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An $\mathfrak{M}_q(\mathbb{T})$-functional calculus for power-bounded operators on certain UMD spaces

by

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Abstract. For $1 \le q < \infty$, let $\mathfrak{M}_q(\mathbb{T})$ denote the Banach algebra consisting of the bounded complex-valued functions on the unit circle having uniformly bounded q-variation on the dyadic arcs. We describe a broad class $\mathcal{I}$ of UMD spaces such that whenever $X \in \mathcal{I}$, the sequence space $\ell^2(\mathbb{Z}, X)$ admits the classes $\mathfrak{M}_q(\mathbb{T})$ as Fourier multipliers, for an appropriate range of values of $q > 1$ (the range of q depending on X). This multiplier result expands the vector-valued Marcinkiewicz Multiplier Theorem in the direction $q > 1$. Moreover, when taken in conjunction with vector-valued transference, this $\mathfrak{M}_q(\mathbb{T})$-multiplier result shows that if $X \in \mathcal{I}$, and U is an invertible power-bounded operator on X, then U has an $\mathfrak{M}_q(\mathbb{T})$-functional calculus for an appropriate range of values of $q > 1$. The class $\mathcal{I}$ includes, in particular, all closed subspaces of the von Neumann–Schatten p-classes $\mathcal{C}_p$ $(1 < p < \infty)$, as well as all closed subspaces of any UMD lattice of functions on a σ-finite measure space. The $\mathfrak{M}_q(\mathbb{T})$-functional calculus result for $\mathcal{I}$, when specialized to the setting of closed subspaces of $L^p(\mu)$ (μ an arbitrary measure, $1 < p < \infty$), recovers a previous result of the authors.

1. Introduction and notation. Throughout what follows, the symbols $\mathbb{R}$, $\mathbb{C}$, $\mathbb{T}$, $\mathbb{N}$, and $\mathbb{Z}$, respectively, will stand for the real line, the complex plane, the unit circle in $\mathbb{C}$, the set of positive integers, and the additive group of all integers. The following notion will be central to our considerations.

DEFINITION 1.1. If $[a, b]$ is a compact interval of $\mathbb{R}$, and $1 \le q < \infty$, the *q-variation* of a function ψ mapping $[a, b]$ into $\mathbb{C}$ is defined by putting

$$\operatorname{var}_q(\psi, [a, b]) = \sup \Big\{ \sum_{k=1}^{N} |\psi(x_k) - \psi(x_{k-1})|^q \Big\}^{1/q},$$

where the supremum is extended over all partitions $a = x_0 < x_1 < \cdots < x_N = b$ of $[a, b]$.

We denote by $V_q([a, b])$ the class of all functions $\psi : [a, b] \to \mathbb{C}$ such that $\operatorname{var}_q(\psi, [a, b]) < \infty$. It is straightforward to see that $V_q([a, b])$ is a Banach

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algebra under pointwise operations and the norm

$$\|\psi\|_{V_q([a,b])} \equiv \sup_{x\in[a,b]} |\psi(x)| + \mathrm{var}_q(\psi, [a,b]).$$

If $\psi \in V_q([a,b])$, then $\lim_{x\to y^+} \psi(x)$ exists for each $y \in [a,b)$, $\lim_{x\to y^-} \psi(x)$ exists for each $y \in (a,b]$, and the set of discontinuities of ψ in $[a,b]$ is countable. Note that $\mathrm{var}_1(\psi, [a,b])$ is identical to the usual total variation $\mathrm{var}(\psi, [a,b])$, and so the Banach algebras $V_1([a,b])$ and $\mathrm{BV}([a,b])$ coincide (with identical norms).

Recall that the dyadic points relevant to the study of 2π-periodic functions are the terms of the sequence $\{s_k\}_{k=-\infty}^{\infty} \subseteq (0, 2\pi)$ given by

$$s_k = \begin{cases} 2^{k-1}\pi & \text{if } k \le 0, \\ 2\pi - 2^{-k}\pi & \text{if } k > 0. \end{cases} \tag{1.1}$$

The corresponding dyadic arcs of $\mathbb{T}$, $\{\Delta_k\}_{k=-\infty}^{\infty}$, are specified by

$$\Delta_k = \{e^{ix} : x \in [s_k, s_{k+1}]\}.$$

Our main theme will be the multiplier actions on vector-valued functions of the *Marcinkiewicz q-classes* $\mathfrak{M}_q(\mathbb{T})$, $1 \le q < \infty$, which are described as follows. Given a function $\phi : \mathbb{T} \to \mathbb{C}$ and $k \in \mathbb{Z}$, we write $\mathrm{var}_q(\phi, \Delta_k)$ to stand for

$$\mathrm{var}_q(\phi(e^{i(\cdot)}), [s_k, s_{k+1}]).$$

For $1 \le q < \infty$, $\mathfrak{M}_q(\mathbb{T})$ is defined as the class of all functions $\phi : \mathbb{T} \to \mathbb{C}$ such that

$$\|\phi\|_{\mathfrak{M}_q(\mathbb{T})} \equiv \sup_{z\in\mathbb{T}} |\phi(z)| + \sup_{k\in\mathbb{Z}} \mathrm{var}_q(\phi, \Delta_k) < \infty.$$

It is easily seen that $\mathfrak{M}_q(\mathbb{T})$ is a Banach algebra under pointwise operations and the norm $\|\cdot\|_{\mathfrak{M}_q(\mathbb{T})}$. For $1 \le r \le q < \infty$, $\mathfrak{M}_r(\mathbb{T}) \subseteq \mathfrak{M}_q(\mathbb{T})$, since $\|\cdot\|_{\mathfrak{M}_q(\mathbb{T})} \le \|\cdot\|_{\mathfrak{M}_r(\mathbb{T})}$. Note that $\mathfrak{M}_1(\mathbb{T})$ is the usual class of Marcinkiewicz multipliers. For key properties of the notion of q-variation and of the Marcinkiewicz q-classes, we refer the reader to [7], [9], [12], and [19]. In particular, when $1 < p < \infty$, both the unweighted and the A_p-weighted ℓ^p-spaces of bilateral complex-valued sequences admit the classes $\mathfrak{M}_q(\mathbb{T})$ as Fourier multipliers for corresponding appropriate ranges of q (Corollary 4.12 of [7], Theorem 9 of [12]). (In the framework of the unweighted as well as of the A_p-weighted Lebesgue spaces of complex-valued functions on $\mathbb{T}$ and $\mathbb{R}$, corresponding multiplier results hold for the relevant notions of the Marcinkiewicz q-classes—see [7], [12], [13], [19].) We remark in passing that, in contrast to $\mathfrak{M}_1(\mathbb{T})$, each function class $\mathfrak{M}_q(\mathbb{T})$, $1 < q < \infty$, contains continuous, nowhere differentiable functions of Weierstrass type (see 1.33 on p. 303 of [23], Theorem (17.7) of [24]).

The natural venue for seeking to extend classical multiplier theorems to Lebesgue spaces of vector-valued functions is the class of UMD spaces, which are characterized as the Banach spaces X such that the Hilbert transform for $\mathbb{R}$, $\mathbb{T}$, or $\mathbb{Z}$ acts as a bounded convolution operator on the corresponding spaces L^p_X when $1 < p < \infty$. (For the salient features of UMD spaces, see, e.g., [14], [16], [18], [29].) In particular, for an arbitrary UMD space X, the classical forms of the Marcinkiewicz Multiplier Theorem (wherein $q = 1$) have been extended to L^p_X, $1 < p < \infty$ (see Theorem 4 of [17] and Theorem (4.5) of [6]). Our first goal is to expand this vector-valued multiplier result beyond $q = 1$ by identifying a broad class $\mathcal{I}$ of UMD spaces so that whenever $X \in \mathcal{I}$, the sequence space $\ell^2(\mathbb{Z}, X)$ admits the classes $\mathfrak{M}_q(\mathbb{T})$ as Fourier multipliers, for an appropriate range of values $q > 1$ (depending on X). More specifically, this class $\mathcal{I}$ is described as follows. If a Hilbert space $\mathfrak{X}_0$ and a UMD space $\mathfrak{X}_1$ constitute a compatible couple for Calderón's complex method of interpolation, we shall denote the corresponding scale of interpolation spaces by $[\mathfrak{X}_0, \mathfrak{X}_1]_t$, $0 \le t \le 1$. By definition, the class $\mathcal{I}$ will consist of all Banach spaces X such that

(1.2) X is isomorphic (in the Banach space sense) to a subspace of $[\mathfrak{X}_0, \mathfrak{X}_1]_t$, for some Hilbert space $\mathfrak{X}_0$, some UMD space $\mathfrak{X}_1$, and some t in the open interval $(0, 1)$.

Clearly $X \in \mathcal{I}$ implies that X is UMD, and that all closed subspaces of X belong to $\mathcal{I}$.

EXAMPLES 1.2. (i) From the standard interpolation properties of L^p-spaces (Theorem 5.1.1 of [1]), it is obvious that for an arbitrary measure μ, and $1 < p < \infty$, any closed subspace of $L^p(\mu)$ belongs to $\mathcal{I}$.

(ii) The Corollary of Theorem 4 in [29] establishes that every UMD lattice of measurable functions on a σ-finite measure space is an intermediate space $[\mathfrak{X}_0, \mathfrak{X}_1]_t$, as described in (1.2), and hence belongs to $\mathcal{I}$. Since a Banach space with an unconditional basis can be equivalently renormed so as to become a Banach lattice by identifying each vector with the coefficient sequence of its expansion, any UMD space with an unconditional basis belongs to $\mathcal{I}$.

(iii) In a direction away from lattices, consider the von Neumann–Schatten classes $\mathcal{C}_p$ for the Hilbert space $\ell^2(\mathbb{N})$. The class $\mathcal{C}_2$, consisting of the Hilbert–Schmidt operators, is itself a Hilbert space, and, as is well known, $\mathcal{C}_p$ is UMD for $1 < p < \infty$ (for the UMD property of $\mathcal{C}_p$ via the Cotlar "bootstrap" method, see, e.g., III.6 of [21], IV.4 of [22]). It is also well known (see, e.g., Proposition 8 on p. 44 of [28]) that any of the spaces $\mathcal{C}_p$, $1 < p < \infty$, is a strictly intermediate interpolation space between $\mathcal{C}_2$ and some $\mathcal{C}_r$, where $1 < r < \infty$. Consequently, for $1 < p < \infty$, $\mathcal{C}_p \in \mathcal{I}$. (However, in contrast to example (ii), for $1 < p < \infty$, $p \neq 2$, $\mathcal{C}_p$ is not isomorphic to a subspace of a UMD lattice, by virtue of Theorem 2.1 in [27].)

(iv) Problem 4 in §III.d of [29] poses the question of whether every UMD space is an intermediate space $[\mathfrak{X}_0, \mathfrak{X}_1]_t$ as described in (1.2). An affirmative answer would, of course, imply that the class UMD coincides with $\mathcal{I}$. However, it appears that this question remains open.

After having established (in Theorem 2.7) the $\mathfrak{M}_q(\mathbb{T})$-multiplier result for $\ell^2(\mathbb{Z}, X)$ ($X \in \mathcal{I}$, and $q > 1$ in a suitable range depending only on X), we apply it, via vector-valued transference ([15]), to an arbitrary invertible power-bounded operator U on X. This procedure yields, for the same values of q, an $\mathfrak{M}_q(\mathbb{T})$-functional calculus for U (Theorem 3.8). The outcome extends to $\mathcal{I}$ the $\mathfrak{M}_q(\mathbb{T})$-functional calculus result in Theorem 4.10 of [7] for $L^p(\mu)$-subspaces (μ an arbitrary measure, $1 < p < \infty$), and also expands in the direction $q > 1$ the $\mathfrak{M}_1(\mathbb{T})$-functional calculus result for arbitrary UMD spaces in Theorem (1.1)-(ii) of [6].

Throughout all that follows, the symbol K with a (possibly empty) set of subscripts will stand for a constant which depends only on those subscripts, and which may change in value from one occurrence to another. For a given Banach space Y, we shall denote by $\mathfrak{B}(Y)$ the Banach algebra of all bounded linear mappings of Y into Y. The identity operator of Y will be symbolized by I. The Fourier transform (respectively, inverse Fourier transform) of a function f will be written as $\widehat{f}$ (respectively, $f^\vee$).

2. The multiplier theorem for $\mathfrak{M}_q(\mathbb{T})$ when $X \in \mathcal{I}$. In treating Fourier multipliers for Lebesgue spaces of vector-valued functions, we shall follow the presentation of this topic in §3 of [6]. If G is a locally compact abelian group with dual group Γ, Y is a Banach space, and $1 \le p < \infty$, then the algebra of Fourier multipliers for $L^p(G, Y)$ (denoted by $M_{p,Y}(\Gamma)$) consists of all bounded, measurable, complex-valued functions ϕ on Γ such that the mapping T_ϕ, initially defined on $\{L^1(G) \cap L^\infty(G)\} \otimes Y$ by putting

$$T_\phi\Big(\sum_{j=1}^N f_j y_j\Big) = \sum_{j=1}^N (\phi\, \widehat{f_j})^\vee y_j,$$

extends to an element of $\mathfrak{B}(L^p(G, Y))$ (also denoted by T_ϕ, and called the *multiplier transform on $L^p(G, Y)$ corresponding to ϕ*). In this case, we define the multiplier norm $\|\phi\|_{M_{p,Y}(\Gamma)}$ of ϕ to be $\|T_\phi\|_{\mathfrak{B}(L^p(G,Y))}$. Regularization of Fourier multipliers for $L^p(G, Y)$ is furnished by Proposition A in §3 of [6]: if $k \in L^1(\Gamma)$ and $\phi \in M_{p,Y}(\Gamma)$, then the convolution $k * \phi$ belongs to $M_{p,Y}(\Gamma)$, and

$$\|k * \phi\|_{M_{p,Y}(\Gamma)} \le \|k\|_{L^1(\Gamma)} \, \|\phi\|_{M_{p,Y}(\Gamma)}. \tag{2.1}$$

Notice that if Z is a closed subspace of Y, then $M_{p,Y}(\Gamma) \subseteq M_{p,Z}(\Gamma)$, and

$$\|\phi\|_{M_{p,Z}(\Gamma)} \le \|\phi\|_{M_{p,Y}(\Gamma)} \quad \text{for all } \phi \in M_{p,Y}(\Gamma). \tag{2.2}$$

It will be convenient to record here, for later use, the following lemma.

LEMMA 2.1. *Suppose G is a locally compact abelian group with dual group Γ, Y is a Banach space, $1 \le p < \infty$, $\{\phi_n\}_{n=1}^{\infty} \subseteq M_{p,Y}(\Gamma)$, and*

$$\sup_{n\in\mathbb{N}} \|\phi_n\|_{M_{p,Y}(\Gamma)} < \infty.$$

If ϕ is a bounded measurable function on Γ such that $\phi_n \to \phi$ a.e. on Γ, then $\phi \in M_{p,Y}(\Gamma)$, and

$$\|\phi\|_{M_{p,Y}(\Gamma)} \le \sup_{n\in\mathbb{N}} \|\phi_n\|_{M_{p,Y}(\Gamma)}.$$

Proof. Apart from the trivial case where $Y = \{0\}$, we have, for each $k \in \mathbb{N}$,

$$|\phi_k| \le \sup_{n\in\mathbb{N}} \|\phi_n\|_{M_{p,Y}(\Gamma)} \quad \text{locally a.e. on } \Gamma. \tag{2.3}$$

It follows from our hypothesis of a.e. convergence and (2.3) that if

$$f \in \{L^1(G) \cap L^\infty(G)\} \otimes Y,$$

then $T_{\phi_n} f \to T_\phi f$ in $L^2(G,Y)$. Consequently, we can choose a subsequence $\{T_{\phi_{n_k}} f\}_{k=1}^{\infty}$ convergent to $T_\phi f$ a.e. An application of Fatou's Lemma completes the proof. ∎

For a given Banach space Y, the multipliers for the sequence spaces $\ell^p(\mathbb{Z},Y)$, $1 \le p < \infty$, are easily seen to have the following simple characterization in terms of convolution operators.

PROPOSITION 2.2. *Suppose Y is a Banach space, and $1 \le p < \infty$. Then a bounded measurable function $\psi : \mathbb{T} \to C$ belongs to $M_{p,Y}(\mathbb{T})$ if and only if its inverse Fourier transform $\psi^\vee$ satisfies the following two conditions:*

(i) *For each sequence $x = \{x_j\}_{j=-\infty}^{\infty} \in \ell^p(\mathbb{Z},Y)$, and each $k \in \mathbb{Z}$, the series*

$$(\psi^\vee * x)(k) \equiv \sum_{j=-\infty}^{\infty} \psi^\vee(k-j)\, x_j$$

converges unconditionally in Y.

(ii) *The mapping $S_\psi : x \in \ell^p(\mathbb{Z},Y) \mapsto \psi^\vee * x$ belongs to $\mathfrak{B}(\ell^p(\mathbb{Z},Y))$.*

If this is the case, then S_ψ is the multiplier transform on $\ell^p(\mathbb{Z},Y)$ corresponding to ψ.

In the case of a UMD space Y, the vector-valued Marcinkiewicz Theorem for $\ell^p(\mathbb{Z},Y)$ takes the following form (see Theorem (4.5) of [6]).

Theorem 2.3. *Suppose Y is a UMD space, and $1 < p < \infty$. If $\phi \in \mathfrak{M}_1(\mathbb{T})$, then $\phi \in M_{p,Y}(\mathbb{T})$, and*

$$\|\phi\|_{M_{p,Y}(\mathbb{T})} \le K_{p,Y} \, \|\phi\|_{\mathfrak{M}_1(\mathbb{T})}. \tag{2.4}$$

The following interpolation result sets the stage for showing that if $X \in \mathcal{I}$, then for suitable values of $q > 1$ we have $\mathfrak{M}_q(\mathbb{T}) \subseteq M_{2,X}(\mathbb{T})$.

Theorem 2.4. *Let $F = [\mathfrak{X}_0, \mathfrak{X}_1]_t$, where $\mathfrak{X}_0$ is a Hilbert space, $\mathfrak{X}_1$ is a UMD space, and $0 < t < 1$. Then for each $\phi \in \mathfrak{M}_1(\mathbb{T})$,*

$$\|\phi\|_{M_{2,F}(\mathbb{T})} \le K_{\mathfrak{X}_1}^t \,(\sup_{z\in\mathbb{T}} |\phi(z)|)^{1-t} \, \|\phi\|_{\mathfrak{M}_1(\mathbb{T})}^t. \tag{2.5}$$

Proof. By Theorem 5.1.2 of [1], we have, with equal norms,

$$[\ell^2(\mathbb{Z}, \mathfrak{X}_0), \ell^2(\mathbb{Z}, \mathfrak{X}_1)]_t = \ell^2(\mathbb{Z}, F). \tag{2.6}$$

Since $\mathfrak{X}_0$ is a Hilbert space and ϕ is bounded, it follows from Proposition 2.2 and the Parseval formula for $L^2(\mathbb{T}, \mathfrak{X}_0)$ that

$$\|\phi\|_{M_{2,\mathfrak{X}_0}(\mathbb{T})} \le \sup_{z\in\mathbb{T}} |\phi(z)|. \tag{2.7}$$

By Theorem 2.3, $\phi \in M_{2,\mathfrak{X}_1}(\mathbb{T})$, and

$$\|\phi\|_{M_{2,\mathfrak{X}_1}(\mathbb{T})} \le K_{\mathfrak{X}_1} \, \|\phi\|_{\mathfrak{M}_1(\mathbb{T})}. \tag{2.8}$$

Now a complex interpolation based on (2.6)–(2.8) concludes the proof. ■

In order to implement Theorem 2.4, we shall require the following version of Lemma 3 in [25]; a proof for the generality stated here can be obtained by suitable modifications to the argument on pp. 209–210 of [26].

Lemma 2.5. *Let $[a, b]$ be a compact interval in $\mathbb{R}$, and suppose that $1 \le q < \infty$. Then for each complex-valued function f on $[a, b]$ such that $\operatorname{var}_q(f, [a, b]) \le 1$, and for each real number ε such that $0 < \varepsilon < 1$, there is a function $f_\varepsilon \in \mathrm{BV}([a, b])$ such that:*

$$\sup_{x\in[a,b]} |f(x) - f_\varepsilon(x)| \le \varepsilon, \tag{2.9}$$

$$\operatorname{var}(f_\varepsilon, [a, b]) \le 4\varepsilon^{1-q}. \tag{2.10}$$

The function f_ε can be chosen so that

$$f_\varepsilon(a) = f(a), \qquad f_\varepsilon(b) = f(b). \tag{2.11}$$

Theorem 2.6. *Let $F = [\mathfrak{X}_0, \mathfrak{X}_1]_t$, where $\mathfrak{X}_0$ is a Hilbert space, $\mathfrak{X}_1$ is a UMD space, and $0 < t < 1$. Suppose that $1 \le q < 1/t$. Then*

$$\mathfrak{M}_q(\mathbb{T}) \subseteq M_{2,F}(\mathbb{T}).$$

For each $\phi \in \mathfrak{M}_q(\mathbb{T})$,

$$\|\phi\|_{M_{2,F}(\mathbb{T})} \le K_{\mathfrak{X}_1,q,t} \, \|\phi\|_{\mathfrak{M}_q(\mathbb{T})}.$$

Proof. (Compare the method of proof used for the Theorem in [25].) Let $\{s_k\}_{k=-\infty}^{\infty}$ be the sequence of dyadic points of $(0, 2\pi)$ specified in (1.1), and suppose that $1 \le q < 1/t$, $\phi \in \mathfrak{M}_q(\mathbb{T})$, and $\|\phi\|_{\mathfrak{M}_q(\mathbb{T})} \le 1$. Let $n \in \mathbb{N}$, and for each $k \in \mathbb{Z}$, apply Lemma 2.5 (including (2.11)) to the function $\phi(e^{i(\cdot)})$ restricted to the interval $[s_k, s_{k+1}]$, taking $\varepsilon = 2^{-n}$. This procedure allows us to construct a function $\phi_n : \mathbb{T} \to \mathbb{C}$ such that $\phi_n(1) = \phi(1)$, and

$$\sup_{z \in \mathbb{T}} |\phi(z) - \phi_n(z)| \le 2^{-n}, \tag{2.12}$$

$$\sup_{k \in \mathbb{Z}} \operatorname{var}_1(\phi_n, \Delta_k) \le 4(2^{n(q-1)}). \tag{2.13}$$

Now we define the functions ψ_n, $n \in \mathbb{N}$, on $\mathbb{T}$ by putting:

$$\psi_1 = \phi_1, \qquad \psi_n = \phi_n - \phi_{n-1} \qquad \text{for } n \ge 2.$$

It is readily seen from (2.12) and (2.13) that for each $n \in \mathbb{N}$,

$$\sup_{z \in \mathbb{T}} |\psi_n(z)| \le 3 \cdot 2^{-n}, \tag{2.14}$$

$$\|\psi_n\|_{\mathfrak{M}_1(\mathbb{T})} \le K_q 2^{n(q-1)}. \tag{2.15}$$

Moreover, it follows from (2.12) that

$$\sum_{n=1}^{\infty} \psi_n \text{ converges to } \phi \text{ uniformly on } \mathbb{T}. \tag{2.16}$$

Applying Theorem 2.4 to (2.14) and (2.15), we see that for each $n \in \mathbb{N}$,

$$\|\psi_n\|_{M_{2,F}(\mathbb{T})} \le K_{\mathfrak{X}_1,q,t}\, 2^{-n(1-qt)}. \tag{2.17}$$

Since $qt < 1$, (2.17) implies that

$$\sup_{N \in \mathbb{N}} \Big\| \sum_{n=1}^{N} \psi_n \Big\|_{M_{2,F}(\mathbb{T})} \le K_{\mathfrak{X}_1,q,t}. \tag{2.18}$$

In view of (2.16) and (2.18), the desired conclusions (for $\|\phi\|_{\mathfrak{M}_q(\mathbb{T})} \le 1$) now follow from Lemma 2.1. ∎

Our main multiplier theorem is now readily deduced.

THEOREM 2.7. *Suppose that X belongs to the class of Banach spaces $\mathcal{I}$ defined by* (1.2). *Then there is a real number q_0, depending only on X, such that $1 < q_0 < \infty$, and for $1 \le q < q_0$,*

$$\mathfrak{M}_q(\mathbb{T}) \subseteq M_{2,X}(\mathbb{T}).$$

Moreover, if $1 \le q < q_0$, and $\phi \in \mathfrak{M}_q(\mathbb{T})$, then

$$\|\phi\|_{M_{2,X}(\mathbb{T})} \le K_{X,q}\, \|\phi\|_{\mathfrak{M}_q(\mathbb{T})}.$$

Proof. Let $t \in (0,1)$ be as in (1.2), set $q_0 = 1/t$, and use Theorem 2.6 together with (2.2). ∎

3. The $\mathfrak{M}_q(\mathbb{T})$-functional calculus. Let $X \in \mathcal{I}$, and suppose that $U \in \mathfrak{B}(X)$ is an invertible operator which is power-bounded, i.e.,

$$c \equiv \sup_{k \in \mathbb{Z}} \|U^k\| < \infty. \tag{3.1}$$

We shall show that for a suitable range of $q > 1$, U has a norm-continuous $\mathfrak{M}_q(\mathbb{T})$-functional calculus. In order to accomplish this it will be necessary to review beforehand relevant background items concerned with spectral decomposability.

Definition 3.1. A *spectral family* in a Banach space Y is an idempotent-valued function $E(\cdot) : \mathbb{R} \to \mathfrak{B}(Y)$ with the following properties:

(i) $E(\lambda)E(\tau) = E(\tau)E(\lambda) = E(\lambda)$ if $\lambda \le \tau$;
(ii) $\sup\{\|E(\lambda)\| : \lambda \in \mathbb{R}\} < \infty$;
(iii) with respect to the strong operator topology of $\mathfrak{B}(Y)$, $E(\cdot)$ is right continuous and has a left-hand limit $E(\lambda^-)$ at each point $\lambda \in \mathbb{R}$;
(iv) $E(\lambda) \to I$ as $\lambda \to \infty$ and $E(\lambda) \to 0$ as $\lambda \to -\infty$, each limit being with respect to the strong operator topology of $\mathfrak{B}(Y)$.

If, in addition, we have $a, b \in \mathbb{R}$ with $a \le b$, $E(\lambda) = 0$ for $\lambda < a$, and $E(\lambda) = I$ for $\lambda \ge b$, then $E(\cdot)$ is said to be *concentrated on* $[a, b]$.

Given a spectral family $E(\cdot)$ in Y concentrated on a compact interval $J = [a, b]$, an associated theory of spectral integration can be developed as follows. For each bounded function $\varphi : J \to \mathbb{C}$ and each partition $\mathcal{P} = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ of J, where $a = \lambda_0 < \lambda_1 < \cdots < \lambda_n = b$, set

$$\mathcal{S}(\mathcal{P}; \varphi, E) = \sum_{k=1}^{n} \varphi(\lambda_k)\{E(\lambda_k) - E(\lambda_{k-1})\}. \tag{3.2}$$

If the net $\{\mathcal{S}(\mathcal{P}; \varphi, E)\}$ converges in the strong operator topology of $\mathfrak{B}(Y)$ as $\mathcal{P}$ increases with respect to refinement through the set of partitions of J, then the strong limit is called the *spectral integral of* φ with respect to $E(\cdot)$ (over J) and is denoted by $\int_J \varphi(\lambda)\, dE(\lambda)$. In this case, we define $\int_J^{\oplus} \varphi(\lambda)\, dE(\lambda)$ by writing

$$\int_J^{\oplus} \varphi(\lambda)\, dE(\lambda) \equiv \varphi(a)E(a) + \int_J \varphi(\lambda)\, dE(\lambda).$$

The Banach algebra $\mathrm{BV}(J)$ consists of the functions $\varphi : J \to \mathbb{C}$ having bounded variation on J, with norm

$$\|\varphi\|_{\mathrm{BV}(J)} = \sup_{x \in \mathcal{J}} |\varphi(x)| + \mathrm{var}(\varphi, J).$$

It can be shown that the spectral integral $\int_J \varphi(\lambda)\, dE(\lambda)$ exists for each

$\varphi \in \mathrm{BV}(J)$, and that the mapping

$$\varphi \in \mathrm{BV}(J) \mapsto \int_J^{\oplus} \varphi(\lambda)\, dE(\lambda)$$

is an identity-preserving algebra homomorphism of $\mathrm{BV}(J)$ into $\mathfrak{B}(Y)$ satisfying

$$\left\| \int_J^{\oplus} \varphi(\lambda)\, dE(\lambda) \right\| \le \|\varphi\|_{\mathrm{BV}(J)} \sup\{\|E(\lambda)\| : \lambda \in \mathbb{R}\}. \tag{3.3}$$

(See [20, Chapter 17], or the simplified account in [5, §2].) We shall also consider the Banach algebra $\mathrm{BV}(\mathbb{T})$, which consists of all $\psi : \mathbb{T} \to \mathbb{C}$ such that the function $\widetilde{\psi}(t) \equiv \psi(e^{it})$ belongs to $\mathrm{BV}([0, 2\pi])$, and which is furnished with the norm $\|\psi\|_{\mathrm{BV}(\mathbb{T})} = \|\widetilde{\psi}\|_{\mathrm{BV}([0,2\pi])}$.

Definition 3.2. Let Y be a Banach space. An operator $V \in \mathfrak{B}(Y)$ is said to be *trigonometrically well-bounded* if there is a spectral family $E(\cdot)$ in Y concentrated on $[0, 2\pi]$ such that $V = \int_{[0,2\pi]}^{\oplus} e^{i\lambda}\, dE(\lambda)$. In this case, it is possible to arrange that $E((2\pi)^-) = I$, and with this additional property the spectral family $E(\cdot)$ is uniquely determined by V, and is called the *spectral decomposition* of V.

The class of trigonometrically well-bounded operators was introduced in [3], and its theory further developed in [4]. Trigonometrically well-bounded operators occur naturally in fundamental structural roles, since every invertible power-bounded operator on a UMD space is trigonometrically well-bounded (Theorem (4.5) of [14]). (In particular, the operator $U \in \mathfrak{B}(X)$ in (3.1) is trigonometrically well-bounded, since X is UMD.) For a variety of natural examples of trigonometrically well-bounded operators which are not power-bounded, see, e.g., §4 of [8]. For some further applications of trigonometrically well-bounded operators to ergodic theory, see [2] and [10]–[12].

The next proposition (Theorem (3.10)-(i) of [5]) provides a vector-valued variant of Fejér's theorem valid for trigonometrically well-bounded operators.

Definition 3.3. Given a function $\psi \in \mathrm{BV}(\mathbb{T})$, we define $\psi^{\#} \in \mathrm{BV}([0, 2\pi])$ by writing

$$\psi^{\#}(t) = \frac{1}{2}\{\lim_{s \to t^+} \psi(e^{is}) + \lim_{s \to t^-} \psi(e^{is})\} \quad \text{for all } t \in [0, 2\pi].$$

Proposition 3.4. *Suppose that Y is a Banach space, and V is a trigonometrically well-bounded operator on Y with spectral decomposition $E(\cdot)$. Then for each $\psi \in \mathrm{BV}(\mathbb{T})$, and each $y \in Y$, we have, in the notation of Definition* 3.3,

$$\Big\| \sum_{j=-n}^{n} \widehat{\kappa}_n(j)\widehat{\psi}(j)V^j y - \int_{[0,2\pi]}^{\oplus} \psi^{\#}(t)\, dE(t)y \Big\| \to 0 \quad \text{as } n \to \infty,$$

where $\{\kappa_n\}_{n=0}^{\infty}$ *is the Fejér kernel for* $\mathbb{T}$:

$$\kappa_n(z) = \sum_{j=-n}^{n} \Big(1 - \frac{|j|}{n+1}\Big) z^j \quad \text{for all } n \ge 0 \text{ and all } z \in \mathbb{T}.$$

The discussion leading up to (3.3) shows that a trigonometrically well-bounded operator V has a norm-continuous $\mathrm{BV}(\mathbb{T})$-functional calculus Ψ_V : $\psi \in \mathrm{BV}(\mathbb{T}) \mapsto \int_{[0,2\pi]}^{\oplus} \psi(e^{it})\, dE(t)$, where $E(\cdot)$ is the spectral decomposition of V. The following theorem (contained in Theorem 14 of [12]) provides a simplifying condition for the existence of values $q > 1$ such that Ψ_V can be extended to a norm-continuous $\mathfrak{M}_q(\mathbb{T})$-functional calculus for V.

THEOREM 3.5. *Suppose that* Y *is a Banach space*, $V \in \mathfrak{B}(Y)$ *is trigonometrically well-bounded*, $E(\cdot)$ *is the spectral decomposition of* V, *and* $1 < \beta < \infty$. *If there is a constant* η *such that* (*with notation as in Definition* 3.3)

$$\Big\| \int_{[0,2\pi]}^{\oplus} \psi^{\#}(t)\, dE(t) \Big\| \le \eta \|\psi\|_{\mathfrak{M}_\beta(\mathbb{T})} \quad \text{for all } \psi \in \mathrm{BV}(\mathbb{T}),$$

then whenever $1 \le q < \beta$, *the integral* $\int_{[0,2\pi]} \phi(e^{it})\, dE(t)$ *exists for each* $\phi \in \mathfrak{M}_q(\mathbb{T})$, *and the mapping* $\phi \in \mathfrak{M}_q(\mathbb{T}) \mapsto \int_{[0,2\pi]}^{\oplus} \phi(e^{it})\, dE(t)$ *is a homomorphism of the Banach algebra* $\mathfrak{M}_q(\mathbb{T})$ *into* $\mathfrak{B}(Y)$ *such that*

$$\Big\| \int_{[0,2\pi]}^{\oplus} \phi(e^{it})\, dE(t) \Big\| \le K\eta \|\phi\|_{\mathfrak{M}_q(\mathbb{T})} \quad \text{for all } \phi \in \mathfrak{M}_q(\mathbb{T}).$$

In order to relate the multiplier result in Theorem 2.7 to the trigonometrically well-bounded operator $U \in \mathfrak{B}(X)$ in (3.1), we shall also require the following vector-valued version of transference (Theorem (2.8) of [15]).

THEOREM 3.6. *Let* $u \mapsto R_u$ *be a strongly continuous representation of a locally compact abelian group* G *in a Banach space* Y *such that*

$$\tau \equiv \sup\{\|R_u\| : u \in G\} < \infty.$$

Let $k \in L^1(G)$, *and let* H_k *denote the bounded linear mapping of* Y *into* Y *defined* (*via Bochner integration with respect to Haar measure* du *on* G) *by*

$$H_k y = \int_G k(u) R_{-u} y\, du \quad \text{for all } y \in Y.$$

Then for $1 \le p < \infty$,

$$\|H_k\| \le \tau^2 N_{p,Y}(k), \tag{3.4}$$

where $N_{p,Y}(k)$ *denotes the norm of convolution by* k *on* $L^p(G, Y)$.

REMARK 3.7. If G is a locally compact abelian group with dual group Γ, Y is a Banach space, $1 \le p < \infty$, and $k \in L^1(G)$, then it is clear that $\widehat{k} \in M_{p,Y}(\Gamma)$, the multiplier transform of $\widehat{k}$ coinciding on $L^p(G, Y)$ with convolution by k. Hence we can replace $N_{p,Y}(k)$ in (3.4) by $\|\widehat{k}\|_{M_{p,Y}(\Gamma)}$.

The stage is now set for establishing the following theorem, which constitutes the $\mathfrak{M}_q(\mathbb{T})$-functional calculus result described at the outset of this section.

THEOREM 3.8. *Suppose that* X *belongs to the class* $\mathcal{I}$ *of Banach spaces defined by* (1.2), *and* $U \in \mathfrak{B}(X)$ *is an invertible operator such that* (3.1) *holds.* (*It follows, in particular, that* U *is trigonometrically well-bounded.*) *Let* $\mathcal{E}(\cdot)$ *denote the spectral decomposition of* U. *Then there is a real number* q_0, *depending only on* X, *satisfying* $1 < q_0 < \infty$, *and such that whenever* $1 \le q < q_0$, *the following assertions are valid*:

(i) *For each* $\phi \in \mathfrak{M}_q(\mathbb{T})$, *the spectral integral* $\int_{[0,2\pi]} \phi(e^{it})\, d\mathcal{E}(t)$ *exists.*

(ii) *The mapping* $\phi \in \mathfrak{M}_q(\mathbb{T}) \mapsto \int_{[0,2\pi]}^{\oplus} \phi(e^{it})\, d\mathcal{E}(t)$ *is a homomorphism of the Banach algebra* $\mathfrak{M}_q(\mathbb{T})$ *into* $\mathfrak{B}(X)$ *such that*

$$\Big\| \int_{[0,2\pi]}^{\oplus} \phi(e^{it})\, d\mathcal{E}(t) \Big\| \le c^2 K_{X,q} \|\phi\|_{\mathfrak{M}_q(\mathbb{T})} \quad \text{for all } \phi \in \mathfrak{M}_q(\mathbb{T}).$$

Proof. Fix the number q_0 furnished by Theorem 2.7. Let $Q : \mathbb{T} \to \mathbb{C}$ be a trigonometric polynomial:

$$Q(z) \equiv \sum_{j=-\infty}^{\infty} \widehat{Q}(j) z^j,$$

where $\widehat{Q}(j) = 0$ for all but finitely many j. In view of (3.1), we can now specialize Theorem 3.6 to the representation R of $\mathbb{Z}$ in X given by

$$R_j = U^j \quad \text{for all } j \in \mathbb{Z},$$

taking $k \in \ell^1(\mathbb{Z})$ to be the sequence $\{Q^\vee(j)\}_{j=-\infty}^{\infty}$. Under these circumstances, the operator H_k appearing in (3.4) is $Q(U)$, and so with the aid of Remark 3.7, we have

$$\|Q(U)\| \le c^2 \|Q\|_{M_{2,X}(\mathbb{T})}. \tag{3.5}$$

If $\psi \in \mathrm{BV}(\mathbb{T})$, and $\{\kappa_n\}_{n=0}^{\infty}$ is the Fejér kernel for $\mathbb{T}$, we infer from (3.5) that for each $n \ge 0$, the trigonometric polynomial $\kappa_n * \psi$ satisfies

$$\Big\| \sum_{j=-n}^{n} \widehat{\kappa}_n(j) \widehat{\psi}(j) U^j \Big\| = \|(\kappa_n * \psi)(U)\| \le c^2 \, \|\kappa_n * \psi\|_{M_{2,X}(\mathbb{T})}. \tag{3.6}$$

We now apply (2.1) and Theorem 2.7 to the right-hand member of (3.6) to get, for $1 \le q < q_0$,

$$\Big\| \sum_{j=-n}^{n} \widehat{\kappa}_n(j)\widehat{\psi}(j)U^j \Big\| \le c^2 K_{X,q} \|\psi\|_{\mathfrak{M}_q(\mathbb{T})}.$$

Using this in Proposition 3.4, we see that for $1 \le q < q_0$ and $\psi \in \mathrm{BV}(\mathbb{T})$,

$$\Big\| \int_{[0,2\pi]}^{\oplus} \psi^{\#}(t)\, d\mathcal{E}(t) \Big\| \le c^2 K_{X,q} \|\psi\|_{\mathfrak{M}_q(\mathbb{T})}. \tag{3.7}$$

The proof is completed by observing that if $1 \le q < q_0$, we can set $\beta = (q+q_0)/2$, apply (3.7) to β in place of q, and then appeal to Theorem 3.5. ∎

REMARK 3.9. If $X \in \mathcal{I}$, and $U_0 \in \mathfrak{B}(X)$ is a trigonometrically well-bounded operator which does not satisfy the power-boundedness assumption described by (3.1), then the conclusions of Theorem 3.8 can fail to hold. Specifically, (5.36) in [6] furnishes the spectral decomposition $E_0(\cdot)$ of a trigonometrically well-bounded operator U_0 on Hilbert space such that for some $\phi \in \mathfrak{M}_1(\mathbb{T})$, the spectral integral $\int_{[0,2\pi]} \phi(e^{it})\, dE_0(t)$ does not exist.

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