STUDIA MATHEMATICA 217 (3) (2013)

The group of L^2-isometries on H_0^1

by

ESTEBAN ANDRUCHOW (Los Polvorines),
EDUARDO CHIUMIENTO (La Plata)
and GABRIEL LAROTONDA (Los Polvorines)

Abstract. Let Ω be an open subset of $\mathbb{R}^n$. Let $L^2 = L^2(\Omega, dx)$ and $H_0^1 = H_0^1(\Omega)$ be the standard Lebesgue and Sobolev spaces of complex-valued functions. The aim of this paper is to study the group $\mathbb{G}$ of invertible operators on H_0^1 which preserve the L^2-inner product. When Ω is bounded and $\partial\Omega$ is smooth, this group acts as the intertwiner of the H_0^1 solutions of the non-homogeneous Helmholtz equation $u - \Delta u = f$, $u|_{\partial\Omega} = 0$. We show that $\mathbb{G}$ is a real Banach–Lie group, whose Lie algebra is (i times) the space of symmetrizable operators. We discuss the spectrum of operators belonging to $\mathbb{G}$ by means of examples. In particular, we give an example of an operator in $\mathbb{G}$ whose spectrum is not contained in the unit circle. We also study the one-parameter subgroups of $\mathbb{G}$. Curves of minimal length in $\mathbb{G}$ are considered. We introduce the subgroups $\mathbb{G}_p := \mathbb{G} \cap (I - \mathcal{B}_p(H_0^1))$, where $\mathcal{B}_p(H_0^1)$ is the Schatten ideal of operators on H_0^1. An invariant (weak) Finsler metric is defined by the p-norm of the Schatten ideal of operators on L^2. We prove that any pair of operators $G_1, G_2 \in \mathbb{G}_p$ can be joined by a minimal curve of the form $\delta(t) = G_1 e^{itX}$, where X is a symmetrizable operator in $\mathcal{B}_p(H_0^1)$.

1. Introduction. Let $\Omega \subset \mathbb{R}^n$ be an open subset. Denote by $L^2 = L^2(\Omega, dx)$ the Lebesgue space of square-integrable functions endowed with its usual inner product $\langle \cdot, \cdot \rangle$. Let $H_0^1 = H_0^1(\Omega)$ be the closure in the Sobolev norm of the C^∞ functions with compact support contained in Ω. In this paper, we study the group $\mathbb{G}$ of invertible operators on H_0^1 that preserve the L^2-inner product:

$$\mathbb{G} = \{G \in Gl(H_0^1) : \langle Gf, Gg \rangle = \langle f, g \rangle\}.$$

In the case where $\Omega = \mathbb{R}^n$, the group $\mathbb{G}$ was introduced in [5] in relation to the geometry of the variational spaces arising in the many-particle Hartree–Fock theory. One could give an abstract definition of $\mathbb{G}$, involving a complex Hilbert space H and a dense and continuously included subspace $E \subset H$ with their respective (non-equivalent) inner products. However, we preferred

2010 *Mathematics Subject Classification*: Primary 47D03; Secondary 22E65, 58B20.
Key words and phrases: group of isometries of a positive form, Sobolev space, symmetrizable operator, one-parameter subgroup, minimal curve.

DOI: 10.4064/sm217-3-1

this concrete setting given by the inclusion $H_0^1 \subset L^2$ because we shall deal mainly with examples.

From the definition of the group $\mathbb{G}$, it is clear that the theory of operators on spaces with two norms will play a central role in the study of this group. This theory was independently initiated by M. G. Krein [10] and P. D. Lax [13]. In Section 2 we recall the results most useful for our purposes.

Our first results on the structure of $\mathbb{G}$ are given in Section 3. We prove that $\mathbb{G}$ is a real Banach–Lie group equipped with the norm of the algebra $\mathcal{B}(H_0^1)$ of bounded operators, and its Lie algebra Γ can be identified with the real Banach space of operators $X \in \mathcal{B}(H_0^1)$ such that $\langle Xf, g\rangle = -\langle f, Xg\rangle$ for any $f, g \in H_0^1$. Thus $i\Gamma$ is a well studied class of operators that naturally arises when one deals with spaces with two norms, which is usually known as the class of *symmetrizable operators*. An alternative description of $\mathbb{G}$ is given by

$$\mathbb{G} = \{G \in Gl(H_0^1) : G^*AG = A\},$$

where A is the positive operator on H_0^1 satisfying $[Af, g] = \langle f, g\rangle$ and $[\cdot,\cdot]$ denotes the inner product on H_0^1. In fact, when Ω is a bounded domain in $\mathbb{R}^n$ and $\partial\Omega$ is smooth, the operator A is the solution operator of the Sturm–Liouville equation. On the other hand, note that any operator belonging to $\mathbb{G}$ may be extended to a unitary operator on L^2. This extension procedure induces a norm continuous representation of the group onto L^2, which does not have a continuous inverse.

In Section 4 we examine different elementary aspects of $\mathbb{G}$ by means of examples. For instance, we show that the norm of an operator in $\mathbb{G}$ can be arbitrarily large. In the general setting of operators on spaces with two norms, it is known that there exist symmetrizable operators with non-real spectrum. Nevertheless, the few known examples of this fact do not apply to our concrete situation (see [4, 6, 10]). We present an example of a symmetrizable operator belonging to $i\Gamma$ with non-real spectrum (Example 4.4). In particular, this implies that the spectrum of an operator in $\mathbb{G}$ may not be contained in the unit circle. Another interesting problem is to determine if $\mathbb{G}$ is an exponential group. It turns out that this depends on the topology of Ω (see Proposition 4.8 and Example 4.9).

In Section 5 we investigate the one-parameter subgroups of $\mathbb{G}$. We construct a norm continuous unitary representation $\mathbb{G} \to U(H_0^1)$, $G \mapsto U_G$, satisfying $U_G A^{1/2} = A^{1/2} G$. Then we study the infinitesimal generators associated with this representation.

The results concerning the metric geometry of $\mathbb{G}$ are presented in Section 6. A natural invariant Finsler metric in $\mathbb{G}$ is provided by the usual operator norm of $\mathcal{B}(L^2)$. If one measures the length of curves with this metric, $\mathbb{G}$ behaves like a unitary group near the identity. Indeed, any operator G in $\mathbb{G}$ such that $\|G-I\| \le 1$ can be joined to I by a minimal curve of the form $\delta(t) = e^{itX}$, where X is a symmetrizable operator and $\|\cdot\|$ stands for the operator norm in $\mathcal{B}(H_0^1)$. Next we consider the following subgroups:

$$\mathbb{G}_p = \mathbb{G} \cap (I - \mathcal{B}_p(H_0^1)),$$

where $\mathcal{B}_p(H_0^1)$ is the Schatten ideal of operators on H_0^1 ($1 \le p \le \infty$). Essentially due to the fact that the logarithm of operators in $\mathbb{G}_p$ is well defined, we are able to extend the aforementioned minimality result to a global result in $\mathbb{G}_p$, where the Finsler metric is now the p-norm of the Schatten ideal $\mathcal{B}_p(L^2)$ (see Theorem 6.3).

We end this introduction by fixing some notation. The Sobolev space H_0^1 is a Hilbert space with the inner product given by

$$[f,g] = \int_\Omega f\bar{g}\,dx + \int_\Omega \nabla f\cdot\nabla\bar{g}\,dx.$$

To avoid confusion among the several norms considered, we denote by $|\cdot|_1$ ($=[\cdot,\cdot]^{1/2}$) the norm on H_0^1, $|\cdot|_2$ ($=\langle\cdot,\cdot\rangle^{1/2}$) the norm on L^2, $\|\cdot\|$ the operator norm in $\mathcal{B}(H_0^1)$, and by $\|\cdot\|_{\mathcal{B}(L^2)}$ the operator norm in $\mathcal{B}(L^2)$. If a given operator X acts both on L^2 and H_0^1, we shall denote by $\sigma_{L^2}(X)$ its spectrum as an operator on L^2, and by $\sigma_{H_0^1}(X)$ its spectrum as an operator on H_0^1.

2. Background on operators on spaces with two norms. Let H be a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\|$. Let $(E, |\cdot|_E)$ be a Banach space. Assume that E is a dense linear subspace of H and suppose that the norms satisfy $\|\cdot\| \le C|\cdot|_E$ for some positive constant C. Throughout this article, we are interested in the special case where $E = H_0^1$ and $H = L^2$.

Let $\mathcal{B}(E)$ (resp. $\mathcal{B}(H)$) denote the algebra of bounded operators on E (resp. H). An operator X in $\mathcal{B}(E)$ is said to be *symmetrizable* if

$$\langle Xf, g\rangle = \langle f, Xg\rangle, \qquad f,g \in E.$$

Given an operator $X \in \mathcal{B}(E)$, we denote by $\sigma_E(X)$ the spectrum of X over E. We use the obvious notation $\sigma_H(X)$ for the spectrum of X over H. In the following theorem we collect the basics results on symmetrizable operators.

THEOREM 2.1 (M. G. Krein [10], P. D. Lax [13]). *Let X be a symmetrizable operator. The following assertions hold:*

(i) *X is bounded as an operator on H.*
(ii) *$\sigma_H(X) \subseteq \sigma_E(X)$.*

(iii) *If λ belongs to the point spectrum of X as an operator on E, then λ belongs to the point spectrum of X as an operator on H. Moreover, the eigenspaces* $\ker(X-\lambda)$ *over E and H are the same.*
(iv) *If X is a compact operator on E, then X is a compact operator on H.*

Remark 2.2. It is not difficult to see that the two possible norms of a symmetrizable operator X satisfy $\|X\|_{\mathcal{B}(H)} \leq \|X\|_{\mathcal{B}(E)}$.

A more general approach to study operators on spaces with two norms can be found in [8]. Since any $f \in H$ determines a continuous functional $\langle \cdot, f\rangle$ of the space E^*, it follows that $E \subseteq H \subseteq E^*$. A bounded operator X on E is called *proper* if $X'(E) \subseteq E$, where X' is the (Banach) adjoint of X. If X is proper, X^+ denotes the restriction of X' to E. It can be shown that X^+ is the restriction to E of the adjoint on H.

Theorem 2.3 (I. C. Gohberg and M. K. Zambickiĭ [8]). *Let X be a proper operator. The following assertions hold:*

(i) *X is bounded as an operator on H.*
(ii) *$\sigma_H(X) \subseteq \sigma_E(X) \cup \overline{\sigma_E(X^+)}$, where the bar indicates complex conjugation.*
(iii) *If X is a compact operator on E, then it is compact on H. Moreover, $\sigma_H(X) = \sigma_E(X)$ and the eigenspaces of X in E and H corresponding to each non-zero eigenvalue coincide.*

If E is also a Hilbert space with an inner product denoted by $[\cdot,\cdot]$, it follows that there is a positive operator A on E such that $[Af, g] = \langle f, g\rangle$. Thus X is symmetrizable if and only if $AX = X^*A$, where the adjoint is taken with respect to E. The following result will be useful.

Theorem 2.4 (J. Dieudonné [6]). *Let A be a positive operator on a Hilbert space E. Let X be a bounded operator on E such $AX = X^*A$. Then there is a unique self-adjoint operator Y on E such that $A^{1/2}X = YA^{1/2}$.*

3. Basic facts on $\mathbb{G}$. From the definition of the group, it follows that any operator in $\mathbb{G}$ extends to an isometry of L^2, which has a dense subset in its range, namely H_0^1. Hence, operators belonging to $\mathbb{G}$ extend to unitary operators onto L^2. Thus one can describe $\mathbb{G}$ alternatively as

$$\mathbb{G} = \{G = U|_{H_0^1} : U \in U(L^2) \text{ and } U(H_0^1) = H_0^1\}.$$

Moreover, there is a third algebraic characterization of $\mathbb{G}$. Note that the sesquilinear form $\langle\cdot,\cdot\rangle$ is bounded and positive definite in H_0^1, thus there exists a positive operator $A \in \mathcal{B}(H_0^1)$ such that

$$\langle f, g\rangle = [Af, g] = [f, Ag].$$

Therefore, a straightforward computation shows that

$$\mathbb{G} = \{G \in Gl(H_0^1) : G^* A G = A\}. \tag{3.1}$$

From this characterization it becomes apparent that $\mathbb{G}$ is a closed subgroup of $Gl(H_0^1)$. We shall see in Section 3.1 that it is a Banach–Lie group, and a submanifold of $\mathcal{B}(H_0^1)$. Its Lie algebra is

$$\varGamma = \{X \in \mathcal{B}(H_0^1) : X^* A + AX = 0\}.$$

Note that $X \in \varGamma$ if

$$\langle Xf, g\rangle = [AXf, g] = -[X^* Af, g] = -[Af, Xg] = -\langle f, Xg\rangle,$$

i.e. if X is antihermitian for the L^2-inner product. Therefore one has the following spatial characterization of $\varGamma$:

$$\varGamma = \{X = Z|_{H_0^1} : Z \in \mathcal{B}(L^2),\ Z^* = -Z,\ Z(H_0^1) \subset H_0^1\}.$$

In fact, if $X = Z|_{H_0^1}$ as above, then the operator $X : H_0^1 \to H_0^1$ satisfies $\langle Xf, g\rangle = -\langle f, Xg\rangle$ for $f, g \in H_0^1$, and so is bounded in H_0^1 by the uniform boundedness principle. Conversely, if $X \in \mathcal{B}(H_0^1)$ satisfies $\langle Xf, g\rangle = -\langle f, Xg\rangle$, then iX lies in $\mathcal{B}(H_0^1)$ and it is symmetric for the L^2-inner product. It follows by Theorem 2.1 that iX extends to a bounded self-adjoint operator on L^2.

In order to understand $\mathbb{G}$, it will be useful to provide some examples of elements in $\mathbb{G}$. As is standard notation, if $f, g \in H_0^1$, denote by $f \otimes g$ the rank one operator in $\mathcal{B}(H_0^1)$ given by $f \otimes g(h) = [h, g]f$. Clearly, $(f \otimes g)^* = g \otimes f$, $\|f \otimes g\| = |f|_1 |g|_1$, and if $B, C \in \mathcal{B}(H_0^1)$, $B(f \otimes g)C = Bf \otimes C^* g$.

Example 3.1. (1) A straightforward verification shows that a unitary operator U on H_0^1 which commutes with A, belongs to $\mathbb{G}$. Conversely, if a unitary operator on H_0^1 belongs to $\mathbb{G}$, then it commutes with A.

(2) Let $f \in H_0^1$ be such that $|f|_2 = 1$. Then $f \otimes Af$ is a rank one idempotent:

$$(f \otimes Af)^2 = (f \otimes Af(f)) \otimes Af = ([f, Af]f) \otimes Af = \langle f, f\rangle f \otimes Af = f \otimes Af.$$

Note that $f \otimes Af$ extends to an orthogonal projection on L^2, $if \otimes Af \in \varGamma$ and

$$e^{if \otimes Af} = e^i f \otimes Af + (1 - f \otimes Af) \in \mathbb{G}.$$

By the above remarks, $f \otimes Af$ is an orthogonal projection on H_0^1 if and only if f is an eigenvector of A.

(3) Let $\mathcal{S}$ be a finite-dimensional subspace of H_0^1, and let $f_1, \ldots, f_k$ a basis of $\mathcal{S}$ which is orthonormal for the L^2-inner product $\langle \cdot, \cdot\rangle$. Then there exists a closed subspace $\mathcal{T}$ of H_0^1 such that $\mathcal{S} + \mathcal{T} = H_0^1$, and $\mathcal{S}, \mathcal{T}$ are orthogonal for $\langle \cdot, \cdot\rangle$. Indeed, let

$$E = \sum_{j=1}^{k} f_j \otimes Af_j \in \mathcal{B}(H_0^1).$$

Note that $E(f) = \sum_{j=i}^{k} \langle f, f_j \rangle f_j$, i.e. E is the L^2-orthogonal projection onto $\mathcal{S}$. Then $\mathcal{T} = \ker(E)$. Let U_0 be an operator in $\mathcal{B}(\mathcal{S})$, which is isometric for the L^2-norm $|\cdot|_2$, and put

$$G : H_0^1 \to H_0^1, \quad G|_{\mathcal{S}} = U_0 \quad \text{and} \quad G|_{\mathcal{T}} = 1_{\mathcal{T}}.$$

Then it is easy to check that $G \in \mathbb{G}$.

The former two examples consist of operators which are of the form 1 + compact (in fact finite rank). Let us show two examples which are not of this form: multiplication and composition operators.

(4) Let $H^{1,\infty}(\Omega)$ be the space of complex-valued functions in $L^\infty(\Omega)$ such that their first partial derivatives in the distributional sense also belong to $L^\infty(\Omega)$. Pick $\theta \in H^{1,\infty}(\Omega)$ satisfying $|\theta(x)| = 1$, and consider M_θ defined by

$$M_\theta f(x) = \theta(x) f(x), \quad x \in \Omega.$$

Then M_θ is a linear operator which acts both in L^2 and H_0^1. It is a unitary operator in L^2, and preserves H_0^1: clearly $M_\theta(H_0^1) \subset H_0^1$, and $(M_\theta)^{-1}(H_0^1) = M_{\bar{\theta}}(H_0^1) \subset H_0^1$, i.e. $M_\theta(H_0^1) = H_0^1$. It follows that $M_\theta \in \mathbb{G}$.

(5) Let $\psi : \Omega \to \Omega$ be a volume-preserving C^1 diffeomorphism such that the partial derivatives $\psi^i_{x_j}$ and $(\psi^{-1})^i_{x_j}$, $i, j = 1, \ldots, n$, are bounded on Ω. It is not difficult to show that the operator $U_\psi : H_0^1 \to H_0^1$, $U_\psi(f) = f \circ \psi$, is an isomorphism (see e.g. [18, Proposition 2.47]). Moreover, it also satisfies

$$\int_\Omega |U_\psi(f)|^2 \, dx = \int_\Omega |f \circ \psi|^2 \, dx = \int_\Omega |f|^2 |\det(D\psi^{-1})| \, dx = \int_\Omega |f|^2 \, dx,$$

where we have used that $|\det(D\psi^{-1})| \equiv 1$. This shows that $U_\psi \in \mathbb{G}$.

3.1. Smooth structure. Now we prove the preceding statement about the Lie algebra of $\mathbb{G}$, defined as the set of velocity vectors $\dot{\alpha}(0)$ of smooth curves α in $\mathcal{B}(H_0^1)$ which satisfy $\alpha(t) \in \mathbb{G}$ and $\alpha(0) = 1$.

LEMMA 3.2. *The Lie algebra of $\mathbb{G}$ is Γ.*

Proof. In order to prove this assertion it suffices to show that

$$\Gamma = \{Y \in \mathcal{B}(H_0^1) : e^{tY} \in \mathbb{G} \text{ for all } t \in \mathbb{R}\}.$$

If $X \in \Gamma$, $(X^*)^k A = (-1)^k A X^k$. Then $(e^{tX})^* A = e^{tX^*} A = A e^{-tX} = A(e^{tX})^{-1}$, i.e. $e^{tX} \in \mathbb{G}$. Conversely, if $e^{tY} \in \mathbb{G}$ for all t, we may differentiate the identity $e^{tY^*} A = A e^{-tY}$ at $t = 0$, to obtain $Y^* A = -AY$. ∎

LEMMA 3.3. *Let $G \in \mathbb{G}$. The following assertions hold:*

(i) *Let $\mathcal{L}$ be a half-line in the complex plane, from 0 to infinity. If $\sigma_{H_0^1}(G) \cap \mathcal{L} = \emptyset$, then there exists $X \in \Gamma$ such that $e^X = G$.*
(ii) *If $\|G - 1\| \le 1$, then there exists $X \in \Gamma$ such that $e^X = G$.*

Proof. (i) We first note that one can consider $e^{i\theta}G$ in place of G, where θ is a suitable angle, to reduce the proof to the case where $\mathcal{L}$ is the negative real axis. Thus we will assume that $\mathcal{L}$ is the negative real axis.

Since $0 \notin \sigma_{H_0^1}(G)$ and $0 \notin \sigma_{H_0^1}(G^{-1})$, it is possible to find a simple closed curve γ which does not intersect $\mathcal{L}$ and contains $\sigma_{H_0^1}(G)$ and $\sigma_{H_0^1}(G^{-1})$ in its interior. In addition, we can choose γ satisfying $\overline{\gamma} = \gamma$. From the assumption $\sigma_{H_0^1}(G) \cap \mathcal{L} = \emptyset$, it follows that there is a well defined branch of the logarithm, and $X = \log(G)$ can be defined using the Riesz functional calculus. If γ is counterclockwise oriented, then

$$\begin{aligned} X^*A &= -\frac{1}{2\pi i}\int_\gamma \overline{\log(z)}(G^* - \overline{z})^{-1}A\,dz = -\frac{1}{2\pi i}\int_\gamma \log(\overline{z})A(G^{-1} - \overline{z})^{-1}\,dz \\ &= \frac{1}{2\pi i}\int_{\overline{\gamma}} \log(z)A(G^{-1} - z)^{-1}\,dz = A\log(G^{-1}) = -AX. \end{aligned}$$

Hence $X \in \Gamma$, and the proof is complete.

(ii) Under the assumption $\|G - 1\| \le 1$, $\sigma_{H_0^1}(G)$ does not intersect the negative real axis (note that $0 \notin \sigma_{H_0^1}(G)$). Then the result can be deduced from (i). ∎

This lemma allows us to exhibit local charts for $\mathbb{G}$, modeled on Γ:

PROPOSITION 3.4. *The group $\mathbb{G}$ is a real Banach–Lie group endowed with the norm topology of $\mathcal{B}(H_0^1)$.*

Proof. Let us consider the open subsets $\mathcal{U} = \{X \in \mathcal{B}(H_0^1) : \sigma_{H_0^1}(X) \subseteq \mathbb{R} + i(-\pi, \pi)\}$ and $\mathcal{W} = \{G \in Gl(H_0^1) : \arg(z) \in (-\pi, \pi), \forall z \in \sigma_{H_0^1}(G)\}$. The exponential map of $Gl(H_0^1)$, i.e.

$$\exp : \mathcal{U} \to \mathcal{W}, \qquad \exp(X) = \sum_{i=0}^{\infty} \frac{X^n}{n!},$$

is a real analytic bijection (see [19, Lemma 2.11]). According to Lemma 3.3(i), it follows that $\exp(\mathcal{U} \cap \mathbb{G}) = \mathcal{W} \cap \Gamma$. Then a standard translation procedure can be used to cover $\mathbb{G}$. The smoothness of the group operations follows from that of the group operations in $Gl(H_0^1)$. ∎

REMARK 3.5. In the case where $\Omega = \mathbb{R}^n$, it was shown in [5] that $\mathbb{G}$ is an algebraic subgroup of $Gl(H_0^1)$. Hence the Banach–Lie structure of $\mathbb{G}$ followed from a general result on algebraic subgroups (see [9, Theorem 1]). It is noteworthy that $\mathbb{G}$ is an algebraic subgroup of $Gl(H_0^1)$ for any open set Ω, and thus by the same result on algebraic subgroups, we have another proof of the smooth structure of $\mathbb{G}$.

3.2. The relationship to the equation $u - \Delta u = h$. In this section we assume that Ω is bounded and $\partial\Omega$ is smooth. Let $f, g : \Omega \to \mathbb{C}$ be C^∞ functions with compact support contained in Ω. They can be smoothly extended to $\mathbb{R}^n$ by setting them equal to zero on the complement of Ω. Then

$$\langle f, g\rangle = [Af, g] = \langle Af, g\rangle + \int_\Omega \nabla Af(x) \cdot \nabla \bar{g}(x)\, dx,$$

and by Green's formula,

$$\langle f, g\rangle = \langle f, Ag\rangle - \int_\Omega Af\, \Delta \bar{g}\, dx = \langle f, A(g - \Delta g)\rangle.$$

Since this holds for any smooth function f, it follows that $A(g - \Delta g) = g$. Thus, if we set $h = g - \Delta g$, then $Ah = g$. In other words, if g is the unique solution of the non-homogeneous Helmholtz equation

$$\begin{cases} u - \Delta u = h, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{3.2}$$

then $Ah = g$. If $h \in H_0^1$, then $Ah = u_h$ is the weak solution of (3.2). That is, $1 - \Delta$ is the unbounded right inverse of A in H_0^2, or equivalently, A is the solution operator of equation (3.2). These facts are certainly well known (see e.g. [18]). Moreover, if Ω is bounded and $\partial\Omega$ is smooth, then A is compact. If $G \in \mathbb{G}$, the equality $G^* A G = A$ can be interpreted as follows: $G^* u_{Gh} = u_h$, or putting $h = G^{-1} f$,

$$G^* u_f = u_{G^{-1} f},$$

which means that G intertwines solutions of (3.2).

Example 3.6. One simple example in which A can be explicitly computed occurs when $\Omega = (0, 1)$. Let us compute its eigenvalues: $u_h = Ah = \lambda h$ implies that $u_h - u_h'' = (1/\lambda) u_h$, i.e.

$$\begin{cases} u_h'' + (1/\lambda - 1) u_h = 0, \\ u_h(0) = u_h(1) = 0. \end{cases}$$

Then, $\lambda = (k^2\pi^2 + 1)^{-1}$ with eigenfunction $\sin(k\pi x)$. If we normalize these eigenfunctions in H_0^1, we get

$$s_k(x) = \frac{\sqrt{2}}{\sqrt{k^2\pi^2 + 1}} \sin(k\pi x) \quad \text{and} \quad A = \sum_{k=1}^\infty \frac{1}{k^2\pi^2 + 1}\, s_k \otimes s_k.$$

Example 3.7. Let $\Omega \subset \mathbb{R}^2$ denote the open disk $x^2 + y^2 < 1$. In this example, the eigenvalues and eigenfunctions of the Laplacian can be expressed in terms of the Bessel functions J_m ($m \geq 0$), which are defined by

$$J_m(s) = \left(\frac{s}{2}\right)^m \sum_{p=0}^\infty \frac{(-1)^p}{\Gamma(p+1)\Gamma(m+p+1)} \left(\frac{s}{2}\right)^{2p},$$

where Γ stands for the Euler Gamma function. We refer the reader to [18, Example 34.2] for a detailed solution of the homogeneous Helmholtz equation in this example. It can be shown that the eigenvalues of the Laplace operator are given by $\lambda = z_{m,j}^2$ for $m = 0, 1, \dots$ and $j = 1, 2, \dots$, where the $z_{m,j}$ are the positive zeros of J_m. It is convenient to express the corresponding eigenfunctions in polar coordinates:

$$e_{m,j}(r,\theta) = \begin{cases} J_m(z_{m,j}r)e^{im\theta} & \text{if } m > 0, \\ J_0(z_{0,j}r) & \text{if } m = 0, \\ J_{-m}(z_{-m,j}r)e^{im\theta} & \text{if } m < 0. \end{cases}$$

According to formulas (5.14.6) and (5.14.9) in [14],

$$\int_0^1 rJ_m(z_{m,j}r)J_m(z_{m,k}r)\,dr = \begin{cases} 0 & \text{if } \neq k, \\ \frac{1}{2}J_{m+1}^2(z_{m,j}) & \text{if } j = k. \end{cases}$$

It follows that

$$|e_{m,j}|_2 = \sqrt{\pi}\,|J_{m+1}(z_{m,j})|.$$

Hence, the solution operator is given by

$$A = \sum_{m=-\infty}^{\infty} \sum_{j=1}^{\infty} \frac{1}{1+z_{m,j}^2} s_{m,j} \otimes s_{m,j},$$

where the eigenfunctions $s_{m,j} = \frac{\sqrt{1+z_{m,j}^2}}{\sqrt{\pi}|J_{m+1}(z_{m,j})|}e_{m,j}$ are normalized in H_0^1.

EXAMPLE 3.8. In the case $\Omega = \mathbb{R}^n$, A can be explicitly computed. It is well known that $H_0^1(\mathbb{R}^n) = H^1(\mathbb{R}^n)$ (see e.g. [18, Proposition 24.9]), where the latter is the space of functions in $L^2(\mathbb{R}^n)$ with first partial (distributional) derivatives also belonging to $L^2(\mathbb{R}^n)$. A function f is in $H^1(\mathbb{R}^n)$ if and only if $(1+|\xi|^2)^{1/2}\hat{f}(\xi) \in L^2(\mathbb{R}^n)$, where $\hat{f}$ denotes the Fourier transform of f, and the inner product is given by

$$[f,g] = \int_{\mathbb{R}^n} (1+|\xi|^2)\hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi.$$

Therefore, the solution operator is given by

$$\widehat{Af}(\xi) = \frac{1}{1+|\xi|^2}\hat{f}(\xi).$$

3.3. The extension map. As stated in the introduction, we may identify $\mathbb{G}$ with the subgroup of the unitary group $U(L^2)$ given by

$$\mathcal{U}_{H_0^1}(L^2) := \{W \in U(L^2) : W(H_0^1) = H_0^1\}.$$

In fact, the map

$$\mathbb{G} \to \mathcal{U}_{H_0^1}(L^2), \quad G \mapsto \bar{G}, \tag{3.3}$$

is a bijection, where $\bar{G}$ denotes the unique unitary operator acting on L^2 which extends the operator $G \in \mathbb{G}$. In what follows, we will endow $\mathcal{U}_{H_0^1}(L^2)$ with the operator norm topology of $\mathcal{B}(L^2)$, while $\mathbb{G}$ will be considered with the operator norm topology of $\mathcal{B}(H_0^1)$.

PROPOSITION 3.9. *Let $G_1, G_2 \in \mathbb{G}$. Then*

$$\|\bar{G}_1 - \bar{G}_2\|_{\mathcal{B}(L^2)} \leq \max\{\|G_1^{-1}\|, \|G_2^{-1}\|\}\|G_1 - G_2\|.$$

In particular, the map in (3.3) *is continuous. Its inverse is not continuous.*

Proof. Note that operators in $\mathbb{G}$ are proper (see Section 2). In particular, $G^+ = G^{-1}$ for any $G \in \mathbb{G}$. Denote by $r_{H_0^1}(X)$ and $r_{L^2}(Y)$, respectively, the spectral radius of $X \in \mathcal{B}(H_0^1)$ and $Y \in \mathcal{B}(L^2)$. Then

$$\begin{aligned}
\|\bar{G}_1 - \bar{G}_2\|_{\mathcal{B}(L^2)} &= \|1 - \bar{G}_1^{-1}\bar{G}_2\|_{\mathcal{B}(L^2)} \\
&= r_{L^2}(1 - \bar{G}_1^{-1}\bar{G}_2) \quad \text{(since } 1 - \bar{G}_1^{-1}\bar{G}_2 \text{ is normal)} \\
&\leq \max\{r_{H^1}(1 - G_1^{-1}G_2), r_{H^1}((1 - G_1^{-1}G_2)^+)\} \\
&= \max\{r_{H^1}(1 - G_1^{-1}G_2), r_{H^1}(1 - G_2^{-1}G_1)\} \\
&\leq \max\{\|1 - G_1^{-1}G_2\|, \|1 - G_2^{-1}G_1\|\} \\
&\leq \max\{\|G_1^{-1}\|, \|G_2^{-1}\|\}\|G_1 - G_2\|,
\end{aligned}$$

where in the first inequality we used Theorem 2.3. Combining the preceding inequality with the continuity of the inversion map on $\mathbb{G}$ shows that the map in (3.3) is continuous.

Now we are going to prove that the inverse of the extension map is not continuous for any open subset Ω of $\mathbb{R}^n$. Let $x \in \Omega$ and $C := (a_1, b_1) \times \cdots \times (a_n, b_n) \subset \Omega$ be a neighborhood of x. Consider the following sequence of smooth functions:

$$\theta_n : \Omega \to \mathbb{C}, \qquad \theta_n(x_1, \ldots, x_n) = e^{i\frac{\sin(nx_1)}{n}}.$$

Then, as remarked in Example 3.1(4), the multiplication operators M_{θ_n} belong to $\mathbb{G}$. Given any $f \in L^2$ such that $|f|_2 = 1$, note that

$$\begin{aligned}
|M_{\theta_n}(f) - f|_2^2 &= \int_\Omega |e^{i\frac{\sin(nx_1)}{n}} - 1|^2 |f(x)|^2 \, dx \\
&= 2\int_\Omega \left(1 - \cos\left(\frac{\sin(nx_1)}{n}\right)\right)|f(x)|^2 \, dx \\
&\leq 2\left\|1 - \cos\left(\frac{\sin(nx_1)}{n}\right)\right\|_\infty \to 0.
\end{aligned}$$

Thus, $\|M_{\theta_n} - I\|_{\mathcal{B}(L^2)} \to 0$. On the other hand, let f be a C^∞ function with compact support such that $f(x) \equiv 1$ for $x \in C$. We have

$$|M_{\theta_n}(f) - f|_1^2 \geq \int\limits_C \nabla\theta_n f \cdot \nabla\bar{\theta}_n \bar{f}\, dx = \int\limits_C \cos^2(nx_1)\, dx$$
$$= (b_2 - a_2)\dots(b_n - a_n)\left(\frac{1}{2n}\cos(nx_1)\sin(nx_1) + \frac{x_1}{2}\Big|_{a_1}^{b_1}\right)$$
$$\to \frac{1}{2}(b_1 - a_1)(b_2 - a_2)\dots(b_n - a_n) > 0,$$

so that $\|M_{\theta_n} - I\| \nrightarrow 0$, and this shows that the inverse of the extension map is not continuous. ∎

4. Norms and spectra of elements in $\mathbb{G}$. Note that if $G \in \mathbb{G}$, the equality $G^*AG = A$ implies that $\|A\| \leq \|A\|\,\|G\|^2$, and thus $\|G\| \geq 1$. Examining the previous examples, it can be shown that there are elements in G with arbitrarily large norm.

EXAMPLE 4.1. (1) Consider $\Omega = (0,1)$ and pick $f(x) = \sin(\pi x) + \sin(k\pi x)$. Clearly $|f|_2 = 1$. Thus, as in Example 3.1(2), $G = e^{if\otimes Af} = e^i f\otimes Af + (1 - f\otimes Af) \in \mathbb{G}$. Clearly,

$$\|G\| \geq \max\{\|f\otimes Af\|, \|1 - f\otimes Af\|\} \geq \|f\otimes Af\| = |f|_1|Af|_1.$$

A straightforward computation shows that

$$|f|_1^2|Af|_1^2 = \frac{1}{4}\Big(2 + \pi^2(k^2+1)\Big)\left(\frac{1}{\pi^2+1} + \frac{1}{k^2\pi^2+1}\right).$$

Therefore, for large k, the norm of G can be arbitrarily large.

(2) Let Ω be an open and bounded subset of $\mathbb{R}^n$ such that $\partial\Omega$ is smooth. Under this assumption, the operator A is compact. Let $f \in H_0^1$ be such that $|f|_1 = 1$ and $Af = \lambda f$ for some $\lambda \neq 0$. Set $\theta : \Omega \subset \mathbb{R}^n \to \mathbb{C}$, $\theta(x) = e^{ik(x_1+\dots+x_n)}$ for some $k \in \mathbb{R}$. Then,

$$|M_\theta f|_1^2 = \langle \theta f, \theta f\rangle + \int\limits_\Omega \nabla\theta f \cdot \nabla\bar{\theta}\bar{f}\, dx$$
$$= \langle f, f\rangle + \int\limits_\Omega (|f|^2|\nabla\theta|^2 + 2\operatorname{Re}\bar{f}\theta\nabla f\cdot\nabla\bar{\theta} + |\nabla f|^2)\, dx$$
$$= [f,f] + \int\limits_\Omega (nk^2|f|^2 + 2f\nabla f\cdot\vec{k})\, dx = 1 + nk^2\lambda + k\sum_{j=1}^n \int\limits_\Omega f\frac{\partial f}{\partial x_j}\, dx,$$

where $\vec{k} = (k,\dots,k)$. Since $f \in H_0^1$, integrating by parts we obtain

$$\int\limits_\Omega f\frac{\partial f}{\partial x_j}\, dx = -\int\limits_\Omega \frac{\partial f}{\partial x_j} f\, dx = 0.$$

Therefore,

$$\|M_\theta\| \geq \sqrt{1+nk^2\lambda}.$$

Since any operator $G \in \mathbb{G}$ can be extended to a unitary operator $\bar{G}$ on L^2 such that $\bar{G}(H_0^1) = H_0^1$, it is clear that operators in $\mathbb{G}$ are proper and $G^+ = G^{-1}$. Thus by Theorem 2.3 we know that

$$\sigma_{L^2}(\bar{G}) \subset \sigma_{H_0^1}(G) \cup \overline{\sigma_{H_0^1}(G^{-1})}. \tag{4.1}$$

Let us now examine the spectra of the operators in Example 3.1.

Example 4.2. (1) If U is a unitary in H_0^1 which commutes with A, then clearly

$$\sigma_{H_0^1}(U) = \sigma_{L^2}(U),$$

and it is a subset of $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

(2) Examples 3.1(2, 3) are constructed as operators G acting on an L^2-orthogonal decomposition of H_0^1, $H_0^1 = \mathcal{S}+\mathcal{T}$, with $\mathcal{S}$ finite-dimensional, and the operators acting as the identity on $\mathcal{T}$. Therefore their spectra in $\mathcal{B}(H_0^1)$ are finite, and consist of eigenvalues, and therefore are also eigenvalues of the extension $\bar{G}$ of G to L^2. In particular they are elements of $\mathbb{T}$. It is apparent by construction that also in this case the two spectra coincide.

(3) As in Example 3.1(4), consider M_θ, where $\theta : \Omega \to \mathbb{C}$ is an element of $H^{1,\infty}(\Omega)$, now with $|\theta(x)| = 1$. Clearly, $\sigma_{L^2}(M_\theta) = \mathcal{R}(\theta) \subset \mathbb{T}$, the essential range of θ. Also in this case the spectra coincide (though none of their elements are eigenvalues). Indeed, if $M_\theta - \lambda = M_{\theta-\lambda}$ is invertible in $\mathcal{B}(L^2)$, then the function $\theta - \lambda$ does not vanish in Ω, and moreover $(\theta-\lambda)^{-1}$ is also in $H^{1,\infty}(\Omega)$. Then the operator $M_{(\theta-\lambda)^{-1}}$, the inverse of $M_\theta - \lambda$ on L^2, defines a bounded operator in H_0^1, and therefore $M_\theta - \lambda$ is invertible on H_0^1. Conversely, suppose that $M_\theta - \lambda$ is invertible in $\mathcal{B}(H_0^1)$, and let B be its inverse. Since $M_\theta - \lambda$ is proper, it follows that B is proper if and only if $M_{\bar\theta} - \bar\lambda = (M_\theta - \lambda)^+$ is invertible in $\mathcal{B}(H_0^1)$ (see [8, p. 148]). Using that $M_\theta - \lambda$ is bijective, it is straightforward to show that $M_{\bar\theta} - \bar\lambda$ is also bijective. By the open mapping theorem, $M_{\bar\theta} - \bar\lambda$ is invertible in $\mathcal{B}(H_0^1)$. Therefore B is proper, and by Theorem 2.3, it has a bounded extension to L^2. Hence $\lambda \notin \sigma_{L^2}(M_\theta)$.

Example 4.3. The examples of elements of $\mathbb{G}$ so far have spectra in $\mathbb{T}$. There is an example by Gohberg and Zambickiĭ [8], adapted by Barnes [4] to the case of a pair of Hilbert space norms, of an operator whose extension is symmetric, but whose spectrum does not lie in the real line. Namely, in this latter form, Barnes considers the Hilbert space ℓ^2, and the dense subspace ℓ_0^2 consisting of sequences $(a_n)_n$ such that $\sum_{n=1}^\infty 4^n a_n^2 < \infty$. Comparing with our situation, one has $[\ ,\ ]$ in ℓ_0^2, given by

$$[a,b] = \sum_{n=1}^\infty 4^n a_n \bar{b}_n,$$

which makes ℓ^2_0 a (complete) Hilbert space, and the usual inner product $\langle\cdot,\cdot\rangle$ of ℓ^2, which is bounded in ℓ^2_0. This latter inner product is implemented by a diagonal compact operator A, whose eigenvalues are $1/4^n$.

The above counterexample does not apply to our situation, where the operator A is the solution operator. Let us reconstruct below the analogue of Barnes' example, and show that in our context, its spectrum is real. Consider $\Omega=(0,1)$ and

$$e_k(x)=\sqrt{2}\sin(k\pi x)\quad\text{and}\quad s_k(x)=\frac{1}{\gamma_k}e_k(x),$$

the eigenvectors of A, normalized, respectively, in L^2 and H^1_0 (where $\gamma_k=\sqrt{k^2\pi^2+1}$). Let $T=S+B$ in L^2, where S is the unilateral shift and B is the backward-shift. Thus T is self-adjoint in L^2, and $\sigma_{L^2}(T)\subset\mathbb{R}$. Moreover, $T(H^1_0)\subset H^1_0$. Indeed,

$$S(s_k)=\frac{1}{\gamma_k}S(e_k)=\frac{1}{\gamma_k}e_{k+1}=\frac{\gamma_{k+1}}{\gamma_k}s_{k+1}.$$

Analogously $B(s_k)=\frac{\gamma_{k-1}}{\gamma_k}s_{k-1}$ (putting $e_0=s_0=0$). Thus if $f=\sum_{k=1}^N c_ks_k$,

$$|Sf|_1^2=\Big|\sum_{k=1}^N c_k\frac{\gamma_{k+1}}{\gamma_k}s_{k+1}\Big|_1^2=\sum_{k=1}^N|c_k|^2\frac{\gamma_{k+1}^2}{\gamma_k^2}.$$

The fractions $\gamma_{k+1}^2/\gamma_k^2$ are bounded by 2. Thus

$$|Sf|_1^2\le 2\sum_{k=1}^N|c_k|^2=2|f|_1^2.$$

It follows that S is bounded in H^1_0 (and $\|S|_{H^1_0}\|\le\sqrt{2}$). Analogously B is bounded in H^1_0 (with $\|B|_{H^1_0}\|\le 1$, because $\gamma_{k-1}^2/\gamma_k^2\le 1$). We claim that the spectrum of T in H^1_0 is real, and coincides with its spectrum in L^2 (the analogue of T in ℓ^2_0 has non-real spectrum). Indeed, let T' in H^1_0 be given by

$$T'(s_k)=s_{k-1}+s_{k+1}.$$

Clearly T' is self-adjoint in H^1_0. Let T'_N be given by setting $T'_N(s_k)$ equal to $T(s_k)$ if $k\le N$, and to $T'(s_k)$ if $k\ge N+1$. Since T' and T'_N differ on a finite-dimensional subspace, their essential spectra coincide: $\sigma_e(T'_N)=\sigma_e(T')\subset\mathbb{R}$. On the other hand $T-T'_N(s_k)=0$ if $k\le N$, and

$$T-T'_N(s_k)=\Big(\frac{\gamma_{k-1}}{\gamma_k}-1\Big)s_{k-1}+\Big(\frac{\gamma_{k+1}}{\gamma_k}-1\Big)s_{k+1}$$

if $k\ge N+1$. In our case, where $\Omega=(0,1)$, these fractions tend to 1. Therefore $\|T-T'_N\|$ tends to 0. By the semicontinuity property of the (essential spectrum), this implies that $\sigma_e(T) \subset \mathbb{R}$. It was proved in [4] that an extendable (or proper) operator whose extension is self-adjoint, such as T, has the property that $\sigma_{H_0^1}(T) \setminus \sigma_e(T)$ consists of isolated eigenvalues of finite multiplicity. As remarked before, these eigenvalues are necessarily real. It follows that $\sigma_{H_0^1}(T) \subset \mathbb{R}$.

However, modifying the example above one can obtain an element of $\mathbb{G}$ whose spectrum as an operator of $H_0^1(\Omega)$ is not contained in $\mathbb{T}$.

EXAMPLE 4.4. Let Ω be a bounded domain in $\mathbb{R}^n$ such that $\partial\Omega$ is smooth. We will show an example of a symmetrizable operator belonging to $i\Gamma$ with non-real spectrum. In particular, this implies the existence of an operator in $\mathbb{G}$ with spectrum not contained in $\mathbb{T}$.

Let A be the solution operator of equation (3.2), whose eigenvalues are related to the eigenvalues of the Laplacian in Ω. It is a classical 1911 result by Hermann Weyl [20] that the eigenvalues of the Laplacian of a bounded domain Ω in $\mathbb{R}^n$ grow as

$$\mu_k \sim 4\pi\left(\frac{\Gamma(n/2+1)}{|\Omega|}\right)^{2/n} k^{2/n}$$

as $k \to \infty$. Since Ω is bounded and $\partial\Omega$ smooth, A is compact, and consequently there exists an orthonormal basis $(e_k)_k$ of L^2 consisting of eigenfunctions. Moreover, the eigenfunctions e_k belong to H_0^1. By a straightforward computation taking into account the relationship between the L^2 and H_0^1 inner products, it follows that $s_k = e_k/\gamma_k$ is an orthonormal basis of H_0^1, where $\gamma_k = \sqrt{1+\mu_k}$.

The orthonormal basis $(e_k)_k$ can be used to define the bounded operator

$$S : L^2 \to L^2, \qquad S(e_k) = e_{2k}.$$

Set $B = S^*$. Note that

$$B(e_k) = \begin{cases} 0 & \text{for } k \text{ odd,} \\ e_{k/2} & \text{for } k \text{ even.} \end{cases}$$

Then $T = S + B$ is a self-adjoint operator on L^2, so that $\sigma_{L^2}(T) \subseteq \mathbb{R}$. On the other hand, for any $f \in H_0^1$, $f = \sum_{k=1}^\infty c_k s_k$, it is easily seen that

$$|Sf|_1^2 = \left|\sum_{k=1}^\infty c_k \frac{\gamma_{2k}}{\gamma_k} s_{2k}\right|_1^2 = \sum_{k=1}^\infty \left|c_k \frac{\gamma_{2k}}{\gamma_k}\right|^2 \le K|f|_1^2,$$

where K is a constant that bounds the convergent sequence $(\gamma_{2k}/\gamma_k)_k$. In a similar fashion, one can see that B is bounded on H_0^1. Hence $T(H_0^1) \subseteq H_0^1$, and T turns out to be bounded on H_0^1. The expression of T in the orthonormal basis of H_0^1 is

$$T(s_k)=\begin{cases}\dfrac{\gamma_{2k}}{\gamma_k}s_{2k} & \text{for } k \text{ odd},\\ \dfrac{\gamma_{2k}}{\gamma_k}s_{2k}+\dfrac{\gamma_{k/2}}{\gamma_k}s_{k/2}, & \text{for } k \text{ even}.\end{cases}$$

We claim that $\sigma_{H_0^1}(T)$ contains all the points inside and on the ellipse

$$\lambda=\sqrt[n]{2}\,e^{i\theta}+\frac{1}{\sqrt[n]{2}}\,e^{-i\theta},\qquad \theta\in[0,2\pi].$$

To this end, note that due to Weyl's asymptotic formula,

$$\lim_{k\to\infty}\frac{\gamma_{2k}}{\gamma_k}=\sqrt[n]{2}\quad\text{and}\quad\lim_{k\to\infty}\frac{\gamma_{k/2}}{\gamma_k}=\frac{1}{\sqrt[n]{2}}.$$

Consider

$$T'(s_k)=\begin{cases}\sqrt[n]{2}\,s_{2k} & \text{for } k \text{ odd},\\ \sqrt[n]{2}\,s_{2k}+\dfrac{1}{\sqrt[n]{2}}s_{k/2} & \text{for } k \text{ even}.\end{cases}$$

Then the operators T'_N defined by $T'_N(s_k)=T(s_k)$ if $k\le N$, and $T'_N(s_k)=T'(s_k)$ if $k\ge N+1$, satisfy $\|T'_N-T\|\to 0$. Given $\epsilon>0$, the semicontinuity of the essential spectrum implies that $\sigma_e(T'_N)\subseteq\sigma_e(T)+\epsilon$ for N large enough. Since $\sigma_e(T'_N)=\sigma_e(T')$, it follows that $\sigma_e(T')\subseteq\sigma_e(T)$. So what is left is to show that the ellipse is contained in $\sigma_e(T')$. To prove the latter, note that the subspace

$$\mathcal{S}=\operatorname{span}\{s_{2^k}:k\ge 0\}$$

reduces T'. Hence $\sigma_e(P_{\mathcal{S}}T'|_{\mathcal{S}})\subseteq\sigma_e(T')$, where $P_{\mathcal{S}}$ denotes the orthogonal projection onto $\mathcal{S}$. But $P_{\mathcal{S}}T'|_{\mathcal{S}}$ is a Toeplitz operator with $\sqrt[n]{2}$ under the diagonal and $1/\sqrt[n]{2}$ over the diagonal. Thus $\sigma_e(P_{\mathcal{S}}T'|_{\mathcal{S}})=\sigma_{H_0^1}(P_{\mathcal{S}}T'|_{\mathcal{S}})$, and according to a result by M. G. Krein [11, Theorem 13.2], the spectrum of this Toeplitz operator is the above defined ellipse.

4.1. Image of the exponential map. The first examples of elements in $\mathbb{G}$ given in Section 3.1 were operators $G=G_0\oplus I_{\mathcal{T}}$, where G_0 acts in $\mathcal{S}$ with $\dim\mathcal{S}<\infty$, and $\mathcal{S}+\mathcal{T}=H_0^1$ is an L^2-orthogonal sum.

PROPOSITION 4.5. *If $G=G_0\oplus I_{\mathcal{T}}\in\mathbb{G}$ as above, then there exists a finite rank operator $Z\in\Gamma$ such that*

$$G=e^Z.$$

Proof. Note that $G=1+F$, where F has finite rank (inside $\mathcal{S}$). This implies that the spectrum of G is finite and consists of eigenvalues of modulus one and finite multiplicity. According to Theorem 2.3(iii), we know that $\sigma_{L^2}(G)=\sigma_{H_0^1}(G)$ and the multiplicities of each non-zero eigenvalue coincide. Therefore there exists a self-adjoint operator Z of finite rank in L^2 such that $\bar{G}=e^{iZ}$. Note that the eigenvectors of $\bar{G}$ are eigenvectors of G, so that the eigenvectors of Z lie in H_0^1 and there are a finite number of them. Then $Z(H_0^1) \subset H_0^1$, and thus $X = iZ|_{H_0^1} \in \Gamma$ with $G = e^X$. ∎

We point out a simple necessary condition on the spectrum for an operator in $\mathbb{G}$ to belong to the image of the exponential map.

REMARK 4.6. If $G = e^X$ with $X \in \Gamma$, we claim that $\sigma_{L^2}(\bar{G}) \subset \sigma_{H_0^1}(G)$. Indeed, let $Z \in \mathcal{B}(L^2)$ be such that $Z^* = -Z$ and $Z|_{H_0^1} = X$. Recall that from [10, Theorem 2] we have $\sigma_{L^2}(Z) \subseteq \sigma_{H_0^1}(X)$. Hence by repeated application of the analytic spectral mapping theorem we find that

$$\sigma_{L^2}(U_G) = \{e^\lambda : \lambda \in \sigma_{L^2}(Z)\} \subseteq \{e^\lambda : \lambda \in \sigma_{H_0^1}(X)\} = \sigma_{H_0^1}(G),$$

which proves our claim.

Next we study multiplication operators when the set Ω is bounded. The following lemma about functions is probably well known, but we give the proof below.

LEMMA 4.7. *Let Ω be a bounded, connected and open subset of $\mathbb{R}^n$. If $\theta \in H^{1,\infty}(\Omega)$, then θ is Lipschitz on Ω.*

Proof. We may suppose that $n = 2$. Take any two points $(x, y), (\bar{x}, \bar{y}) \in \Omega$. There exists a continuous curve $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = (x, y)$ and $\gamma(1) = (\bar{x}, \bar{y})$. Then it possible to approximate γ by a polygonal line with segments parallel to coordinate axes. Moreover, each segment can be chosen inside of Ω. Let $(x, y) = (x_1, y_1), (x_2, y_1), (x_2, y_2), \ldots, (x_m, y_m) = (\bar{x}, \bar{y})$ denote the vertices of the polygonal line. Now, the function θ is locally Lipschitz because $\theta \in H^{1,\infty}(\Omega)$ (see [7, p. 131]). Therefore θ has partial derivatives almost everywhere, and also it can be locally written as the integral of these partial derivatives. Thus

$$|\theta(x_{j+1}, y_j) - \theta(x_j, y_j)| \le \int_{x_j}^{x_{j+1}} \left| \frac{\partial\theta}{\partial x}(x, y_j)\, dx \right| \le \left\| \frac{\partial\theta}{\partial x} \right\|_{\Omega,\infty} (x_{j+1} - x_j).$$

A similar estimate holds for the other partial derivative. Since there are always a finite number of steps from (x, y) to $(\bar{x}, \bar{y})$, we get

$$\begin{aligned} |\theta(x, y) - \theta(\bar{x}, \bar{y})| &\le \sum_{j=1}^{m-1} |\theta(x_j, y_j) - \theta(x_{j+1}, y_j)| + \sum_{j=2}^{m} |\theta(x_j, y_{j-1}) - \theta(x_j, y_j)| \\ &\le \left\| \frac{\partial\theta}{\partial x} \right\|_{\Omega,\infty} |\bar{x} - x| + \left\| \frac{\partial\theta}{\partial y} \right\|_{\Omega,\infty} |\bar{y} - y| \\ &\le \sqrt{2}\, \mathrm{diam}(\Omega) \max\left\{ \left\| \frac{\partial\theta}{\partial x} \right\|_{\Omega,\infty}, \left\| \frac{\partial\theta}{\partial y} \right\|_{\Omega,\infty} \right\} \|(x, y) - (\bar{x}, \bar{y})\|. \end{aligned}$$ ∎

Proposition 4.8. *If Ω is bounded and connected, its closure $\overline{\Omega}$ is simply connected and $\theta \in H^{1,\infty}(\Omega)$ is such that $|\theta(x)| = 1$ for all $x \in \Omega$, then there exists a real function $\alpha \in H^{1,\infty}(\Omega)$ such that $e^{i\alpha} = \theta$, i.e.*

$$M_\theta = e^{iM\alpha},$$

with $M_{i\alpha} \in \Gamma$.

Proof. Consider the commutative Banach algebra $\mathcal{A} = C(\overline{\Omega}, \mathbb{C})$ of complex continuous maps in $\overline{\Omega}$, with the norm

$$|f|_\infty = \sup_{x\in\overline{\Omega}} |f(x)|.$$

Let $G_{\mathcal{A}}$ be the invertible group of $\mathcal{A}$. The maximal ideal spectrum of $\mathcal{A}$ is $\overline{\Omega}$, which by hypothesis is simply connected. Therefore, by the Arens–Royden Theorem [16],

$$G_{\mathcal{A}} = \{e^g : g \in \mathcal{A}\}. \tag{4.2}$$

According to Lemma 4.7 the function θ is Lipschitz. Thus, θ can be extended to a continuous function in $\overline{\Omega}$. By (4.2) it follows that $\theta = e^g$ for some continuous function g on $\overline{\Omega}$. Since $|\theta(x)| \equiv 1$, it follows that $g = i\alpha$ with α real. In particular, note that α is bounded on Ω.

Moreover, we claim that $\alpha \in H^{1,\infty}(\Omega)$, and then it clearly follows that $M_{i\alpha}$ belongs to Γ. To prove our claim, recall that θ is continuous, so that $|\theta(x) - \theta(y)| < 2$ if $\|x - y\| < \delta$ for some $\delta > 0$. Therefore there is an analytic branch of the logarithm for all the points close enough to a fixed $x \in \Omega$. Then, $e^{i\alpha(y)} = e^{i\log(\theta(y))}$, and $\alpha(y) = \log(\theta(y)) + 2k\pi$ by connectedness. Thus, α has partial derivatives almost everywhere, and

$$\left|\frac{\partial\alpha}{\partial x_j}(y)\right| = \left|\frac{1}{\theta(y)}\frac{\partial\alpha}{\partial x_j}(y)\right| \le \|\theta\|_{1,\infty}.$$

Since this bound is the same for any point, we conclude that $\alpha \in H^{1,\infty}(\Omega)$. ∎

The same idea provides an example of an element in $\mathbb{G}$ which is not in the range of the exponential, but it should be noted that in this example we consider Ω a compact manifold (rather than an open subset), namely $\Omega = \mathbb{T}$. See [18, p. 232] for the definition of H^1 in this context.

Example 4.9. Consider $\Omega = \mathbb{T}$, and the function z. We claim that $M_z \in \mathbb{G}$ does not belong to the range of the exponential map. Suppose that $M_z = e^X$ for some $X \in \Gamma$. Then $X = M_g$ for some $g \in H^1$. Indeed, put $g = X1 \in H^1$. Since X commutes with $e^X = M_z$,

$$Xz^n = X(M_z)^n 1 = (M_z)^n X1 = z^n g = M_g z^n$$

for any integer n. It follows that $X = M_g$. Therefore $z = e^X 1 = e^g$, with g continuous in $\mathbb{T}$, which is a contradiction.

5. Stone's theorem in $\mathbb{G}$. Clearly, the positive operator $A \in \mathcal{B}(H_0^1)$ such that $[Af,g] = \langle f,g\rangle$ is symmetrizable. According to Theorem 2.1, it extends to a bounded operator on L^2. Note that A has dense range, both regarded as an operator on L^2 or H_0^1. The next elementary remark shows that $A(L^2) \subset H_0^1$. More precisely:

REMARK 5.1. If A is regarded as an operator in $\mathcal{B}(L^2)$, then $A^{1/2}(L^2) = H_0^1$. To this end, let $f \in L^2$ and $(g_n)_n$ be a sequence in H_0^1 such that $|g_n - f|_2 \to 0$. Then $A^{1/2}g_n \to A^{1/2}f$ in L^2. Note that $A^{1/2}g_n$ is a Cauchy sequence in H_0^1. Indeed, we have

$$|A^{1/2}(g_n - g_m)|_1^2 = [g_n - g_m, A(g_n - g_m)] = \langle g_n - g_m, g_n - g_m\rangle = |g_n - g_m|_2^2.$$

It follows that $A^{1/2}f \in H_0^1$, and thus $A^{1/2}(L^2) \subset H_0^1$. On the other hand, if $g \in H_0^1$, since $A^{1/2}$ has dense range, there exists a sequence $(f_n)_n$ in H_0^1 such that $|A^{1/2}f_n - g|_1 \to 0$. The same computation above shows that $(f_n)_n$ is a Cauchy sequence in L^2:

$$|f_n - f_m|_2^2 = |A^{1/2}(f_n - f_m)|_1^2.$$

Therefore there exists $f \in L^2$ such that $|f_n - f|_2 \to 0$. Then $A^{1/2}f = g$. Moreover, we have

$$|A^{1/2}f|_1^2 = [f, Af] = \langle f, f\rangle = |f|_2^2.$$

Hence $A^{1/2} : L^2 \to H_0^1$ is a surjective isometry.

REMARK. The surjective isometry $A^{1/2} : (L^2, |\cdot|_2) \to (H_0^1, |\cdot|)$ will be denoted by $\mathcal{A}^{1/2}$ to distinguish it from the operator $A^{1/2}$ acting on L^2 or H_0^1.

There is yet another characterization of $\mathbb{G}$:

PROPOSITION 5.2. *Let G be an invertible operator on H_0^1. There is a unique unitary operator $U_G \in U(H_0^1)$ such that $U_G A^{1/2} = A^{1/2}G$ if and only if $G \in \mathbb{G}$. The map $G \mapsto U_G$ is a group isomorphism from $\mathbb{G}$ onto the group*

$$\mathcal{U}_{R(A^{1/2})}(H_0^1) = \{U \in U(H_0^1) : U(R(A^{1/2})) = R(A^{1/2})\}.$$

Moreover, if $\mathrm{Ad}_{\mathcal{A}^{1/2}} : \mathcal{B}(L^2) \to \mathcal{B}(H_0^1)$ denotes the C^*-algebra isomorphism implemented by the unitary transformation $\mathcal{A}^{1/2}$, $\mathrm{Ad}_{\mathcal{A}^{1/2}}(X) = \mathcal{A}^{1/2}X\mathcal{A}^{-1/2}$, then

$$\mathrm{Ad}_{\mathcal{A}^{1/2}}(\{U \in U(L^2) : U(H_0^1) = H_0^1\}) = \mathcal{U}_{R(A^{1/2})}(H_0^1).$$

Proof. The "only if" part is algebraic: if there exists such a U_G, then

$$G^*AG = A^{1/2}U_G^*U_GA^{1/2} = A,$$

thus $G \in \mathbb{G}$.

To prove the other implication, note that when $G \in \mathbb{G}$, the operator $A^{1/2}G$ is injective and has dense range. Therefore the isometric part in its polar decomposition $U|A^{1/2}G|$ extends to a unitary operator, which we denote $U = U_G$. Note that

$$|A^{1/2}G| = ((A^{1/2}G)^*A^{1/2}G)^{1/2} = (G^*AG)^{1/2} = A^{1/2},$$

and thus $U_GA^{1/2} = A^{1/2}G$. The unitary U_G is clearly unique with this property, and the mapping $G \mapsto U_G$ is a group homomorphism:

$$A^{1/2}G_1G_2 = U_{G_1}U_{G_2}A^{1/2} = U_{G_1G_2}A^{1/2}.$$

Clearly $U_GA^{1/2} = A^{1/2}G$ implies that $U_G(R(A^{1/2})) \subset R(A^{1/2})$. Since the same is true for $U_{G^{-1}} = U_G^{-1}$, equality holds. Pick a unitary operator U on H_0^1 such that $U(R(A^{1/2})) = R(A^{1/2})$. For $f \in H_0^1$, $UA^{1/2}f \in R(A^{1/2})$, since $A^{1/2}$ is injective, there exists $g_f \in H_0^1$ such that $UA^{1/2}f = A^{1/2}g_f$. Thus a map $f \mapsto g_f$ is defined, which is clearly a linear bijection of H_0^1. Moreover, since U is unitary in H_0^1,

$$|f|_2 = |A^{1/2}f|_1 = |UA^{1/2}f|_1 = |A^{1/2}g_f|_1 = |g_f|_2.$$

Thus $f \mapsto g_f$ extends to a unitary operator in L^2, which by construction fixes H_0^1, thus G defined $Gf = g_f$ belongs to $\mathbb{G}$, and clearly $UA^{1/2} = GA^{1/2}$.

Finally, a straightforward verification shows that $\mathcal{A}^{-1/2}U_G\mathcal{A}^{1/2}$ is a unitary operator of L^2, which extends G. ∎

Corollary 5.3. *The map $\mathbb{G} \ni G \mapsto U_G \in \mathcal{U}_{R(A^{1/2})}$ from Proposition* 5.2 *is a continuous group isomorphism* (*in the topology induced by the norm of $\mathcal{B}(H_0^1)$*). *Its inverse is not continuous.*

Proof. Modulo the automorphism $\mathrm{Ad}_{\mathcal{A}}$, this homomorphism is the extension map $G \mapsto \bar{G}$ (see Proposition 3.9). ∎

Let $G(t)$ be a strongly continuous one-parameter group in $\mathbb{G}$, i.e. for $t \in \mathbb{R}$, $G(t) \in \mathbb{G}$, $G(0) = 1$, $G(t+s) = G(t)G(s)$, and for each $f \in H_0^1$, the map $t \mapsto G(t)f \in H_0^1$ is continuous. By Proposition 5.2, this gives rise to a one-parameter group of unitaries $U_{G(t)}$. Let us see first that $U_{G(t)}$ is also strongly continuous.

Proposition 5.4. *Let $G(t)$, $t \in \mathbb{R}$, be a strongly continuous one-parameter group in $\mathbb{G}$. Then $U_{G(t)}$ is a strongly continuous group of unitaries in H_0^1.*

Proof. Since the map $G \mapsto U_G$ is a group homomorphism, it is clear that $U_{G(t)}$ is a one-parameter group of unitaries. The fact that $U_{G(t)}A^{1/2} = A^{1/2}G(t)$ implies that $t \mapsto U_{G(t)}f$ is continuous for any $f \in R(A^{1/2})$, which is dense in H_0^1. By von Neumann's extension of Stone's theorem (see for instance [15, Theorem VIII.9]), which states that a one-parameter group in a separable Hilbert space, which is weakly measurable, is strongly continuous, our result follows. Indeed, if $f \in H_0^1$, let (f_n) be a sequence in $R(A^{1/2})$ such that $|f_n - f|_1 \to 0$. Then, for each $t \in \mathbb{R}$ and $g \in H_0^1$,

$$\phi_n(t) = [U_{G(t)} f_n, g] \to \phi(t) = [U_{G(t)} f, g].$$

Since the ϕ_n are continuous, it follows that ϕ is measurable. ∎

According to Stone's theorem, there exists a (possibly unbounded) self-adjoint operator $S : D(S) \subset H_0^1 \to H_0^1$ such that

$$U_{G(t)} = e^{itS}.$$

Let us now relate S to the Lie algebra Γ.

Remark 5.5. If $G(t)$ is strongly continuous differentiable, i.e. if $t \mapsto G(t)f$ is continuously differentiable for every $f \in H_0^1$, then the identity $U_{G(t)} A^{1/2} = A^{1/2} G(t)$ implies that the function $t \mapsto U_{G(t)} f$ is differentiable for any $f \in R(A^{1/2})$. Thus $R(A^{1/2}) \subset D(S)$, and moreover $iSA^{1/2} f = A^{1/2} \dot{G}(0) f$.

On the other hand, $\dot{G}(0)$ is an everywhere defined operator in H_0^1, which has an adjoint. Indeed, $G^*(t)$ is weakly differentiable, because

$$t \mapsto [G^*(t)f, g] = [f, G(t)g]$$

is differentiable, and therefore its weak derivative $\dot{G}^*(0)$ is an adjoint for $\dot{G}(0)$. It follows that $\dot{G}(0)$ is bounded. Thus, differentiating the identity

$$G^*(t) A G(t) f = Af$$

at $t = 0$ for any $f \in H_0^1$ yields

$$\dot{G}(0)^* Af + A\dot{G}(0) f = 0.$$

Therefore $\dot{G}(0) \in \Gamma$. Moreover, $X = i\dot{G}(0)$ satisfies $X^* A = AX$. By Theorem 2.4 there exists a self-adjoint bounded operator S_0 in H_0^1 such that $S_0 A^{1/2} = A^{1/2} X$. Therefore, $S_0 = S$, that is, S is bounded, and satisfies $S(R(A^{1/2})) \subset R(A^{1/2})$.

In the general case ($G(t)$ strongly continuous), we have the following result. Let $C_0^\infty(\mathbb{R})$ denote the space of smooth functions with compact support on $\mathbb{R}$. Let $D \subset H_0^1$ be the linear span of the vectors

$$f_\varphi = \int_{-\infty}^{\infty} \varphi(t) G(t) f \, dt,$$

where $f \in H_0^1$ and $\varphi \in C_0^\infty(\mathbb{R})$. This space D was used in the proof of Stone's theorem due to Gårding and Wightmann (see [15]). The fact that D is dense in H_0^1 is a general property of the space D for any underlying Hilbert space.

Proposition 5.6. *With the above notations, the following assertions hold:*

(i) *The subspace D is dense in H_0^1, satisfies $G(t)(D) \subset D$ for all $t \in \mathbb{R}$, and for $f \in D$, $t \mapsto G(t)f$ is differentiable.*
(ii) *The subspace $A^{1/2}(D)$ is dense in H_0^1, satisfies $U_G(t)(A^{1/2}(D)) \subset A^{1/2}(D)$, and for $f \in A^{1/2}(D)$, the map $t \mapsto U_{G(t)}f$ is differentiable.*
(iii) *$S(A^{1/2}(D)) \subset A^{1/2}(D)$, $A^{1/2}(D)$ is a core for S, and if $f \in D$,*

$$iSA^{1/2}f = A^{1/2}\dot{G}(0)f.$$

Proof. Pick $\varphi \in C_0^\infty(\mathbb{R})$ and $f \in H_0^1$. Then

$$G(t)f_\varphi = \int_{-\infty}^{\infty} \varphi(s)G(s+t)f\,ds = \int_{-\infty}^{\infty} \varphi(s-t)G(s)f\,ds = f_{\varphi(\cdot\,-t)} \in D,$$

and clearly $t \mapsto G(t)f_\varphi$ is differentiable. Since $D \subset H_0^1$ is dense, and A has dense range, we see that $A^{1/2}(D)$ is dense in H_0^1. Moreover,

$$U_{G(t)}A^{1/2}f_\varphi = A^{1/2}G(t)f_\varphi = A^{1/2}f_{\varphi(\ \,-t)} \in A^{1/2}(D),$$

and it is also clearly differentiable as a function in t. Therefore

$$e^{itS}(A^{1/2}(D)) = U_{G(t)}(A^{1/2}(D)) \subset A^{1/2}(D)$$

for all $t \in \mathbb{R}$. This implies that $A^{1/2}(D)$ is a core for S (see [15, Theorem VIII.11]). Finally, differentiating $U_{G(t)}A^{1/2}f_\varphi = A^{1/2}G(t)f_\varphi$ at $t = 0$, one obtains $iSA^{1/2}f_\varphi = A^{1/2}\dot{G}(0)f_\varphi$. ■

6. Invariant Finsler metrics in $\mathbb{G}$. The group $\mathbb{G}$ preserves the norms in L^2, the usual spectral norm and the Schatten p-norms. Therefore it is natural, from a geometric standpoint, to use these norms to endow $\mathbb{G}$ with a Finsler metric. The tangent space $(T\mathbb{G})_G$ of $\mathbb{G}$ at G identifies with

$$(T\mathbb{G})_G = g\Gamma = \{GX : X \in \Gamma\}.$$

Since the elements $G \in \mathbb{G}$ preserve the 2-norm $|\ |_2$, it is natural to consider, in each tangent space, the norm

$$\|V\|_G = \|V\|_{\mathcal{B}(L^2)}.$$

Note that if $V = GX$, for $X \in \Gamma$, then

$$\|V\|_G = \|GX\|_{\mathcal{B}(L^2)} = \|\bar{G}X\|_{\mathcal{B}(L^2)} = \|X\|_{\mathcal{B}(L^2)},$$

because $\bar{G}$ is a unitary operator in L^2. This implies that this metric is bi-invariant for the left and right action of $\mathbb{G}$ on itself. Note that the tangent spaces are not complete with this norm.

We measure the length of a differentiable curve γ in $\mathbb{G}$, parametrized on the interval I, as is usual, by

$$L_2(\gamma) = \int_I \|\dot{\gamma}(t)\|_{\gamma(t)}\,dt = \int_I \|\dot{\gamma}(t)\|_{\mathcal{B}(L^2)}\,dt.$$

The rectifiable distance d_2 induced by the infimum of the L_2-lengths of the paths joining given endpoints is a continuous map when we give $\mathbb{G}$ the natural topology as an open subset of the bounded linear operators on H_0^1. However, the topology induced by this rectifiable distance on $\mathbb{G}$ is finer, thus what we have introduced is a weak Finsler metric on the manifold $\mathbb{G}$.

PROPOSITION 6.1. *Suppose that $G \in \mathbb{G}$ with $\|G-1\|_{\mathcal{B}(H^1)} \le 1$. Then there exists a curve $\delta(t) = e^{tX}$ with $X \in \Gamma$ such that $\delta(1) = G$, which has minimal length among all curves in $\mathbb{G}$ joining 1 and G, and in particular $d_2(1, G) = \|X\|_{\mathcal{B}(L^2)}$.*

Proof. By Lemma 3.3, there exists $X \in \Gamma$ such that $e^X = G$. Moreover, $X = \log(G)$, with log being the branch of the logarithm with singularities in the negative real axis. By the formula (4.1),

$$\sigma_{L^2}(\bar{G}) \subset \sigma_{H_0^1}(G) \cup \overline{\sigma_{H_0^1}(G^{-1})}.$$

Note that $\|G-1\| \le 1$ implies that $\sigma_{H_0^1}(G) \subset \{z \in \mathbb{C} : \mathrm{Re}(z) \ge 0\}$. Then, if $\lambda \in \sigma_{H_0^1}(G^{-1})$, we have $\lambda = \mu^{-1}$ with $\mu \in \sigma_{H_0^1}(G)$, and thus $\mathrm{Re}(\lambda) \ge 0$. It follows that

$$\sigma_{L^2}(\bar{G}) \subset \{z \in \mathbb{C} : \mathrm{Re}(z) \ge 0\} \cap \mathbb{T} = \{e^{i\theta} : |\theta| \le \pi/2\},$$

and therefore $\|X\|_{\mathcal{B}(L^2)} \le \pi/2$. Note that $L_2(\gamma)$ equals the length of the curve of unitaries in L^2, measured with the Finsler metric given by the usual operator norm on $\mathcal{B}(L^2)$. It is a known fact that one-parameter groups of unitaries e^{tX} have minimal length along their paths, for time t such that $|t|\,\|X\|_{\mathcal{B}(L^2)} \le \pi$ (see for instance [1]). Therefore $\delta(t)$ remains minimal for $|t| \le 2$, which proves our assertion. ∎

6.1. The subgroups $\mathbb{G}_p$. Let $\mathcal{B}_p(H^1)$, $1 \le p \le \infty$, be the Schatten ideal of operators on H_0^1. As usual, $\mathcal{B}_\infty(H_0^1)$ stands for the compact operators on H_0^1. We introduce the following subgroups:

$$\mathbb{G}_p := \mathbb{G} \cap (I - \mathcal{B}_p(H_0^1)).$$

Clearly, $\mathbb{G}_p \subset \mathbb{G}_\infty$ properly. Apparently, the Banach–Lie algebra Γ_p of $\mathbb{G}_p$ is

$$\Gamma_p = \Gamma \cap \mathcal{B}_p(H_0^1).$$

For the subgroup $\mathbb{G}_\infty$ there is a stronger result on the minimality of curves. First note that using Lemma 3.3, we find that any $G \in \mathbb{G}_\infty$ is of the form $G = e^X$ for some compact operator $X \in \Gamma$.

PROPOSITION 6.2. *An operator G belongs to $\mathbb{G}_\infty$ if and only if there exists a compact operator $X \in \Gamma$ such that $e^X = G$.*

Proof. The sufficiency part is clear. Note that the spectrum $\sigma_{H_0^1}(G)$ is finite, or a sequence in $\mathbb{T}$ converging to 1. In particular one can always find a half-line $\mathcal{L}$ connecting 0 and infinity, which does not intersect $\sigma_{H_0^1}(G)$. Thus by Lemma 3.3, there exists $X \in \Gamma$ such that $e^X = G$, namely

$$X = \frac{1}{2\pi i} \int_\alpha \log(z)(G - z1)^{-1}\, dz,$$

where α is a simple closed curve which does not intersect $\mathcal{L}$ and encompasses $\sigma_{H_0^1}(G)$. Note that since $1 \in \sigma_{H_0^1}(G)$, one can adjust the definition of log in a way such that $0 \in \sigma_{H_0^1}(X)$ (trimming eigenvalues which are multiples of $2\pi i$).

It remains to prove that X is compact. Note that $G - z1$ belongs to the Banach algebra $\mathbb{C}1 + \mathcal{B}_\infty(H_0^1)$, the unitization of the algebra $\mathcal{B}_\infty(H_0^1)$ of compact operators. Therefore (since $\log(z)(G - z1)^{-1}$ is a map continuous in z, defined on a neighborhood of $\sigma_{H_0^1}(G)$), it follows that $X \in \mathbb{C}1 + \mathcal{B}_\infty(H_0^1)$, i.e. $X = \lambda + K$. Since X is non-invertible, it must be that $\lambda = 0$. ∎

In particular, if $G_1, G_2 \in \mathbb{G}_\infty$, then there exists a compact operator $X \in \Gamma$ such that $G_2 = e^X G_1$.

Theorem 6.3. *Let $G_1, G_2 \in \mathbb{G}_\infty$. Then there exists a compact operator $X \in \Gamma$ such that the curve $\delta(t) = e^{tX} G_1$ satisfies $\delta(1) = G_2$ and has minimal length among all curves joining the same endpoints in $\mathbb{G}_\infty$ (and in $\mathbb{G}$).*

Proof. By the above proposition, there exists $X \in \Gamma$, which is compact and satisfies $e^X G_1 = G_2$. If $\|X\|_{\mathcal{B}(L^2)} \le \pi$, the result follows by using the same argument as in Proposition 6.1, with the (unitary extension of the) curve $\delta(t) = e^{tX} G_1$. Suppose otherwise; then there exist finitely many eigenvalues λ of X such that $|\lambda| > \pi$. Pick one such λ. If P is the spectral projection in L^2 associated to λ, then $P(H_0^1) \subset H_0^1$. Indeed, the eigenvectors of the extension of X to L^2 belong to H_0^1 (see Theorem 2.1(iii)). Therefore, $iP|_{H_0^1} \in \Gamma$. There exists an integer m such that $|\lambda - 2m\pi i| \le \pi$. Then $X' = X - 2m\pi i P \in \Gamma$ is compact, and clearly satisfies $e^{X'} = e^X$. Replacing in this fashion all the eigenvalues (finite in number) which lie outside $[-\pi, \pi]$ yields a compact operator X_0 such that $e^{X_0} G_1 = G_2$. ∎

Proposition 6.4. *If $G \in \mathbb{G}_p$, then there exists $X \in \Gamma_p$ such that $\|X\| \le \pi$ and $e^X = G$.*

Proof. By Proposition 6.2, there exists a compact operator $X \in \Gamma$ with $\|X\| \le \pi$ such that $e^X = G$. It remains to prove that $X \in \mathcal{B}_p(H_0^1)$. The proof is similar to that case: consider now the Banach algebra $\mathbb{C}1 + \mathcal{B}_p(H_0^1)$, the unitization of $\mathcal{B}_p(H_0^1)$, which is a Banach algebra with the p-norm. Since $\log(z)(G - z1)^{-1}$ is continuous in the p-norm topology, it follows that $X = \lambda 1 + K$, with $K \in \mathcal{B}_p(H_0^1)$. Again, since $0 \in \sigma_{H_0^1}(X)$, we have $\lambda = 0$. ∎

Since $\Gamma_p \subset \mathcal{B}_p(H_0^1)$, a natural metric to consider in $\mathbb{G}_p$, which takes account of the specific spectral properties of the elements in $\mathbb{G}_p$, should be related to the p-norm. On the other hand, as remarked at the beginning of this section, we want the metric to be invariant under the action of the group. By Theorem 2.1, the operators $X \in \Gamma_p$, when extended to L^2, are compact and antihermitian. Moreover, the eigenvalues and multiplicities of the extension remain the same as for X. By a classical inequality of Lalesco [12] (see also [17]), the p-norm of the sequence of eigenvalues of X is bounded by the p-norm of the sequence of singular values of X. The former equals the p-norm of the extension of X to L^2 (because X is antihermitian there), the latter is the p-norm of X in H_0^1. Thus,

$$\|X\|_{p,\mathcal{B}(L^2)} \leq \|X\|_p. \tag{6.1}$$

We define the following metric in $\mathbb{G}_p$: if $X \in (T\mathbb{G}_p)_G$, then

$$\|X\|_{p,G} = \|X\|_{p,\mathcal{B}(L^2)}.$$

THEOREM 6.5. *Let $G_1, G_2 \in \mathbb{G}_p$. Then there exists $X \in \Gamma_p$ such that the curve $\delta(t) = e^{tX}G_1$ in $\mathbb{G}_p$ satisfies $\delta(1) = G_2$ and has minimal length for the above defined metric among all smooth curves joining the same endpoints in $\mathbb{G}_p$.*

Proof. By Proposition 6.4, there exists $X \in \Gamma_p$ with $\|X\|_{\mathcal{B}(L^2)} \leq \pi$ such that $G_2 = e^X G_1$. The result now follows just as for $\mathbb{G}_\infty$, using the fact that in the classical unitary groups

$$U_p(L^2) = \{G \in U(L^2) : G - 1 \in \mathcal{B}_p(L^2)\},$$

curves of the form e^{tX}, where X is antihermitian, have minimal length for the p-norm for $|t| \leq 1$ provided that $\|X\|_{\mathcal{B}(L^2)} \leq \pi$ (see [2] for $p \geq 2$ or [3] for the general case). ∎

REMARK 6.6. Using Lalesco's inequality, one may prove the inequality in (6.1) for any symmetric norm in the sense of [17]. Moreover, our last result on minimality of curves can be carried out in the general setting of symmetrically normed ideals.

Acknowledgements. This research was partly supported by CONICET (PIP 0757) and ANPCyT (PICT 2478).

References

[1] E. Andruchow and G. Larotonda, *The rectifiable distance in the unitary Fredholm group*, Studia Math. 196 (2010), 151–178.

[2] E. Andruchow, G. Larotonda and L. Recht, *Finsler geometry and actions of the p-Schatten unitary groups*, Trans. Amer. Math. Soc. 362 (2010), 319–344.

[3] J. Antezana, G. Larotonda and A. Varela, *Optimal paths for symmetric actions in the unitary group*, arXiv:1107.2439, 2011.

[4] B. A. Barnes, *The spectral properties of certain linear operators and their extensions*, Proc. Amer. Math. Soc. 128 (2000), 1371–1375.

[5] E. Chiumiento and M. Melgaard, *Stiefel and Grassmann manifolds in quantum chemistry*, J. Geom. Phys. 62 (2012), 1866–1881.

[6] J. Dieudonné, *Quasi-hermitian operators*, Proc. Int. Sympos. Linear Spaces (Jerusalem, 1960), Jerusalem Academic Press, Jerusalem, and Pergamon, Oxford, 115–122.

[7] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, FL, 1992.

[8] I. C. Gohberg and M. K. Zambickiĭ, *On the theory of linear operators in spaces with two norms*, Ukrain. Mat. Zh. 18 (1966), 11–23 (in Russian).

[9] L. A. Harris and W. Kaup, *Linear algebraic groups in infinite dimensions*, Illinois J. Math. 21 (1977), 666–674.

[10] M. G. Krein, *Compact linear operators on functional spaces with two norms*, Integral Equations Operator Theory 30 (1998), 140–162.

[11] M. G. Krein, *Integral equations on a half-line with kernels depending upon the difference of arguments*, Uspekhi Mat. Nauk 13 (1958), no. 5 (83), 3–120 (in Russian); English transl.: Amer. Math. Soc. Transl. (2) 22 (1962) 163–288.

[12] T. Lalesco, *Une théorème sur les noyaux composés*, Bull. Acad. Sci. Roumaine 3 (1915), 271–272.

[13] P. D. Lax, *Symmetrizable linear transformations*, Comm. Pure Appl. Math. 7 (1954), 633–647.

[14] N. N. Lebedev, *Special Functions and Their Applications*, Prentice-Hall, Englewood Cliffs, NJ, 1965.

[15] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I. Functional Analysis*, 2nd ed., Academic Press, New York, 1980.

[16] H. L. Royden, *Function algebras*, Bull. Amer. Math. Soc. 69 (1963), 281–298.

[17] B. Simon, *Trace Ideals and Their Applications*, 2nd ed., Math. Surveys Monogr. 120, Amer. Math. Soc., Providence, RI, 2005.

[18] F. Trèves, *Basic Linear Partial Differential Equations*, Pure Appl. Math. 62, Academic Press, New York, 1975.

[19] H. Upmeier, *Symmetric Banach Manifolds and Jordan C^*-Algebras*, North-Holland Math. Stud. 104, North-Holland, Amsterdam, 1985.

[20] H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung)*, Math. Ann. 71 (1912), 441–479.

Esteban Andruchow, Gabriel Larotonda
Instituto de Ciencias
Universidad Nacional de General Sarmiento
and
Instituto Argentino de Matemática
'Alberto P. Calderón', CONICET
J.M. Gutierrez 1150 (B1613GSX)
Los Polvorines, Argentina
E-mail: eandruch@ungs.edu.ar
glaroton@ungs.edu.ar

Eduardo Chiumiento
Departamento de Matemática
FCE-Universidad Nacional de La Plata
and
Instituto Argentino de Matemática
'Alberto P. Calderón', CONICET
Calles 50 y 115 (1900)
La Plata, Argentina
E-mail: eduardo@mate.unlp.edu.ar

Received June 23, 2012 (7547)