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Ideals and hereditary subalgebras in operator algebras

by

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Abstract. This paper may be viewed as having two aims. First, we continue our study of algebras of operators on a Hilbert space which have a contractive approximate identity, this time from a more Banach-algebraic point of view. Namely, we mainly investigate topics concerned with the ideal structure, and *hereditary subalgebras* (or HSA's, which are in some sense a generalization of ideals). Second, we study properties of operator algebras which are hereditary subalgebras in their bidual, or equivalently which are 'weakly compact'. We also give several examples answering natural questions that arise in such an investigation.

1. Introduction. For us, an *operator algebra* is a norm closed algebra of operators on a Hilbert space (see e.g. [4, 7]). This paper may be viewed as having two aims. The first is the continuation of our study of the structure of operator algebras which have a contractive approximate identity (cai). In this paper we adopt a slightly more Banach-algebraic point of view. Indeed this paper may be seen as a collection of general results growing out of topics raised in [3] concerning ideals and *hereditary subalgebras* (HSA's) of operator algebras. We recall that HSA's are in some sense a generalization of ideals (a HSA D in A must satisfy $DAD \subset D$). Some of these general results are of a technical nature, and so this paper should serve in part as a repository that will be useful in later development of the themes of interest here. Second, we study properties of operator algebras which are hereditary subalgebras in their bidual (this is equivalent to A being 'weakly compact', see e.g. Lemma 5.1 below). We give several examples answering natural questions that arise in such an investigation, for instance, involving semisimplicity or semiprimeness (we recall that an algebra A is *semisimple* if its Jacobson radical satisfies $J(A) = (0)$, and is *semiprime* if (0) is the only (closed) ideal with square (0)).

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In Section 2 (resp. Section 4) of the paper we present some general results about ideals (resp. HSA's) in operator algebras. In Section 3 we discuss adjoining a square root to an operator algebra, and use this to answer some natural questions. We also discuss in Section 3 whether for an approximately unital operator algebra, being semisimple (resp. semiprime, radical) implies or is implied by A^{**} being semisimple (or semiprime, radical). (We do not think we know yet if A^{**} semisimple implies A semisimple if A is noncommutative.) In Section 5 we study operator algebras A which are hereditary subalgebras of their bidual, which as we said above is equivalent to the multiplication $x \mapsto axb$ being weakly compact on A for all $a, b \in A$. Some of the properties of such algebras are similar to operator algebras which are one-sided ideals in their bidual, which were studied in [34, 3]. See e.g. [19, 21, 36] for some related theory of Banach algebras that are ideals in their bidual. We also study the more general class of algebras that we call *nc-discrete*, which means that all the open projections are also closed (or equivalently lie in the multiplier algebra $M(A)$). Any compact operator algebra is a HSA in its bidual; and we show that any operator algebra which is a HSA in its bidual is nc-discrete. Neither of these two implications are reversible though, as may be seen in Subsections 6.1, 6.2, and Theorem 6.4. Indeed in Section 6 we present examples of operator algebras exhibiting various properties illustrating the topics of interest in this paper. In particular we give what is as far as we can see the first explicit example in the literature of an interesting (i.e. not reflexive in the Banach space sense) commutative algebra whose multiplication is weakly compact but not compact. In Section 7 we discuss the diagonal of a quotient algebra.

We now turn to notation and more precise definitions. The reader is referred for example to [7, 6, 9] for more details on some of the topics below if needed. By an *ideal* of an operator algebra A we shall always mean a closed two-sided ideal in A, unless the contrary is clear from the context. For us a *projection* is always an orthogonal projection, and an *idempotent* merely satisfies $x^2 = x$. If X, Y are sets, then XY denotes the closure of the span of products of the form xy for $x \in X$, $y \in Y$. We recall that by a theorem due to Ralf Meyer, every operator algebra A has a unique unitization A^1 (see [7, Section 2.1]). Below, 1 always refers to the identity of A^1 if A has no identity. If A is a nonunital operator algebra represented (completely) isometrically on a Hilbert space H then one may identify A^1 with $A + \mathbb{C}I_H$. The second dual A^{**} is also an operator algebra with its (unique) Arens product; this is also the product inherited from the von Neumann algebra B^{**} if A is a subalgebra of a C^*-algebra B. Meets and joins in B^{**} of projections in A^{**} remain in A^{**}, as can be readily seen for example by inspecting some of the classical formulae for meets and joins of Hilbert space projections, or by noting that these meets and joins may be computed in the biggest von Neumann algebra contained inside A^{**}. Note that A has a cai iff A^{**} has an identity $1_{A^{**}}$ of norm 1, and then A^1 is sometimes identified with $A+\mathbb{C}1_{A^{**}}$. In this case the multiplier algebra $M(A)$ is identified with the *idealizer* of A in A^{**}, that is, the set of elements $\alpha \in A^{**}$ such that $\alpha A \subset A$ and $A\alpha \subset A$.

The set of compact operators on a Hilbert space is often called an *elementary C^*-algebra*. We call a c_0-direct sum of elementary C^*-algebras an *annihilator C^*-algebra.*

The *diagonal* $\Delta(A)$ is defined to be $A \cap A^*$; it is a C^*-algebra which is well defined independently of the particular (completely isometric) representation of A. Most of our algebras and ideals are *approximately unital*, i.e. have a contractive approximate identity (cai), although for some results this is probably not necessary. We recall that an *r-ideal* is a right ideal with a left cai, and an *ℓ-ideal* is a left ideal with a right cai. We say that an operator algebra D with cai, which is a subalgebra of another operator algebra A, is a HSA (*hereditary subalgebra*) in A if $DAD \subset D$. See [6] for the theory of HSA's (a few more results may be found in [3, 9]). HSA's in A are in an order preserving, bijective correspondence with the r-ideals in A, and also with the *open projections* $p \in A^{**}$, by which we mean that there is a net $x_t \in A$ with $x_t = px_tp \to p$ weak*. These are also the open projections p in the sense of Akemann [1] in B^{**} (see also e.g. [30]), where B is a C^*-algebra containing A, such that $p \in A^{\perp\perp}$. The complement ('perp') of an open projection is called a *closed projection.* We spell out some of the correspondences above: if D is a HSA in A, then DA (resp. AD) is the matching r-ideal (resp. ℓ-ideal), and $D = (DA)(AD) = (DA) \cap (AD)$. The weak* limit of a cai for D, or of a left cai for an r-ideal, is an open projection, and is called the *support projection.* Conversely, if p is an open projection in A^{**}, then $pA^{**} \cap A$ and $pA^{**}p \cap A$ is the matching r-ideal and HSA pair in A.

It is a well-known fact that if J is an ideal of an operator algebra A, then the quotient algebra A/J is isometrically isomorphic to an operator algebra [7, Proposition 2.3.4]. Of course there is a 'factor theorem': if $u : A \to B$ is a completely bounded homomorphism between operator algebras, and if J is an ideal in A contained in $\mathrm{Ker}(u)$, then the canonical map $\tilde{u} : A/J \to B$ is also completely bounded with completely bounded norm $\|\tilde{u}\|_{\mathrm{cb}} = \|u\|_{\mathrm{cb}}$. If $J = \mathrm{Ker}(u)$, then u is a complete quotient map if and only if $\tilde{u}$ is a completely isometric isomorphism.

Let A be an operator algebra. The set $\mathfrak{F}_A = \{x \in A : \|1 - x\| \leq 1\}$ equals $\{x \in A : \|1 - x\| = 1\}$ if A is nonunital, whereas if A is unital then $\mathfrak{F}_A = 1 + \mathrm{Ball}(A)$. Many properties of $\mathfrak{F}_A$ are developed in [9, 10, 33]. If A is a closed subalgebra of an operator algebra B then it is easy to see, using the uniqueness of the unitization, that $\mathfrak{F}_A = A \cap \mathfrak{F}_B$.

We write $J(A)$ for the Jacobson radical (see e.g. [29]). It is a fact in pure algebra that an algebra is semiprime (resp. semisimple) iff its unitization is semiprime (resp. semisimple). Indeed $J(A) = J(A^1)$ (see [29, 4.3.3]). One trap to beware of is that the C^*-algebra generated by a HSA D in an operator algebra A need not be a HSA in a C^*-algebra generated by A. In particular $C^*(D)$ need not be an HSA in $C^*_{\max}(A)$ or in $C^*_{\mathrm{e}}(A)$. An example is $\mathcal{U}(M_2)$, the subalgebra of $M_2(A)$ with 0 in the 2-1-entry, and scalar multiples of the identity on the main diagonal, in the case $A = M_2$.

2. General results on ideals in operator algebras. The first two results below are obvious, and follow from the analogous results for general operator spaces.

THEOREM 2.1 (First Isomorphism Theorem). *Let $u : A \to B$ be a complete quotient map which is a homomorphism between operator algebras. Then* $\mathrm{Ker}(u)$ *is an ideal in A and $A/\mathrm{Ker}(u) \cong B$ completely isometrically isomorphically. Conversely, every ideal of A is of the form* $\mathrm{Ker}(u)$ *for a complete quotient map $u : A \to B$, where A and B are operator algebras.*

THEOREM 2.2 (Second Isomorphism Theorem). *Let A be an approximately unital operator algebra, let J be an ideal in A, and suppose that I is an ideal in J. Then $(A/I)/(J/I) \cong A/J$ completely isometrically isomorphically* (*as operator algebras*).

THEOREM 2.3 (Third Isomorphism Theorem). *Let A be an approximately unital operator algebra, and suppose that J and K are ideals in A, where J has a cai. Then $J/(J \cap K) \cong (J + K)/K$ completely isometrically isomorphically. In particular, $(J + K)/K$ is closed.*

Proof. Note that by [17, Proposition 2.4], $J + K$ is closed. Define a map $u : J/(J \cap K) \to (J + K)/K$ by $u(j + J \cap K) = j + K$. This is a well-defined map and u is one-to-one since $\mathrm{Ker}(u) = (0_{J/(J\cap K)})$. Moreover, u is onto since $x + K \in (J + K)/K$ implies that $x = j + k$, where $j \in J$, $k \in K$ and $x + K = j + K = u(j + J \cap K)$.

Since $\inf\{\|j + k\| : k \in K\} \le \inf\{\|j + k\| : k \in J \cap K\}$, u is a contraction. Let (e_t) be the cai for J and let $k \in K$. Then,

$$\|j + k\| \ge \|e_t j + e_t k\| \ge \|e_t j + J \cap K\|.$$

After taking the limit, we get $\|j + J \cap K\| \le \|j + k\|$, and so $\|j + J \cap K\| \le \|j + K\|$. Hence, u is an isometry. Similarly, u is a complete isometry. ∎

For Banach algebras the 'Correspondence Theorem' states that for a Banach algebra A and a closed ideal J in A, every closed subalgebra K of A/J is of the form I/J, where I is a closed subalgebra of A with $J \subset I \subset A$.

Also, every ideal K of A/J is of the form I/J, where I is an ideal of A with $J \subset I \subset A$. Indeed $I = q^{-1}(K)$ where $q : A \to A/J$ is the canonical map.

THEOREM 2.4. *Let A be an approximately unital operator algebra, let I be an approximately unital ideal in A and let J be an approximately unital ideal in I. Then I/J is an approximately unital ideal in A/J. Conversely, every approximately unital ideal of A/J is of the form I/J, where I is an approximately unital ideal in A with $J \subset I \subset A$.*

Proof. The first assertion is easy. The second assertion follows from [11, Proposition 3.1] (as in e.g. [9, Section 6], where the analogue of the above result is proved for HSA's and certain one-sided ideals). ∎

REMARK. Note that since [11, Proposition 3.1] is also valid for Arens regular Banach algebras, the previous result can be stated for such Banach algebras. Similarly for Corollary 2.6 below.

LEMMA 2.5. *Suppose that A is an operator algebra such that A^{**} is semiprime, and that J is a closed ideal in A such that J^2 has a cai. Then $J = J^2$.*

Proof. Let p be the support projection of J^2 in A^{**}. We have $J^2(1-p) = 0$. If $\zeta, \eta \in J^{\perp\perp}$ and if $a_t \to \eta$ weak* and $b_s \to \zeta$ weak*, then since $a_t b_s(1-p) = 0$ we have $a_t\zeta(1-p) = 0$ and $0 = \eta\zeta(1-p) = \eta(1-p)\zeta(1-p)$. So $(J^{\perp\perp}(1-p))^2 = (0)$. Since A^{**} is semiprime we have $J^{\perp\perp}(1-p) = (0)$, so that $J^{\perp\perp}$ is unital, so that J has a cai. ∎

COROLLARY 2.6. *Suppose that A is an approximately unital operator algebra, and that J is an approximately unital ideal in A. Suppose that J has the property that if I is an ideal with square J then $I = J$. Then A/J is semiprime. In particular, if A^{**} is semiprime then A/J is semiprime for every approximately unital ideal J in A.*

Proof. Let K be an ideal in A/J such that $K^2 = (0)_{A/J}$. There exists an ideal I in A such that $J \subset I \subset A$ and $K = I/J$ (namely, the inverse image of K in A, see Theorem 2.4). Since $K^2 = (I/J)(I/J) = I^2/J = (0)_{A/J}$, we conclude that $I^2 = J$. Under our hypotheses this forces $I = J$ (the 'in particular' assertion uses Lemma 2.5 here). That is, $K = I/J = (0)_{A/J}$, and so A/J is semiprime. ∎

REMARK. (1) In view of the last results, and independently, it is of interest to know whether every approximately unital ideal J in a semisimple or semiprime algebra A has the property that if I is an ideal with square J then $I = J$. We will see in Corollary 3.2 that this is false.

(2) Algebras whose square (or nth power) is approximately unital are discussed in [10, Section 4].

Proposition 2.7. *Let A be a Banach algebra with no nonzero left annihilators, and let $\{I_\alpha\}$ be an increasing family of ideals in A such that $A = \overline{\bigcup I_\alpha}$. If each I_α is semiprime* (*resp. semisimple*), *then A is semiprime* (*resp. semisimple*).

Proof. Assume that I_α is semiprime for each α. Let J be an ideal in A with $J^2 = (0)$. Then $(J \cap I_\alpha)^2 = (0)$ for each α. Since I_α is semiprime, $J \cap I_\alpha = (0)$. Hence $JI_\alpha \subset J \cap I_\alpha = (0)$ for each α, so that $JA = (0)$ and $J = (0)$. Hence A is semiprime.

Now suppose that each I_α is semisimple. Notice that $J = \mathrm{Rad}(A)$ is an ideal in A and $J \cap I_\alpha$ is an ideal in I_α for each α. Then $J \cap I_\alpha = \mathrm{Rad}(I_\alpha) = (0)$ by [29, Theorem 4.3.2]. Hence as in the first paragraph, $J = (0)$ and A is semisimple. ∎

Proposition 2.8. *Suppose that A is an operator algebra with a bai, and that I and J are ideals in A. If $A/I \cong A/J$ isomorphically as A-bimodules, then $I = J$.*

Proof. If A is unital then this follows from the analogous result in pure algebra. If A contains a bai, then its bidual A^{**} is unital. If $\pi : A/I \to A/J$ is an A-bimodule isomorphism, then π^{**} maps $(A/I)^{**} \cong A^{**}/I^{\perp\perp}$ into $(A/J)^{**} \cong A^{**}/J^{\perp\perp}$. Moreover, π^{**} is an A^{**}-bimodule isomorphism by the separate weak*-continuity of the Arens product. Since A^{**} is unital, by the unital case we have $I^{\perp\perp} = J^{\perp\perp}$. Thus $I = A \cap I^{\perp\perp} = A \cap J^{\perp\perp} = J$. ∎

Remark. Note that the previous proposition is not true for general operator algebras; the existence of a bai is needed. For example, let $A = \mathrm{Span}(x, y)$ where $xy = x^2 = y^2 = 0$, and $x \neq y$. If $I = \mathrm{Span}(x)$ and $J = \mathrm{Span}(y)$, then $A/I \cong A/J$ as A-bimodules, but $I \neq J$.

3. Example: adjoining a root to an algebra. In this section we show how to create examples of operator algebras by adjoining a root. We then use this to answer several basic questions regarding operator algebras with cai.

If A is an algebra, and S is in the center of A, we define an algebra

$$A_S = \left\{ \begin{bmatrix} x & y \\ Sy & x \end{bmatrix} : x \in A,\, y \in A^1 \right\} \subset M_2(A^1).$$

We identify A with the main diagonal of this algebra, and we set T to be the matrix with rows $0, 1$ and S, 0. Then any element of A_S may be written as $x + yT$ for $x \in A$, $y \in A^1$. In this notation, $T^2 = S$, and so now S has a square root even if it did not have one before. A good example to bear in mind is the case that A is the approximately unital ideal in the disk algebra $A(\mathbb{D})$ of functions vanishing at 1, and $S = z(1 - z) \in A$, which has no root in A.

It is obvious that if A is an operator algebra then so is A_S, and if A is commutative then so is A_S. If A has a cai but no identity then A_S has no cai, but $A_S^2 = \{x + yT : x, y \in A\}$ does have a cai.

We say that an element a in an algebra A *has no rational square root* if there exist no $b, c \in A$ with $ac^2 = b^2 \neq 0$.

LEMMA 3.1. *Suppose that A is a commutative algebra, and $S \in A$, $S \neq 0$. If A is an integral domain then A_S is semiprime. On the other hand, if A is semisimple and S has no rational square root, and is not a divisor of zero, then A_S is semisimple.*

Proof. If $(x+yT)^2 = x^2+y^2S+2xyT = 0$, and A is an integral domain, then $x = 0$ or $y = 0$. Since $x^2 + y^2S = 0$ we have $x = y = 0$.

In the semisimple case, suppose that all characters of A_S vanish at $x+yT \in A_S$. If χ is a character of A^1, define $\chi'(a + bT) = \chi(a) + \alpha\chi(b)$ for $a \in A$, $b \in A^1$, where α is a square root of $\chi(S)$. This defines a character on A_S, and so we have $\chi(x)+\alpha\chi(y) = 0$. Thus x^2-Sy^2 is in the kernel of every character of A^1, so that $Sy^2 = x^2$. Therefore $x^2 = 0$, and since A is semiprime we have $x = y = 0$. ∎

COROLLARY 3.2. *There exists a semisimple commutative operator algebra A with no cai, such that A^2 has a cai.*

Proof. In our disk algebra example mentioned above, the element $z(1-z) \in A$ has no rational square root. To see this note that of course $1 - z$ does, and so we are asking if $zg^2 = h^2$ is possible with $g, h \in A(\mathbb{D})$. By Riemann's Theorem in basic complex analysis this equation implies that h/g has an analytic continuation k to $\mathbb{D}$ such that $k(z)^2 = z$ on $\mathbb{D}$, which is well known to be impossible ($2k(z)k'(z) = 1$ so k' is unbounded at 0). We deduce from Lemma 3.1 that A_S is semisimple if $S = z(1 - z)$. Here A_S is a semisimple commutative operator algebra with no cai, but A_S^2 has a cai. ∎

In [15] it is shown that semisimple $B(\ell^p)$ fails to have a semisimple second dual if $p \neq 2$. This can also happen for operator algebras:

PROPOSITION 3.3. *Let A be an operator algebra.*

(1) *If A^{**} is semiprime* (*resp. radical, semisimple and commutative*) *then A is semiprime* (*resp. radical, semisimple*).
(2) *If A is semiprime* (*resp. approximately unital and radical, unital and semisimple*) *then A^{**} need not be semiprime* (*resp. radical, semiprime and hence not semisimple*).

Proof. (1) If A is commutative and A^{**} is semisimple, then A is semisimple by e.g. [13, Proposition 2.6.25(iv)]. If A^{**} is semiprime and if $J^2 = (0)$ in A then $(J^{\perp\perp})^2 = (0)$ in A^{**}, so that $(0) = J^{\perp\perp} = J$. So A is semiprime.

It follows from [13, Proposition 2.6.25(iii)] that if A^{**} is radical then A is radical.

(2) If A is radical then A^{**} is not radical (indeed A^{**} is unital).

Suppose that the second dual of every unital semiprime operator algebra A was semiprime. Then by Lemma 2.5, if J is an ideal in a unital semisimple operator algebra such that J^2 has a cai, then J has a cai. However this is not true, as may be seen from the disk algebra example two paragraphs above (one may take $J = A_S$, $A = J^1$ here). Hence the second dual of a commutative unital semisimple operator algebra need not be semiprime. ∎

We shall see in Corollary 5.3 that the situation in the last result improves if A is a HSA in its bidual.

4. General facts about HSA's

PROPOSITION 4.1. *If D is a HSA in an operator algebra A, then $x \in D$ is quasi-invertible in D iff x is quasi-invertible in A. Thus $\sigma_D(x) \setminus \{0\} = \sigma_A(x) \setminus \{0\}$. In particular, D is a spectral subalgebra of A in the sense of e.g.* [29, p. 245].

Proof. By [13, Proposition 2.6.25], x is quasi-invertible in D (resp. A) iff x is quasi-invertible in D^{**} (resp. A^{**}). Now $D^{**} = pA^{**}p$, where p is the support projection of D. It is a simple algebraic exercise that in an algebra A with an idempotent e, an element of eAe is quasi-invertible in eAe iff it is quasi-invertible in A. So x is quasi-invertible in D iff x is quasi-invertible in $D^{**} = pA^{**}p$, iff x is quasi-invertible in A^{**}, and hence iff x is quasi-invertible in A.

The last assertion follows from the usual formulation of the spectrum in terms of quasi-invertible elements. ∎

The following answers a question posed in [3]. The first assertion is well known with HSA's replaced by ideals [29].

THEOREM 4.2. *If D is a HSA in an operator algebra A, then $J(D) = D \cap J(A)$. In particular, semisimplicity passes to HSA's.*

Proof. Suppose that A is an operator algebra and D is a HSA in A with cai (f_t). We recall that $J(A)$ may be characterized (see e.g. [29]) as the set of $a \in A$ with $r(ab) = 0$ for all $b \in A^1$. Here $r(\cdot)$ denotes the spectral radius. Let $x \in J(D)$. Since $J(D)$ is a nondegenerate D-module, by Cohen's factorization there exist $d \in D$ and $y \in J(D)$ with $x = dy$. Now $yf_t\,ad \in J(D)$ for all $a \in A^1$ (since D is a HSA in A^1). Since $J(D)$ is closed we have $yad \in J(D)$. Thus $0 = r(yad) = r(dya) = r(xa)$ for all $a \in A^1$. Hence $x \in J(A)$. So $J(D) \subset D \cap J(A)$. The converse follows from [29, Theorem 4.3.6(c),(e)]: if $x \in D \cap J(A)$, then A^1x consists of quasi-invertibles in A. Hence D^1x consists of quasi-invertibles in A, hence of quasi-invertibles in D by Proposition 4.1. So $x \in J(D)$. ∎

We have a generalization of the last result:

COROLLARY 4.3. *Suppose that D is a HSA in an operator algebra A, and that I is an approximately unital ideal in A. Then*

(1) *$D \cap I = DID$ is a HSA in A, and $J(D \cap I) = J(D) \cap J(I)$.*
(2) *$(D \cap I)^{\perp\perp} = D^{\perp\perp} \cap I^{\perp\perp}$.*
(3) *$D + I$ is closed, and is a HSA in A.*

Proof. (1) We have $(D \cap I)A(D \cap I) \subset (DAD) \cap (IAI) \subset D \cap I$. Note that $DID \subset I \cap D$. Conversely, since D has a cai we have $I \cap D \subset DID$. So $DID = I \cap D$. If (f_s) is a cai for I and (e_λ) is a cai for D, then $(e_\lambda f_s e_\lambda)$ is easily seen to yield a cai for DID, by routine techniques. So $I \cap D = DID$ is a HSA in A. By Theorem 4.2 we have

$$J(D \cap I) = D \cap I \cap J(A) = D \cap J(A) \cap I \cap J(A) = J(D) \cap J(A)$$

as desired.

(3) Write (e_λ) for the cai of D. If $r \in I$ then $e_\lambda r e_\lambda \in D \cap I$. Moreover if $a \in D$ then

$$\|a - r\| \geq \|e_\lambda a e_\lambda - e_\lambda r e_\lambda\| \geq \|a - e_\lambda r e_\lambda\| - \|a - e_\lambda a e_\lambda\|.$$

Also, $\|a - e_\lambda r e_\lambda\| \geq \|a + (I \cap D)\|$. The above constitute the modifications of the proof of [17, Proposition 2.4] that need to be made so that as in that proof we may deduce that $D + I$ is closed. Clearly $(D + I)A(D + I) \subset DAD + I = D + I$.

(2) This follows from (3) and the fact from e.g. [12, Appendices A.3, A.5] that for closed subspaces E, F of any Banach space X,

$$(E \cap F)^{\perp\perp} = (E^\perp + F^\perp)^\perp = E^{\perp\perp} \cap F^{\perp\perp}$$

if $E + F$ is closed, or equivalently, if $E^{\perp\perp} + F^{\perp\perp}$ is closed. ∎

REMARK. If I is any ideal in a HSA D of an operator algebra A, and if $I \subset J(A)$, then $J(D/I) = J(D)/I$. This follows from Theorem 4.2 and [29, Theorem 4.3.2(b)].

5. Algebras that are HSA's in their bidual. We write $M_{a,b} : A \to A : x \mapsto axb$, where $a, b \in A$. Recall that a Banach algebra is *compact* if the map $M_{a,a}$ is compact for all $a \in A$. We say that A is *weakly compact* if $M_{a,a}$ is weakly compact for all $a \in A$. We are concerned here mostly with operator algebras A that are HSA's in their bidual. That is, A has a cai, and $AA^{**}A \subset A$. For algebras A that do not have a cai, one could pass to the algebra A_H described in [10], the biggest subalgebra with a cai.

Lemma 5.1. *An operator algebra A with cai is a HSA in its bidual iff the map $M_{a,b} : A \to A : x \mapsto axb$ is weakly compact for all $a, b \in A$, and iff A is weakly compact in the sense just defined.*

Similarly, $M_{a,b}$ is compact on A for all $a, b \in A$ iff A is compact.

If in addition A is commutative, then A is compact (*resp. weakly compact*) *iff multiplication by a is compact* (*resp. weakly compact*) *on A for all $a \in A$.*

Proof. The first 'iff' follows by basic functional analysis (namely, the well-known fact that an operator $T : X \to Y$ is weakly compact iff $T^{**}(X^{**}) \subset Y$). To see the second and third 'iff' we use the fact that the compact (resp. weakly compact) operators constitute a norm closed ideal. From this, first, if (e_t) is a cai for A and M_{e_t,e_t} is compact (resp. weakly compact), then so is M_{ae_t,e_tb} for all $a, b \in A$. Second, $M_{a,b}$ is compact (resp. weakly compact) since $M_{ae_t,e_tb} \to M_{a,b}$ in norm. ∎

Clearly then compact operator algebras are HSA's in their bidual. It is easy to find Banach space reflexive examples showing that the converse is not true (see Subsection 6.1). Note that the class of unital operator algebras which are HSA's in their bidual is the same as the class of unital operator algebras which are Banach space reflexive. It is of interest to find nonreflexive weakly compact algebras which are not compact, and we shall do this later in Subsection 6.4. In this connection we remark that semisimple annihilator Banach algebras in the sense of [29, Chapter 8] are compact, and are *ideals* in their bidual [29, Corollary 8.7.14].

Remark. In any commutative operator algebra A, two natural ideals to consider are those constituting the elements $a \in A$ with multiplication by a being compact or weakly compact on A.

The property of being a HSA in the bidual passes to subalgebras and quotients:

Lemma 5.2. *Let A be an operator algebra which is weakly compact. If B is a closed subalgebra of A, then B is weakly compact. If I is a closed ideal in A, then A/I is weakly compact.*

Proof. We leave this as an exercise for the reader. ∎

Remark. Similarly, if A is an approximately unital ideal in its bidual then so is any closed subalgebra, or quotient by a closed ideal (see [34]).

Corollary 5.3. *Suppose that A is an operator algebra which is a HSA in its bidual. Then A is semisimple* (*resp. semiprime*) *iff A^{**} is semisimple* (*resp. semiprime*).

Proof. One direction follows from Theorem 4.2 (resp. [3, Proposition 2.5]). If A is semisimple, let $0 \neq \eta \in J(A^{**})$. Then $A\eta A \subset J(A^{**}) \cap A \subset J(A) = (0)$ (using Theorem 4.2). So $\eta = 0$, and A^{**} is semisimple. If A is semiprime, and if J is an ideal in A^{**} with $J^2 = (0)$, then $(J \cap A)^2 = (0)$, so that $J \cap A = (0)$. Hence $AJA = (0)$ since $AJA \subset J \cap A$. Since a cai of A converges weak* to the identity of A^{**} we deduce that $J = (0)$. So A^{**} is semiprime. ∎

Proposition 5.4. *If A is an operator algebra which is a HSA in its bidual, and if A has no ideals* (*resp. no closed ideals, no closed ideals with a cai*), *then every ideal* (*resp. closed ideal, closed ideal with a cai*) *in A^{**} contains A.*

Proof. If J is a nontrivial ideal in A^{**}, then as in the proof of Corollary 5.3, $AJA \subset J \cap A = A$ or (0), and the latter is impossible. Similarly for the closed ideal case. Similarly for the case of a closed ideal J with a cai, because by Corollary 4.3 the ideal $J \cap A$ of A is also a HSA in A^{**}, so has a cai. ∎

An operator algebra A with cai is *nc-discrete* if every closed right ideal which has a left cai is of the form eA for a projection $e \in M(A)$. Equivalently, all the open projections are also closed (or equivalently are in $M(A)$). The first part of the following was independently noticed recently in [28], and no doubt by others:

Proposition 5.5. *A C^*-algebra which is a HSA in its bidual, or is nc-discrete, is an annihilator C^*-algebra.*

Proof. One well-known characterization of annihilator C^*-algebras is that every commutative C^*-subalgebra D has maximal ideal space which is topologically discrete. Thus the HSA case of the proposition follows by Lemma 5.2 and the fact that if a $C_0(K)$ space is an ideal in its bidual, then K is topologically discrete. The nc-discrete case for C^*-algebras is another well-known characterization of annihilator C^*-algebras. ∎

Proposition 5.6. *If an operator algebra A is a HSA in its bidual* (*resp. is compact*), *then so is $\mathbb{K}_I(A)$ for any cardinal I* (*here $\mathbb{K}_I(A)$ is the spatial tensor product of A and the compact operators on $\ell^2(I)$*). *Also, the c_0-direct sum of operator algebras which are HSA's in their bidual* (*resp. nc-discrete, Δ-dual*), *is a HSA in its bidual* (*resp. is nc-discrete, Δ-dual*).

Proof. We leave this as an exercise. ∎

Remark. We said earlier that compact approximately unital Banach algebras are HSA's in their bidual. We recall that a semisimple Banach algebra A is a modular annihilator algebra iff no element of A has a nonzero limit point in its spectrum [29, Theorem 8.6.4]), and iff for every $a \in A$ multiplication on A by a is a Riesz operator (see [29, Chapter 8]). If A is also commutative then this is equivalent to the Gelfand spectrum of A being discrete [25, p. 400]. By [29, Chapter 8], compact semisimple algebras are modular annihilator algebras. We note that any radical semiprime algebra is a modular annihilator algebra by [29, Theorem 8.7.2]. There are some interesting commutative radical algebras in [16] which are modulator annihilator algebras, but they are probably not approximately unital nor are ideals in their bidual. See also e.g. [26]. One may ask if for algebras that are HSA's in their bidual (or even ideals in their bidual), the spectrum of every element is finite or countable. We have examples of algebras which are HSA's in their bidual with elements having spectrum which does have nonzero limit points (see Subsection 6.1). Such algebras are not modular annihilator algebras, but are *Duncan modular annihilator algebras* in the sense of [29, Chapter 8] (see also [18]). Any semisimple operator algebra with the spectrum of any element finite or countable is a Duncan modular annihilator algebra [29]. If A is a commutative approximately unital operator algebra which is an ideal in its bidual, one may ask if the spectrum of A (e.g. the set of characters of A) is scattered. In this case, and if A is not reflexive in the Banach space sense, then the spectrum of A^{**} equals the one-point compactification of the spectrum of A (see Theorem 5.10(4)). See also [27, 36] for some related Banach-algebraic topics.

In the converse direction, Duncan modular annihilator algebras, or semisimple operator algebras with the spectrum of every element finite or countable, need not be nc-discrete. An example is the space c. We are not sure if every (approximately unital) semisimple modular annihilator operator algebra is nc-discrete, or is a HSA in its bidual, although this seems unlikely, even in the commutative case. We hope to investigate this, and some of the other questions above, elsewhere.

THEOREM 5.7. *If an operator algebra A is a HSA in A^{**}, and if $\Delta(A)$ acts nondegenerately on A, then $\Delta(A)^{**} = \Delta(A^{**}) = \Delta(M(A))$. In particular, every projection in A^{**} is both open and closed.*

Proof. That $\Delta(A)$ acts nondegenerately on A implies that A has a positive cai (e_t) say. If A is a HSA in A^{**} then $e_t\eta e_t \in \Delta(A)$ for all $\eta \in \Delta(A^{**})_+$. If we represent A^{**} as a weak* closed subalgebra of H containing I_H, then A is represented nondegenerately on H (via [7, Lemma 2.1.9] say), and so $e_t\zeta \to \zeta$ for all $\zeta \in H$. It follows that $e_t\eta e_t \to \eta$ WOT, hence weak*. Thus $\Delta(A^{**}) \subset \Delta(A)^{\perp\perp}$. Since the converse inclusion is obvious we have $\Delta(A)^{**} = \Delta(A^{**})$. This equals $\Delta(M(A))$ by [3, Proposition 2.11]. For the last part, note that $\Delta(A)$ is a HSA in its bidual by Lemma 5.13, and hence is an annihilator C^*-algebra by Proposition 5.5. Thus any projection in $\Delta(A^{**}) = \Delta(A)^{**}$ is open and closed with respect to $\Delta(A)^{**}$ and hence is also open with respect to A. ∎

THEOREM 5.8. *If an operator algebra A is a HSA in A^{**} then A is nc-discrete.*

Proof. We follow some ideas in the proof of [6, Proposition 5.1], which the reader might wish to consult. By Lemma 5.2, $\Delta(A)$ is a HSA in $\Delta(A)^{\perp\perp} = \Delta(A)^{**}$. Hence $\Delta(A)$ is an annihilator C^*-algebra by Proposition 5.5.

Next, let p be an open projection in A^{**}. Suppose that A is a subalgebra of a C^*-algebra B, generating B as a C^*-algebra. Then $ApA \subset A$, and by Cohen's factorization $BpB = BApAB \subset B$. There is an increasing net $x_t \nearrow p$, with $x_t \in B$ for all t. If $b \in B_+$, then $bx_tb \nearrow bpb \in B$. Therefore, by Mazur's Theorem, replacing x_t by convex combinations of the x_t, we may assume that $bx_tb \to bpb$ in norm, and $0 \leq x_t \leq p$. Then $\|\sqrt{p - x_t}b\|^2 = \|b(p - x_t)b\| \to 0$. Hence $(p - x_t)b \to 0$, so that $pb \in B$. Therefore p is a left multiplier of B. However any projection which is a left multiplier is a two-sided multiplier. Consequently, $pA \in B \cap A^{\perp\perp} = A$, so p is a left multiplier of A. Similarly, p is a right multiplier of A, hence $p \in M(A)$. ∎

Remark. In this case there are bijective correspondences between the right ideals in A with left cai, HSA's in A, and orthogonal projections in the multiplier algebra $M(A)$. This will follow from the previous theorem and the basic facts about HSA's (see [6, Section 2]).

Lemma 5.9. *Suppose that an approximately unital operator algebra A is a HSA in its bidual, and that $\pi : A \to B(H)$ is a nondegenerate completely isometric representation. Then $A^{**} \cong \overline{\pi(A)}^{w^*}$ as dual operator algebras. Also, A^{**} is an* essential extension *of A (that is, every completely contractive linear map $T : A^{**} \to B(H)$ which restricts to a complete isometry on A is a complete isometry), and A^{**} embeds as a unital subalgebra of a C^*-algebra $I(A)$ which is an injective envelope of A.*

Proof. Most of this is essentially in [23], and follows standard ideas (see [7, Section 2.6]), but for completeness we sketch a proof. Define $QM_\pi(A) = \{T \in B(H) : \pi(A)T\pi(A) \subset \pi(A)\}$. It is easy to see using [7, Lemma 2.1.6] that $\pi(A)T\pi(A) = (0)$ implies that $T = 0$. The canonical weak* continuous representation $\tilde{\pi} : A^{**} \to \overline{\pi(A)}^{w^*}$ maps into $QM_\pi(A)$, since $\pi(a)\tilde{\pi}(\eta)\pi(b) = \tilde{\pi}(a\eta b) \in \pi(A)$. Clearly $\tilde{\pi}$ is one-to-one, since $\tilde{\pi}(\eta) = 0$ implies $a\eta b = 0$ for all $a, b \in A$, so that $\eta = 0$. In fact $\tilde{\pi}(A^{**}) = QM_\pi(A)$. To see this suppose that $T \in QM_\pi(A)$, $\|T\| \leq 1$. Suppose that $\pi(e_t)T\pi(e_s) = \pi(a_{t,s})$ for each s, t. For fixed s the net $(a_{t,s})$ has a subnet converging to $\eta_s \in \mathrm{Ball}(A^{**})$. Suppose that η is a weak* limit point for (η_s) in $\mathrm{Ball}(A^{**})$. Then

$$\pi(a)\tilde{\pi}(\eta)\pi(b) = \lim_{\mu} \pi(a)\tilde{\pi}(\eta_{s_\mu})\pi(b), \qquad a, b \in A,$$

where this limit and the ones below are weak* limits. However, if $s = s_\mu$ is fixed, there is a net (t_ν) such that

$$\begin{aligned}\pi(a)\tilde{\pi}(\eta_s)\pi(b) &= \lim_{\nu} \pi(a)\tilde{\pi}(a_{t_\nu,s}\pi(b) = \pi(a)\pi(e_{t_\nu})T\pi(e_s)\pi(b) \\ &= \pi(a)T\pi(e_s)\pi(b).\end{aligned}$$

Therefore

$$\pi(a)\tilde{\pi}(\eta)\pi(b) = \lim_{\mu} \pi(a)\tilde{\pi}(\eta_{s_\mu})\pi(b) = \lim_{\mu} \pi(a)T\pi(e_{s_\mu})\pi(b) = \pi(a)T\pi(b).$$

So $\tilde{\pi}(\eta) = T$. Thus $\tilde{\pi}$ is isometric, hence its range is weak* closed, hence $QM_\pi(A) = \overline{\pi(A)}^{w^*}$. Once we know that $\tilde{\pi}$ is isometric, applying this in the setting of $M_n(A^{**}) \cong M_n(A)^{**}$ shows that $\tilde{\pi}$ is completely isometric.

Suppose that $z \in \mathrm{Ball}(QM(A))$. By the main theorem in [24] (see also the quicker proof of [8, Theorem 5.2]), z corresponds to a unique element $w \in \mathrm{Ball}(I(A))$ such that $awb = azb$ for all $a, b \in A$. This defines a contractive one-to-one unital map $\rho : QM(A) \to I(A)$ which extends the identity map on A. If this w has norm κ then $\|e_t z e_s\| \le \kappa$ for each s, t, so that $\|z\| \le \kappa$. Hence $\rho : A^{**} \to I(A)$ is a unital isometry. By the usual trick (using the isometry applied on $M_n(A^{**}) = M_n(A)^{**}$, and the fact that $I(M_n(A)) = M_n(I(A))$ by 4.2.10 in [7]), ρ is a complete isometry. Since $I(A)$ is an essential extension of A, we deduce that A^{**} is an essential extension of A. Now suppose that $I(A^{**})$ is an injective envelope of A^{**} containing A^{**} as a unital subalgebra. Any complete contraction on $I(A^{**})$ which restricts to a complete isometry on A, must be a complete isometry on A^{**} by the last part, hence is a complete isometry on $I(A^{**})$ by rigidity. So $I(A^{**})$ is rigid for A, hence is an injective envelope of A with the desired property. ∎

Remark. The bulk of the first paragraph of the last proof shows that $\tilde{\pi}$ is completely isometric. However this follows immediately from the second paragraph (the fact there that A^{**} is an essential extension). Nonetheless we felt it worthwhile to include a more elementary argument.

Theorem 5.10. *Let A be an operator algebra which is a HSA in its bidual.*

1. *A is an* Asplund space (*that is, A^* has the RNP*). *Also, A^* has no proper subspace that norms A.*
2. *A^{**} is a* rigid extension *of A in the Banach space category* (*that is, there is only one contractive linear map from A^{**} to itself extending I_A*).
3. *Every surjective linear complete isometry $A^{**} \to A^{**}$ is weak* continuous.*
4. *There is a unique completely contractive extension $\tilde{\pi} : A^{**} \to B(H)$ of any nondegenerate completely contractive representation $\pi : A \to B(H)$, namely the canonical weak* continuous extension. In particular, every character of A^{**} is weak* continuous.*

Proof. (1) and (2) follow from [6, Theorem 2.10], which says that such A is 'Hahn–Banach smooth', and well-known properties of 'Hahn–Banach smooth' spaces due to Godefroy and coauthors, and others (see e.g. [20]).

(3) This follows from the Remark at the end of Section 5 in [3].

(4) In fact this is true even if π is a linear complete contraction with $\pi(e_t) \to I_H$ weak*. Note that if $\tilde{\pi}$ is a completely contractive extension of π, then for any unit vector $\zeta \in H$, $\langle \tilde{\pi}(\cdot)\zeta, \zeta\rangle$ is the unique (and hence necessarily weak* continuous) extension from [6, Theorem 2.10] of the state $\langle \tilde{\pi}(\cdot)\zeta, \zeta\rangle$. Thus $\langle \tilde{\pi}(1)\zeta, \zeta\rangle = 1$. Hence $\tilde{\pi}(1) = I$, and we may now appeal to [6, Proposition 2.11]. ∎

Remark. Being an Asplund space is hereditary, so any closed subalgebra C of an operator algebra A which is an Asplund space has $\Delta(C)$ an Asplund space. But this does not imply that $\Delta(C)$ is an annihilator C^*-algebra (a C^*-algebra which is an Asplund space need not be annihilator, certainly $C_0(K)^*$ may be a separable ℓ^1 space without K being discrete: consider K the one-point compactification of $\mathbb{N}$).

As in [34, Proposition 3.14] we obtain:

Corollary 5.11. *If an operator algebra A is a HSA in its bidual, and if A is not reflexive, then it contains a copy of c_0. Similarly, every approximately unital subalgebra of A, and every quotient algebra of A which is not reflexive, contains a copy of c_0.*

The following is a variant of the 'Wedderburn Theorem' for operator algebras from [3]:

Corollary 5.12. *A separable operator algebra A is σ-matricial in the sense of* [3] *iff A is semiprime, a HSA in its bidual, and every HSA D in A with* $\dim(D) > 1$ *contains a nonzero projection which is not an identity for D.*

Proof. Follows from [3, Theorem 4.23(vii)] together with Theorem 5.8. ∎

The property of being nc-discrete also passes to subalgebras (and to quotients by closed ideals having cai):

Lemma 5.13. *Let A be a nc-discrete operator algebra with cai. If B is a closed subalgebra of A with a cai, then B is nc-discrete. If I is a closed ideal in A, and if I has a cai, then A/I is nc-discrete.*

Proof. If A is nc-discrete, a subalgebra of a C^*-algebra B, and D is a closed approximately unital subalgebra of A, with p an open projection in B^{**} which lies in $D^{\perp\perp}$, then $p \in A^{\perp\perp}$, so $p \in M(A)$. Then $pd \in D^{\perp\perp} \cap A = D$ for all $d \in D$. Similarly $dp \in D$, so $p \in M(D)$. Thus D is nc-discrete.

Next, suppose that A is nc-discrete, and that I is an approximately unital ideal in A. If B is any C^*-algebra generated by A, then the C^*-algebra generated in B by I is an ideal J in B [11, Lemma 2.4]. Let $q^\perp$ be the (central) support projection of J, which equals the support projection of I.

We identify $(A/I)^{**}$ and $A^{**}/I^{\perp\perp} = A^{**}/A^{**}(1-q) = A^{**}q$; and similarly for B/J. Then

$$A/I \subset A^{**}/I^{\perp\perp} \cong A^{**}q \subset B^{**}q.$$

The map $A/I \to B/J$ is a completely isometric embedding, since its composition with the 'canonical inclusion' $B/J \subset B^{**}q$ is the complete isometry in the displayed equation above. An open projection e in $(A/I)^{**}$ which is open with respect to A/I can thus be identified with a projection $p \in B^{**}q$ such that there exists a net $(x_t) \subset A$ with $x_t q \to p$ weak*, and $qx_t = px_t$ for all t. By hypothesis, q is open with respect to A, so that there is a net $(y_s) \subset A$ with $y_s \to q$ weak* and $qy_s = y_s$ for all s. Then $x_t y_s = x_t q y_s \to pq = p$, and $px_t y_s = qx_t y_s = x_t y_s q = x_t y_s$. It follows that p is open in A^{**}. Thus $p \in M(A)$. Since $paq = ap \in A \cap Aq$ for any $a \in A$, it is clear that $e \in M(A/I)$. ∎

Proposition 5.14. *If A is a nc-discrete approximately unital operator algebra, and is an integral domain, then A has no nontrivial r-ideals.*

Proof. This is clear: the support projection p of any r-ideal is in $M(A)$, as is $p^{\perp}$, and $Ap^{\perp}pA = (0)$. ∎

Any uniform algebra in the sense of e.g. [35] is a commutative operator algebra.

Proposition 5.15. *A closed ideal with cai in a uniform algebra which is nc-discrete, is isometrically isomorphic to $c_0(I)$ for some set I.*

Proof. If A is a nc-discrete ideal with cai in a uniform algebra, which we can take to be A^1, then by Lemma 5.13 and Proposition 5.5, $\Delta(A)$ is a commutative annihilator C^*-algebra. Thus $\Delta(A)$ is densely spanned by its minimal projections. If f is the sup of these minimal projections in $\Delta(A)^{**}$, then obviously f is open with respect to $\Delta(A)$, hence open with respect to A, and therefore it is also closed and is in $M(A)$ (since A is nc-discrete). Then $J = Af^{\perp}$ is an ideal with cai in a uniform algebra and J possesses no projections (for these would have to be in $\Delta(A)$). On the other hand, if $f \neq 1$ then J has proper closed 1-regular ideals by Theorem 3.3 in [3], and since J is nc-discrete (by Lemma 5.13) it contains nontrivial projections by Proposition 3.5 in [3]. So $f = 1$.

If e is a minimal projection in $\Delta(A)$, then eA is a uniform algebra containing no nontrivial projections, hence containing no proper nonzero closed ideals with cai (or else the support projection f for such an ideal would satisfy $f = fe \in A$, contradicting minimality of e). However every nontrivial uniform algebra contains proper closed ideals with cai (for example those associated with Choquet boundary points). Thus $eA = \mathbb{C}e$. Hence 1_{A^1} is the sum of a family $\{e_i : i \in I\}$ of mutually orthogonal algebraically minimal projections in A, and so $A \cong c_0(I)$. ∎

In [3] it is conjectured that C^*-algebras are exactly the operator algebras satisfying conditions of the type: every closed left ideal has a right cai. The following is a complement to Theorem 5.1 of [3]. The hypotheses can be weakened further, we just state a simple representative form of the result:

PROPOSITION 5.16. *A semisimple operator algebra A which is nc-discrete, and such that every right ideal in A has a left cai, is an annihilator C^*-algebra.*

Proof. It is obvious that A is a left annihilator algebra. Thus A has dense socle (see e.g. [29, Chapter 8]). The rest is as in Theorem 5.1 of [3]. ∎

6. Examples of operator algebras that are ideals in their biduals. In this section we list several examples answering natural questions that arise when investigating some of the topics of this paper.

6.1. A reflexive semisimple operator algebra. A unitization of some operator algebra structure on ℓ^2 with pointwise product will be a unital reflexive commutative semisimple noncompact operator algebra. One such operator algebra structure on ℓ^2 may be explicitly represented as follows. Identify $\mathbb{N}$ with two disjoint copies of $\mathbb{N}$, and consider the span of $E_{1k} + E_{kk}$, with k in the second copy of $\mathbb{N}$, and 1 here from the first copy. This example is not a modular annihilator algebra (because the canonical maximal ideal has no nonzero annihilator), but it is a Duncan modular annihilator algebra in the sense of [29, Chapter 8]. This example has no nontrivial r-ideals, since it is reflexive and has no nontrivial projections. Its spectrum is the one point compactification of $\mathbb{N}$, which is a scattered topological space.

6.2. A nc-discrete semisimple operator algebra which is not a HSA in its bidual. It is known that ℓ^1 with pointwise product is isomorphic to an operator algebra A say (actually this may be done in many ways, see e.g. [7, Chapter 5]), and it is semisimple. Let A^1 be the unitization of A. Then A^1 is commutative, unital, semisimple, and it is not an ideal in its bidual, since it is unital but not reflexive. We claim that the total number of orthogonal projections in A^1 is finite. To see this, let e_j be the minimal idempotents in A coming from the canonical basis for ℓ^1. If $p = \lambda 1 + a$ is a projection in A^1 then λ is either 1 or 0, so either p or $1 - p$ is in A. Also, the only projections in A are finite sums of some of the e_j. The sup of a finite family of orthogonal projections is an orthogonal projection, so if there were infinitely many distinct orthogonal projections in A then there would be arbitrarily large finite sums of the e_j represented in the operator algebra as norm 1 projections. This is impossible, because in ℓ^1 the norm of a sum of n of the e_j is n. So we have just finitely many orthogonal projections in A, and the orthogonal projections in A^1 are these projections and their complements. Depending on the choice of representation, any finite ring of projections of ℓ^1 can be the ring of orthogonal projections in A (by basic similarity theory such a finite family of idempotents are simultaneously similar to orthogonal projections); and the r-ideals of A^1 include the unital ideals pA and $(1-p)A^1$ for each projection p in the chosen ring.

We claim that the just mentioned ideals are the only r-ideals of A^1, so that A^1 is nc-discrete. To see this, suppose that J is an r-ideal.

Case 1: $x+1 \notin J$ for all $x \in A$. In this case, J is an ideal in A. Setting $E = \{j \in \mathbb{N} : e_j \in J\}$, it is easy to see that $J = J_E$, where J_E consists of the members of A with 'jth coordinate' zero for all $j \notin E$. This is isomorphic to $\ell^1(E)$. If E is finite then J is finite-dimensional, hence $J = eA^1$ for a projection $e \in A^1$ as desired. However if E is infinite then $\ell^1(E)$ with pointwise product, or equivalently ℓ^1 or A, cannot have a bai. Indeed if A had a bai, then $A \subset B(A)$ via the regular representation, and then the argument at the start of [3, Section 4] gives the contradiction that $(\sum_{j=1}^n e_j)$ is uniformly bounded.

Case 2: $x+1 \in J$ for some $x \in A$. By the argument above, $J \cap A = J_E$ for some set $E \subset \mathbb{N}$. If $y+1 \in J$ for some $y \in A$, then $x-y \in J_E$. It follows that $J = \mathbb{C}(1+x) + J_E$. If $j \notin E$ then since $e_j + xe_j \in J$ we must have $xe_j = -e_j$. This can happen for at most a finite number of j; that is, $\mathbb{N} \setminus E$ is finite. Let q be the sum of the e_j for $j \in \mathbb{N} \setminus E$. Then $x+q \in J_E \subset J$, so that $1-q = 1+x-(x+q) \in J$. Since $q(1+x) = 0$ we see that J has an identity $1-q$, which necessarily has norm 1 since J is approximately unital.

Note that A^1 is not a modular annihilator algebra since A has no annihilator, but it is a Duncan modular annihilator algebra in the sense of [29, Section 8.6].

The remainder of our examples are commutative and radical operator algebras. In this connection we remark that there are quite a number of papers on commutative radical operator algebras in the literature, but most of these algebras are not approximately unital. See for example [16] and [31]. Indeed in [31] and several related papers by Wogen, Larson, and others, one aim is to study an operator T in terms of the norm closed algebra $\mathrm{oa}(T)$ generated by T, particularly in cases where the latter algebra is radical. The following is one of the best studied examples:

6.3. The Volterra operator and a subquotient of the disk algebra. Let V be the Volterra operator on $[0,1]$. Let A_V be the norm closed algebra generated by V. We may write $A_V = \mathrm{oa}(T)$ for an operator T with $\|I-T\| \leq 1$. Indeed let $T = I-(I+V)^{-1}$ (it is well-known that the norm of $(I+V)^{-1}$ is 1 and its spectrum is $\{1\}$). We have $\mathrm{oa}(T) \subset A_V$, and the converse inclusion holds since $V = (I-T)^{-1}T$. This algebra has been studied extensively, for example in [31] and [14, Corollary 5.11]. It is commutative, approximately unital, compact, radical, and is an ideal in its bidual. Indeed, $M(A_V) = V' \cong A_V^{**}$ [14, Corollary 5.11]. As we said in [9, Section 5], it has no r-ideals; indeed all the closed ideals in this algebra are known. The algebra A_V is nc-discrete but not semiprime, in fact it has a dense ideal consisting of nilpotent elements.

Jean Esterle suggested to the second author in 2009 to look at the example of $D = B/[gB]$ where B is the ideal of functions in the disk algebra which vanish at the point 1, and $g(z) = \exp((z+1)/(z-1))$. Although g is not in the disk algebra, it is well-known from the theory of inner functions that $gB \subset B$, and that gB is a closed proper ideal in B (see top of p. 84 in [22]). By [7, Proposition 2.3.4], D is a commutative operator algebra, and it has a cai since B does. In fact it turns out that D is completely isometrically isomorphic to the algebra A_V above generated by the Volterra operator. See e.g. [31], where it is pointed out that this leads to mutual insight into the operator theory both in A_V and its commutant, and the function theory on the disk associated with an interesting class of ideals of the disk algebra. For example we see from this that D is compact as a Banach algebra, a fact that seems difficult to prove by direct computations in D.

6.4. Weighted convolution algebras which are ideals in their bidual. In this section we consider operator algebras formed from weighted convolution algebras $L^1(\mathbb{R}_+, \omega)$. By a *weight* we will mean a measurable function $\omega : \mathbb{R}_+ = [0, \infty) \to (0, \infty)$ with $\omega(0) = 1$ which is submultiplicative in the sense that $\omega(s+t) \leq \omega(s)\omega(t)$ for all $s, t \geq 0$. Then for $1 \leq p < \infty$, the set $L^p(\omega) = L^p(\mathbb{R}_+, \omega)$ of equivalence classes of measurable functions $f : \mathbb{R}_+ \to \mathbb{C}$ such that $\|f\|_p = (\int_0^\infty |f(t)|^p \omega(t)^p \, dt)^{1/p} < \infty$ is a Banach space with norm $\|f\|_p$. As in [13, Section 4.7], $L^1(\omega)$ with convolution product is a Banach algebra, and it is radical iff $\lim_{t\to\infty} \omega(t)^{1/t} = 0$. Otherwise it is semisimple. We write $R_x f$ for the right translation of f by x; in other notation this is $\delta_x * f$. Similarly, we write $L_x f$ for the left translation of f by x. As in [9, Section 5], convolution induces a contractive homomorphism $f \mapsto M_f$ from $L^1(\omega)$ into $B(L^2(\omega))$, and we define $\mathcal{A} = \mathcal{A}(\omega)$ to be the norm closure of the set of operators M_f for $f \in L^1(\omega)$. This is an operator algebra. We write $\|\cdot\|_{\rm op}$ for the operator norm on $\mathcal{A}(\omega)$ or more generally in $B(L^2(\omega))$. Whenever we refer below to 'the operator norm' it is this one.

It is known that $1/\omega$ is bounded on compact intervals, so the arguments in [9, Corollary 5.3] work to show that $\mathcal{A}(\omega)$ is an integral domain, and in particular is semiprime, and is not an annihilator algebra. Being an integral domain, it has no nontrivial idempotents, hence has zero socle. If it is radical then it is a modular annihilator algebra in the sense of [29], by [29, Theorem 8.7.2]. If ω is right continuous at 0 then there is a nonnegative cai for $L^1(\omega)$, and hence for $\mathcal{A}(\omega)$, consisting of constant multiples of characteristic functions of a sequence of compact intervals shrinking to 0 (by e.g. [13, 4.7.41]). In this case, since $L^1(\omega) \cap L^2(\omega)$ is dense in $L^2(\omega)$, it is clear that $\mathcal{A}(\omega)$ acts nondegenerately on $L^2(\omega)$. Hence if also $\mathcal{A}(\omega)$ is an ideal in its bidual then $\mathcal{A}(\omega)^{**}$ may be identified with the weak* closure of $\mathcal{A}(\omega)$ in $B(L^2(\omega))$ by Lemma 5.9, and $\mathcal{A}(\omega)$ possesses no nontrivial r-ideals by Proposition 5.14 (and $\mathcal{A}(\omega)^{**}$ contains no nontrivial projections). We remark in passing that [5, Theorem 2.2] states that if $L^1(\omega)$ contains a nonzero compact element then ω is radical.

Lemma 6.1. *If a weight $\omega : [0,\infty) \to (0,\infty)$ is right continuous at 0, and is* not regulated *at some $x > 0$ (that is, if $\omega(x+t)/\omega(t) \nrightarrow 0$), then $\mathcal{A}(\omega)$ is not compact.*

Proof. Since $\limsup_{t\to\infty} \omega(x+t)/\omega(t) > 0$, there exist an $\epsilon > 0$ and an unbounded increasing sequence of numbers (a_n) such that $\omega(x+a_n)/\omega(a_n) > \epsilon$. Since ω is continuous at 0, it is bounded near 0. By submultiplicativity of ω it is bounded on any compact interval. If $y \in [0,x]$ then $\omega(x+a_n) \le \omega(y+a_n)\omega(x-y)$, and it follows that (changing ϵ if necessary)

$$\frac{1}{\epsilon} \le \frac{\omega(y+a_n)}{\omega(a_n)} \le \epsilon, \qquad y \in [0,x]. \tag{6.1}$$

If f is a nonzero nonnegative C^∞ function supported on $[0,\frac{x}{3}]$, we claim that multiplication by f is not compact on $\mathcal{A}(\omega)$. To see this define $f_n = \frac{1}{\omega(a_n)} R_{a_n} f$, which is supported on $[a_n, a_n + x/3]$. We have

$$\|f_n\|_{L^1(\omega)} = \int_0^{x/3} |f(t)| \frac{\omega(t+a_n)}{\omega(a_n)}\, dt \le \int_0^{x/3} |f(t)|\omega(t)\, dt < \infty,$$

and $\|f_n\|_{L^1(\omega)} \in [\|f\|_{L^1(\mathbb{R}_+)}/\epsilon, \epsilon\|f\|_{L^1(\mathbb{R}_+)}]$ by (6.1). Similarly

$$\|f_n\|^2_{L^2(\omega)} = \int_0^{x/3} |f(t)|^2\, \frac{\omega(t+a_n)^2}{\omega(a_n)^2}\, dt \in [\|f\|^2_{L^2(\mathbb{R}_+)}/\epsilon^2, \epsilon^2\|f\|^2_{L^2(\mathbb{R}_+)}],$$

and also $\|f * f_n\|_{L^2(\omega)} \ge \|f * f\|^2_{L^2(\mathbb{R}_+)}/\epsilon$. Since $f * f_n$ is supported on $[a_n,\infty)$, this sequence converges weakly to zero in $L^2(\omega)$. Thus $(f * f_n)$ does not have a norm convergent subsequence in $L^2(\omega)$. Similarly, $(f * f_n * f)$ does not have a norm convergent subsequence in $L^2(\omega)$, and so $(f * f_n)$ does not have a norm convergent subsequence in $\mathcal{A}(\omega)$. Yet (f_n) is a norm bounded sequence in $L^1(\omega)$ and hence in $\mathcal{A}(\omega)$. ■

Corollary 6.2. *If ω is any weight and if $\mathcal{A}(\omega)$ is compact, then $L^1(\omega)$ is compact.*

Proof. By Lemma 6.1 we deduce that ω is regulated at all $x > 0$. Now apply [5, Theorem 2.7]. ∎

REMARK. The converse of Corollary 6.2 is false: $L^1(\omega)$ may be compact without $\mathcal{A}(\omega)$ being compact. This follows from Theorem 6.4 below and [5, Theorem 2.9].

We say that a radical weight ω satisfies *Domar's criterion* if the function $\eta(t) = -\log \omega(t)$ is a convex function on $(0, \infty)$, and for some $\epsilon > 0$ we have $\eta(t)/t^{1+\epsilon} \to \infty$ as $t \to \infty$. An obvious example of such a weight is $\omega(t) = e^{-t^2}$. In [9, Section 5] we studied $\mathcal{A}(\omega)$ in the case that ω satisfies Domar's criterion.

PROPOSITION 6.3. *If $\omega : [0, \infty) \to (0, \infty)$ is a radical weight satisfying Domar's criterion, then $\mathcal{A}(\omega)$ is compact.*

Proof. The collection of compact operators on $\mathcal{A}$ is closed, hence it suffices to show that $g \mapsto h * g$ is compact on $\mathcal{A}$ for $h \in L^1(\omega)$. Since the embedding of $L^1(\mathbb{R}_+)$ in $\mathcal{A}$ is continuous, we may assume further that h is bounded (since simple functions are dense in $L^1(\omega)$), and h has support contained in $[\epsilon, N]$ say, for a fixed $\epsilon > 0$. Indeed as $L^1(\omega)$ is an approximately unital Banach algebra, the convolution is continuous and $L^1(\omega) * L^1(\omega) = L^1(\omega)$; hence we may assume that $h = f * g$ for bounded $f, g \in L^1(\omega)$ both with support contained in $[\epsilon, N]$. Clearly such g belong to $L^2(\omega)$ since g and ω are bounded, and so $a * g \in L^2(\omega)$ for $a \in \mathrm{Ball}(\mathcal{A})$, and $\|a * g\|_{L^2(\omega)} \leq \|a\|_{\mathcal{A}} \|g\|_{L^2(\omega)}$. By [9, Corollary 5.6], if $\delta_{\epsilon/2} * a * g$ is a shift of $a * g$ to the right by $\epsilon/2$, then this is in $L^1(\omega)$, with norm there dominated by $\|a * g\|_{L^2(\omega)} \leq C \|g\|_{L^2(\omega)}$ for a constant C. On the other hand, if f_2 is a shift of f to the left by $\epsilon/2$, then on any compact subinterval of $(0, \infty)$ the function $|f_2|\omega$ is dominated by a constant times a left shift of $|f|\omega$, since ω is continuous. As f has compact support it follows that $f_2 \in L^1(\omega)$.

Let $(x_n) \subset \mathrm{Ball}(\mathcal{A})$. Then $x_n * g \in L^2(\omega)$, and $(\delta_{\epsilon/2} * (x_n * g))$ is a bounded sequence in $L^1(\omega)$, by the inequality involving C in the last paragraph. Since the latter algebra is compact by [5], there is a convergent subsequence of $(f_2 * (\delta_{\epsilon/2} * (x_n * g)))$ in $L^1(\omega)$, and hence in $\mathcal{A}$. However $f_2 * (\delta_{\epsilon/2} * (x_n * g)) = f * (x_n * g) = h * x_n$. Thus multiplication by h is compact on $\mathcal{A}$ as desired. ∎

Henceforth we take the weight ω to be a "staircase weight", namely on each interval $[n, n+1)$ we assume that ω is a constant $1/a_n$, where (a_n) is a strictly and rapidly increasing sequence of positive integers with $a_0 = 1$. So long as $a_{n+m} \geq a_n a_m$ for $n, m \in \mathbb{N}$ then ω is a weight function, and $L^1(\omega)$ and $\mathcal{A}(\omega)$ are commutative Banach algebras with cai (see e.g. [5, 13]). It seems to be quite difficult to find examples of commutative Banach algebras (which are not reflexive in the Banach space sense) which are weakly compact (hence ideals in their biduals by Lemma 5.1) but not compact. In fact this is mentioned as an open problem in [36]. The following gives a commutative approximately unital operator algebra with this property.

THEOREM 6.4. *Let (ϵ_n) be a decreasing null sequence, and let (a_n) be a sequence chosen so that*

$$a_0 = 1, \qquad a_n \geq \max\left\{ \frac{2^k}{\epsilon_{n-k}} a_k \, a_{n-k} : k = 1, \ldots, n-1 \right\}. \tag{6.2}$$

Then the operator algebra $\mathcal{A}(\omega)$ for the associated staircase weight ω is not compact, but is weakly compact, and hence is an ideal in its bidual. Also, $\mathcal{A}(\omega)$ is a commutative approximately unital radical operator algebra which is not reflexive in the Banach space sense, and which is topologically singly generated.

Clearly such a staircase weight ω is not regulated at numbers in $(0,1)$, and hence $\mathcal{A}(\omega)$ is not compact by Lemma 6.1. To show that it is weakly compact, by the Eberlein–Šmulian theorem it suffices to show that if $F \in \mathcal{A}$ and (G_n) is a norm bounded sequence in $\mathcal{A}$, then (FG_n) has a weakly convergent subsequence in $B(L^2(\omega))$. Since the weakly compact operators are closed, it is enough to show this in the case that $F = \pi(f)$ and $G_n = \pi(g_n)$ for f, g_n continuous functions of compact support on $\mathbb{R}_+$ (since such functions are dense in $L^1(\omega)$ by the usual arguments, and hence in $\mathcal{A}$). Here (G_n) is uniformly bounded in the operator norm, and so has a weak* convergent subsequence. By passing to this subsequence we may assume that $G_n \to G$ weak*, and $FG_n \to FG$ weak*. We will show that a subsequence FG_{k_n} tends to FG weakly. Set $H_n = G_n - G$. We may, and will henceforth, assume that $\|H_n\|_{\mathrm{op}} \leq 1$ for all n.

Let P_n denote the orthogonal projection onto the functions in $L^2(\omega)$ supported on $[0,n]$, with $P_0 = 0$, and set $\Delta_n = P_{n+1} - P_n$. We have $P_n\pi(g) = P_n\pi(g)P_n$ for each n and $g \in L^1(\omega)$, hence also $P_n u = P_n u P_n$ for u in $\mathcal{A}$ or in $\bar{\mathcal{A}}^{w^*}$.

LEMMA 6.5. *Assume the hypotheses and notation of Theorem* 6.4 *and the discussion below it, and fix $m \in \mathbb{N}$.*

(1) *Left multiplying by $P_m\pi(g) = P_m\pi(g)P_m$ on $\mathcal{A}(\omega)$ is a compact operator on $\mathcal{A}(\omega)$ for each $g \in L^1(\omega)$.*
(2) *There exists $k_1 < k_2 < \cdots$ with $\|P_m F H_{k_n}\|_{\mathrm{op}} \to 0$ as $n \to \infty$.*

Proof. (1) Since the compact operators are closed it suffices to prove (1) for a dense set of $g \in L^1(\omega)$, as in Proposition 6.3, and as in that proof we may assume that $g = g_1 * g_2$ for bounded g_1, g_2 with support in $[\epsilon, N]$ where $0 < \epsilon < N$. We may also assume that g_1, g_2 are continuous, by the same idea.

For functions supported on $[0,m]$ the $L^1(\omega)$ norm is equivalent to the usual L^1 norm. We view $L^1([0,m]) \subset L^2([0,1])$ as a subspace of $L^1(\omega) \cap L^2(\omega)$ in this way. Suppose that (a_n) is a bounded sequence in $\mathcal{A}$. Then $(g_2 * a_n)$ is a bounded sequence in $L^2(\omega)$, and so if $b_n = P_m(g_2 * a_n)$ then (b_n) is a bounded sequence in $L^1([0,m])$. We then note that left multiplying by $P_m\pi(g_1)$ may be viewed as the operator $T : L^1([0,m]) \to C([0,m]) : h \mapsto P_m(g_1 * h)$. Indeed T takes $\mathrm{Ball}(L^1([0,m]))$ into a uniformly bounded and equicontinuous subset of $C([0,m])$. By the Arzelà–Ascoli theorem, T is compact. Now

$$T(b_n) = P_m(g_1 * P_m(g_2 * a_n)) = P_m(g_1 * g_2 * a_n) = P_m(g * a_n).$$

As T is compact and (b_n) bounded, there will be a subsequence $(P_m(g*a_{k_n}))$ that converges to a function in $C([0,m])$ in the uniform norm. Since the uniform norm on $C([0,m])$ dominates the L^1 norm, which is equivalent to the norm on $L^1(\omega)$, which dominates the operator norm, we deduce that $(P_m(g * a_{k_n}))$ converges in the operator norm too.

(2) By (1), (P_mFG_n) has a norm convergent subsequence, and the limit must be P_mFG since $FG_n \to FG$ weak*. The last assertion is now obvious. ∎

LEMMA 6.6. *Under the hypotheses of Theorem* 6.4, *if* $u \in \overline{\mathcal{A}(\omega)}^{w^*}$ *and* $r \in \mathbb{N}$, *then*

$$\Big\|u(I-P_r) - \sum_{m=r}^{\infty} \varDelta_m u \varDelta_m\Big\| \le \epsilon_r\|u\|.$$

Further, $\varDelta_m u \varDelta_m$ *on* $\mathrm{Ran}(\varDelta_m)$ *is unitarily equivalent to* P_1uP_1 *on* $\mathrm{Ran}(P_1)$, *and hence* $\lim_{r\to\infty} \|u(I-P_r)\| = \|P_1uP_1\|$. *All norms here are the operator norm.*

Proof. To establish the inequality, we recall that $P_nu = P_nuP_n$, and so

$$u(I-P_r) = \sum_{m=r}^{\infty} u\varDelta_m = \sum_{m=r}^{\infty} \varDelta_m u \varDelta_m + \sum_{m=r}^{\infty}\sum_{k=1}^{\infty} \varDelta_{m+k} u \varDelta_m, \tag{6.3}$$

where these sums converge weak*. We next estimate $\|\varDelta_{m+k}u\varDelta_m\|$. If $\eta \in \mathrm{Ran}(\varDelta_m)$ then η is supported on $[m, m+1]$, and so $\|\eta\|_{L^2(\omega)} = a_m^{-1}\|\eta\|_{L^2}$. Then $\varDelta_{m+k}u\eta$ is supported on $[m+k, m+k+1]$, and so $\|\varDelta_{m+k}u\eta\|_{L^2(\omega)} = a_{m+k}^{-1}\|\varDelta_{m+k}u\eta\|_{L^2}$. Hence $\|\varDelta_{m+k}u\varDelta_m\|$ is a_m/a_{m+k} times the norm of $\varDelta_{m+k}u\varDelta_m$ as an operator on $L^2(\mathbb{R}_+)$. However the last norm equals the norm of $\varDelta_k u\varDelta_0$ as an operator on $L^2(\mathbb{R}_+)$, since these operators are unitarily equivalent on their supports (since it is easy to check that $L_m\varDelta_{m+k}u\varDelta_mR_m = \varDelta_k u\varDelta_0$, and the right shift R_m by m is an isometry with adjoint L_m). By the norm identity we just established for $\|\varDelta_{m+k}u\varDelta_m\|$ in the case $m=0$, we deduce that

$$\|\varDelta_{m+k}u\varDelta_m\| = \frac{a_m}{a_{m+k}}\,\frac{a_k}{a_0}\|\varDelta_k u\varDelta_0\| \le \frac{a_m a_k}{a_{m+k}}\|u\|.$$

For varying m, the operators $\Delta_{m+k}u\Delta_m$ have mutually orthogonal left and right supports, and so

$$\Big\|\sum_{m=r}^{\infty}\Delta_{m+k}u\Delta_m\Big\| = \sup_{m\geq r}\|\Delta_{m+k}u\Delta_m\| \leq \|u\|\sup_{m\geq r}\frac{a_m a_k}{a_{m+k}} \leq \|u\|\frac{\epsilon_r}{2^k},$$

the last inequality following from (6.2). Hence

$$\sum_{k=1}^{\infty}\Big\|\sum_{m=r}^{\infty}\Delta_{m+k}u\Delta_m\Big\| \leq \|u\|\epsilon_r.$$

By straightforward operator theory, by the observation above about mutually orthogonal left and right supports, we can interchange the double summation in (6.3) to obtain

$$\Big\|u(I-P_r) - \sum_{m=r}^{\infty}\Delta_m u\Delta_m\Big\| \leq \sum_{k=1}^{\infty}\Big\|\sum_{m=r}^{\infty}\Delta_{m+k}u\Delta_m\Big\| \leq \|u\|\epsilon_r,$$

as desired.

Define $Jg(t) = a_m g(t-m)$ if $m \leq t \leq m+1$, and $Jg(t) = 0$ otherwise. Then J is an isometry on $\mathrm{Ran}(P_1)$, with kernel $\mathrm{Ran}(I-P_1) = \mathrm{Ran}(P_1)^{\perp}$. Thus J is a partial isometry, and its final space is clearly $\mathrm{Ran}(\Delta_m)$. The reader can check that $JP_1\pi(f)P_1J^* = \Delta_m\pi(f)\Delta_m$ for $f \in L^1(\omega)$, and the same will be true with $\pi(f)$ replaced by u, for u in $\mathcal{A}$ or $\bar{\mathcal{A}}^{w^*}$. From this it is clear that $\Delta_m u\Delta_m$ is unitarily equivalent to P_1uP_1 as stated. The final assertion of the lemma is obvious since we are adding elements with mutually orthogonal supports. ∎

LEMMA 6.7. *Under the hypotheses of Theorem* 6.4, *and in the notation above, there is a sequence of integers* $1 = b_1 < c_1 < b_2 < c_2 < \cdots$ *such that for each* n,

$$\|(I-P_{b_{n+1}})FH_{c_n}\| < 2^{-n}, \qquad \|P_{b_n}FH_{c_n}\| < 2^{-n}.$$

The norms here are the operator norm.

Proof. We claim that left multiplication by FP_m is a compact operator on $L^2(\omega)$. Since F has compact support there exists an N such that $FP_m = P_NFP_m$. Suppose that (g_n) is a bounded sequence in $L^2(\omega)$. Then (P_mg) is a bounded sequence in $L^1([0,m])$, and by the method in the proof of Lemma 6.5(1) we deduce that (P_NFP_mg) has a subsequence that converges to a function in $C([0,N])$ in the uniform norm. Since the uniform norm on $C([0,N])$ dominates a constant multiple of the $L^2(\omega)$ norm, the subsequence converges in the latter norm. This proves the claim. It follows that $FH_nP_m = H_nFP_m$ is also compact on $L^2(\omega)$ for any fixed n. Thus $(I-P_s)FH_nP_m \to 0$ in the operator norm as $s \to \infty$ for any fixed n, m.

Assume that $1 = b_1 < c_1 < \cdots < b_k$ have already been chosen. By Lemma 6.5(2), a subsequence of $(P_{b_k} F H_n)$ converges in norm to 0. Thus we may choose $c_k > b_k$ with $\|P_{b_k} F H_{c_k}\| < 2^{-k-1}$. Hence $\|P_1 F H_{c_k} P_1\| < 2^{-k-1}$. By Lemma 6.6 we have $\lim_{r\to\infty} \|F H_{c_k}(I - P_r)\| = \|P_1 F H_{c_k} P_1\| < 2^{-n-1}$. Choose a particular r with $\|F H_{c_k}(I - P_r)\| < 2^{-k-1}$. If $s > r$ we have

$$\|(I - P_s) F H_{c_k}\| < 2^{-k-1} + \|(I - P_s) F H_{c_k} P_r\|.$$

We know from the last line of the first paragraph of the present proof that $\|(I - P_s) F H_{c_k} P_r\| \to 0$ as $s \to \infty$, so we may choose $b_{k+1} > c_k$ with $\|(I - P_{b_{k+1}}) F H_{c_k}\| < 2^{-k}$. This completes the inductive step. ∎

Proof of Theorem 6.4. If we choose b_n, c_n as in Lemma 6.7 we have $\|F H_{c_n} + (P_{b_n} - P_{b_{n+1}}) F H_{c_n}\| < 2^{1-n}$ for all n. Thus to show that $F H_{c_n} \to 0$ weakly, which concludes our proof that $\mathcal{A}(\omega)$ is weakly compact, it is enough that $R_n = (P_{b_n} - P_{b_{n+1}}) F H_{c_n} \to 0$ weakly. But this is clear since any bounded family (R_n) of operators on a Hilbert space H with mutually orthogonal ranges converges weakly to 0. Indeed if $\sum_{n=1}^{\infty} R_n R_n^* \le 1$ then $R_n \to 0$ weakly. To see this note that for any state φ on $B(H)$, we have $\sum_{n=1}^{\infty} \varphi(R_n R_n^*) \le 1$. Thus by the Cauchy–Schwarz inequality, $|\varphi(R_n)|^2 \le \varphi(R_n R_n^*) \to 0$.

The assertion about the second dual follows from Lemma 5.1. To see that $\mathcal{A}(\omega)$ is radical, we first note that by mathematical induction on the case $k = n-1$ in (6.2), we have $a_n \ge (a_1^n/\epsilon_1^{n-1}) 2^{1+2+3+\cdots+(n-1)}$. Thus $a_n^{1/n} \to \infty$ and so $\lim_{t\to\infty} \omega(t)^{1/t} = 0$. Finally, $\mathcal{A}(\omega)$ is not reflexive since if it were then it would have an identity of norm 1, which cannot happen for radical algebras. It is topologically singly generated by the constant function 1, by [13, Theorem 4.7.26]. ∎

7. The diagonal of a quotient algebra. We remark that it is easy to see that if A and B are closed subalgebras of $B(H)$ then $\Delta(A \cap B) = \Delta(A) \cap \Delta(B)$.

PROPOSITION 7.1. *If J is an inner ideal in an operator algebra A (i.e. $JAJ \subset J$), then $J \cap \Delta(A) = \Delta(J)$.*

Proof. It is trivial that $\Delta(J)$ is a subalgebra of $J \cap \Delta(A)$. Conversely, if $JAJ \subset J$, then $(J \cap \Delta(A))\Delta(A)(J \cap \Delta(A)) \subset J \cap \Delta(A)$. So $J \cap \Delta(A)$ is a HSA in a C^*-algebra, hence it is selfadjoint. So $x \in J \cap \Delta(A)$ implies that $x^* \in J \cap \Delta(A) \subset J$, and so $x \in \Delta(J)$. ∎

We will use the fact that the diagonal of an ideal J in A is an ideal in $\Delta(A)$ if it is nonzero. Indeed, $\Delta(J) = \Delta(A) \cap J$ and $\Delta(J)\Delta(A) \subset \Delta(A) \cap (JA) \subset \Delta(A) \cap J = \Delta(J)$. Similarly, since J is a two-sided ideal, $\Delta(A)\Delta(J) \subset \Delta(J)$. We sometimes will silently use this fact. However if J is an approximately unital ideal in an approximately unital operator algebra A, then it is not always true that $\Delta(A/J) \cong \Delta(A)/\Delta(J)$. A counterexample is given by the ideal of functions in the disk algebra vanishing at two points on the circle, inside the ideal of functions vanishing at one point. Most of the rest of this section is an attempt to understand this phenomenon, and to give some conditions ensuring that $\Delta(A/J) \cong \Delta(A)/\Delta(J)$.

PROPOSITION 7.2. *Let A be an approximately unital operator algebra and let J be an ideal in A. Then $\Delta(A)/\Delta(J) \subset \Delta(A/J)$.*

Proof. Let $u : A \to A/J$ be the canonical complete quotient map defined as $u(x) = x + J$. The restriction of u to $\Delta(A) \subset A$, u', is a complete contraction. Since u' is a contractive homomorphism, it maps into $\Delta(A/J)$ by 2.1.2 of [7]. Hence, we have a completely contractive map $u' = u|_{\Delta(A)}$: $\Delta(A) \to \Delta(A/J)$, where $\mathrm{Ker}(u') = \Delta(A) \cap J = \Delta(J)$. By the fact mentioned above the proposition, $\Delta(J)$ is an approximately unital ideal in $\Delta(A)$, and we deduce that $\Delta(A)/\Delta(J) \subset \Delta(A/J)$ completely isometrically. ∎

LEMMA 7.3. *Let J be an approximately unital ideal with positive cai in an approximately unital operator algebra A. Then $\Delta(A/J) \cong \Delta(A)/\Delta(J)$ canonically iff every positive element of A/J lifts to an element $b \in A$ such that $b\Delta(J) \subset \Delta(J)$.*

Proof. One direction is obvious since as we said earlier, $\Delta(J)$ is an ideal in $\Delta(A)$.

For the other direction, let p be the support projection of J. The element b in the statement satisfies $b = bp + bp^{\perp}$. Now the canonical isometric homomorphism $A/J \to A^{**}p^{\perp} \subset A^{**}$ takes the diagonal of A/J into $\Delta(A^{**})$. Thus $bp^{\perp} \in \Delta(A^{**})$. If (e_t) is a cai for $\Delta(J)$ then $be_t \in \Delta(J)$, and in the limit we also have $bp \in \Delta(J)^{\perp\perp} \subset \Delta(A^{**})$ (the latter since $\Delta(J) \subset \Delta(A^{**})$). So $b \in \Delta(A^{**}) \cap A = \Delta(A)$. From this the result is evident. ∎

For any operator algebra A, the diagonal $\Delta(A)$ acts nondegenerately on A if and only if A has a positive cai, and if and only if $1_{\Delta(A)^{\perp\perp}} = 1_{A^{**}}$. The latter is equivalent to $1_{A^{**}} \in \Delta(A)^{\perp\perp}$. Hence, we may use the statements '$\Delta(A)$ acts nondegenerately on A' and 'A has a positive cai' interchangeably.

REMARK. If $\Delta(A)$ acts nondegenerately on A, then this does not imply that $\Delta(J)$ acts nondegenerately on J if J is an ideal of A. To see this, take any approximately unital operator algebra J such that $\Delta(J)$ does not act nondegenerately on J. Then J is an ideal in $A = M(J)$, and $\Delta(A)$ acts nondegenerately on A. However, if $\Delta(A)$ acts nondegenerately on A, then $\Delta(A/J)$ acts nondegenerately on A/J. Indeed, it is fairly evident by e.g. 2.1.2 in [7] that if J is an ideal in an operator algebra A, and if A has a positive cai, then A/J has a positive cai.

PROPOSITION 7.4. *Let A be an operator algebra. If J is a left ideal in A with a selfadjoint right cai, then $\Delta(J) = \Delta(A) \cap J$ is a left ideal in $\Delta(A)$, $\Delta(A) + J$ is closed, and $(J \cap \Delta(A))^{\perp\perp} = J^{\perp\perp} \cap \Delta(A)^{\perp\perp}$.*

Proof. The first statement is evident from Proposition 7.1. Write (e_λ) for the selfadjoint right cai of J. If $r \in \Delta(A)$ then $re_\lambda \in \Delta(A) \cap J$. This is what is needed to make the idea in the proof of [17, Proposition 2.4] work, as in the proof of Corollary 4.3, showing that $\Delta(A) + J$ is closed. By the proof of [12, Lemma 5.29 and Appendix A.1.5], $(\Delta(A) \cap J)^{\perp\perp} = (\Delta(A)^{\perp} + J^{\perp})^{\perp} = \Delta(A)^{\perp\perp} \cap J^{\perp\perp}$. ∎

PROPOSITION 7.5. *Let A be an approximately unital operator algebra such that $\Delta(A^{**}) = \Delta(A)^{**}$. If J is an ideal in A that contains a positive cai, then $\Delta(J^{\perp\perp}) = \Delta(J)^{\perp\perp}$ and $\Delta((A/J)^{**}) = (\Delta(A)/\Delta(J))^{**}$.*

Proof. By Proposition 7.4, $\Delta(J)^{\perp\perp} = \Delta(A)^{\perp\perp} \cap J^{\perp\perp}$. Clearly $\Delta(A^{**}) \cap J^{\perp\perp} = \Delta(J^{\perp\perp})$ by Proposition 7.1, and so $\Delta(J^{\perp\perp}) = \Delta(J)^{\perp\perp}$. Let $p \in A^{**}$ be the support projection of J. Then

$$\begin{aligned}\Delta((A/J)^{**}) &= \Delta(A^{**}(1-p)) = \Delta(A^{**})(1-p) = \Delta(A)^{**}(1-p)\\ &= (\Delta(A)/\Delta(J))^{**},\end{aligned}$$

where we have used Remark 2.10(ii) of [3]. ∎

COROLLARY 7.6. *Let A be an operator algebra with a positive cai that is a HSA in its bidual. If J is an ideal in A that possesses a positive cai, then $\Delta(A/J) = \Delta(A)/\Delta(J)$.*

Proof. Since A is a HSA in its bidual and it contains a positive cai, we have $\Delta(A^{**}) = \Delta(A)^{**}$ by Theorem 5.7. Also, A/J is a HSA in its bidual $(A/J)^{**}$ by Lemma 5.2, and it can easily be seen by e.g. 2.1.2 in [7] to have a positive cai. Hence $\Delta((A/J)^{**}) = \Delta(A/J)^{**}$ by Theorem 5.7. Moreover, since A/J is a HSA in its bidual, $\Delta(A/J)$ is an annihilator C^*-algebra. We know by Proposition 7.2 that $\Delta(A)/\Delta(J) \subset \Delta(A/J)$ completely isometrically. Hence $\Delta(A)/\Delta(J)$ is an annihilator C^*-algebra as well. Using Proposition 7.5, we have $(\Delta(A)/\Delta(J))^{**} = \Delta((A/J)^{**}) = \Delta(A/J)^{**}$. Hence, $\Delta(A)/\Delta(J) = \Delta(A/J)$. ∎

We end with another result on the diagonal related to Corollary 4.3:

PROPOSITION 7.7. *If A is an approximately unital operator algebra, if D is a HSA in A and if J is an approximately unital ideal in A, then $\Delta(D \cap J)^{\perp\perp} = \Delta(D)^{\perp\perp} \cap \Delta(J)^{\perp\perp}$. If D and J have positive cais, then $D \cap J$ has a positive cai as well.*

Proof. A modification of the proof of Proposition 7.4 or Corollary 4.3 shows that $\Delta(D \cap J)^{\perp\perp} = \Delta(D)^{\perp\perp} \cap \Delta(J)^{\perp\perp}$.

If D and J have positive cais, then

$$1_{(D\cap J)^{\perp\perp}} = 1_{D^{\perp\perp}\cap J^{\perp\perp}} = 1_{D^{\perp\perp}}1_{J^{\perp\perp}} \in \varDelta(D)^{\perp\perp} \cap \varDelta(J)^{\perp\perp} = \varDelta(D\cap J)^{\perp\perp}.$$

That is, $1_{(D\cap J)^{\perp\perp}} \in \varDelta(D\cap J)^{\perp\perp}$. Hence, $D\cap J$ has a positive cai. ■

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Some of the topics in the earlier parts of this paper and in Section 7 were investigated in the first author's Ph.D. thesis [2]. Some related results and topics may be found there too. Others of our results related to HSA's were presented at various venues in 2010.

References

[1] C. A. Akemann, *Left ideal structure of C^*-algebras*, J. Funct. Anal. 6 (1970), 305–317.
[2] M. Almus, *Structure in operator algebras*, Ph.D. thesis, Univ. of Houston, 2011.
[3] M. Almus, D. P. Blecher, and S. Sharma, *Ideals and structure of operator algebras*, J. Operator Theory 67 (2012), 397–436.
[4] W. B. Arveson, *Subalgebras of C^*-algebras*, Acta Math. 123 (1969), 141–224.
[5] W. G. Bade and H. G. Dales, *Norms and ideals in radical convolution algebras*, J. Funct. Anal. 41 (1981), 77–109.
[6] D. P. Blecher, D. M. Hay, and M. Neal, *Hereditary subalgebras of operator algebras*, J. Operator Theory 59 (2008), 333–357.
[7] D. P. Blecher and C. Le Merdy, *Operator Algebras and Their Modules—An Operator Space Approach*, Oxford Univ. Press, Oxford, 2004.
[8] D. P. Blecher and M. Neal, *Metric characterizations II*, Illinois J. Math., to appear.
[9] D. P. Blecher and C. J. Read, *Operator algebras with contractive approximate identities*, J. Funct. Anal. 261 (2011), 188–217.
[10] D. P. Blecher and C. J. Read, *Operator algebras with contractive approximate identities II*, J. Funct. Anal., to appear.
[11] D. P. Blecher and M. K. Royce, *Extensions of operator algebras I*, J. Math. Anal. Appl. 339 (2008), 1451–1467.
[12] D. P. Blecher and V. Zarikian, *The calculus of one-sided M-ideals and multipliers in operator spaces*, Mem. Amer. Math. Soc. 179 (2006), no. 842.
[13] H. G. Dales, *Banach Algebras and Automatic Continuity*, London Math. Soc. Monogr. 24, Clarendon Press, New York, 2000.
[14] K. R. Davidson, *Nest Algebras. Triangular Forms for Operator Algebras on Hilbert Space*, Pitman Res. Notes Math. 191, Longman, Harlow, 1988.
[15] M. Daws and C. Read, *Semisimplicity of $\mathcal{B}(E)''$*, J. Funct. Anal. 219 (2005), 177–204.
[16] P. G. Dixon, *Radical Q-algebras*, Glasgow Math. J. 17 (1976), 119–126.
[17] P. G. Dixon, *Non-closed sums of closed ideals in Banach algebras*, Proc. Amer. Math. Soc. 128 (2000), 3647–3654.
[18] J. Duncan, *$B^{\#}$ modular annihilator algebras*, Proc. Edinburgh Math. Soc. 15 (1966), 89–102.
[19] F. Ghahramani, *Weighted group algebra as an ideal in its second dual space*, Proc. Amer. Math. Soc. 90 (1984), 71–76.

[20] G. Godefroy and P. D. Saphar, *Duality in spaces of operators and smooth norms on Banach spaces*, Illinois J. Math. 32 (1988), 672–695.
[21] P. Harmand, D. Werner, and W. Werner, *M-ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Math. 1547, Springer, Berlin, 1993.
[22] K. Hoffman, *Banach Spaces of Analytic Functions*, Dover, New York, 1988.
[23] M. Kaneda, *Quasi-multipliers and algebrizations of an operator space*, J. Funct. Anal. 251 (2007), 346–359.
[24] M. Kaneda and V. I. Paulsen, *Quasi-multipliers of operator spaces*, J. Funct. Anal. 217 (2004), 347–365.
[25] K. B. Laursen and M. M. Neumann, *An Introduction to Local Spectral Theory*, London Math. Soc. Monogr. (N.S.) 20, Clarendon Press, New York, 2000.
[26] R. J. Loy, C. J. Read, V. Runde, and G. A. Willis, *Amenable and weakly amenable Banach algebras with compact multiplication*, J. Funct. Anal. 171 (2000), 78–114.
[27] G. S. Mustafaev, *On the discreteness of the structure space of weakly completely continuous Banach algebras*, Ukraïn. Mat. Zh. 56 (2004), 411–418 (in Russian); English transl.: Ukrainian Math. J. 56 (2004), 504–511.
[28] T. Oikhberg and E. Spinu, *Domination of operators in the noncommutative setting*, preprint, 2012.
[29] T. W. Palmer, *Banach Algebras and the General Theory of ∗-Algebras, Vol. I. Algebras and Banach Algebras*, Encyclopedia Math. Appl. 49, Cambridge Univ. Press, Cambridge, 1994.
[30] G. K. Pedersen, *C^*-algebras and Their Automorphism Groups*, Academic Press, London, 1979.
[31] J. R. Peters and W. R. Wogen, *Commutative radical operator algebras*, J. Operator Theory 42 (1999), 405–424.
[32] C. J. Read, *C^*-algebras with dense nilpotent subalgebras*, Bull. London Math. Soc. 35 (2003), 362–366.
[33] C. J. Read, *On the quest for positivity in operator algebras*, J. Math. Anal. Appl. 381 (2011), 202–214.
[34] S. Sharma, *Operator spaces which are one-sided M-ideals in their bidual*, Studia Math. 196 (2010), 121–141.
[35] E. L. Stout, *The Theory of Uniform Algebras*, Bogden & Quigley, Tarrytown-on-Hudson, NY, 1971.
[36] A. Ülger, *Some results about the spectrum of commutative Banach algebras under the weak topology and applications*, Monatsh. Math. 121 (1996), 353–379.

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