = \mathcal{R}(A_1^*)
\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}((B^*A^*)^\dagger) = \mathcal{R}((A^*)^\dagger(B^*)^\dagger) &\Leftrightarrow \mathcal{N}((AB)^\dagger) = \mathcal{N}(B^\dagger A^\dagger) = \mathcal{N}((AB)^*)\\ &\Leftrightarrow \mathcal{N}(B_1^*A_1^*) = \mathcal{N}(B_1^{-1}A_1^*D^{-1})\\ &\Leftrightarrow \mathcal{N}(A_1^*) = \mathcal{N}(A_1^*D^{-1})\\ &\Leftrightarrow \mathcal{R}(A_1) = \mathcal{R}(D^{-1}A_1),\end{aligned}$$

which together are equivalent to (a1), so we have just proved (g2)⇔(a1). ∎

Now we formulate an analogous result for the weighted Moore–Penrose inverse.

THEOREM 2.2. *Let $X$, $Y$, $Z$ be Hilbert spaces, and let $A \in \mathcal{L}(Y,Z)$ and $B \in \mathcal{L}(X,Y)$ be operators such that $A$, $B$ and $AB$ have closed ranges. Suppose $M \in \mathcal{L}(Z)$ and $\mathcal{N} \in \mathcal{L}(X)$ are positive definite invertible operators. The following statements are equivalent:*

(a1) $(AB)^\dagger_{M,N} = B^\dagger_{I,N}(A^\dagger_{M,I}ABB^\dagger_{I,N})^\dagger A^\dagger_{M,I}$;

(a2) $(AB)^\dagger_{M,N} = N^{-1}B^*(A^*MABN^{-1}B^*)^\dagger A^*M$;

(a3) $(AB)^\dagger_{M,N} = B^\dagger_{I,N}A^\dagger_{M,I} - B^\dagger_{I,N}((I-BB^\dagger_{I,N})(I-A^\dagger_{M,I}A))^\dagger A^\dagger_{M,I}$;

(b1) $((A^*)^\dagger_{I,M^{-1}}B)^\dagger_{M^{-1},N} = B^\dagger_{I,N}(A^\dagger_{M,I}ABB^\dagger_{I,N})^\dagger A^*$;

(b2) $((A^*)^\dagger_{I,M^{-1}}B)^\dagger_{M^{-1},N} = N^{-1}B^*((A^*MA)^\dagger(BN^{-1}B^*))^\dagger A^\dagger_{M,I}M^{-1}$;

(b3) $((A^*)^\dagger_{I,M^{-1}}B)^\dagger_{M^{-1},N} = B^\dagger_{I,N}A^* - B^\dagger_{I,N}((I-BB^\dagger_{I,N})(I-A^\dagger_{M,I}A))^\dagger A^*$;

(c1) $(A(B^*)^\dagger_{N^{-1},I})^\dagger_{M,N^{-1}} = B^*(A^\dagger_{M,I}ABB^\dagger_{I,N})^\dagger A^\dagger_{M,I}$;

(c2) $(A(B^*)^\dagger_{N^{-1},I})^\dagger_{M,N^{-1}} = NB^\dagger_{I,N}((A^*MA)(BN^{-1}B^*)^\dagger)^\dagger A^*M$;

(c3) $(A(B^*)^\dagger_{N^{-1},I})^\dagger_{M,N^{-1}} = B^*A^\dagger_{M,I} - B^*((I-BB^\dagger_{I,N})(I-A^\dagger_{M,I}A))^\dagger A^\dagger_{M,I}$;

(d1) $(B^\dagger_{I,N}A^\dagger_{M,I})^\dagger_{N,M} = A(BB^\dagger_{I,N}A^\dagger_{M,I}A)^\dagger B$;

(d2) $(B^\dagger_{I,N}A^\dagger_{M,I})^\dagger_{N,M} = M^{-1}(A^*)^\dagger_{I,M^{-1}}((BN^{-1}B^*)^\dagger(A^*MA)^\dagger)^\dagger(B^*)^\dagger_{N^{-1},I}N$;

(d3) $(B^\dagger_{I,N}A^\dagger_{M,I})^\dagger_{N,M} = AB - A((I-A^\dagger_{M,I}A)(I-BB^\dagger_{I,N}))^\dagger B$;

(e1) $(A^\dagger_{M,I}AB)^\dagger_{I,N}A^\dagger_{M,I} = B^\dagger_{I,N}(ABB^\dagger_{I,N})^\dagger_{M,I}$;

(e2) $(A^\dagger_{M,I}AB)^\dagger_{I,N}A^* = B^\dagger_{I,N}((A^*)^\dagger_{I,M^{-1}}BB^\dagger_{I,N})^\dagger_{M^{-1},I}$;

(e3) $(A^\dagger_{M,I}A(B^*)^\dagger_{N^{-1},I})^\dagger_{I,N^{-1}}A^\dagger_{M,I} = B^*(ABB^\dagger_{I,N})^\dagger_{M,I}$;

(e4) $(BB^\dagger_{I,N}A^\dagger_{M,I})^\dagger_{I,M}B = A(B^\dagger_{I,N}A^\dagger_{M,I}A)^\dagger_{N,I}$;

(e5) $N(A^*MAB)^\dagger_{I,N}A^*M = B^*(ABN^{-1}B^*)^\dagger_{M,I}$;

(e6) $N((A^*MA)^\dagger B)^\dagger_{I,N}A^\dagger_{M,I} = B^*((A^*)^\dagger_{I,M^{-1}}BN^{-1}B^*)^\dagger_{M^{-1},I}M$;

(e7) $(A^*MA(B^*)^\dagger_{N^{-1},I})^\dagger_{I,N^{-1}}A^*M = NB^\dagger_{I,N}(A(BN^{-1}B^*)^\dagger)^\dagger_{M,I}$;

(e8) $NB^\dagger_{I,N}((A^*)^\dagger_{I,M^{-1}}(BN^{-1}B^*)^\dagger)^\dagger_{M^{-1},I}M = ((A^*MA)^\dagger(B^*)^\dagger_{N^{-1},I})^\dagger_{I,N^{-1}}A^\dagger_{M,I}$;

(e9) $(AA^*MABN^{-1}B^*B)^\dagger_{M,N} = B^\dagger_{I,N}(A^*MABN^{-1}B^*)^\dagger A^\dagger_{M,I}$;

(f1) $(A^\dagger_{M,I}AB)^\dagger_{I,N} = B^\dagger_{I,N}(A^\dagger_{M,I}ABB^\dagger_{I,N})^\dagger$ *and*
$(ABB^\dagger_{I,N})^\dagger_{M,I} = (A^\dagger_{M,I}ABB^\dagger_{I,N})^\dagger A^\dagger_{M,I}$;

(f2) $(A^\dagger_{M,I}AB)^\dagger_{I,N} = N^{-1}B^*(A^\dagger_{M,I}ABN^{-1}B^*)^\dagger$ *and*
$(ABB^\dagger_{I,N})^\dagger_{M,I} = (A^*MABB^\dagger_{I,N})^\dagger A^*M$;

(f3) $(A^\dagger_{M,I}AB)^\dagger_{I,N} = B^\dagger_{I,N}A^\dagger_{M,I}A - B^\dagger_{I,N}((I-BB^\dagger_{I,N})(I-A^\dagger_{M,I}A))^\dagger A^\dagger_{M,I}A$ *and*
$(ABB^\dagger_{I,N})^\dagger_{M,I} = BB^\dagger_{I,N}A^\dagger_{M,I} - BB^\dagger_{I,N}((I-BB^\dagger_{I,N})(I-A^\dagger_{M,I}A))^\dagger A^\dagger_{M,I}$;

(g1) $\mathcal{R}((AB)^\dagger_{M,N}) = \mathcal{R}(B^\dagger_{I,N}(A^\dagger_{M,I}ABB^\dagger_{I,N})^\dagger A^\dagger_{M,I})$ *and*
$\mathcal{R}(((AB)^\dagger_{M,N})^*) = \mathcal{R}((B^\dagger_{I,N}(A^\dagger_{M,I}ABB^\dagger_{I,N})^\dagger A^\dagger_{M,I})^*)$;

(g2) $\mathcal{R}((AB)^\dagger_{M,N}) = \mathcal{R}(B^\dagger_{I,N}A^\dagger_{M,I})$ *and*
$\mathcal{R}((B^*A^*)^\dagger_{N^{-1},M^{-1}}) = \mathcal{R}((A^*)^\dagger_{I,M^{-1}}(B^*)^\dagger_{N^{-1},I})$;

(g3) $\mathcal{R}(AA^*MAB) = \mathcal{R}(AB)$ *and* $\mathcal{R}((ABN^{-1}B^*B)^*) = \mathcal{R}((AB)^*)$.

*Proof.* If we use the basic relation between ordinary and weighted Moore–Penrose inverse

$$A^\dagger_{M,N} = N^{-1/2}(M^{1/2}AN^{-1/2})^\dagger M^{1/2},$$

and the substitutions

$$\tilde{A} = M^{1/2}A, \qquad \tilde{B} = BN^{-1/2},$$

all statements from this theorem reduce to statements of the already proven Theorem 2.1. For example, let us prove (e6)⇔(g2):

$$\begin{aligned}
\text{(e6)} \;\Leftrightarrow\;& N((A^*MA)^\dagger B)^\dagger_{I,N}A^\dagger_{M,I} = B^*((A^*)^\dagger_{I,M^{-1}}BN^{-1}B^*)^\dagger_{M^{-1},I}M \\
\Leftrightarrow\;& N^{1/2}((A^*MA)^\dagger BN^{-1/2})^\dagger(M^{1/2}A)^\dagger M^{1/2} \\
&= B^*((A^*M^{-1/2})^\dagger BN^{-1}B^*)^\dagger M^{1/2} \\
\Leftrightarrow\;& ((\tilde{A}^*\tilde{A})^\dagger\tilde{B})^\dagger\tilde{A}^\dagger = \tilde{B}^*((\tilde{A}^*)^\dagger\tilde{B}\tilde{B}^*)^\dagger,
\end{aligned}$$

which is actually (e6) from Theorem 2.1.

On the other hand, for (g2) we obtain

$$\begin{aligned}
\text{(g2.1)} \;&\Leftrightarrow\; \mathcal{R}((AB)^{\dagger}_{M,N}) = \mathcal{R}(B^{\dagger}_{I,N}A^{\dagger}_{M,I})\\
&\Leftrightarrow\; \mathcal{R}(N^{-1/2}(M^{1/2}ABN^{-1/2})^{\dagger}M^{1/2})\\
&\qquad = \mathcal{R}(N^{-1/2}(BN^{-1/2})^{\dagger}(M^{1/2}A)^{\dagger}M^{1/2})\\
&\Leftrightarrow\; \mathcal{R}(N^{-1/2}(\tilde{A}\tilde{B})^{\dagger}M^{1/2}) = \mathcal{R}(N^{-1/2}\tilde{B}^{\dagger}\tilde{A}^{\dagger}M^{1/2})\\
&\Leftrightarrow\; \mathcal{R}(N^{-1/2}(\tilde{A}\tilde{B})^{\dagger}) = \mathcal{R}(N^{-1/2}\tilde{B}^{\dagger}\tilde{A}^{\dagger})\\
&\Leftrightarrow\; \mathcal{R}((\tilde{A}\tilde{B})^{\dagger}) = \mathcal{R}(\tilde{B}^{\dagger}\tilde{A}^{\dagger}),
\end{aligned}$$

and

$$\begin{aligned}
\text{(g2.2)} \;&\Leftrightarrow\; \mathcal{R}((B^*A^*)^{\dagger}_{N^{-1},M^{-1}}) = \mathcal{R}((A^*)^{\dagger}_{I,M^{-1}}(B^*)^{\dagger}_{N^{-1},I})\\
&\Leftrightarrow\; \mathcal{R}(M^{1/2}(N^{-1/2}B^*A^*M^{1/2})^{\dagger}N^{-1/2})\\
&\qquad = \mathcal{R}(M^{1/2}(A^*M^{1/2})^{\dagger}(N^{-1/2}B^*)^{\dagger}N^{-1/2})\\
&\Leftrightarrow\; \mathcal{R}(M^{1/2}(\tilde{B}^*\tilde{A}^*)^{\dagger}N^{-1/2}) = \mathcal{R}(M^{1/2}(\tilde{A}^*)^{\dagger}(\tilde{B}^*)^{\dagger}N^{-1/2})\\
&\Leftrightarrow\; \mathcal{R}(M^{1/2}(\tilde{B}^*\tilde{A}^*)^{\dagger}) = \mathcal{R}(M^{1/2}(\tilde{A}^*)^{\dagger}(\tilde{B}^*)^{\dagger})\\
&\Leftrightarrow\; \mathcal{R}((\tilde{B}^*\tilde{A}^*)^{\dagger}) = \mathcal{R}((\tilde{A}^*)^{\dagger}(\tilde{B}^*)^{\dagger}),
\end{aligned}$$

which means we have (g2) from Theorem 2.1. Since Theorem 2.1 is already proven, the current theorem follows immediately. ∎

**3. Conclusions.** In this paper we consider a number of necessary and sufficient conditions for the reverse order law $(AB)^{\dagger} = B^{\dagger}(A^{\dagger}ABB^{\dagger})A^{\dagger}$ to hold for operators on Hilbert spaces. Applying this result we obtain conditions equivalent to the reverse order rule for the weighted Moore–Penrose inverse of operators. Although these results are already known for complex matrices, we demonstrated a new technique of proof. In the theory of complex matrices various authors used matrix rank to prove equivalent conditions related to this reverse order law. In the case of bounded linear operators on Hilbert spaces, we applied the method of operator matrices. It would be interesting to extend this work to the Moore–Penrose inverse and weighted Moore–Penrose inverse of a triple product.

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