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A condition equivalent to uniform ergodicity

by

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Abstract. Let T be a linear operator on a Banach space X with $\sup_n \|T^n/n^w\| < \infty$ for some $0 \le w < 1$. We show that the following conditions are equivalent: (i) $n^{-1}\sum_{k=0}^{n-1} T^k$ converges uniformly; (ii) $\operatorname{cl}(I-T)X = \{z\in X : \lim_n \sum_{k=1}^{n} T^k z/k \text{ exists}\}$.

1. Introduction. Let X be a Banach space and let $\mathcal{L}(X)$ denote the Banach algebra of bounded linear operators from X to itself. An operator $T \in \mathcal{L}(X)$ is called *uniformly ergodic* if the averages

$$A_n(T) = \frac{1}{n}\sum_{k=0}^{n-1} T^k$$

converge in the uniform operator topology.

M. Lin [5] showed that when $\lim_{n\to\infty} \|T^n/n\| = 0$, T is uniformly ergodic if and only if $(I-T)X$ is closed. From this it is easy to see that a power-bounded T (that is, $\sup_n \|T^n\| < \infty$) is uniformly ergodic if and only if $\{z \in X : \sup_n \|\sum_{k=0}^{n} T^k z\| < \infty\}$ is closed.

V. Fonf, M. Lin and A. Rubinov [2] proved that if X is separable and does not contain an isomorphic copy of an infinite-dimensional dual Banach space, then the uniform ergodicity of a power-bounded T is equivalent to

$$(I-T)X = \Big\{z \in X : \sup_n \Big\|\sum_{k=0}^{n} T^k z\Big\| < \infty\Big\}.$$

In [3], S. Grabiner and J. Zemánek give the following generalization of Lin's theorem. Under the hypothesis of boundedness of $A_n(T)$ or convergence to zero of T^n/n in some operator topology, they prove that if $(I-T)^n X$ is closed for some $n \ge 2$ ($n \ge 1$ if T^n/n converges to zero in the uniform operator topology) or if $(I-T)X + \operatorname{Ker}(I-T)$ is closed for some $n \ge 1$, then X is the direct sum of the closed subspaces $(I-T)X$ and $\operatorname{Ker}(I-T)$. In this case the sequence $A_n(T)$ converges in some operator topology if and only if T^n/n converges to zero in the same operator topology.

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Very recently, E. Ed-dari [1] obtained an improvement of a result of T. Yoshimoto [6] about the uniform ergodic theorem with Cesàro means of order α. Ed-dari proved that for every $\alpha > 0$ the sequence

$$M_n^\alpha(T) = \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1}\, T^k,$$

where A_n^α, $n = 0, 1, \ldots$, are the (C, α) coefficients of order α, converges in the uniform operator topology to an operator $E \in \mathcal{L}(X)$ if and only if

$$\|(\lambda - 1)R(\lambda, T) - E\| \to 0 \quad \text{as } \lambda \to 1^+ \quad \text{and} \quad \lim_{n\to\infty} \frac{\|T^n\|}{n^\alpha} = 0.$$

Let $T \in \mathcal{L}(X)$ be such that $\sup_n \|T^n/n^w\| < \infty$ for some $0 \le w < 1$. We shall denote $\{z \in X : \lim_{n\to\infty} \sum_{k=1}^n T^k z/k \text{ exists}\}$ by X_1. In this paper we study the relationship between X_1 and $(I - T)X$, and prove that $X_1 = \operatorname{cl}(I - T)X$ is equivalent to the uniform ergodicity of T.

2. Results. Throughout this section, T is a linear operator in $\mathcal{L}(X)$ with $\sup_n \|T^n/n^w\| < \infty$ for some $0 \le w < 1$.

LEMMA. $(I - T)X \subset X_1 \subset \operatorname{cl}(I - T)X$. *Therefore X_1 is closed if and only if $X_1 = \operatorname{cl}(I - T)X$.*

Proof. Let $z \in (I - T)X$. Then $z = (I - T)x$. From

$$\Bigl\| \sum_{k=n+1}^{n+p} \frac{T^k z}{k} \Bigr\| = \Bigl\| \frac{T^{n+1}x}{n+1} - \frac{T^{n+p+1}x}{n+p} - \sum_{k=n+2}^{n+p} \frac{T^k x}{k(k-1)} \Bigr\|$$

and the boundedness of $\|T^n/n^w\|$, we see that $(\sum_{k=1}^n T^k z/k)_n$ is a Cauchy sequence, and thus $z \in X_1$.

Now, let $z \in X_1$ and $u \in \operatorname{Ker}(I - T^*)$, where T^* is the adjoint operator of T. Let z_0 denote the limit of $\sum_{k=1}^n T^k z/k$. Then we have

$$\langle u, z_0\rangle = \lim_{n\to\infty} \Bigl\langle u, \sum_{k=1}^{n} \frac{T^k z}{k} \Bigr\rangle = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k}\, \langle u, z\rangle.$$

We conclude that $\langle u, z\rangle = 0$. Hence $z \in \operatorname{cl}(I - T)X$ by the Hahn–Banach theorem. ∎

We can now state our result.

THEOREM. *The following conditions are equivalent*:

(i) *T is uniformly ergodic.*
(ii) *X_1 is closed.*

Proof. (i)⇒(ii) by Lin's theorem [5] and the previous lemma.

(ii)⇒(i). By the lemma, $X_1 = \operatorname{cl}(I - T)X$ and therefore X_1 is invariant under T. Let S be the restriction of T to X_1. We define $S_n = \sum_{k=1}^n S^k/k$,

$n \geq 1$. By the principle of uniform boundedness, there exists a constant $K > 0$ such that $\sup_n \|S_n\| \leq K$. For each positive integer n we define on X_1 the operator B_n by

$$B_n = \frac{1}{a_n} S_n, \quad \text{where} \quad a_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Then B_n converges to 0 as $n \to \infty$ in the uniform operator topology.

Put $A_n = n^{-1} \sum_{k=1}^{n} S^k$. Making use of the partial summation formula of Abel, we obtain

$$A_n = \frac{n+1}{n} S_n - \frac{1}{n} \sum_{k=1}^{n} S_k.$$

Thus there exists a constant $C > 0$ such that $\sup_n \|A_n\| \leq C$.

Since $\|S^n\| \leq \|T^n\|$, we also have

$$\lim_{n\to\infty} \|A_n S^j - A_n\| = 0 \quad \text{for any } j \geq 0. \tag{1}$$

Now, for any given $\varepsilon > 0$, choose an integer k so large that $\|B_k\| < \varepsilon$. By (1) we may find an N such that $\|A_n S^j - A_n\| < \varepsilon$ holds for $n \geq N$ and $j = 1, \ldots, k$. As B_k is a convex combination of S^j, $1 \leq j \leq k$, for $n \geq N$ we therefore obtain $\|A_n B_k - A_n\| < \varepsilon$. Hence

$$\|A_n\| \leq \|A_n - A_n B_k\| + \|A_n B_k\| < \varepsilon(1 + C).$$

Consequently, S is uniformly ergodic. Fix n such that $\|A_n\| < 1$. Then $I - A_n$ is invertible, and hence so is $I - S$. Therefore

$$\mathrm{cl}\,(I - T)X = X_1 = (I - S)X_1 \subset (I - T)X.$$

Thus, $(I - T)X$ is closed and we may apply Lin's theorem [5] again to conclude that T is uniformly ergodic. This completes the proof of the theorem. ■

Remark 1. For T power-bounded, the uniform ergodicity of S in the proof of the theorem follows from Krengel [4, p. 88].

Remark 2. Inspecting the above proofs we see that for an operator T which satisfies $\lim_{n\to\infty} \|T^n/n\| = 0$, we just used the condition $\sup_n \|T^n/n^w\| < \infty$ to prove $(I-T)X \subset X_1$. Therefore the theorem is also valid for an operator $T \in \mathcal{L}(X)$ satisfying $(I-T)X \subset X_1$ together with $\lim_{n\to\infty} \|T^n/n\| = 0$.

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