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*-Representations, seminorms and structure properties of normed quasi *-algebras

by

Camillo Trapani (Palermo)

Abstract. The class of *-representations of a normed quasi *-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ is investigated, mainly for its relationship with the structure of $(\mathfrak{X}, \mathfrak{A}_0)$. The starting point of this analysis is the construction of GNS-like *-representations of a quasi *-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ defined by invariant positive sesquilinear forms. The family of bounded invariant positive sesquilinear forms defines some seminorms (in some cases, C^*-seminorms) that provide useful information on the structure of $(\mathfrak{X}, \mathfrak{A}_0)$ and on the continuity properties of its *-representations.

1. Introduction. A *quasi *-algebra* is a couple $(\mathfrak{X}, \mathfrak{A}_0)$, where $\mathfrak{X}$ is a vector space with involution *, $\mathfrak{A}_0$ is a *-algebra and a vector subspace of $\mathfrak{X}$, and $\mathfrak{X}$ is an $\mathfrak{A}_0$-bimodule whose module operations and involution extend those of $\mathfrak{A}_0$. This notion was first introduced by G. Lassner in the early 80's of the last century ([13, 14], see also [17]).

The simplest way to construct such an object consists in taking the completion of a locally convex *-algebra $(\mathfrak{A}_0, \tau)$ whose multiplication is separately but not jointly continuous. This puts on the stage *locally convex* quasi *-algebras, both for their intrinsic interest and for their potential applications [1, 20]. Even though a certain number of results on general locally convex quasi *-algebras can be found in the literature (see [1, 17] and references therein), rather limited attention has been focused on the case where the locally convex topology of $\mathfrak{X}$ is a norm topology. In this paper we continue the study, undertaken in [22, 23], of the structure properties of a *normed quasi *-algebra* $(\mathfrak{X}, \mathfrak{A}_0)$. The latter means, roughly speaking, that $\mathfrak{X}$ is a normed space with norm $\|\cdot\|$ and this norm satisfies certain coupling properties related to the *partial* multiplication of $(\mathfrak{X}, \mathfrak{A}_0)$. A special case of this situation is that of the so-called *CQ^*-algebras* considered in a series of papers [2, 4–6].

The main subject of this paper is the study of the class of all *-representations of a normed quasi *-algebra $(\mathfrak{X}, \mathfrak{A}_0)$. A *-representation of $(\mathfrak{X}, \mathfrak{A}_0)$

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is a *-homomorphism π of $(\mathfrak{X}, \mathfrak{A}_0)$ into the *partial* *-algebra $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ of closable linear operators defined on a dense domain $\mathcal{D}$ of Hilbert space $\mathcal{H}$. A *-representation of a normed or Banach quasi *-algebra may be *unbounded* in the sense that for some $x \in \mathfrak{X}$, $\pi(x)$ can be a true unbounded operator in $\mathcal{H}$. Since $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ carries a number of topologies making it a *locally convex* partial *-algebra, it makes sense to consider the problem of continuity of *-representations. It turns out that this problem is closely linked with the structure of the normed quasi *-algebra under consideration: thus, just as in the case of Banach *-algebras, a certain amount of information on the structure of a Banach quasi *-algebra can be obtained from the knowledge of the properties of the family of its *-representations.

A relevant role in our study is played by two seminorms $\mathfrak{p}$, $\mathfrak{q}$ that emulate the Gel'fand–Naĭmark seminorm on a Banach *-algebra (but $\mathfrak{q}$ is only defined on a domain $D(\mathfrak{q}) \subseteq \mathfrak{X}$: it is actually an *unbounded* C^*-seminorm in the sense of [8, 3, 11, 24]). This approach, already extensively used in the study of locally convex *-algebras [8, 7, 10], has given a quite deep insight into their structure, in particular concerning the existence of *well-behaved* *-representations (see also [18]). For this reason, extensions to possibly more general situations (like that considered here) is highly desirable.

The seminorms $\mathfrak{p}$, $\mathfrak{q}$ were introduced in [22] where the family of *bounded elements* of a normed quasi *-algebra was studied. An element $x \in \mathfrak{X}$ is said to be bounded if both the maps $L_x : a \in \mathfrak{A}_0 \mapsto xa \in \mathfrak{X}$ and $R_x : a \in \mathfrak{A}_0 \mapsto ax \in \mathfrak{X}$ are bounded linear maps. In some special situations, namely when $\mathfrak{p}(x) = \|x\|$ for every $x \in \mathfrak{X}$, $D(\mathfrak{q})$ is a C^*-algebra and coincides with the set of bounded elements of $\mathfrak{X}$. Even if this is quite a particular case, it makes it clear that $\mathfrak{p}$ and $\mathfrak{q}$ (and through them the class of all *-representations of $(\mathfrak{X}, \mathfrak{A}_0)$) constitute a very useful tool in studying the basic structure properties of Banach quasi *-algebras and they may also be used for a first classification of Banach quasi *-algebras.

The paper is organized as follows.

In Section 2, after giving some preliminaries and basic results, we discuss some properties of the family of positive sesquilinear forms on $\mathfrak{X}$ satisfying certain *invariance* properties. The possibility of obtaining a Hilbert space *-representation, starting either from such a form or from a positive linear functional on $\mathfrak{X}$, following the classical procedure of Gel'fand–Naĭmark–Segal (GNS), is examined in Section 3, where no topology is a priori given on $\mathfrak{X}$.

In Section 4 we consider the case where $(\mathfrak{X}, \mathfrak{A}_0)$ is a normed or Banach quasi *-algebra and we study the properties of the two seminorms $\mathfrak{p}$, $\mathfrak{q}$, which are actually defined through families of bounded invariant positive sesquilinear forms on $\mathfrak{X}$, and analyze their relationship with some classes of *-representations of $(\mathfrak{X}, \mathfrak{A}_0)$ (*regular* and *completely regular*). Moreover, some result on automatic continuity of invariant positive sesquilinear forms is given.

Section 5 is devoted to the study of the continuity of *-representations in terms of the seminorms $\mathfrak{p}$ and $\mathfrak{q}$. In particular, we look for a characterization of the strong*-continuity of any regular (but possibly unbounded) *-representation.

2. Preliminary definitions and basic facts. To begin, we give a series of definitions and preliminary results. For general properties of locally convex *-algebras and representation theory we refer to [9, 16, 17]. More details on partial *-algebras and their representations can be found in [1].

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{D}$ a dense subspace of $\mathcal{H}$. We denote by $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ the set of all (closable) linear operators X such that $\mathcal{D}(X)=\mathcal{D}$ and $\mathcal{D}(X^*)\supseteq\mathcal{D}$. The set $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is a *partial *-algebra* with respect to the following operations: the usual sum X_1+X_2, the scalar multiplication λX, the involution $X\mapsto X^\dagger = X^*\lceil\mathcal{D}$ and the (*weak*) partial multiplication $X_1 \,\square\, X_2 = X_1^{\dagger *}X_2$, defined whenever X_2 is a weak right multiplier of X_1 (equivalently, X_1 is a weak left multiplier of X_2), that is, iff $X_2\mathcal{D}\subset\mathcal{D}(X_1{}^{\dagger *})$ and $X_1^\dagger\mathcal{D}\subset\mathcal{D}(X_2^*)$ (we write $X_2\in R^{\mathrm{w}}(X_1)$ or $X_1\in L^{\mathrm{w}}(X_2)$).

Let

$$\mathcal{L}^\dagger(\mathcal{D})=\{X\in\mathcal{L}^\dagger(\mathcal{D},\mathcal{H}): X\mathcal{D}\subseteq D,\ X^\dagger\mathcal{D}\subseteq D\}.$$

Then $\mathcal{L}^\dagger(\mathcal{D})$ is a *-algebra with respect to $\square$ and $X_1\,\square\,X_2\xi = X_1(X_2\xi)$ for every $\xi\in\mathcal{D}$ [17].

We will consider the following locally convex topologies on $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$:

- the *strong topology* τ_s, defined by the family of seminorms $\{p_\xi\}$ with $p_\xi(X)=\|X\xi\|$ for $X\in\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, $\xi\in\mathcal{D}$;
- the *strong* topology* τ_{s^*}, defined by the family of seminorms $\{p^*_\xi\}$ with $p^*_\xi(X)=\max\{\|X\xi\|,\|X^\dagger\xi\|\}$ for $X\in\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, $\xi\in\mathcal{D}$.

Clearly, τ_{s^*} is finer than τ_s in general.

A **-representation* of a quasi *-algebra $(\mathfrak{X},\mathfrak{A}_0)$ is a *-homomorphism of $\mathfrak{X}$ into $\mathcal{L}^\dagger(\mathcal{D}_\pi,\mathcal{H}_\pi)$ for some pair $(\mathcal{D}_\pi,\mathcal{H}_\pi)$, where $\mathcal{D}_\pi$ is a dense subspace of the Hilbert space $\mathcal{H}_\pi$, that is, a linear map $\pi:\mathfrak{X}\to\mathcal{L}^\dagger(\mathcal{D}_\pi,\mathcal{H}_\pi)$ such that: (i) $\pi(x^*)=\pi(x)^\dagger$ for every $x\in\mathfrak{X}$; (ii) if $x\in\mathfrak{X}$ and $a\in\mathfrak{A}_0$ then $\pi(x)\in L^{\mathrm{w}}(\pi(a))$ and $\pi(x)\,\square\,\pi(a)=\pi(xa)$. If $(\mathfrak{X},\mathfrak{A}_0)$ has a unit e, we assume that $\pi(e)=\mathbb{I}$.

For an arbitrary *-representation π the inclusion $\pi(\mathfrak{A}_0)\subseteq\mathcal{L}^\dagger(\mathcal{D}_\pi)$ does not hold in general. However, if we put

$$\mathcal{D}_{\widehat{\pi}}=\Big\{\xi_0+\sum_{k=1}^n\pi(a_k)\xi_k : \xi_0,\ \xi_k\in\mathcal{D}_\pi,\ a_k\in\mathfrak{A}_0,\ k=1,\dots,n\Big\},$$

the map

$$\widehat{\pi}(x)\Big(\xi_0 + \sum_{k=1}^{n} \pi(a_k)\xi_k\Big) := \pi(x)\xi_0 + \sum_{k=1}^{n} (\pi(x) \,\square\, \pi(a_k))\xi_k$$

defines a *-representation on $\mathcal{D}_{\widehat{\pi}}$ having the property $\widehat{\pi}(\mathfrak{A}_0) \subset \mathcal{L}^\dagger(\mathcal{D}_{\widehat{\pi}})$. Thus, without loss of generality, we will always assume that $\pi(\mathfrak{A}_0) \subseteq \mathcal{L}^\dagger(\mathcal{D}_\pi)$ for a *-representation π of $(\mathfrak{X}, \mathfrak{A}_0)$.

A *-representation π of $(\mathfrak{X}, \mathfrak{A}_0)$ is called

- *cyclic* if there exists $\eta \in \mathcal{D}_\pi$ such that $\pi(\mathfrak{A}_0)\eta$ is dense in $\mathcal{H}_\pi$;
- *faithful* if $\pi(x) = 0$ implies $x = 0$.

If π is a *-representation of $(\mathfrak{X}, \mathfrak{A}_0)$ in $\mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi)$, then the *closure* $\widetilde{\pi}$ of π is defined, for every $x \in \mathfrak{X}$, as the restriction of $\overline{\pi(x)}$ to the domain $\widetilde{\mathcal{D}}_\pi$, which is the completion of $\mathcal{D}_\pi$ under the *graph topology* [1, 17] defined by the seminorms $\xi \in \mathcal{D} \mapsto \|\pi(x)\xi\|$, $x \in \mathfrak{X}$. If $\pi = \widetilde{\pi}$, the representation is said to be *closed*.

Definition 2.1. Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a quasi *-algebra. We denote by $\mathcal{Q}(\mathfrak{X})$ the set of all sesquilinear forms on $\mathfrak{X} \times \mathfrak{X}$ such that

(i) $\varphi(x, x) \geq 0$, $\forall x \in \mathfrak{X}$;
(ii) $\varphi(xa, b) = \varphi(a, x^*b)$, $\forall x \in \mathfrak{X}$, $a, b \in \mathfrak{A}_0$.

Let $\varphi \in \mathcal{Q}(\mathfrak{X})$. Then the positivity of φ implies that

$$\begin{aligned} \varphi(x, y) &= \overline{\varphi(y, x))}, && \forall x, y \in \mathfrak{X}; \\ |\varphi(x, y)|^2 &\leq \varphi(x, x)\varphi(y, y), && \forall x, y \in \mathfrak{X}. \end{aligned}$$

Hence

$$N_\varphi := \{x \in \mathfrak{X} : \varphi(x, x) = 0\} = \{x \in \mathfrak{X} : \varphi(x, y) = 0,\ \forall y \in \mathfrak{X}\},$$

and so N_φ is a left submodule of $\mathfrak{A}$. For each $x \in \mathfrak{X}$, we denote by $\lambda_\varphi(x)$ the coset of $\mathfrak{X}/N_\varphi$ which contains x, and define an inner product $\langle \cdot \,|\, \cdot \rangle$ on $\lambda_\varphi(\mathfrak{X}) = \mathfrak{X}/N_\varphi$ by

$$\langle \lambda_\varphi(x) \,|\, \lambda_\varphi(y) \rangle = \varphi(x, y), \qquad x, y \in \mathfrak{X}.$$

We let $\mathcal{H}_\varphi$ be the Hilbert space completion of the pre-Hilbert space $\lambda_\varphi(\mathfrak{X})$.

Proposition 2.2. *Let $\varphi \in \mathcal{Q}(\mathfrak{X})$. The following statement are equivalent*:

(i) $x \in \mathfrak{X}$ *and* $\varphi(x, a) = 0$ *for every* $a \in \mathfrak{A}_0$ *implies* $\varphi(x, x) = 0$.
(ii) $\lambda_\varphi(\mathfrak{A}_0)$ *is dense in* $\mathcal{H}_\varphi$.

Proof. (i)$\Rightarrow$(ii): Let $\eta \in \mathcal{H}_\varphi$ and assume that $\langle \eta \,|\, \lambda_\varphi(a) \rangle = 0$ for every $a \in \mathfrak{A}_0$. Without loss of generality, we may assume that $\eta = \lambda_\varphi(x)$ for some $x \in \mathfrak{X}$. Then

$$\varphi(x, a) = \langle \lambda_\varphi(x) \,|\, \lambda_\varphi(a) \rangle = 0, \qquad \forall a \in \mathfrak{A}_0.$$

Then, by assumption, $\varphi(x, x) = 0$, i.e. $\lambda_\varphi(x) = 0$.

(ii)⇒(i): Let $\lambda_\varphi(\mathfrak{A}_0)$ be dense in $\mathcal{H}_\varphi$ and assume that there exists $x \in \mathfrak{X}$ such that $\varphi(x, a) = 0$ for every $a \in \mathfrak{A}_0$. By assumption, there exists a sequence $\{a_n\} \subset \mathfrak{A}_0$ such that $\|\lambda_\varphi(a_n - x)\| \to 0$. Then

$$\varphi(x, x) = \|\lambda_\varphi(x)\|^2 = \lim_{n\to\infty} \langle \lambda_\varphi(x) \mid \lambda_\varphi(a_n)\rangle = \lim_{n\to\infty} \varphi(x, a_n) = 0. \blacksquare$$

We denote by $\mathcal{I}(\mathfrak{X})$ the subset of all forms $\varphi \in \mathcal{Q}(\mathfrak{X})$ satisfying (i) or (ii) of Proposition 2.2. An element of $\mathcal{Q}(\mathfrak{X}) \setminus \mathcal{I}(\mathfrak{X})$ will be called a *singular form*.

EXAMPLE 2.3. In general $\mathcal{Q}(\mathfrak{X}) \supset \mathcal{I}(\mathfrak{X})$ and singular forms really exist. We construct an example as follows. Let $\mathcal{A}$ be a C^*-algebra with norm $\|\cdot\|$ and unit e. Let $\mathcal{J}$ be a proper dense *-ideal of $\mathcal{A}$. Let $w_0 \in \mathcal{A}$ with $w_0 = w_0^*$ be an element satisfying the following condition:

$$\alpha e + \beta w_0 + \gamma w_0^2 \in \mathcal{J} \;\Leftrightarrow\; \alpha = \beta = \gamma = 0. \tag{1}$$

Let

$$\mathfrak{X} = \{\lambda e + \mu w_0 + u : \lambda, \mu \in \mathbb{C},\ u \in \mathcal{J}\}.$$

From (1) it follows that for every $x \in \mathfrak{X}$ the decomposition $x = \lambda_x e + \mu_x w_0 + u_x$ with $\lambda_x, \mu_x \in \mathbb{C}$ and $u_x \in \mathcal{J}$ is unique. Put

$$\mathfrak{A}_0 = \{\lambda e + u : \lambda \in \mathbb{C},\ u \in \mathcal{J}\}.$$

From (1) it also follows that there is no $a \in \mathfrak{A}_0$ such that $w_0 a = w_0^2$, hence $(\mathfrak{X}, \mathfrak{A}_0)$ is a true quasi*-algebra with respect to the multiplication inherited from $\mathcal{A}$. The norm of $\mathcal{A}$ makes of $(\mathfrak{X}, \mathfrak{A}_0)$ a normed quasi *-algebra. Now we define

$$\varphi(\lambda_x e + \mu_x w_0 + u_x, \lambda_y e + \mu_y w_0 + u_y) = \mu_x \overline{\mu}_y \omega(w_0^* w_0)$$

where ω is a positive linear functional on $\mathcal{A}$ such that $\omega(w_0^* w_0) > 0$. Then it is easily seen that $\varphi \in \mathcal{Q}(\mathfrak{X})$. But $\varphi \notin \mathcal{I}(\mathfrak{X})$, since $N_\varphi = \mathfrak{A}_0$, and so $\mathfrak{X}/N_\varphi \sim \mathbb{C}$, while $\lambda_\varphi(\mathfrak{A}_0) = \{0\}$.

3. Representations of quasi *-algebras. The Gel'fand–Naĭmark–Segal (GNS) construction for positive linear functionals is one of the most relevant tools for the study of (locally convex) *-algebras [9, 16, 17]. As is customary when a partial multiplication is involved [1, 12], we consider, as a starting point for the construction, a positive sesquilinear form enjoying certain *invariance* properties (see also [6]). An analogous GNS construction for positive *linear* functionals will be discussed later (Theorem 3.5). In this section we will not suppose that $\mathfrak{X}$ has a topology given a priori.

PROPOSITION 3.1. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a quasi *-algebra with unit e and φ a sesquilinear form on $\mathfrak{X} \times \mathfrak{X}$. The following statements are equivalent*:

(i) $\varphi \in \mathcal{I}(\mathfrak{X})$.

(ii) *There exist a Hilbert space* $\mathcal{H}_\varphi$, *a dense domain* $\mathcal{D}_\varphi$ *of* $\mathcal{H}_\varphi$ *and a closed cyclic* **-representation* π_φ *of* $(\mathfrak{X}, \mathfrak{A}_0)$ *in* $\mathcal{L}^\dagger(\mathcal{D}_\varphi, \mathcal{H}_\varphi)$ *with cyclic vector* ξ_φ *(i.e.* $\pi_\varphi(\mathfrak{A}_0)\xi_\varphi$ *dense in* $\mathcal{H}_\varphi$*) such that*

$$\varphi(x, y) = \langle \pi_\varphi(x)\xi_\varphi \mid \pi_\varphi(y)\xi_\varphi \rangle. \tag{2}$$

Proof. (i)⇒(ii): Let $\varphi \in \mathcal{I}(\mathfrak{X})$. Put

$$\pi^\circ_\varphi(x)\lambda_\varphi(a) = \lambda_\varphi(xa), \qquad x \in \mathfrak{X},\ a \in \mathfrak{A}_0. \tag{3}$$

First we prove that, for every $x \in \mathfrak{X}$, the map $\pi^\circ_\varphi(x)$ is well-defined. Assume that $\lambda_\varphi(a) = 0$ for some $a \in \mathfrak{A}_0$. If $x \in \mathfrak{X}$, we then get $\varphi(a, x^*b) = 0$ for every $b \in \mathfrak{A}_0$. For each $y \in \mathfrak{X}$, there exists a sequence $\{b_n\} \subset \mathfrak{A}_0$ such that $\|\lambda_\varphi(y) - \lambda_\varphi(b_n)\| \to 0$. This clearly implies that $\varphi(xa, y) = 0$ for every $y \in \mathfrak{X}$. Hence $xa \in N_\varphi$. Thus, for every $x \in \mathfrak{X}$, the map $\pi^\circ_\varphi(x)$ is a well-defined linear operator from $\lambda_\varphi(\mathfrak{A}_0)$ into $\mathcal{H}_\varphi$. The properties of φ listed in Definition 2.1 now easily imply that π°_φ is a *-representation and the restriction of π°_φ to $\mathfrak{A}_0$ maps $\lambda_\varphi(\mathfrak{A}_0)$ into itself. If $(\mathfrak{X}, \mathfrak{A}_0)$ has a unit e then (i) and (ii) follow from the very definitions. Denote by π_φ the closure of π°_φ. Then, as is easily seen, π_φ satisfies (2). It is also clear by the definition of π that, if $(\mathfrak{X}, \mathfrak{A}_0)$ has a unit e, then $\xi_\varphi := \lambda_\varphi(e)$ is a cyclic vector for π.

(ii)⇒(i). From (2) it follows easily that

$$\begin{aligned} &\varphi(x, x) \geq 0, && \forall x \in \mathfrak{X};\\ &\varphi(xa, b) = \varphi(a, x^*b), && \forall x \in \mathfrak{X},\ a, b \in \mathfrak{A}_0. \end{aligned}$$

Since $\pi_\varphi(\mathfrak{A}_0)\xi_\varphi$ is dense in $\mathcal{H}_\varphi$, for every $x \in \mathfrak{X}$ there exists a sequence $\{a_n\} \subset \mathfrak{A}_0$ such that $\|\pi_\varphi(x)\xi_\varphi - \pi_\varphi(a_n)\xi_\varphi\| \to 0$ as $n \to \infty$. Therefore

$$\|\lambda_\varphi(x) - \lambda_\varphi(a_n)\|^2 = \varphi(x - a_n, x - a_n) = \|\pi_\varphi(x)\xi_\varphi - \pi_\varphi(a_n)\xi_\varphi\|^2 \to 0$$

as $n \to \infty$. This implies that $\lambda_\varphi(\mathfrak{A}_0)$ is dense in $\mathcal{H}_\varphi$. ∎

Definition 3.2. The triple $(\pi_\varphi, \lambda_\varphi, \mathcal{H}_\varphi)$ is called the *GNS construction* for φ, and π_φ is called the *GNS representation* of $\mathfrak{X}$ constructed from φ.

The proof of the following proposition is similar to the classical one and we omit it.

Proposition 3.3. *Let* $(\mathfrak{X}, \mathfrak{A}_0)$ *be a quasi* **-algebra with unit* e *and* $\varphi \in \mathcal{I}(\mathfrak{X})$. *Then the GNS construction* $(\pi_\varphi, \lambda_\varphi, \mathcal{H}_\varphi)$ *is unique up to unitary equivalence.*

It is easy to prove that

Proposition 3.4. *The* **-representation* π_φ *is bounded if, and only if,* φ *is admissible, i.e., for every* $x \in \mathfrak{X}$, *there exists* $\gamma_x > 0$ *such that*

$$\varphi(xa, xa) \leq \gamma_x \varphi(a, a), \qquad \forall a \in \mathfrak{A}_0.$$

Let now ω be a linear functional on $\mathfrak{X}$ satisfying the following conditions:

(L.1) $\omega(a^*a) \geq 0$, $\forall a \in \mathfrak{A}_0$;
(L.2) $\omega(b^*x^*a) = \overline{\omega(a^*xb)}$, $\forall a, b \in \mathfrak{A}_0$, $x \in \mathfrak{X}$;
(L.3) for every $x \in \mathfrak{X}$, there exists $\gamma_x > 0$ such that

$$|\omega(x^*a)| \leq \gamma_x \omega(a^*a)^{1/2}, \quad \forall a \in \mathfrak{A}_0.$$

We define $N_\omega = \{a \in \mathfrak{A}_0 : \omega(a^*a) = 0\}$. Then N_ω is a left ideal of $\mathfrak{A}_0$ and the quotient $\mathfrak{A}_0/N_\omega$ is a pre-Hilbert space with inner product

$$\langle \lambda_\omega(a) \,|\, \lambda_\omega(b) \rangle = \omega(b^*a), \quad a, b \in \mathfrak{A}_0,$$

where $\lambda_\omega(a)$, $a \in \mathfrak{A}_0$, denotes the coset containing a. Let $\mathcal{H}_\omega$ be the completion of $\lambda_\omega(\mathfrak{A}_0)$.

If $x \in \mathfrak{X}$, we put $x^\omega(\lambda_\omega(a)) = \omega(x^*a)$. Then, by (L.3), x^ω is a well-defined linear functional on $\lambda_\omega(\mathfrak{A}_0)$ and

$$|x^\omega(\lambda_\omega(a))| = |\omega(x^*a)| \leq \gamma_x \omega(a^*a)^{1/2} = \gamma_x \|\lambda_\omega(a)\|, \quad \forall a \in \mathfrak{A}_0.$$

Thus, x^ω is bounded and by Riesz's lemma, there exists a unique $\xi_\omega(x) \in \mathcal{H}_\omega$ such that

$$x^\omega(\lambda_\omega(a)) = \langle \lambda_\omega(a) \,|\, \xi_\omega(x) \rangle, \quad \forall a \in \mathfrak{A}_0.$$

Now, we put

$$\pi_\omega(x)\lambda_\omega(a) = \xi_\omega(xa), \quad a \in \mathfrak{A}_0.$$

Since

$$\begin{aligned} \langle \lambda_\omega(b) \,|\, \pi_\omega(x)\lambda_\omega(a) \rangle &= \langle \lambda_\omega(b) \,|\, \xi_\omega(xa) \rangle = (xa)^\omega(\lambda_\omega(b)) \\ &= \omega(a^*x^*b) = \overline{\omega(b^*xa)}, \quad \forall b \in \mathfrak{A}_0, \end{aligned}$$

it follows from (L.3) that $\pi_\omega(x)$ is well-defined and maps $\lambda_\omega(\mathfrak{A}_0)$ into $\mathcal{H}_\omega$. In a similar way one can prove the equality

$$\langle \pi_\omega(x^*)\lambda_\omega(b) \,|\, \lambda_\omega(a) \rangle = \overline{\omega(b^*xa)}, \quad \forall a, b \in \mathfrak{A}_0.$$

This implies that $\pi(x) \in \mathcal{L}^\dagger(\lambda_\omega(\mathfrak{A}_0), \mathcal{H}_\omega)$ and $\pi_\omega(x)^\dagger = \pi_\omega(x^*)$.

With analogous computations and taking into account that

$$\pi_\omega(a)\lambda_\omega(b) = \lambda_\omega(ab), \quad \forall a, b \in \mathfrak{A}_0,$$

we also get, for $x \in \mathfrak{A}$ and $a \in \mathfrak{A}_0$, the equality

$$\langle \pi_\omega(xa)\lambda_\omega(b) \,|\, \lambda_\omega(c) \rangle = \langle \pi_\omega(a)\lambda_\omega(b) \,|\, \pi_\omega(x^*)\lambda_\omega(c) \rangle, \quad \forall b, c \in \mathfrak{A}_0.$$

This implies that $\pi_\omega(x) \,\square\, \pi_\omega(a)$ is well-defined and

$$\pi_\omega(xa) = \pi_\omega(x) \,\square\, \pi_\omega(a), \quad \forall x \in \mathfrak{X},\ a \in \mathfrak{A}_0.$$

Thus, π_ω is a *-representation. It is clear that $\pi_\omega(\mathfrak{A}_0)\eta_\omega$, where $\eta_\omega := \lambda_\omega(e)$, is dense in $\mathcal{H}_\varphi$. Now we can take the closure $\widetilde{\pi}_\omega$ of π_ω to obtain

THEOREM 3.5. *Let* $(\mathfrak{X}, \mathfrak{A}_0)$ *be a quasi* **-algebra with unit* e *and let* ω *be a linear functional on* $\mathfrak{X}$ *satisfying the conditions* (L.1)–(L.3). *Then there* *exists a closed cyclic* **-representation* $\widetilde{\pi}_\omega$ *of* $(\mathfrak{X}, \mathfrak{A}_0)$ *with cyclic vector* η_ω *such that*

$$\omega(x) = \langle \widetilde{\pi}_\omega(x)\eta_\omega \,|\, \eta_\omega \rangle, \qquad \forall x \in \mathfrak{X}.$$

This representation is unique up to unitary equivalence.

The uniqueness statement comes again from a slight modification of the classical argument.

In the light of Theorem 3.5 it is natural to call a linear functional ω on $\mathfrak{X}$ satisfying (L1)–(L3) *representable*.

REMARK 3.6. We notice that π_ω is bounded if, and only if, for every $x \in \mathfrak{X}$, there exists $\gamma_x > 0$ such that

$$|\omega(b^* x a)| \le \gamma_x \omega(a^* a)^{1/2} \omega(b^* b)^{1/2}, \qquad \forall a, b \in \mathfrak{A}_0.$$

REMARK 3.7. It is not difficult to show that, for every $x \in \mathfrak{X}$, $\widetilde{\pi}_\omega(x)$ maps $\lambda_\omega(\mathfrak{A}_0)$ into $\bigcap_{a \in \mathfrak{A}_0} \mathcal{D}(\pi_\omega(a)^*)$.

EXAMPLE 3.8. Let $\mathcal{D}$ be a dense domain in a Hilbert space $\mathcal{H}$ and $\|\cdot\|_1$ a norm on $\mathcal{D}$, stronger than the Hilbert norm $\|\cdot\|$. Let $\mathsf{B}(\mathcal{D}, \mathcal{D})$ denote the vector space of all *jointly* continuous sesquilinear forms on $\mathcal{D} \times \mathcal{D}$. The map $\varphi \mapsto \varphi^*$, where

$$\varphi^*(\xi, \eta) = \overline{\varphi(\eta, \xi)},$$

defines an involution in $\mathsf{B}(\mathcal{D}, \mathcal{D})$.

We denote by $\mathfrak{L}^\dagger(\mathcal{D})$ the *-subalgebra of $\mathcal{L}^\dagger(\mathcal{D})$ consisting of all operators $A \in \mathcal{L}^\dagger(\mathcal{D})$ such that both A and $A^\dagger$ are continuous from $\mathcal{D}[\|\cdot\|_1]$ into itself.

Every $A \in \mathfrak{L}^\dagger(\mathcal{D})$ defines a sesquilinear form $\varphi_A \in \mathsf{B}(\mathcal{D}, \mathcal{D})$ by

$$\varphi_A(\xi, \eta) = \langle A\xi \,|\, \eta \rangle, \qquad \xi, \eta \in \mathcal{D}.$$

Indeed, for all $\xi, \eta \in \mathcal{D}$,

$$|\varphi_A(\xi, \eta)| = |\langle A\xi \,|\, \eta \rangle| \le \|A\xi\| \, \|\eta\| \le \gamma \|A\xi\|_1 \|\eta\|_1 \le \gamma' \|\xi\|_1 \|\eta\|_1.$$

We put

$$\mathsf{B}^\dagger(\mathcal{D}) = \{\varphi_A : A \in \mathfrak{L}^\dagger(\mathcal{D})\}.$$

It is easily seen that $\varphi_A^* = \varphi_{A^\dagger}$ for every $A \in \mathfrak{L}^\dagger(\mathcal{D})$.

For $\varphi \in \mathsf{B}(\mathcal{D}, \mathcal{D})$, $\varphi_A \in \mathsf{B}^\dagger(\mathcal{D})$, we define

$$(\varphi \circ \varphi_A)(\xi, \eta) = \varphi(A\xi, \eta), \qquad \xi, \eta \in \mathcal{D},$$
$$(\varphi_A \circ \varphi)(\xi, \eta) = \varphi(\xi, A^\dagger \eta), \qquad \xi, \eta \in \mathcal{D}.$$

With these operations and involution, $(\mathsf{B}(\mathcal{D}, \mathcal{D}), \mathsf{B}^\dagger(\mathcal{D}))$ is a quasi *-algebra (see also [1, Ch. 10] for a complete discussion).

For every $\xi \in \mathcal{D}$, we define

$$\omega_\xi(\varphi) = \varphi(\xi, \xi), \qquad \varphi \in \mathsf{B}(\mathcal{D}, \mathcal{D}).$$

Then ω_ξ is a linear functional on $\mathsf{B}(\mathcal{D},\mathcal{D})$. Moreover,

$$\omega_\xi(\varphi_A \circ \varphi_A) = (\varphi_A \circ \varphi_A)(\xi,\xi) = \langle A\xi \,|\, A\xi\rangle \geq 0,$$
$$\omega_\xi(\varphi_{B^\dagger} \circ \varphi \circ \varphi_A) = \varphi(A\xi, B\xi) = \overline{\omega_\xi(\varphi_{A^\dagger} \circ \varphi^* \circ \varphi_A)}.$$

Hence, ω_ξ satisfies (L1) and (L2).

The functional ω_ξ satisfies (L3) if, and only if, for every $\varphi \in \mathsf{B}(\mathcal{D},\mathcal{D})$, there exists $\gamma_\varphi > 0$ such that

$$|\varphi(A\xi,\xi)| \leq \gamma_\varphi \|A\xi\|, \quad \forall A \in \mathfrak{L}^\dagger(\mathcal{D}).$$

Indeed, ω_ξ satisfies (L3) if, and only if, for every $\varphi \in \mathsf{B}(\mathcal{D},\mathcal{D})$, there exists $\gamma_\varphi > 0$ such that

$$|\omega_\xi(\varphi \circ \varphi_A)| = |(\varphi \circ \varphi_A)(\xi,\xi)| = |\varphi(A\xi,\xi)| \leq \gamma_\varphi \omega_\xi(\varphi_A^* \circ \varphi_A)^{1/2} = \gamma_\varphi \|A\xi\|.$$

This condition is clearly satisfied if, and only if, φ is bounded in the first variable on the subspace $\mathcal{M}_\xi = \{A\xi : A \in \mathfrak{L}^\dagger(\mathcal{D})\}$. If this is the case, then there exists $\zeta \in \mathcal{H}$ such that

$$\omega_\xi(\varphi \circ \varphi_A) = \langle A\xi \,|\, \zeta\rangle, \quad \forall A \in \mathfrak{L}^\dagger(\mathcal{D}).$$

Hence, not every ω_ξ is representable.

Let now $\varphi \in \mathcal{I}(\mathfrak{X})$. Then the linear functional ω_φ with $\omega_\varphi(x) = \varphi(x,e)$ for $x \in \mathfrak{X}$ satisfies the conditions (L.1)–(L.3), i.e. it is representable. Thus Theorem 3.5 can be applied to get the *-representation $\widetilde{\pi}_{\omega_\varphi}$ constructed as above. On the other hand, we can also build, as in Proposition 3.1, the closed *-representation π_φ with cyclic vector ξ_φ. Since

$$\omega_\varphi(x) = \varphi(x,e) = \langle \pi_\varphi(x)\xi_\varphi \,|\, \xi_\varphi\rangle, \quad \forall x \in \mathfrak{X},$$

it turns out that $\widetilde{\pi}_{\omega_\varphi}$ and π_φ are unitarily equivalent.

Let $\varphi \in \mathcal{Q}(\mathfrak{X})$. Then the linear functional ω_φ with $\omega_\varphi(x) = \varphi(x,e)$ for $x \in \mathfrak{X}$ is representable. Let $\widetilde{\pi}_{\omega_\varphi}$ be the corresponding *-representation. If we define

$$\Omega_\varphi(x,y) = \langle \widetilde{\pi}_{\omega_\varphi}(x)\xi_{\omega_\varphi} \,|\, \widetilde{\pi}_{\omega_\varphi}(y)\xi_{\omega_\varphi}\rangle, \quad x,y \in \mathfrak{X},$$

then, as is easily seen, $\Omega_\varphi \in \mathcal{Q}(\mathfrak{X})$. But, in general, $\Omega_\varphi \neq \varphi$.

Proposition 3.9. *The following statements hold*:

(i) *For every $\varphi \in \mathcal{Q}(\mathfrak{X})$, $\Omega_\varphi \in \mathcal{I}(\mathfrak{X})$.*

(ii) *For every $\varphi \in \mathcal{Q}(\mathfrak{X})$, there exist $\varphi_0 \in \mathcal{I}(\mathfrak{X})$ and a singular form $s_\varphi \in \mathcal{Q}(\mathfrak{X}) \setminus \mathcal{I}(\mathfrak{X})$ such that*

$$\varphi(x,y) = \varphi_0(x,y) + s_\varphi(x,y), \quad \forall x,y \in \mathfrak{X}.$$

(iii) *$\Omega_\varphi = \varphi$ if, and only if, $\lambda_\varphi(\mathfrak{A}_0)$ is dense in $\mathcal{H}_\varphi$, i.e. if, and only if, $\varphi \in \mathcal{I}(\mathfrak{X})$.*

Proof. (i) Since $\widetilde{\pi}_{\omega_\varphi}$ is cyclic, for every $x \in \mathfrak{X}$ there exists a sequence $\{a_n\} \subset \mathfrak{A}_0$ such that $\|\widetilde{\pi}_{\omega_\varphi}(x-a_n)\xi_{\omega_\varphi}\| \to 0$. Then

$$\begin{aligned}\|\lambda_\varphi(x)-\lambda_\varphi(a_n)\|^2 &= \varphi(x-a_n, x-a_n)\\ &= \langle \widetilde{\pi}_{\omega_\varphi}(x-a_n)\xi_{\omega_\varphi} \,|\, \widetilde{\pi}_{\omega_\varphi}(x-a_n)\xi_{\omega_\varphi}\rangle \to 0.\end{aligned}$$

(ii) Put $\varphi_0 = \Omega_\varphi$ and $s_\varphi = \varphi - \Omega_\varphi$.

(iii) is clear. ∎

If π is a *-representation of $(\mathfrak{X}, \mathfrak{A}_0)$ then, for every $\xi \in \mathcal{D}_\pi$, the *vector form* φ_ξ defined by

$$\varphi_\xi(x,y) = \langle \pi(x)\xi \,|\, \pi(y)\xi\rangle, \qquad x, y \in \mathfrak{X}. \tag{4}$$

is an element of $\mathcal{Q}(\mathfrak{X})$ but it need not belong to $\mathcal{I}(\mathfrak{X})$. For this reason, we say that π is *regular* if $\varphi_\xi \in \mathcal{I}(\mathfrak{X})$ for every $\xi \in \mathcal{D}_\pi$.

PROPOSITION 3.10. *Let π be a *-representation of $(\mathfrak{X}, \mathfrak{A}_0)$. The following statements are equivalent*:

(i) *π is regular.*

(ii) *$\pi(x)\xi \in \overline{\pi(\mathfrak{A}_0)\xi}$ for all $x \in \mathfrak{X}$ and $\xi \in \mathcal{D}_\pi$.*

(iii) *For every $\xi \in \mathcal{D}_\pi$, $\pi_0 := \pi_{\restriction \mathcal{M}_\xi}$ is a *-representation of $(\mathfrak{X}, \mathfrak{A}_0)$ into $\mathcal{L}^\dagger(\mathcal{M}_\xi, \overline{\mathcal{M}}_\xi)$, where $\mathcal{M}_\xi = \pi(\mathfrak{A}_0)\xi$.*

Proof. (i)⇒(ii): Let π be regular and $\xi \in \mathcal{D}_\pi$. Consider the vector form φ_ξ. For every $x \in \mathfrak{X}$, there exists a sequence $\{a_n\} \subset \mathfrak{A}_0$ such that $\|\lambda_{\varphi_\xi}(x-a_n)\| \to 0$. Then

$$\|(\pi(x)-\pi(a_n))\xi\|^2 = \|\lambda_{\varphi_\xi}(x-a_n)\|^2 \to 0.$$

This proves that $\pi(x)\xi \in \overline{\pi(\mathfrak{A}_0)\xi}$.

(ii)⇒(iii): The assumption implies that, for every $x \in \mathfrak{X}$ and $\xi \in \mathcal{D}_\pi$, $\pi_0(x)$ maps $\pi(\mathfrak{A}_0)\xi$ into $\overline{\pi(\mathfrak{A}_0)\xi}$. Some simple calculations, which make use of the fact that π is a *-representation and that the module associativity holds, show that $\pi_0(x^*) = (\pi_0(x))^*_{\restriction \pi(\mathfrak{A}_0)\xi}$ and that π_0 preserves the partial multiplication of $(\mathfrak{X}, \mathfrak{A}_0)$.

(iii)⇒(i): The assumption implies that $\pi(x)\xi \in \overline{\mathcal{M}}_\xi$ for all $\xi \in \mathcal{D}_\pi$ and $x \in \mathfrak{X}$. Therefore, for every $x \in \mathfrak{X}$, there exists a sequence $\{a_n\} \subset \mathfrak{A}_0$ such that $\|(\pi(x)-\pi(a_n))\xi\| \to 0$. Then

$$\varphi_\xi(x-a_n, x-a_n) = \|\lambda_{\varphi_\xi}(x-a_n)\|^2 = \|(\pi(x)-\pi(a_n))\xi\|^2 \to 0.$$

Hence, π is regular. ∎

If φ is a sesquilinear form on $\mathfrak{X} \times \mathfrak{X}$ and $a \in \mathfrak{A}_0$, we denote by φ_a the sesquilinear form on $\mathfrak{X} \times \mathfrak{X}$ defined by

$$\varphi_a(x,y) = \varphi(xa, ya), \qquad x, y \in \mathfrak{X}, \tag{5}$$

and by ω_{φ_a} the corresponding linear functional on $\mathfrak{X}$ defined by

$$\omega_{\varphi_a}(x) = \varphi(xa, a), \qquad x \in \mathfrak{X}. \tag{6}$$

It is readily checked that if $\varphi \in \mathcal{Q}(\mathfrak{X})$, then $\varphi_a \in \mathcal{Q}(\mathfrak{X})$ for all $a \in \mathfrak{A}_0$.

PROPOSITION 3.11. *Let* $(\mathfrak{X}, \mathfrak{A}_0)$ *be a quasi* **-algebra with unit* e, $\varphi \in \mathcal{I}(\mathfrak{X})$ *and* π_φ° *the* **-representation defined in* (3). *The following statements are equivalent*:

(i) π_φ° *is regular.*
(ii) $\varphi_a \in \mathcal{I}(\mathfrak{X})$ *for all* $a \in \mathfrak{A}_0$.

Proof. If $\eta \in \mathcal{D}_\varphi = \lambda_\varphi(\mathfrak{A}_0)$, then $\eta = \lambda_\varphi(a)$ for some $a \in \mathfrak{A}_0$. Hence

$$\varphi_\eta(x, y) = \langle \pi_\varphi^\circ(x)\lambda_\varphi(a) \mid \pi_\varphi^\circ(y)\lambda_\varphi(a)\rangle = \varphi(xa, ya) = \varphi_a(x, y), \qquad \forall x, y \in \mathfrak{X}.$$

Thus $\varphi_\eta = \varphi_a$. This clearly implies the desired equivalence. ∎

It is now useful to introduce the notation

$$\mathcal{I}_s(\mathfrak{X}) := \{\varphi \in \mathcal{I}(\mathfrak{X}) : \varphi_a \in \mathcal{I}(\mathfrak{X}),\ \forall a \in \mathfrak{A}_0\}.$$

It is clear that for every $\varphi \in \mathcal{I}_s(\mathfrak{X})$, the corresponding *-representation π_φ° is regular.

REMARK 3.12. For the GNS representation π_φ constructed from φ (i.e. for the closure of π_φ°) the implication (i)⇒(ii) still holds, in the obvious way; however, (ii) does not imply (i) in general.

EXAMPLE 3.13. Let π be a regular *-representation of $(\mathfrak{X}, \mathfrak{A}_0)$ in $\mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi)$. Let $\xi \in \mathcal{D}_\pi$ and let φ_ξ be the corresponding vector form (in the sense of (4)). Then, by definition, $\varphi_\xi \in \mathcal{I}(\mathfrak{X})$. Since $\pi(a)\xi \in \mathcal{D}_\pi$ for every $a \in \mathfrak{A}_0$, also $(\varphi_\xi)_a \in \mathcal{I}(\mathfrak{X})$ for every $a \in \mathfrak{A}_0$. Thus, in this case, $\varphi_\xi \in \mathcal{I}_s(\mathfrak{X})$.

We notice that a bounded *-representation (i.e. $\overline{\pi(x)} \in \mathcal{B}(\mathcal{H})$ for every $x \in \mathfrak{X}$) need not be regular.

$\mathrm{Rep}(\mathfrak{X})$ and $\mathrm{Rep}_r(\mathfrak{X})$ will denote, respectively, the families of all *-representations and of all regular *-representations of $(\mathfrak{X}, \mathfrak{A}_0)$.

4. The case of normed quasi ***-algebras.** In this section, we will consider the case where $\mathfrak{X}$ is endowed with a norm topology making $(\mathfrak{X}, \mathfrak{A}_0)$ a *normed quasi* **-algebra* in the following sense.

DEFINITION 4.1. A quasi *-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ is called a *normed quasi* **-algebra* if a norm $\|\cdot\|$ is defined on $\mathfrak{X}$ with the properties:

(i) $\|x^*\| = \|x\|$ for all $x \in \mathfrak{X}$;
(ii) $\mathfrak{A}_0$ is dense in $\mathfrak{X}$;
(iii) for every $a \in \mathfrak{A}_0$, the map $R_a : x \in \mathfrak{X} \mapsto xa \in \mathfrak{X}$ is continuous in $\mathfrak{X}$.

If $(\mathfrak{X}, \|\cdot\|)$ is a Banach space, we say that $(\mathfrak{X}, \mathfrak{A}_0)$ is a *Banach quasi* **-algebra*.

The continuity of the involution implies that

(iii′) for every $a \in \mathfrak{A}_0$, the map $L_a : x \in \mathfrak{X} \mapsto ax \in \mathfrak{X}$ is continuous in $\mathfrak{X}$.

If $(\mathfrak{X}, \mathfrak{A}_0)$ has a unit e (i.e., $e \in \mathfrak{A}_0$ and $xe = ex = x$ for every $x \in \mathfrak{X}$), we will assume (without loss of generality) that $\|e\| = 1$. If $(\mathfrak{X}, \mathfrak{A}_0)$ has no unit, it can always be embedded in a normed quasi *-algebra with unit e in the standard fashion.

In what follows, we will always assume that if $xa = 0$ for every $a \in \mathfrak{A}_0$, then $x = 0$ (of course, this is automatically true if $(\mathfrak{X}, \mathfrak{A}_0)$ has a unit).

In this case, if $(\mathfrak{X}, \mathfrak{A}_0)$ is a normed quasi *-algebra, a norm topology can be defined on $\mathfrak{A}_0$ in the following way. Define

$$\|a\|_L = \sup_{\|x\|\leq 1} \|xa\| \quad \text{and} \quad \|a\|_R = \sup_{\|x\|\leq 1} \|ax\|.$$

Finally, we put

$$\|a\|_0 = \max\{\|a\|, \|a\|_L, \|a\|_R\}.$$

Then we have

Proposition 4.2. $(\mathfrak{A}_0, \|\,\|_0)$ *is a normed* **-algebra. Moreover*,

$$\|ab\| \leq \|a\|\,\|b\|_0, \qquad \|ba\| \leq \|a\|\,\|b\|_0, \qquad \forall a, b \in \mathfrak{A}_0.$$

Clearly, $\|b\| \leq \|b\|_0$ for every $b \in \mathfrak{A}_0$.

Remark 4.3. If $(\mathfrak{X}, \mathfrak{A}_0)$ has a unit e, then $\|a\|_0 = \max\{\|a\|_L, \|a\|_R\}$.

Example 4.4. Many Banach function spaces provide examples of Banach quasi *-algebras since they often contain a dense *-algebra of functions. For instance, if $I = [0, 1]$ then $(L^p(I), C(I))$, where $C(I)$ denotes the C^*-algebra of all continuous functions on I and $p \geq 1$, is a Banach quasi *-algebra (more precisely, a proper CQ^*-algebra in the sense of [4]). Similarly $(L^p(\mathbb{R}), C_0^0(\mathbb{R}))$ is a Banach quasi *-algebra without unit ($C_0^0(\mathbb{R})$ is the *-algebra of continuous functions in $\mathbb{R}$ with compact support). Similarly, *non-commutative* L^p*-spaces* constructed from a von Neumann algebra $\mathfrak{M}$ and a normal semifinite faithful trace τ on $\mathfrak{M}$ can be cast into the framework of Banach quasi *-algebras [15, 19].

Example 4.5. Consider again the situation of Example 3.8. Let $\mathcal{D}^\times$ denote the Banach conjugate dual space of $\mathcal{D}[\|\cdot\|_1]$ endowed with the dual norm $\|\cdot\|_{-1}$, i.e.,

$$\|f\|_{-1} = \sup_{\|\xi\|_1\leq 1} |f(\xi)|, \qquad f \in \mathcal{D}^\times.$$

The Hilbert space $\mathcal{H}$ is canonically identified with a subspace of $\mathcal{D}^\times$ by the map $\xi \mapsto f_\xi$, where $f_\xi(\eta) = \langle \xi \,|\, \eta \rangle$ for every $\eta \in \mathcal{D}$. The form $b(\cdot,\cdot)$ which puts $\mathcal{D}$ and $\mathcal{D}^\times$ in conjugate duality is an extension of the inner product of $\mathcal{D}$ and therefore we adopt the same symbol for both. The space $\mathfrak{L}(\mathcal{D}, \mathcal{D}^\times)$

of all continuous linear maps from $\mathcal{D}[\|\cdot\|_1]$ into $\mathcal{D}^\times[\|\cdot\|_{-1}]$ carries a natural involution $X \mapsto X^\dagger$, defined by

$$\langle \xi \,|\, X^\dagger \eta\rangle = \overline{\langle \eta \,|\, X\xi\rangle}, \qquad \xi, \eta \in \mathcal{D},$$

and the norm

$$\|X\|_{\mathfrak{L}} = \sup_{\|\xi\|_1 \le 1} \|X\xi\|_{-1}.$$

The involution $X \mapsto X^\dagger$ is isometric, and as $\mathcal{D}^\times$ is complete, $\mathfrak{L}(\mathcal{D}, \mathcal{D}^\times)[\|\cdot\|_{\mathfrak{L}}]$ is a Banach space.

The space $\mathsf{B}(\mathcal{D}, \mathcal{D})$ also has a natural norm $\|\cdot\|_{\mathsf{B}}$ defined by

$$\|\varphi\|_{\mathsf{B}} = \sup\{|\varphi(\xi, \eta)| : \|\xi\|_1 = \|\eta\|_1 = 1\}.$$

With this norm, $\mathsf{B}(\mathcal{D}, \mathcal{D})$ is a Banach space. Moreover, $\|\varphi^*\|_{\mathsf{B}} = \|\varphi\|_{\mathsf{B}}$ for every $\varphi \in \mathsf{B}(\mathcal{D}, \mathcal{D})$.

For every $\varphi \in \mathsf{B}(\mathcal{D}, \mathcal{D})$, there exists $X \in \mathfrak{L}(\mathcal{D}, \mathcal{D}^\times)$ such that $\varphi = \varphi_X$, where

$$\varphi_X = \langle \xi \,|\, X\eta\rangle, \qquad \xi, \eta \in \mathcal{D}.$$

It is easy to prove that, in this case, $\|\varphi\|_{\mathsf{B}} = \|X\|_{\mathfrak{L}}$.

If $\mathcal{D}[\|\cdot\|_1]$ is a Banach space, then the converse is also true, i.e., if $X \in \mathfrak{L}(\mathcal{D}, \mathcal{D}^\times)$, then $\varphi_X \in \mathsf{B}(\mathcal{D}, \mathcal{D})$ and the map $X \mapsto \varphi_X$ is an isometric *-isomorphism [1, Ch. 10].

Let $\mathfrak{M}$ be an O^*-algebra on $\mathcal{D}$ (i.e., $\mathfrak{M}$ is a *-subalgebra of $\mathcal{L}^\dagger(\mathcal{D})$) with the property that each $X \in \mathfrak{M}$ is continuous from $\mathcal{D}[\|\cdot\|_1]$ into itself (this is always true if $\mathcal{D}[\|\cdot\|_1]$ is a reflexive space). Then $(\overline{\mathfrak{M}}, \mathfrak{M})$, where $\overline{\mathfrak{M}}$ denotes the closure of $\mathfrak{M}$ in $\mathcal{L}(\mathcal{D}, \mathcal{D}^\times)[\|\cdot\|_{\mathcal{L}}]$, is a Banach quasi *-algebra.

For instance, let $\mathcal{D} = D(S)$, where S is a positive self-adjoint operator with domain $D(S)$ dense in $\mathcal{H}$. If $S \ge \mathbb{I}$, then $\mathcal{D}$ is a Hilbert space with norm $\|\cdot\|_S$ defined by $\|\xi\|_S = \|S\xi\|$. In this case, $\mathfrak{L}(\mathcal{D}, \mathcal{D}^\times) \simeq \mathsf{B}(\mathcal{D}, \mathcal{D})$ and $\mathfrak{L}^\dagger(\mathcal{D}) = \mathcal{L}^\dagger(\mathcal{D})$. If $A \in \mathfrak{L}^\dagger(\mathcal{D})$, then

$$\|\varphi_A\|_{\mathsf{B}} = \sup\{|\langle A\xi \,|\, \eta\rangle| : \|S\xi\| = \|S\eta\| = 1\} = \|S^{-1}AS^{-1}\|. \tag{7}$$

For every O^*-algebra $\mathfrak{M}$ on $\mathcal{D}$, $(\overline{\mathfrak{M}}, \mathfrak{M})$ is a Banach quasi *-algebra.

Now we check that, in general, $\overline{\mathfrak{M}}$ is not a *-algebra. From the above discussion it follows that the set $S^{-1}\mathfrak{M}S^{-1}$ is a *-invariant vector space of bounded operators in $\mathcal{H}$. We denote by $\mathcal{M}_S$ its norm closure in $\mathcal{B}(\mathcal{H})$. Let

$$\mathfrak{M}_S^\dagger = \{\varphi \in \mathsf{B}^\dagger(\mathcal{D}) : \varphi = \varphi_A,\ A \in \mathfrak{M}\}$$

and $\overline{\mathfrak{M}_S^\dagger}$ its closure in $\mathsf{B}(\mathcal{D}, D)[\|\cdot\|_B]$. Then

$$\overline{\mathfrak{M}_S^\dagger} \subseteq \{\varphi \in \mathsf{B}(\mathcal{D}, D) : \varphi(\xi, \eta) = \langle YS\xi \,|\, S\eta\rangle \text{ for some } Y \in \mathcal{M}_S,\ \forall \xi, \eta \in \mathcal{D}\}.$$

Indeed, if $\varphi \in \overline{\mathfrak{M}_S^\dagger}$, then there exists a sequence $\{\varphi_n\} \subset \mathfrak{M}_S^\dagger$ converging to φ. Clearly, $\varphi_n = \varphi_{A_n}$ for some $A_n \in \mathcal{L}^\dagger(\mathcal{D})$. The sequence $\{\varphi_n\}$ being

Cauchy, by (7) we have $\|S^{-1}(A_n - A_m)S^{-1}\| \to 0$. Hence $S^{-1}A_nS^{-1} \to Y$ for some $Y \in \mathcal{B}(\mathcal{H})$. Clearly, $Y \in \mathcal{M}_S$. We have

$$\sup_{\|S\xi\|=\|S\eta\|=1} |\varphi(\xi,\eta) - \langle YS\xi \mid S\eta\rangle| \leq \sup_{\|S\xi\|=\|S\eta\|=1} |\varphi(\xi,\eta) - \langle A_n\xi \mid \eta\rangle| + \sup_{\|S\xi\|=\|S\eta\|=1} |\langle A_n\xi \mid \eta\rangle - \langle YS\xi \mid S\eta\rangle| \to 0,$$

since

$$\sup_{\|S\xi\|=\|S\eta\|=1} |\langle A_n\xi \mid \eta\rangle - \langle YS\xi \mid S\eta\rangle| = \sup_{\|\xi'\|=\|\eta'\|=1} |\langle A_nS^{-1}\xi' \mid S^{-1}\eta'\rangle - \langle Y\xi' \mid \eta'\rangle| = \|S^{-1}A_nS^{-1} - Y\|.$$

On the other hand, it is easily seen that $\overline{\mathfrak{M}_S^\dagger} \simeq \overline{\mathfrak{M}}$. Hence, if $X \in \overline{\mathfrak{M}}$, then, for some Y in $\mathcal{M}_S$, we have

$$\langle X\xi \mid \eta\rangle = \langle YS\xi \mid S\eta\rangle, \quad \forall \xi,\eta \in \mathcal{D}.$$

Thus, if $YS(\mathcal{D})$ is not a subset of $\mathcal{D}$, then X is neither an element of $\mathfrak{M}$ nor an operator in a Hilbert space but a *true* element of $\mathfrak{L}(\mathcal{D},\mathcal{D}^\times)$.

Now we come to the main topic of this section. We will define some seminorms (one of them is in fact an unbounded C^*-seminorm), closely related to families of sesquilinear forms [21, 24], and examine their interplay with the family of *-representations of a given quasi *-algebra $(\mathfrak{X},\mathfrak{A}_0)$. In the case where $(\mathfrak{X},\mathfrak{A}_0)$ is a normed quasi *-algebra, the seminorms provide some information on the structure of $(\mathfrak{X},\mathfrak{A}_0)$. First let us fix some terminology.

If σ is a seminorm on $\mathfrak{X}$, a sesquilinear form φ on $\mathfrak{X}\times\mathfrak{X}$ is said to be *σ-bounded* if there exists a positive constant γ such that

$$|\varphi(x,y)| \leq \gamma\sigma(x)\sigma(y), \quad \forall x,y \in \mathfrak{X}.$$

In this case, we put

$$\|\varphi\|_\sigma := \sup_{\sigma(x)=\sigma(y)=1} |\varphi(x,y)| = \sup_{\sigma(x)=1} \varphi(x,x).$$

If σ is exactly the norm of $\mathfrak{X}$, we say *bounded* instead of *σ-bounded* and we write $\|\varphi\|$ instead of $\|\varphi\|_\sigma$.

Let us now define

$$\mathfrak{q}_{\mathcal{I}}(x) = \sup\{\varphi(xa,xa)^{1/2} : \varphi \in \mathcal{I}(\mathfrak{X}),\ a \in \mathfrak{A}_0,\ \varphi(a,a)=1\} \tag{8}$$

and

$$\mathcal{D}(\mathfrak{q}_{\mathcal{I}}) = \{x \in \mathfrak{X} :\ \mathfrak{q}_{\mathcal{I}}(x) < \infty\}.$$

REMARK 4.6. If $(\mathfrak{X},\mathfrak{A}_0)$ has a unit e, then one can easily check that

$$\mathfrak{q}_{\mathcal{I}}(x) = \sup\{\varphi(x,x)^{1/2} : \varphi \in \mathcal{I}(\mathfrak{X}),\ \varphi(e,e)=1\}.$$

Proposition 4.7. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a quasi *-algebra. For each $\varphi \in \mathcal{I}(\mathfrak{X})$, let π_φ denote the corresponding GNS representation. Then*

$$\mathcal{D}(\mathfrak{q}_\mathcal{I}) = \{x \in \mathfrak{X} : \overline{\pi_\varphi(x)} \in \mathcal{B}(\mathcal{H}_\varphi),\ \forall \varphi \in \mathcal{I}(\mathfrak{X}),\ \textit{and} \ \sup_{\varphi \in \mathcal{I}(\mathfrak{X})} \|\overline{\pi_\varphi(x)}\| < \infty\}$$
$$= \{x \in \mathfrak{X} : \pi(x) \ \textit{is bounded}, \forall \pi \in \mathrm{Rep}_r(\mathfrak{X}),\ \textit{and} \ \sup_{\pi \in \mathrm{Rep}_r(\mathfrak{X})} \|\overline{\pi(x)}\| < \infty\}$$

and

$$\mathfrak{q}_\mathcal{I}(x) = \sup_{\varphi \in \mathcal{I}(\mathfrak{X})} \|\overline{\pi_\varphi(x)}\| = \sup_{\pi \in \mathrm{Rep}_r(\mathfrak{X})} \|\overline{\pi(x)}\|, \qquad \forall x \in \mathcal{D}(\mathfrak{q}). \tag{9}$$

Proof. We may assume that $(\mathfrak{X}, \mathfrak{A}_0)$ has a unit e. For brevity we put

$$\mathfrak{X}_0 = \{x \in \mathfrak{X} : \overline{\pi_\varphi(x)} \in \mathcal{B}(\mathcal{H}_\varphi),\ \forall \varphi \in \mathcal{I}(\mathfrak{X}),\ \text{and} \ \sup_{\varphi \in \mathcal{I}(\mathfrak{X})} \|\overline{\pi_\varphi(x)}\| < \infty\}$$

and

$$\mathfrak{X}_1 = \{x \in \mathfrak{X} : \pi(x) \text{ is bounded }, \forall \pi \in \mathrm{Rep}_r(\mathfrak{X}),\ \text{and} \ \sup_{\pi \in \mathrm{Rep}_r(\mathfrak{X})} \|\overline{\pi(x)}\| < \infty\}.$$

Let $x \in \mathcal{D}(\mathfrak{q}_\mathcal{I})$. If $\varphi \in \mathcal{I}(\mathfrak{X})$, then

$$\varphi(xa, xa) \le \mathfrak{q}_\mathcal{I}(x)^2 \varphi(a, a), \qquad \forall a \in \mathfrak{A}_0.$$

Hence, $\pi_\varphi(x)$ is bounded and $\|\overline{\pi_\varphi(x)}\| \le \mathfrak{q}_\mathcal{I}(x)$. Therefore, $x \in \mathfrak{X}_0$ and

$$\sup_{\varphi \in \mathcal{I}(\mathfrak{X})} \|\overline{\pi_\varphi(x)}\| \le \mathfrak{q}_\mathcal{I}(x).$$

Let $x \in \mathfrak{X}_0$. Clearly, $\pi_\varphi(x)$ is bounded if, and only if, $\pi_\varphi^\circ(x)$ is bounded. Since π_φ° is regular (Proposition 3.11), this implies that

$$\sup_{\varphi \in \mathcal{I}(\mathfrak{X})} \|\overline{\pi_\varphi(x)}\| \le \sup_{\pi \in \mathrm{Rep}_r(\mathfrak{X})} \|\overline{\pi(x)}\|. \tag{10}$$

On the other hand, if $\pi \in \mathrm{Rep}_r(\mathfrak{X})$ then, for every $\xi \in \mathcal{D}_\pi$, we consider the corresponding vector form φ_ξ. The regularity of π implies that $\varphi_\xi \in \mathcal{I}(\mathfrak{X})$ and so the *-representation $\pi^\circ_{\varphi_\xi}$ with cyclic vector $\xi_\varphi = \lambda_{\varphi_\xi}(e)$ can be constructed. By assumption, the operator $\pi^\circ_{\varphi_\xi}(x)$ is bounded. Then

$$\|\pi(x)\xi\|^2 = \|\pi^\circ_{\varphi_\xi}(x)\lambda_{\varphi_\xi}(e)\|^2 \le \|\pi^\circ_{\varphi_\xi}(x)\|^2 \|\xi\|^2,$$

which implies that $\pi(x)$ is bounded and that the opposite inequality to (10) holds, i.e., $\mathfrak{X}_0 \subseteq \mathfrak{X}_1$. Therefore, it is sufficient to prove $\mathcal{D}(\mathfrak{q}_\mathcal{I}) = \mathfrak{X}_1$ and

$$\mathfrak{q}_\mathcal{I}(x) = \sup_{\pi \in \mathrm{Rep}_r(\mathfrak{X})} \|\overline{\pi(x)}\|, \qquad \forall x \in \mathcal{D}(\mathfrak{q}_\mathcal{I}).$$

Now, let $\pi \in \mathrm{Rep}_r(\mathfrak{X})$ and, for $\xi \in \mathcal{D}_\pi$, define φ_ξ as above. From (11) it follows that

$$\varphi_\xi(xa, xa) \le \mathfrak{q}_\mathcal{I}(x)^2 \varphi_\xi(a, a), \qquad \forall x \in \mathcal{D}(\mathfrak{q}_\mathcal{I}),\ a \in \mathfrak{A}_0.$$

This implies that

$$\|\pi(x)\xi\|^2 = \varphi_\xi(x,x) \le \mathfrak{q}_{\mathcal{I}}(x)^2\varphi_\xi(e,e) = \mathfrak{q}_{\mathcal{I}}(x)^2\|\xi\|^2, \quad \forall x \in \mathcal{D}(\mathfrak{q}_{\mathcal{I}}).$$

Thus, for every $x \in \mathcal{D}(\mathfrak{q}_{\mathcal{I}})$, $\pi(x)$ is a bounded operator and

$$\sup_{\pi\in \mathrm{Rep}_r(\mathfrak{X})} \|\overline{\pi(x)}\| \le \mathfrak{q}_{\mathcal{I}}(x) < \infty.$$

Conversely, if $\pi(x)$ is bounded for every $\pi \in \mathrm{Rep}_r(\mathfrak{X})$ and

$$\sup_{\pi\in \mathrm{Rep}_r(\mathfrak{X})} \|\overline{\pi(x)}\| < \infty,$$

then, in particular, for any $\varphi \in \mathcal{I}(\mathfrak{X})$,

$$\begin{aligned}\varphi(xa,xa) = \|\pi_\varphi^\circ(x)\lambda_\varphi(a)\|^2 &\le \sup_{\pi\in \mathrm{Rep}_r(\mathfrak{X})} \|\overline{\pi(x)}\|^2 \cdot \|\lambda_\varphi(a)\|^2 \\ &= \sup_{\pi\in \mathrm{Rep}_r(\mathfrak{X})} \|\overline{\pi(x)}\|^2 \cdot \varphi(a,a), \quad \forall x \in \mathfrak{X},\ a \in \mathfrak{A}_0.\end{aligned}$$

Therefore $x \in D(\mathfrak{q}_{\mathcal{I}})$ and

$$\mathfrak{q}_{\mathcal{I}}(x) \le \sup_{\pi\in \mathrm{Rep}_r(\mathfrak{X})} \|\overline{\pi(x)}\|.$$

This concludes the proof. ■

Let now $(\mathfrak{X}, \mathfrak{A}_0)$ be a normed quasi *-algebra. We put

$$\mathcal{P}(\mathfrak{X}) = \{\varphi \in \mathcal{Q}(\mathfrak{X}) : \varphi \text{ is bounded}\}.$$

Then $\mathcal{P}(\mathfrak{X}) \subseteq \mathcal{I}(\mathfrak{X})$. In fact, if $\varphi \in \mathcal{P}(\mathfrak{X})$, then the subspace $\lambda_\varphi(\mathfrak{A}_0)$ is dense in $\mathcal{H}_\varphi$. Indeed, if $x \in \mathfrak{X}$, there exists a sequence $\{a_n\} \subset \mathfrak{A}_0$ such that $a_n \to x$ in $\mathfrak{X}$. Then

$$\|\lambda_\varphi(x) - \lambda_\varphi(a_n)\|^2 = \varphi(x-a_n, x-a_n) \le \|\varphi\|^2\|x-a_n\|^2 \to 0.$$

Finally, we define

$$\mathcal{S}(\mathfrak{X}) = \{\varphi \in \mathcal{P}(\mathfrak{X}) : \|\varphi\| \le 1\}.$$

REMARK 4.8. Of course, the possibility that $\mathcal{S}(\mathfrak{X}) = \{0\}$ is not excluded (see Example 4.18 below).

EXAMPLE 4.9. We give an example where $\mathcal{I}(\mathfrak{X}) \supset \mathcal{P}(\mathfrak{X})$. Consider the CQ^*-algebra $(L^1(I), L^\infty(I))$, $I = [0,1]$. For $x \in L^1(I)$ we denote by x_0 its restriction to $I_a := [0,a]$ with $0 < a < 1$. Define

$$\mathfrak{X} = \{x \in L^1(I) : x_0 \in L^2(I_a)\}.$$

Clearly $(\mathfrak{X}, L^\infty(I))$, when $\mathfrak{X}$ is endowed with the norm induced by $L^1(I)$, is a normed quasi *-algebra. It is easily shown that the positive sesquilinear form φ defined by

$$\varphi(x,y) = \int_0^a x_0(t)\overline{y_0(t)}\,dt$$

is an element of $\mathcal{I}(\mathfrak{X})$. In this case, in fact, $\mathfrak{X}/N_\varphi \sim L^2(I \setminus I_a)$ and $\lambda_\varphi(L^\infty(I)) \sim L^\infty(I \setminus I_a)$, which is dense in $L^2(I \setminus I_a)$. As shown in [5], $\mathcal{P}(L^1(I)) = \{0\}$; thus $\mathcal{P}(\mathfrak{X}) = \{0\}$ too. Therefore $\varphi \notin \mathcal{P}(\mathfrak{X})$.

Example 4.10. Consider again a Banach quasi *-algebra of the type $(\overline{\mathfrak{M}}, \mathfrak{M})$ constructed in Example 4.5. We put

$$\mathcal{D}_0(\mathfrak{M}) = \{\xi \in \mathcal{D} : X\xi \in \mathcal{H},\ \forall X \in \overline{\mathfrak{M}}\}.$$

For $\xi \in \mathcal{D}_0(\mathfrak{M})$, we define

$$\varphi_\xi(X, Y) = \langle X\xi \,|\, Y\eta\rangle, \qquad X, Y \in \overline{\mathfrak{M}}.$$

Then, as is easy to see, $\varphi_\xi \in \mathcal{Q}(\overline{\mathfrak{M}})$.

From the definitions, it is easily seen that:

- $\varphi_\xi \in \mathcal{I}(\overline{\mathfrak{M}}) \Leftrightarrow \xi \in \mathcal{D}_0(\mathfrak{M})$ and $X\xi \in \overline{\mathfrak{M}\xi}$ for all $X \in \overline{\mathfrak{M}}$, where $\overline{\mathfrak{M}\xi}$ denotes the closure of $\mathfrak{M}\xi$ in $\mathcal{H}$;
- $\varphi_\xi \in \mathcal{P}(\overline{\mathfrak{M}}) \Leftrightarrow \xi \in \mathcal{D}_0(\mathfrak{M})$ and $\sup_{\|X\|_{\mathcal{L}} \le 1} \|X\xi\| < \infty$.

It is worth mentioning that one can construct examples where $\mathcal{D}_0(\mathfrak{M}) = \{0\}$. For instance, let $\mathcal{H} = L^2(I)$ where $I = [0, 1]$ and, for $p > 2$, let $\mathcal{D} = L^p(I)$. If η is a measurable function, denote by M_η the operator of multiplication by η. Take for $\mathfrak{M}$ the O^*-algebra of multiplication operators by a function $\phi \in L^\infty(I)$, i.e.,

$$\mathfrak{M} = \{M_\phi : \phi \in L^\infty(I)\}.$$

Then it is easily seen that

$$\overline{\mathfrak{M}} = \{M_\phi : \phi \in L^{p/(p-2)}(I)\} \quad \text{and} \quad \|M_\phi\|_{\mathcal{L}} = \|\phi\|_{p/(p-2)}.$$

Then we have the following situation.

- If $2 < p < 4$, then $\mathcal{D}_0(\mathfrak{M}) = L^{2p/(4-p)}(I)$ and every φ_ξ, $\xi \in \mathcal{D}_0(\mathfrak{M})$, is bounded.
- If $p = 4$ then $\mathcal{D}_0(\mathfrak{M}) = L^\infty(I)$ and, again, every φ_ξ, $\xi \in \mathcal{D}_0(\mathfrak{M})$, is bounded.
- If $p > 4$, then $\mathcal{D}_0(\mathfrak{M}) = \{0\}$.

The following lemma (whose proof is based on Kaplansky's inequality) will often be used in what follows. We recall that an *m^*-seminorm* on a *-algebra $\mathfrak{A}_0$ is a seminorm σ satisfying:

(i) $\sigma(a^*) = \sigma(a)$, $\forall a \in \mathfrak{A}_0$;
(ii) $\sigma(ab) \le \sigma(a)\sigma(b)$, $\forall a, b \in \mathfrak{A}_0$.

Lemma 4.11. *Let $\mathfrak{A}_0$ be a *-algebra and ω a positive linear functional on $\mathfrak{A}_0$. Assume that there exists an m^*-seminorm σ on $\mathfrak{A}_0$ such that*

$$\forall b \in \mathfrak{A}_0,\ \exists \gamma_b > 0 : \quad |\omega(b^*ab)| \le \gamma_b \sigma(a), \quad \forall a \in \mathfrak{A}_0.$$

Then

$$|\omega(b^*ab)| \leq \sigma(a)\omega(b^*b), \qquad \forall a \in \mathfrak{A}_0.$$

Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a normed quasi *-algebra. We put

$$\mathfrak{p}(x) = \sup_{\varphi \in \mathcal{S}(\mathfrak{X})} \varphi(x,x)^{1/2}.$$

Then $\mathfrak{p}$ is a seminorm on $\mathfrak{X}$ with $\mathfrak{p}(x) \leq \|x\|$ for every $x \in \mathfrak{X}$.

Let us now define a second seminorm $\mathfrak{q}$ as follows:

$$\mathfrak{q}(x) = \sup\{\varphi(xa,xa)^{1/2} : \varphi \in \mathcal{P}(\mathfrak{X}),\ a \in \mathfrak{A}_0,\ \varphi(a,a) = 1\} \tag{11}$$

and

$$\mathcal{D}(\mathfrak{q}) = \{x \in \mathfrak{X} :\ \mathfrak{q}(x) < \infty\}.$$

Clearly, $\mathcal{D}(\mathfrak{q}_{\mathcal{I}}) \subseteq \mathcal{D}(\mathfrak{q})$ and $\mathfrak{q}(x) \leq \mathfrak{q}_{\mathcal{I}}(x)$ for every $x \in \mathcal{D}(\mathfrak{q}_{\mathcal{I}})$. Just as for $\mathfrak{q}_{\mathcal{I}}$ one can easily check that if $(\mathfrak{X}, \mathfrak{A}_0)$ has a unit e, then

$$\mathfrak{q}(x) = \sup\{\varphi(x,x)^{1/2} : \varphi \in \mathcal{P}(\mathfrak{X}),\ \varphi(e,e) = 1\}.$$

In order to obtain a description of $\mathcal{D}(\mathfrak{q})$ similar to that of $\mathcal{D}(\mathfrak{q}_{\mathcal{I}})$, we give the following

Definition 4.12. Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a normed quasi *-algebra and π a *-representation of $(\mathfrak{X}, \mathfrak{A}_0)$ with domain $\mathcal{D}_\pi$. We say that π is *completely regular* if, for every $\xi \in \mathcal{D}_\pi$, the positive sesquilinear form φ_ξ is bounded. The set of all completely regular *-representations of $(\mathfrak{X}, \mathfrak{A}_0)$ is denoted by $\mathrm{Rep}_{\mathrm{cr}}(\mathfrak{X})$.

Clearly, if π is completely regular, then it is regular.

Proposition 4.13. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a normed quasi *-algebra. For each $\varphi \in \mathcal{P}(\mathfrak{X})$, let π_φ denote the corresponding GNS representation. Then*

$$\begin{aligned}\mathcal{D}(\mathfrak{q}) &= \{x \in \mathfrak{X} : \overline{\pi_\varphi(x)} \in \mathcal{B}(\mathcal{H}_\varphi),\ \forall \varphi \in \mathcal{P}(\mathfrak{X}),\ \text{and} \sup_{\varphi \in \mathcal{P}(\mathfrak{X})} \|\overline{\pi_\varphi(x)}\| < \infty\} \\ &= \{x \in \mathfrak{X} : \pi(x) \text{ is bounded}, \forall \pi \in \mathrm{Rep}_{\mathrm{cr}}(\mathfrak{X}),\ \text{and} \sup_{\pi \in \mathrm{Rep}_{\mathrm{cr}}(\mathfrak{X})} \|\overline{\pi(x)}\| < \infty\}\end{aligned}$$

and

$$\mathfrak{q}(x) = \sup_{\varphi \in \mathcal{P}(\mathfrak{X})} \|\overline{\pi_\varphi(x)}\| = \sup_{\pi \in \mathrm{Rep}_{\mathrm{cr}}(\mathfrak{X})} \|\overline{\pi(x)}\|, \qquad \forall x \in \mathcal{D}(\mathfrak{q}). \tag{12}$$

Proof. The proof is very similar to that of Proposition 4.7, so we do not repeat the details. The only point to be taken into account is that if $\varphi \in \mathcal{P}(\mathfrak{X})$ then the corresponding representation π_φ° is completely regular. Indeed, if $\xi = \lambda_\varphi(a)$ then

$$\varphi_\xi(x,x) = \langle \pi_\varphi^\circ(x)\lambda_\varphi(a) \mid \pi_\varphi^\circ(x)\lambda_\varphi(a)\rangle = \varphi(xa,xa) \leq \gamma\|a\|_0^2\|x\|^2.$$

Hence φ_ξ is bounded. ∎

We will show that $\mathfrak{q}$, together with $\mathfrak{p}$, plays a crucial role for the structure of a normed or Banach quasi *-algebra.

The following preliminary proposition has been given in [22]. Statement (i) follows from Lemma 4.11, while (ii) can be easily deduced from Proposition 4.13.

PROPOSITION 4.14. *The following statements hold*:

(i) $\mathfrak{A}_0 \subseteq \mathcal{D}(\mathfrak{q})$ *and* $\mathfrak{q}(a) \leq \|a\|_0$, $\forall a \in \mathfrak{A}_0$.
(ii) $\mathfrak{q}$ *is an extended* C^**-seminorm on* $(\mathfrak{X}, \mathfrak{A}_0)$ [*i.e.* $\mathfrak{q}(x^*) = \mathfrak{q}(x)$, $\forall x \in \mathcal{D}(\mathfrak{q})$; $\mathfrak{q}(a^*a) = \mathfrak{q}(a)^2$, $\forall a \in \mathfrak{A}_0$, *see* [21]].
(iii) $\mathfrak{p}(xa) \leq \mathfrak{q}(x)\mathfrak{p}(a)$, $\forall x \in D(\mathfrak{q})$, $a \in \mathfrak{A}_0$.
(iv) $\mathfrak{p}(ax) \leq \|a\|_0 \mathfrak{p}(x)$, $\forall x \in \mathfrak{X}$, $a \in \mathfrak{A}_0$.

REMARK 4.15. If $(\mathfrak{X}, \mathfrak{A}_0)$ has a unit e, then from (iii) it follows that $\mathfrak{p}(x) \leq \mathfrak{q}(x)$ for every $x \in D(\mathfrak{q})$.

Now we put
$$N(\mathfrak{p}) = \{x \in \mathfrak{X} : \mathfrak{p}(x) = 0\}.$$
By Proposition 4.14(iv), $N(\mathfrak{p})$ is a left submodule of $\mathfrak{X}$.

Set $N_0(\mathfrak{p}) = N(\mathfrak{p}) \cap \mathfrak{A}_0$. Then the quotient $\mathfrak{A}_0^{\mathfrak{p}} := \mathfrak{A}_0/N_0(\mathfrak{p})$ is a normed space with norm $\|a + N_0(\mathfrak{p})\|_{\mathfrak{p}} = \mathfrak{p}(a)$, $a \in \mathfrak{A}_0$. Denote by $\mathfrak{X}_{\mathfrak{p}}$ the completion of $(\mathfrak{A}_0/N_0(\mathfrak{p}), \|\cdot\|_{\mathfrak{p}})$.

PROPOSITION 4.16. *The quotient* $\mathfrak{X}/N(\mathfrak{p})$ *can be identified with a dense subspace of* $\mathfrak{X}_{\mathfrak{p}}$. *Moreover*:

(i) *If* $\mathfrak{p}(x) = \mathfrak{p}(x^*)$ *for every* $x \in \mathfrak{X}$, *then* $\mathfrak{X}_{\mathfrak{p}}$ *is a Banach space with isometric involution extending the natural involution of* $\mathfrak{A}_0^{\mathfrak{p}}$; $\mathfrak{A}_0^{\mathfrak{p}}$ *is a* **-algebra and* $(\mathfrak{X}_{\mathfrak{p}}, \mathfrak{A}_0^{\mathfrak{p}})$ *can be made into a Banach quasi* **-algebra.*
(ii) *If* $\mathfrak{p}(ax) \leq \mathfrak{p}(a)\mathfrak{p}(x)$ *for every* $a \in \mathfrak{A}_0$ *and* $x \in \mathfrak{X}$, *then* $\mathfrak{X}_{\mathfrak{p}}$ *is a Banach algebra.*
(iii) *If* $\mathfrak{p}$ *is an* m^**-seminorm on* $\mathfrak{A}_0$, *then* $\mathfrak{X}_{\mathfrak{p}}$ *is a Banach* **-algebra.*
(iv) *If* $\mathfrak{p}$ *is a* C^**-seminorm on* $\mathfrak{A}_0$, *then* $\mathfrak{X}_{\mathfrak{p}}$ *is a* C^**-algebra.*

Proof. Let $x \in \mathfrak{X}$. Then there exists a sequence $\{a_n\} \subset \mathfrak{A}_0$ such that $\|x - a_n\| \to 0$ as $n \to \infty$. This implies that $\mathfrak{p}(x - a_n) \to 0$ as $n \to \infty$. We define $\widehat{x} = \|\cdot\|_{\mathfrak{p}}\text{-}\lim_{n\to\infty}(a_n + N_0(\mathfrak{p}))$. By the construction of the completion, $\widehat{x}$ does not depend on the choice of the sequence $\{a_n\}$. Moreover, the map
$$j : x + N(\mathfrak{p}) \in \mathfrak{X}/N(\mathfrak{p}) \mapsto \widehat{x} \in \mathfrak{X}_{\mathfrak{p}}$$
is well-defined. Indeed, if $x, x' \in \mathfrak{X}$, $x - x' \in N(\mathfrak{p})$ and $b_n \to x - x'$ with respect to the norm of $\mathfrak{X}$, then $\mathfrak{p}(b_n) \to 0$ and so $j(x - x') = 0$. Finally, j is injective. Indeed, assume that $\widehat{x} = 0$ and let $\{a_n\} \subset \mathfrak{A}_0$ be such that $\|x - a_n\| \to 0$ as $n \to \infty$. Then $\|\cdot\|_{\mathfrak{p}}\text{-}\lim_{n\to\infty}(a_n + N_0(\mathfrak{p})) = 0$. Hence $\mathfrak{p}(a_n) \to 0$ as $n \to \infty$. This, in turn, implies that $\mathfrak{p}(x) = 0$ and so $x \in N(\mathfrak{p})$.

The proofs of (ii), (iii) and (iv) are simple checks. For (i), notice that Proposition 4.14(iv) implies that $N_0(\mathfrak{p})$ is a *-ideal of $\mathfrak{A}_0$, and so $\mathfrak{A}_0/N_0(\mathfrak{p})$ is a *-algebra. To define the multiplication that makes $(\mathfrak{X}_\mathfrak{p}, \mathfrak{A}_0^\mathfrak{p})$ a quasi *-algebra, we proceed as follows: if $z \in \mathfrak{X}_\mathfrak{p}$, $z = \|\cdot\|_\mathfrak{p}\text{-}\lim_{n\to\infty}(a_n + N_0(\mathfrak{p}))$, $a_n \in \mathfrak{A}_0$ and $a \in \mathfrak{A}_0$, Proposition 4.14(iv) shows that the sequence $(aa_n + N_0(\mathfrak{p}))$ is $\|\cdot\|_\mathfrak{p}$-Cauchy and its limit does not depend on the particular choice of (a_n). Thus we can define $az = \|\cdot\|_\mathfrak{p}\text{-}\lim_{n\to\infty}(aa_n + N_0(\mathfrak{p}))$. ∎

Remark 4.17. We will show that (iii) and (iv) are indeed equivalent.

Example 4.18. Consider the Banach quasi *-algebra $(L^p(I), C(I))$ of Example 4.4 with $I = [0,1]$. In this case (see [5]),

$$\mathcal{P}(L^p(I)) = \begin{cases} \{\varphi_w : w \in L^{p/(p-2)}(I),\ w \geq 0\} & \text{if } p \geq 2, \\ \{0\} & \text{if } 1 \leq p < 2, \end{cases}$$

where

$$\varphi_w(x,y) = \int_I x(t)\overline{y(t)}w(t)\,dt, \qquad x, y \in L^p(I).$$

If $1 \leq p < 2$ both $\mathfrak{p}$ and $\mathfrak{q}$ are identically zero. If $p \geq 2$, then one can prove that $\mathfrak{p}(x) = \|x\|_p$ and $\mathfrak{q}(x) = \sup\{\varphi_w(x,x)^{1/2} : w \in L^{p/(p-2)}(I),\ \|w\|_1 \leq 1\}$, which is finite if, and only if, $x \in L^\infty(I)$. In fact,

$$\mathfrak{q}(x) = \|x\|_\infty, \qquad \forall x \in L^\infty(I).$$

Example 4.19. A *Hilbert algebra* [16, Section 11.7] is a *-algebra $\mathfrak{A}_0$ which is also a pre-Hilbert space with inner product $\langle \cdot \,|\, \cdot \rangle$ such that:

(i) The map $b \mapsto ab$ is continuous with respect to the norm defined by the inner product.
(ii) $\langle ab \,|\, c\rangle = \langle b \,|\, a^* c\rangle$ for all $a, b, c \in \mathfrak{A}_0$.
(iii) $\langle a \,|\, b\rangle = \langle b^* \,|\, a^*\rangle$ for all $a, b \in \mathfrak{A}_0$.
(iv) $\mathfrak{A}_0^2$ is total in $\mathfrak{A}_0$.

Let $\mathcal{H}$ denote the Hilbert space which is the completion of $\mathfrak{A}_0$ with respect to the norm defined by the inner product. The involution of $\mathfrak{A}_0$ extends to the whole of $\mathcal{H}$, since (iii) implies that * is isometric. Thus $(\mathcal{H}, \mathfrak{A}_0)$ is a Banach quasi *-algebra.

Since the inner product of $\mathcal{H}$ is an element of $\mathcal{P}(\mathcal{H})$, one has $\mathfrak{p}(x) = \|x\|$ for every $x \in \mathcal{H}$. As for $\mathfrak{q}$, it is easily seen that

$$\mathcal{D}(\mathfrak{q}) = \{x \in \mathfrak{X} : L_x \text{ is bounded}\} = \{x \in \mathfrak{X} : R_x \text{ is bounded}\}$$

and

$$\mathfrak{q}(x) = \|L_x\| = \|R_x\|, \qquad \forall x \in D(\mathfrak{q}),$$

where $L_x : a \in \mathfrak{A}_0 \mapsto xa \in \mathcal{H}$ and $R_x : a \in \mathfrak{A}_0 \mapsto ax \in \mathcal{H}$. Thus $D(\mathfrak{q})$ coincides, as already shown in [22], with the set of *bounded* elements of $\mathfrak{X}$.

We conclude this section by discussing, in the case where $(\mathfrak{X}, \mathfrak{A}_0)$ is a Banach quasi *-algebra, some results on the automatic continuity for the classes of positive linear functionals and positive sesquilinear forms introduced so far. For this we need some lemmas (4.20 and 4.21 below) which are already known in slightly different situations. For the sake of completeness we give the proofs adapted to the cases under consideration.

LEMMA 4.20. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra and φ a positive sesquilinear form on $\mathfrak{X} \times \mathfrak{X}$. Assume that φ is lower semicontinuous*, *i.e.*, *if $\{x_n\} \subset \mathfrak{X}$ is a sequence converging to $x \in \mathfrak{X}$ with respect to $\|\cdot\|$*, *one has*

$$\varphi(x, x) \le \liminf_{n\to\infty} \varphi(x_n, x_n).$$

Then φ is bounded.

Proof. We will show that $\mathfrak{X}$ is also complete with respect to the norm $\|\cdot\|_\varphi$ defined by

$$\|x\|_\varphi = \sqrt{\|x\|^2 + \varphi(x, x)}.$$

This implies that $\|\cdot\|$ and $\|\cdot\|_\varphi$ are equivalent, and therefore φ is bounded.

Let $\{x_n\}$ be a Cauchy sequence with respect to $\|\cdot\|_\varphi$. Then, for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\|x_n - x_m\|^2 + \varphi(x_n - x_m, x_n - x_m) < \varepsilon^2, \quad \forall n, m > n_\varepsilon.$$

The completeness of $\mathfrak{X}[\|\cdot\|]$ implies the existence of an element $x \in \mathfrak{X}$ such that $\lim_{n\to\infty} \|x - x_n\| = 0$. Now, fix $m > n_\varepsilon$ and let $n \to \infty$. We get

$$\|x - x_m\|^2 + \varphi(x - x_m, x - x_m) \le \|x - x_m\|^2 + \liminf_{n\to\infty} \varphi(x_n - x_m, x_n - x_m) \le \varepsilon^2.$$

Hence $\|x - x_m\|_\varphi \to 0$. This proves that $\mathfrak{X}$ is complete with respect to $\|\cdot\|_\varphi$. ∎

Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra. We denote by $\mathfrak{A}_0^+$ the set of *positive* elements of $\mathfrak{A}_0$, i.e.,

$$\mathfrak{A}_0^+ = \Big\{ \sum_{k=1}^n a_k^* a_k : a_k \in \mathfrak{A}_0,\ k = 1, \ldots, n;\ n \in \mathbb{N} \Big\}.$$

We put $\mathfrak{X}^+ = \overline{\mathfrak{A}_0^+}$, the closure of $\mathfrak{A}_0^+$ in the norm topology of $\mathfrak{X}$. Elements of $\mathfrak{X}^+$ will also be called *positive*. A linear functional ω on $\mathfrak{X}$ is called *positive* if $\omega(x) \ge 0$ for every $x \in \mathfrak{X}^+$.

LEMMA 4.21. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra. Then every positive linear functional ω on $\mathfrak{X}$ is bounded on positive elements*, *i.e.*, *there exists $\gamma > 0$ such that*

$$\omega(x) \le \gamma \|x\|, \quad \forall x \in \mathfrak{X}^+.$$

Proof. Were it not so, there would exist a sequence $\{x_n\}$ of positive elements of $\mathfrak{X}$ such that $\|x_n\| \le 2^{-n}$ and $\omega(x_n) \to \infty$. Let $y = \sum_{k=1}^\infty x_k$.

Then

$$\omega(y) = \omega\Big(\sum_{k=1}^{\infty} x_k\Big) \geq \omega\Big(\sum_{k=1}^{n} x_k\Big) = \sum_{k=1}^{n} \omega(x_k) \to \infty. \ \blacksquare$$

THEOREM 4.22. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra satisfying the following condition:*

(D) *Every $x = x^* \in \mathfrak{X}$ can be uniquely decomposed as $x = x_+ - x_-$ with $x_+, x_- \in \mathfrak{X}^+$ and $\|x\| = \|x_+\| + \|x_-\|$.*

Then every $\varphi \in \mathcal{I}(\mathfrak{X})$ such that ω_{φ_a} is positive for every $a \in \mathfrak{A}_0$ is bounded.

Proof. Let $a \in \mathfrak{A}_0$. Since ω_{φ_a}, defined as in (6), is positive, Lemma 4.21 shows that ω_{φ_a} is bounded on positive elements, i.e., there exists $\gamma > 0$ such that

$$\omega_{\varphi_a}(x) \leq \gamma \|x\|, \qquad \forall x \in \mathfrak{X}^+.$$

Condition (D) then implies that, for every $x = x^* \in \mathfrak{X}$,

$$|\omega_{\varphi_a}(x)| = |\omega_{\varphi_a}(x_+ - x_-)| \leq \omega_{\varphi_a}(x_+) + \omega_{\varphi_a}(x_-) \leq \gamma(\|x_+\| + \|x_-\|) = \gamma\|x\|.$$

The general statement is easily obtained by decomposing every $z \in \mathfrak{X}$ as $z = x + iy$ with $x = x^*$, $y = y^*$. Using the polarization identity, one proves easily that, for every $a, b \in \mathfrak{A}_0$, the linear functional $L_{a,b}(x) = \varphi(xa, b)$ is bounded.

Let now $\{x_n\} \subset \mathfrak{X}$ and $x \in \mathfrak{X}$ with $\lim_{n\to\infty} \|x_n - x\| = 0$. For every $b \in \mathfrak{A}_0$, by the Cauchy–Schwarz inequality, we have

$$|\varphi(x_n, b)| \leq \varphi(x_n, x_n)^{1/2} \varphi(b, b)^{1/2}.$$

Taking the lim inf of both sides, we get

$$|\varphi(x, b)| \leq \liminf_{n\to\infty} \varphi(x_n, x_n)^{1/2} \varphi(b, b)^{1/2}.$$

Now, since $\varphi \in \mathcal{I}(\mathfrak{X})$, there exists a sequence $\{a_k\} \subset \mathfrak{A}_0$ such that $\varphi(x - a_k, x - a_k) \to 0$. This implies that

$$\lim_{k\to\infty} \varphi(x, a_k) = \varphi(x, x) \quad \text{and} \quad \lim_{k\to\infty} \varphi(a_k, a_k) = \varphi(x, x).$$

Then from

$$|\varphi(x, a_k)| \leq \liminf_{n\to\infty} \varphi(x_n, x_n)^{1/2} \varphi(a_k, a_k)^{1/2}$$

we obtain, for $k \to \infty$,

$$\varphi(x, x) \leq (\liminf_{n\to\infty} \varphi(x_n, x_n))^{1/2} \varphi(x, x)^{1/2}.$$

Hence,

$$\varphi(x, x) \leq \liminf_{n\to\infty} \varphi(x_n, x_n),$$

i.e., φ is lower semicontinuous. The statement then follows from Lemma 4.20. $\blacksquare$

5. Continuity of *-representations. The seminorms $\mathfrak{p}$ and $\mathfrak{q}$, introduced in the previous section, play an interesting role also in the study of the continuity of a *-representation. As we will see at the end of this section, the most favorable situation occurs when $\mathfrak{p}$ is an m^*-seminorm. In this case, in fact, $\mathfrak{X}$ may be viewed (up to a quotient) as a subspace of the C^*-algebra $\mathfrak{X}_\mathfrak{p}$ and (as expected) any regular *-representation is bounded and norm-continuous. But this is, in a sense, a rather extreme situation rarely realized in practice, and Banach quasi *-algebras having unbounded *-representations do really exist. For this reason, we begin by looking for conditions that guarantee the strong continuity of any regular *-representation.

Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a normed quasi *-algebra with unit e and π a *-representation of $(\mathfrak{X}, \mathfrak{A}_0)$ into $\mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi)$. We say that π is

- *strongly continuous* if π is continuous from $\mathfrak{X}[\|\cdot\|]$ into $\mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi)[\tau_s]$.
- *strongly* continuous* if π is continuous from $\mathfrak{X}[\|\cdot\|]$ into $\mathcal{L}^\dagger(\mathcal{D}_\pi, \mathcal{H}_\pi)[\tau_{s^*}]$.

It is easy to prove that a *-representation is strongly continuous if, and only if, it is strongly* continuous.

Proposition 5.1. *The following statements hold*:

(i) *Every strongly continuous *-representation is regular.*
(ii) *Every completely regular *-representation is strongly continuous.*

Proof. (i) Let π be strongly continuous. Then, for every $\xi \in \mathcal{D}_\pi$, there exists $\gamma_\xi > 0$ such that

$$\|\pi(x)\xi\| \le \gamma_\xi \|x\|, \qquad \forall x \in \mathfrak{X}. \tag{13}$$

From this inequality and from the denseness of $\mathfrak{A}_0$ in $\mathfrak{X}$ it follows that $\pi(x)\xi \in \overline{\pi(\mathfrak{A}_0)\xi}$ for every $x \in \mathfrak{X}$. The statement then follows from Proposition 3.10.

(ii) Let π be a completely regular *-representation of $(\mathfrak{X}, \mathfrak{A}_0)$. Then, for every $\xi \in \mathcal{D}_\pi$, the vector form φ_ξ is bounded. Therefore, for some $\gamma_\xi > 0$,

$$\|\pi(x)\xi\|^2 = \varphi_\xi(x, x) \le \gamma_\xi \|x\|^2, \qquad \forall x \in \mathfrak{X}.$$

Hence π is strongly continuous. ∎

However, complete regularity and regularity are not equivalent unless $\mathcal{I}_s(\mathfrak{X}) = \mathcal{P}(\mathfrak{X})$, as the next theorem shows.

Theorem 5.2. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a normed quasi *-algebra with unit e. The following statements are equivalent*:

(i) *Every $\varphi \in \mathcal{I}_s(\mathfrak{X})$ is bounded, i.e. $\mathcal{I}_s(\mathfrak{X}) = \mathcal{P}(\mathfrak{X})$.*
(ii) *Every regular *-representation is completely regular.*
(iii) *Every regular *-representation π of $(\mathfrak{X}, \mathfrak{A}_0)$ is strongly continuous.*

If $(\mathfrak{X},\mathfrak{A}_0)$ *is a Banach quasi* **-algebra, then* (i)–(iii) *are equivalent to*

(iv) *Every* $\varphi\in\mathcal{I}_s(\mathfrak{X})$ *is lower semicontinuous.*

Proof. (i)⇒(ii): Let π be a regular *-representation of $(\mathfrak{X},\mathfrak{A}_0)$. Then $\varphi_\xi\in\mathcal{I}_s(\mathfrak{X})$ for every $\xi\in\mathcal{D}_\pi$. Thus, by assumption, φ_ξ is bounded.

(ii)⇒(iii): This follows immediately from Proposition 5.1(ii).

(iii)⇒(i): Let $\varphi\in\mathcal{I}_s(\mathfrak{X})$. Then π°_φ is a regular *-representation. Hence it is strongly continuous. Thus, for some $\gamma_\varphi>0$,

$$\varphi(x,x)=\langle\pi^\circ_\varphi(x)\lambda_\varphi(e)\,|\,\pi^\circ_\varphi(x)\lambda_\varphi(e)\rangle=\|\pi^\circ_\varphi(x)\lambda_\varphi(e)\|^2\le\gamma_\varphi\|x\|^2,\qquad\forall x\in\mathfrak{X}.$$

Finally, if $\mathfrak{X}$ is complete under $\|\cdot\|$, then (iv)⇒(i) follows from Lemma 4.20. The implication (i)⇒(iv) is obvious. ∎

If π is strongly continuous, then, by (13), we can define a new norm on $\mathcal{D}_\pi$ by putting

$$\|\xi\|_\pi=\sup_{\|x\|\le1}\|\pi(x)\xi\|,\qquad\xi\in\mathcal{D}_\pi.$$

Since $(\mathfrak{X},\mathfrak{A}_0)$ has a unit, $\|\xi\|\le\|\xi\|_\pi$ for every $\xi\in\mathcal{D}_\pi$. With this definition one has, of course,

$$\|\pi(x)\xi\|\le\|x\|\,\|\xi\|_\pi,\qquad\forall x\in\mathfrak{X},\ \xi\in\mathcal{D}_\pi.$$

We put

$$|\!|\!|\pi(x)|\!|\!|=\sup_{\|\xi\|_\pi\le1}\|\pi(x)\xi\|.$$

By the very definition, $|\!|\!|\pi(x)|\!|\!|\le\|x\|$ for every $x\in\mathfrak{X}$.

We denote by $\mathrm{Rep}_{\mathrm{sc}}(\mathfrak{X})$ the set of all strongly continuous *-representations of $(\mathfrak{X},\mathfrak{A}_0)$.

The next proposition shows that the seminorm $\mathfrak{p}$ on $\mathfrak{X}$ is determined by $\mathrm{Rep}_{\mathrm{sc}}(\mathfrak{X})$.

Theorem 5.3. *Let* $(\mathfrak{X},\mathfrak{A}_0)$ *be a normed quasi* **-algebra with unit* e*. Then*

$$\mathfrak{p}(x)=\sup_{\pi\in\mathrm{Rep}_{\mathrm{sc}}(\mathfrak{X})}|\!|\!|\pi(x)|\!|\!|.$$

Proof. Let $\pi\in\mathrm{Rep}_{\mathrm{sc}}(\mathfrak{X})$. For any $\xi\in\mathcal{D}_\pi$ we define, as before,

$$\varphi_\xi(x,y)=\langle\pi(x)\xi\,|\,\pi(y)\xi\rangle,\qquad x,y\in\mathfrak{X}.$$

Then $\varphi_\xi\in\mathcal{P}(\mathfrak{X})$ since

$$|\varphi_\xi(x,y)|\le\|x\|\,\|y\|\,\|\xi\|_\pi,\qquad\forall x,y\in\mathfrak{X}.$$

Clearly, if $\|\xi\|_\pi\le1$, then $\varphi_\xi\in\mathcal{S}(\mathfrak{X})$. Therefore,

$$|\!|\!|\pi(x)|\!|\!|^2=\sup_{\|\xi\|_\pi\le1}\|\pi(x)\xi\|^2=\sup_{\|\xi\|_\pi\le1}\varphi_\xi(x,x)\le\mathfrak{p}(x)^2,\qquad\forall x\in\mathfrak{X}.$$

On the other hand, let $\varphi \in \mathcal{S}(\mathfrak{X})$ and π_φ be the corresponding GNS representation. Then

$$\|\pi^\circ_\varphi(x)\lambda_\varphi(a)\|^2 = \varphi(xa, xa) = \varphi_a(x,x) \le \|\varphi_a\|\,\|x\|^2.$$

Therefore, π°_φ is strongly continuous. Hence,

$$\|\lambda_\varphi(a)\|_{\pi_\varphi} = \sup_{\|x\|\le 1} \|\pi^\circ_\varphi(x)\lambda_\varphi(a)\| = \|\varphi_a\|,$$

and so

$$\|\pi^\circ_\varphi(x)\| = \sup\{\|\pi^\circ_\varphi(x)\lambda_\varphi(a)\| : a \in \mathfrak{A}_0,\ \|\varphi_a\| \le 1\}.$$

This equality implies that

$$\begin{aligned}\sup_{\varphi\in\mathcal{P}(\mathfrak{X})} \|\pi^\circ_\varphi(x)\| &= \sup\{\varphi_a(x,x) : a \in \mathfrak{A}_0,\ \|\varphi_a\| \le 1\}\\ &= \sup_{\varphi\in\mathcal{S}(\mathfrak{X})} \varphi(x,x) = \mathfrak{p}(x), \qquad \forall x \in \mathfrak{X}.\end{aligned}$$

Therefore,

$$\mathfrak{p}(x) = \sup_{\varphi\in\mathcal{P}(\mathfrak{X})} \|\pi^\circ_\varphi(x)\| \le \sup_{\pi\in\mathrm{Rep}_{\mathrm{sc}}(\mathfrak{X})} \|\pi(x)\|, \qquad \forall x \in \mathfrak{X}.$$

This concludes the proof. ∎

Example 5.4. A simple example of a Banach quasi *-algebra having a strongly continuous unbounded *-representation is provided by $(L^p(I), C(I))$, $I = [0,1]$, with $p \ge 2$. If we put

$$(\pi(x)\xi)(t) = x(t)\xi(t), \qquad x \in L^p(I),\ \xi \in C(I),$$

then π is a *-representation of $(L^p(I), C(I))$ in the Hilbert space $L^2(I)$. It is easily seen that $\pi(x)$ is bounded if, and only if, $x \in L^\infty(I)$. This *-representation is strongly continuous. Indeed,

$$\|\pi(x)\xi\|_2 \le \|\xi\|_\infty \|x\|_2 \le \|\xi\|_\infty \|x\|_p, \qquad x \in L^p(I),\ \xi \in C(I).$$

A criterion for $(\mathfrak{X}, \mathfrak{A}_0)$ to have only bounded strongly continuous *-representations is given by the following

Proposition 5.5. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a Banach quasi *-algebra with unit e. The following statements are equivalent*:

(i) $\mathcal{D}(\mathfrak{q}) = \mathfrak{X}$.
(ii) *Every strongly continuous *-representation π of $(\mathfrak{X}, \mathfrak{A}_0)$ is bounded.*
(iii) *Every $\varphi \in \mathcal{P}(\mathfrak{X})$ is admissible.*

Proof. (i)⇒(ii): Assume that there exists a strongly continuous unbounded representation π of $(\mathfrak{X}, \mathfrak{A}_0)$. Then, for some $x \in \mathfrak{X}$, $\pi(x)$ is an unbounded operator. This implies that there exists a sequence $\{\xi_n\} \subset \mathcal{D}_\pi$ with

$$\|\xi_n\| = 1, \qquad \|\pi(x)\xi_n\| \to \infty.$$

As before, put $\varphi_{\xi_n}(y,z) = \langle \pi(y)\xi_n \,|\, \pi(y)\xi_n\rangle$ for $y,z \in \mathfrak{X}$, $n \in \mathbb{N}$. The strong continuity of π implies the existence of $\gamma > 0$ for which

$$\varphi_{\xi_n}(y,y) = \|\pi(y)\xi_n\|^2 \le \gamma\|y\|^2\,\|\xi_n\|^2 = \gamma\|y\|^2, \qquad \forall y \in \mathfrak{X}.$$

Thus, for every $n \in \mathbb{N}$, φ_{ξ_n} is bounded. As is easily seen, $\varphi_{\xi_n}(e,e) = 1$. Then

$$\mathfrak{q}(x)^2 \ge \sup_{n\in\mathbb{N}} \varphi_{\xi_n}(x,x) = \|\pi(x)\xi_n\|^2 \to \infty.$$

Hence $x \notin D(\mathfrak{q})$.

(ii)⇒(iii): Let $\varphi \in \mathcal{P}(\mathfrak{X})$. Then, for some $\gamma > 0$, $\varphi(x,x) \le \gamma\|x\|^2$ for every $x \in \mathfrak{X}$. Let π_φ be the corresponding GNS representation. Then

$$\|\pi_\varphi(x)\lambda_\varphi(a)\|^2 = \varphi(xa,xa) \le \gamma\|x\|^2\|a\|_0^2, \qquad \forall x \in \mathfrak{X},\ a \in \mathfrak{A}_0.$$

Hence π_φ is strongly continuous and therefore bounded. The statement then follows from Proposition 3.4.

(iii)⇒(i): This follows from the definition of $\mathfrak{q}$. ∎

Remark 5.6. If $\mathfrak{q}(x) = 0$ implies that $x = 0$, then by the previous proposition it follows that $\mathfrak{X}$ is contained in the C^*-algebra $\mathfrak{X}_\mathfrak{q}$ obtained by completing $\mathfrak{A}_0$ with respect to the norm $\mathfrak{q}(\cdot)$. We do not know if the identity map is necessarily continuous from $(\mathfrak{X}, \|\cdot\|)$ into $(\mathfrak{X}_\mathfrak{q}, \mathfrak{q}(\cdot))$. If this is the case, then the next proposition shows that $\mathfrak{p}(x) = \mathfrak{q}(x)$ for every $x \in \mathfrak{X}$.

Proposition 5.7. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a normed quasi *-algebra with unit e. The following statements are equivalent*:

(i) *$\mathfrak{p}$ is an m^*-seminorm on $\mathfrak{A}_0$.*
(ii) *For each $\varphi \in \mathcal{P}(\mathfrak{X})$, $\|\varphi\| = \varphi(e,e)$.*
(iii) *$D(\mathfrak{q}) = \mathfrak{X}$ and $\mathfrak{p}(x) = \mathfrak{q}(x)$ for every $x \in \mathfrak{X}$.*
(iv) *$\mathfrak{p}$ is a C^*-seminorm on $\mathfrak{A}_0$.*
(v) *$D(\mathfrak{q}) = \mathfrak{X}$ and $\mathfrak{q}(x) \le \|x\|$ for every $x \in \mathfrak{X}$.*

Proof. (i)⇒(ii): Let $\varphi \in \mathcal{P}(\mathfrak{X})$. We define a linear functional $\widehat{\omega}_\varphi$ on $\mathfrak{A}_0 + N_0(\mathfrak{p})$ by

$$\widehat{\omega}_\varphi(a + N_0(\mathfrak{p})) = \varphi(a,e), \qquad a \in \mathfrak{A}_0.$$

Then $\widehat{\omega}_\varphi$ is $\|\cdot\|_\mathfrak{p}$-bounded and positive, since

$$\widehat{\omega}_\varphi((a + N_0(\mathfrak{p}))^*(a + N_0(\mathfrak{p}))) = \widehat{\omega}_\varphi(a^*a + N_0(\mathfrak{p})) = \varphi(a^*a,e) = \varphi(a,a) \ge 0.$$

We denote by $\widetilde{\omega}_\varphi$ the unique $\|\ \|_\mathfrak{p}$-bounded extension of $\widehat{\omega}_\varphi$ to $\mathfrak{X}_\mathfrak{p}$. If $b \in \mathfrak{X}_\mathfrak{p}$ then there exists a sequence $\{a_n\} \subset \mathfrak{A}_0$ such that $b = \|\cdot\|_\mathfrak{p}\text{-}\lim(a_n + N_0(\mathfrak{p}))$. Then

$$\widetilde{\omega}_\varphi(b^*b) = \lim_{n\to\infty} \widehat{\omega}_\varphi((a_n + N_0(\mathfrak{p}))^*(a_n + N_0(\mathfrak{p}))) \ge 0.$$

Hence $\widetilde{\omega}_\varphi$ is positive on $\mathfrak{X}_\mathfrak{p}$. Then the norm $\|\widetilde{\omega}_\varphi\|^\sharp_\mathfrak{p}$ of the linear functional $\widetilde{\omega}_\varphi$ satisfies $\|\widetilde{\omega}_\varphi\|^\sharp_\mathfrak{p} = \widetilde{\omega}_\varphi(e) = \varphi(e,e)$. Therefore, for all $a, b \in \mathfrak{A}_0$,

$$\begin{aligned}|\varphi(a,b)| = |\widehat{\omega}_\varphi(b^*a + N_0(\mathfrak{p}))| &\le \varphi(e,e)\mathfrak{p}(b^*a)\\ &\le \varphi(e,e)\mathfrak{p}(a)\mathfrak{p}(b) \le \varphi(e,e)\|a\|\,\|b\|.\end{aligned}$$

This implies that $\|\varphi\| \le \varphi(e,e)$, and since $\varphi(e,e) \le \|\varphi\|$, we get equality.

(ii)⇒(iii): Follows immediately from the definition of $\mathfrak{q}$ and from (ii).

(iii)⇒(iv): Follows from the equality $\mathfrak{p} = \mathfrak{q}$.

(iv)⇒(v): From (i)⇒(ii) it follows that $\|\varphi\| = \varphi(e,e)$ for every $\varphi \in \mathcal{P}(\mathfrak{X})$. This implies that $\mathfrak{p} = \mathfrak{q}$ and then, from the properties of $\mathfrak{p}$, one finally gets (v).

(v)⇒(i): Since

$$\varphi(xa, xa) \le \mathfrak{q}(x)^2\varphi(a,a), \qquad \forall x \in D(\mathfrak{q}),\ a \in \mathfrak{A}_0,$$

we have

$$\varphi(xa, xa) \le \|x\|^2\varphi(a,a), \qquad \forall x \in \mathfrak{X},\ a \in \mathfrak{A}_0.$$

For $a = e$ this gives

$$\varphi(x,x) \le \|x\|^2\varphi(e,e), \qquad \forall x \in \mathfrak{X}.$$

Therefore $\varphi(e,e) \ge \|\varphi\|$. Since always $\varphi(e,e) \le \|\varphi\|$, we conclude that $\|\varphi\| = \varphi(e,e)$. Thus, if $\varphi(e,e) = 1$, then $\varphi \in \mathcal{S}(\mathfrak{X})$. This implies that $\mathfrak{q}(x) \le \mathfrak{p}(x)$ for every $x \in \mathfrak{X}$. Hence $\mathfrak{q}(x) = \mathfrak{p}(x)$ for every $x \in \mathfrak{X}$ and $\mathfrak{p}$ is an m^*-seminorm on $\mathfrak{A}_0$. ∎

Corollary 5.8. *Let $(\mathfrak{X}, \mathfrak{A}_0)$ be a normed quasi *-algebra with unit e. The following statements are equivalent*:

(i) $\mathfrak{p}$ *is an m^*-seminorm on $\mathfrak{A}_0$.*

(ii) *Every regular *-representation π of $(\mathfrak{X}, \mathfrak{A}_0)$ in a Hilbert space $\mathcal{H}$ is bounded and continuous from $\mathfrak{X}$ into $\mathcal{B}(\mathcal{H})$ and $\|\pi(x)\| \le \|x\|$ for every $x \in \mathfrak{X}$.*

Proof. This follows immediately from Propositions 4.13 and 5.7(iv). ∎

As seen in previous examples, there exist Banach quasi *-algebras $(\mathfrak{X}, \mathfrak{A}_0)$ (with unit) for which $\mathcal{P}(\mathfrak{X}) = \{0\}$. If this is the case, there is no strongly continuous *-representation of $(\mathfrak{X}, \mathfrak{A}_0)$, apart from the trivial one. This unpleasant feature is avoided if we require that $\mathcal{P}(\mathfrak{X})$ is *sufficient*, by which we mean that if $x \in \mathfrak{X}$ and $\varphi(x,x) = 0$ for every $\varphi \in \mathcal{P}(\mathfrak{X})$, then $x = 0$. Clearly, if $\mathcal{P}(\mathfrak{X})$ is sufficient, then $(\mathfrak{X}, \mathfrak{A}_0)$ has a sufficient family of strongly continuous *-representations, where *sufficient* means in this case that for every $x \in \mathfrak{X}$, $x \ne 0$, there exists a strongly continuous *-representation π such that $\pi(x) \ne 0$. In particular, if $\mathfrak{p}(x) = \|x\|$ for every $x \in \mathfrak{X}$, then $\mathcal{P}(\mathfrak{X})$ is clearly sufficient, and as shown in [22, Theorem 3.26], $\mathcal{D}(\mathfrak{q})$ is a C^*-algebra, consisting of all *bounded* elements of $\mathfrak{X}$ (see Example 4.19). In this case, from Proposition 5.5 it follows that a non-trivial (in the sense that $\mathfrak{X}$ is not an algebra) Banach quasi *-algebra $(\mathfrak{X}, \mathfrak{A}_0)$ necessarily has strongly continuous *unbounded* *-representations.

Banach quasi *-algebras with $\mathcal{P}(\mathfrak{X})$ sufficient have been studied in more detail in [6] and [22]. There remains the open question of characterizing, in terms of the original norm of $\mathfrak{X}$, the existence of sufficiently many positive invariant sesquilinear forms.

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References

[1] J.-P. Antoine, A. Inoue and C. Trapani, *Partial *-Algebras and Their Operator Realizations*, Math. Appl. 553, Kluwer, Dordrecht, 2002.

[2] F. Bagarello, A. Inoue and C. Trapani, *Some classes of topological quasi *-algebras*, Proc. Amer. Math. Soc. 129 (2001), 2973–2980.

[3] —, —, —, *Unbounded C^*-seminorms and *-representations of partial *-algebras*, Z. Anal. Anwend. 20 (2001), 295–314.

[4] F. Bagarello and C. Trapani, *States and representations of CQ^*-algebras*, Ann. Inst. H. Poincaré 61 (1994), 103–133.

[5] —, —, *L^p-spaces as quasi *-algebras*, J. Math. Anal. Appl. 197 (1996), 810–824.

[6] —, —, *CQ^*-algebras: structure properties*, Publ. RIMS Kyoto Univ. 32 (1996), 85–116.

[7] S. J. Bhatt, M. Fragoulopoulou and A. Inoue, *Existence of well-behaved $*$-representations of locally convex $*$-algebras*, Math. Nachr. 279 (2006), 86–100.

[8] S. J. Bhatt, A. Inoue and H. Ogi, *Unbounded C^*-seminorms and unbounded C^*-spectral algebras*, J. Operator Theory 45 (2001), 53–80.

[9] M. Fragoulopoulou, *Topological Algebras with Involution*, North-Holland Math. Stud. 200, North-Holland, Amsterdam, 2005.

[10] M. Fragoulopoulou, A. Inoue and K.-D. Kürsten, *On the completion of a C^*-normed algebra under a locally convex algebra topology*, in: Topological Algebras and Applications, Contemp. Math. 427, Amer. Math. Soc., Providence, RI, 2007, 155–166.

[11] A. Inoue and K.-D. Kürsten, *On C^*-like locally convex $*$-algebras*, Math. Nachr. 235 (2002), 51–58.

[12] K.-D. Kürsten and E. Wagner, *On representations of partial *-algebras based on B-weights*, Z. Anal. Anwend. 19 (2000), 623–638.

[13] G. Lassner, *Topological algebras and their applications in quantum statistics*, Wiss. Z. KMU-Leipzig Math.-Naturwiss. R. 30 (1981), 572–595.

[14] —, *Algebras of unbounded operators and quantum dynamics*, Phys. A 124 (1984), 471–480.

[15] E. Nelson, *Note on non-commutative integration*, J. Funct. Anal. 15 (1974), 103–116.

[16] T. W. Palmer, *Banach Algebras and the General Theory of *-Algebras*, Vol. II, Cambridge Univ. Press, Cambridge, 2001.

[17] K. Schmüdgen, *Unbounded Operator Algebras and Representation Theory*, Akademie-Verlag, Berlin, 1990.

[18] —, *On well-behaved unbounded representations of $*$-algebras*, J. Operator Theory 48 (2002), 487–502.

[19] I. E. Segal, *A noncommutative extension of abstract integration*, Ann. of Math. 57 (1953), 401–457.

[20] C. Trapani, *Quasi *-algebras of operators and their applications*, Rev. Math. Phys. 7 (1995), 1303–1332.

[21] —, *Some seminorms on quasi *-algebras*, Studia Math. 158 (2003), 99–115; Erratum/Addendum, ibid. 160 (2004), 101–101.

[22] —, *Bounded elements and spectrum in Banach quasi *-algebras*, ibid. 172 (2006), 289–305.

[23] —, *Bounded and strongly bounded elements in Banach quasi $*$-algebras*, in: Topological Algebras and Applications, Contemp. Math. 427, Amer. Math. Soc., Providence, RI, 2007, 417–424.

[24] —, *Unbounded C^*-seminorms, biweights and *-representations of partial *-algebras: a review*, Int. J. Math. Math. Sci. 2006 (2006), 1–34.

[25] B. Yood, *C^*-seminorms*, Studia Math. 118 (1996), 19–26.

Dipartimento di Matematica ed Applicazioni
Università di Palermo
I-90123 Palermo, Italy
E-mail: trapani@unipa.it

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