The Banach space S is complementably minimal and subsequentially prime

by

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Abstract. We first include a result of the second author showing that the Banach space S is complementably minimal. We then show that every block sequence of the unit vector basis of S has a subsequence which spans a space isomorphic to its square. By the Pełczyński decomposition method it follows that every basic sequence in S which spans a space complemented in S has a subsequence which spans a space isomorphic to S (i.e. S is a subsequentially prime space).

1. Introduction. The Banach space S was introduced by the second author as an example of an arbitrarily distortable Banach space [14]. In [8] the space S was used to construct a Banach space which does not contain any unconditional basic sequence. In this paper we are concerned with the question whether or not S is a prime space. We present two partial results: In Section 2 we include a result proved by the second author some time ago but not published until now [15], that S is complementably minimal, and thereby answer a question of P. G. Casazza, who asked whether or not ℓ_p, $1 \le p \le \infty$, and c_0 are the only complementably minimal spaces. In Section 3 we prove that S is subsequentially prime.

Let us recall the above notions. A Banach space X is called *prime* [12] if every complemented infinite-dimensional subspace of X is isomorphic to X. A. Pełczyński [13] showed that the spaces c_0 and ℓ_p ($1 \le p < \infty$) are prime, and J. Lindenstrauss [11] showed that this is also true for the space ℓ_∞. New prime spaces were constructed by W. T. Gowers and B. Maurey [9]. But it is still open whether or not ℓ_p, $1 \le p \le \infty$, and c_0 are the only prime spaces with an unconditional basis.

A space X is called *minimal* (a notion due to H. Rosenthal) if every infinite-dimensional subspace of X contains a subspace isomorphic to X, and X is called *complementably minimal* [5] if every infinite-dimensional

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subspace of X contains a subspace which is isomorphic to X and complemented in X. P. G. Casazza and E. Odell [4] showed that Tsirelson's space T (see [16]), as described in [6], fails to have a minimal subspace. On the other hand it was shown by P. G. Casazza, W. B. Johnson and L. Tzafriri [3] that the space T^* is minimal but not complementably minimal. Since S is complementably minimal, either S is prime, or there exists a complemented subspace X of S such that X and S give a negative solution to the Schroeder–Bernstein problem for Banach spaces (see [2] for a detailed discussion of this question): if two spaces are isomorphic to complemented subspaces of each other must they be isomorphic? Negative solutions to the Schroeder–Bernstein problem for Banach spaces are given by W. T. Gowers [7], and W. T. Gowers and B. Maurey [9], but to our knowledge it is open whether or not there are two Banach spaces X and Y, both having an unconditional basis, so that X is complemented in Y and Y is complemented in X, but so that X and Y are not isomorphic.

The following terminology was suggested to us by D. Kutzarova.

DEFINITION. A Banach space X with a basis is called *subsequentially prime* if for every basic sequence (x_i) of X such that the closed linear span of (x_i) is complemented in X, there exists a subsequence (y_i) such that the closed linear span of (y_i) is isomorphic to X.

As mentioned above, we will show that the space S is subsequentially prime. We do not know if S is prime, we even do not know whether or not the closed linear span of a block basis which is complemented in S is isomorphic to S.

We will need some notations. Let c_{00} be the linear span of finitely supported real sequences, and let (e_i) denote its standard basis. For $x \in c_{00}$, $\operatorname{supp}(x) = \{i \in \mathbb{N} : x_i \neq 0\}$ denotes the *support* of x. For a finite set A the cardinality of A is denoted by $\#A$. If $E, F \subset \mathbb{N}$ we write $E < F$ if $\max E < \min F$, and we write $x < y$ for $x, y \in c_{00}$ if $\operatorname{supp}(x) < \operatorname{supp}(y)$. A sequence $(x_i)_i$ in c_{00} is a *block sequence* of (e_i) if $x_1 < x_2 < \ldots$ For $x = \sum_{i \in \mathbb{N}} x_i e_i \in c_{00}$ and $E \subset \mathbb{N}$, $E(x)$ is the projection of x onto the span of $(e_i)_{i \in E}$, i.e.

$$E(x) = \sum_{i \in E} x_i e_i.$$

Recall [14] that the norm of S is the unique norm on the completion of c_{00} which satisfies the implicit equation

$$(1) \qquad \|x\| = \|x\|_{\ell_\infty} \vee \sup_{\substack{2 \leq n,\, E_i \subseteq \mathbb{N},\, i=1,\ldots,n \\ E_1 < \ldots < E_n}} \frac{1}{f(n)} \sum_{i=1}^n \|E_i(x)\|$$

where $\|\cdot\|_{\ell_\infty}$ denotes the norm of ℓ_∞ and $f(n) = \log_2(n+1)$ for $n \in \mathbb{N}$. For

$x \in S$ and $l \in \mathbb{N}$, $l \geq 2$, we define

$$\|x\|_l := \sup_{E_1 < \ldots < E_l} \frac{1}{f(l)} \sum_{i=1}^{l} \|E_i(x)\|.$$

We note that $\|\cdot\|_l$, $2 \leq l < \infty$, is an equivalent norm on S and we observe that for $x \in S$ and $2 \leq l < \infty$ we have

$$\frac{1}{f(l)} \|x\| \leq \|x\|_l \leq \|x\| \quad \text{and} \quad \|x\| = \sup_{2 \leq l \leq \infty} \|x\|_l.$$

Finally for any $2 \leq r < \infty$ and $x \in S$ we put

$$\|x\|_r := \sup_{\substack{l \geq r \\ l \in \mathbb{N} \cup \{\infty\}}} \|x\|_l.$$

Two sequences (x_i), (y_i) in S are called *c-equivalent*, for some $c \geq 1$, and we write $(x_i) \approx_c (y_i)$, if $\|\sum a_i x_i\| \overset{c}{\approx} \|\sum a_i y_i\|$ for all $(a_i) \in c_{00}$, where for $c \geq 1$ and $a, b \geq 0$ we write $a \overset{c}{\approx} b$ to denote that $(1/c)a \leq b \leq ca$. If (x_i) and (y_i) are *c*-equivalent for some $c > 1$ we write $(x_i) \approx (y_i)$. A basic sequence (x_n) is called *c-subsymmetric* if it is *c*-unconditional and *c*-equivalent to all of its subsequences. For two Banach spaces X and Y we write $X \approx_c Y$ if there is an isomorphism T between X and Y with $\|T\| \cdot \|T^{-1}\| \leq c$, and we write $X \approx Y$ if $X \approx_c Y$ for some $c \geq 1$.

If (x_n) is a sequence in a Banach space then $[x_n : n \in \mathbb{N}]$ denotes the closed linear span of (x_n). If not said otherwise, all statements in the following sections refer to the space S.

We would like to thank P. G. Casazza and D. Kutzarova for valuable discussions.

2. The Banach space S is complementably minimal. The goal of this section is the proof of the following theorem.

THEOREM 2.1. *S is complementably minimal.*

First recall the following result which follows from Lemma 5 of [14].

PROPOSITION 2.2. *ℓ_1 is block finitely represented in each block basis of (e_i), i.e., if $\varepsilon > 0$ and $m \in \mathbb{N}$, and if (y_n) is a block basis of (e_i), then there is a block basis $(z_i)_{i=1}^{m}$ of (y_n) which is $(1 + \varepsilon)$-equivalent to the unit basis of ℓ_1^m, i.e.,*

$$\left\| \sum_{i=1}^{m} \alpha_i z_i \right\| \geq \frac{1}{1 + \varepsilon} \sum_{i=1}^{m} |\alpha_i|$$

for $(\alpha_i)_{i=1}^{m} \subset \mathbb{R}$.

The proof of the following statement can be compiled from the proof of Theorem 3 of [14]. Since the statement is crucial for our main result we include its proof.

LEMMA 2.3. *Let $\varepsilon > 0$ and $l \in \mathbb{N}$. Then there is an $n = n(\varepsilon, l) \in \mathbb{N}$ with the following property*: *If $m \geq n$ and if $y = m^{-1} \sum_{i=1}^{m} x_i$, where $(x_i)_{i=1}^{m}$ is a normalized block basis of (e_i) which is $(1 + \varepsilon/2)$-equivalent to the unit basis of ℓ_1^m, then*

$$\sup_{E_1 < \ldots < E_l} \sum_{i=1}^{l} \|E_i(y)\| \leq \|y\| + \varepsilon \leq 1 + \varepsilon.$$

Proof. Let $n \in \mathbb{N}$ so that $4l/n \leq \varepsilon$ and assume $m \geq n$ and $(x_i)_{i=1}^{m}$ are as in the statement. Furthermore, let $E_1 < \ldots < E_l$ be finite subsets of $\mathbb{N}$. Since (e_i) is 1-unconditional we can assume that the E_j's are intervals in $\mathbb{N}$. This implies that for each $j \in \{1, \ldots, l\}$ there are at most two elements $i_1, i_2 \in \{1, \ldots, m\}$ so that $E_j \cap \text{supp}(x_{i_s}) \neq \emptyset$ and $\text{supp}(x_{i_s}) \setminus E_j \neq \emptyset$, $s = 1, 2$. For $j = 1, \ldots, l$, let

$$\widetilde{E}_j = \bigcup \{\text{supp}(x_i) : i \leq m \text{ and } \text{supp}(x_i) \subset E_j\}.$$

It follows that if $y = m^{-1} \sum_{i=1}^{m} x_i$, then $\|E_i(y) - \widetilde{E}_i(y)\| \leq 2/m$ and from the assumption that $(x_i)_{i=1}^{m}$ is $(1 + \varepsilon/2)$-equivalent to the ℓ_1^m unit basis we deduce that

$$\sum_{j=1}^{l} \|E_j(y)\| \leq \frac{2l}{m} + \sum_{j=1}^{l} \|\widetilde{E}_j(y)\| = \frac{2l}{m} + \frac{1}{m} \sum_{j=1}^{l} \left\| \sum_{\text{supp}(x_i) \subset \widetilde{E}_j} x_i \right\|$$

$$\leq \frac{2l}{m} + \frac{1}{m} \left\| \sum_{i=1}^{m} x_i \right\| (1 + \varepsilon/2) \leq \varepsilon + \|y\| \leq \varepsilon + 1. \ \blacksquare$$

The following theorem essentially proves that S is minimal. We postpone its proof.

THEOREM 2.4. *Let $(\varepsilon_n) \subset \mathbb{R}^+$ with $\sum \varepsilon_n < \infty$ and let (y_n) be a normalized block basis of (e_n) with the following properties*: *There is a sequence $k_n \uparrow \infty$ in $\mathbb{N}$ so that for all $n \in \mathbb{N}$,*

$$(2) \qquad \sup_{\substack{k \leq k_{n-1} \\ E_1 < \ldots < E_k}} \sum_{i=1}^{k} \|E_i(y_n)\| \leq 1 + \varepsilon_n,$$

$$(3) \qquad \max \text{supp}(y_n) \leq \varepsilon_n f(k_n/3).$$

Then (y_n) is equivalent to (e_n).

Proof of Theorem 2.1. By the usual perturbation argument we only have to show that every block basis (z_n) of (e_n) has a further block basis which is equivalent to (e_n). Letting for example $\varepsilon_i = 2^{-i}$, $i = 1, 2, \ldots$, we have to

find a normalized block (y_n) of (z_n) and a sequence (k_n) in $\mathbb{N}$ so that (2) and (3) of Theorem 2.4 are satisfied. Indeed, put $k_0 = 1$ and assume that $k_0 < k_1 < \ldots < k_n$ and $y_1 < \ldots < y_n$ are already defined for some $n \geq 0$. By Remark 2.2 and Lemma 2.3 we can choose $y_{n+1} > y_n$ in the linear span of (z_i) so that condition (2) of Theorem 2.4 is satisfied. Since $\lim_{i \to \infty} f(i) = \infty$ we can then choose k_{n+1} so that (3) is true.

In order to show that S is complementably minimal we first observe that (1) implies that every normalized block basis (y_n) of (e_n) dominates (e_n), i.e., that $\|\sum_{i=1}^\infty \alpha_i y_i\| \geq \|\sum_{i=1}^\infty \alpha_i e_i\|$ for all $(\alpha_i) \in c_{00}$. Secondly, we apply the following more general proposition which finishes the proof of Theorem 2.1.

PROPOSITION 2.5. *Let Z be a Banach space with a c_u-unconditional basis (e_n), $c_u \geq 1$. Assume furthermore that there is a $c_d > 0$ so that every normalized block basis (y_n) of (e_n) c_d-dominates (e_n) (i.e., $c_d\|\sum \alpha_i y_i\| \geq \|\sum \alpha_i e_i\|$ for all $(\alpha_i) \in c_{00}$). Then a subspace of Z generated by a normalized block of (e_n) which is equivalent to (e_n) is complemented in Z.*

Proof. Without loss of generality we can assume that (e_n) is a normalized and bimonotone basis of Z (i.e. $\|[m,n](z)\| \leq \|z\|$ for all $z = \sum_{i=1}^\infty z_i e_i \in Z$, and $1 \leq m \leq n$ in $\mathbb{N}$). Assume that (y_n) is a block of (e_i) which is c_e-equivalent to (e_i). Using the assumption that (e_n) is normalized and bimonotone we find $y_n^* \in Z^*$, for $n \in \mathbb{N}$, with $1 = \|y_n^*\| = y_n^*(y_n)$ and $\mathrm{supp}(y_n^*) \subset [1 + \max \mathrm{supp}(y_{n-1}), \max \mathrm{supp}(y_n)]$ (where $y_0 = 0$ and $\max(\emptyset) = 0$). Define $T = \sum y_n^* \otimes y_n$, $x \mapsto \sum y_n y_n^*(x)$. We have to show that T is well defined and bounded on Z; then it easily follows that it is a projection on $[y_n : n \in \mathbb{N}]$. Let $x = \sum a_i e_i$ with $(a_i) \in c_{00}$. We can write $x = \sum x_i = \sum \|x_i\| u_i$ with $x_i = [1 + \max \mathrm{supp}(y_{i-1}), \max \mathrm{supp}(y_i)](x)$ and $u_i = x_i / \|x_i\|$ if $x_i \neq 0$, and $u_i = e_{\max \mathrm{supp}(y_i)}$ otherwise. Then it follows that

$$T(x) = \left\| \sum y_n y_n^*(x_n) \right\| \leq c_u \left\| \sum \|x_n\| y_n \right\| \leq c_u c_e \left\| \sum \|x_n\| e_n \right\|$$
$$\leq c_u c_e c_d \left\| \sum \|x_n\| u_n \right\| = c_u c_e c_d \|x\|. \quad \blacksquare$$

Let $x \in S$. If l is the smallest element of $\mathbb{N}$ so that $\|x\| = \|x\|_l$, we call l the *character* of x and write $\mathrm{char}(x) = l$. If $\|x\| = \|x\|_{\ell_\infty}$ we write $\mathrm{char}(x) = \infty$.

The next lemma makes the following qualitative statement precise: If $x \in c_{00}$, if $r > 1$ is "big enough", and if $E_1 < \ldots < E_l$, $l \geq r$, are subsets of $\mathbb{N}$ so that

$$\|x\|_r = \frac{1}{f(l)} \sum_{i=1}^l \|E_i(x)\|,$$

then for "most of the E_i's" the character of $E_i(x)$ is "much bigger than r".

LEMMA 2.6. *There is a constant $d > 1$ so that for all $r \in \mathbb{R}_+$ with $f(r) > d^2$,*

$$\|x\|_r \leq \left[\frac{1}{1 - d/\sqrt{f(r)}}\right] \sup_{\substack{l \geq r \\ E_1 < \ldots < E_l}} \frac{1}{f(l)} \sum_{i=1}^{l} \|E_i(x)\|_{rf(r)}$$

if $x \in c_{00}$ with $\|x\|_r \neq \|x\|_{\ell_\infty}$.

Proof. From the logarithmic behavior of f we deduce that there is a constant $c > 2$ so that the following inequalities hold:

(4) $\quad f(\xi) - 1 \geq f(\xi)/c \qquad$ whenever $\xi \geq 2$,

(5) $\quad cf(\xi) \geq f(\xi\xi') - f(\xi) \quad$ whenever $\xi, \xi' \geq c$,

(6) $\quad f(\xi^{1/\sqrt{f(\xi)}}) \leq c\sqrt{f(\xi)} \qquad$ whenever $\xi \geq c$,

(7) $\quad f(\xi^\nu) \leq c\nu f(\xi) \qquad$ whenever $\xi \geq c$ and $\nu \geq 1$.

Choose $d = 4c^3$, and let $r \in \mathbb{R}_+$ be such that $f(r) > d^2$. In order to verify that this choice works let $x \in S$ with $\|x\|_r \neq \|x\|_\infty$. Let $l \geq r$ and $E_1 < \ldots < E_l$ be such that

$$\|x\|_r = \frac{1}{f(l)} \sum_{i=1}^{l} \|E_i(x)\|.$$

For $\widetilde{r}, \widetilde{R} \in \mathbb{R}$ with $2 \leq \widetilde{r} < \widetilde{R}$, let $M = M(\widetilde{r}, \widetilde{R}) := \{i \leq l : \mathrm{char}(E_i(x)) \in [\widetilde{r}, \widetilde{R})\}$, and for $i \in M$ let $l_i \in [\widetilde{r}, \widetilde{R})$ be the character of $E_i(x)$. We choose for each $i \in M$ finite subsets $E_1^i < \ldots < E_{l_i}^i$ of E_i so that

$$\|E_i(x)\| = \frac{1}{f(l_i)} \sum_{j=1}^{l_i} \|E_j^i(x)\|.$$

Now we observe that the set $\{E_i : i \notin M\} \cup \bigcup_{i \in M}\{E_j^i : 1 \leq j \leq l_i\}$ is well ordered by $<$ and its cardinality is $l - \#M + \sum_{i \in M} l_i$, which is at least l and at most $l\widetilde{R}$. Thus we deduce:

(8) $\quad \|x\|_r = \frac{1}{f(l)} \sum_{i=1}^{l} \|E_i(x)\|$

$$\geq \frac{1}{f(l - \#M + \sum_{i \in M} l_i)}\Big[\sum_{i=1,\, i \notin M}^{l} \|E_i(x)\| + \sum_{i \in M}\sum_{j=1}^{l_i} \|E_j^i(x)\|\Big]$$

$$\geq \frac{1}{f(l\widetilde{R})}\Big[\sum_{i=1,\, i \notin M}^{l} \|E_i(x)\| + \sum_{i \in M} f(l_i)\|E_i(x)\|_{l_i}\Big]$$

$$\geq \frac{1}{f(l\widetilde{R})}\Big[\sum_{i=1}^{l}\|E_i(x)\| + \sum_{i\in M}(f(\widetilde{r})-1)\|E_i(x)\|\Big]$$

$$\geq \frac{1}{f(l\widetilde{R})}\Big[\sum_{i=1}^{l}\|E_i(x)\| + \frac{1}{c}\,f(\widetilde{r})\sum_{i\in M}\|E_i(x)\|\Big] \quad \text{(by (4))}.$$

Solving for $(1/f(l))\sum_{i\in M}\|E_i(x)\|$ leads to the inequalities

$$(9) \quad \frac{1}{f(l)}\sum_{i\in M}\|E_i(x)\| \leq \frac{1}{f(l)}\Big[\frac{1}{f(l)}-\frac{1}{f(l\widetilde{R})}\Big]\frac{cf(l\widetilde{R})}{f(\widetilde{r})}\sum_{i=1}^{l}\|E_i(x)\|$$

$$= \frac{c}{f(\widetilde{r})}\,\frac{f(l\widetilde{R})-f(l)}{f(l)}\,\|x\|_r$$

$$\leq c^2\,\frac{f(\widetilde{R})}{f(\widetilde{r})f(l)}\,\|x\|_r \leq c^2\,\frac{f(\widetilde{R})}{f(\widetilde{r})f(r)}\,\|x\|_r \quad \text{(by 5)}.$$

Choosing for the pair $(\widetilde{r},\widetilde{R})$ the values $(2,r^{1/\sqrt{f(r)}})$, $(r^{1/\sqrt{f(r)}},r)$, $(r,r^{\sqrt{f(r)}})$, and $(r^{\sqrt{f(r)}},r^{f(r)})$ we deduce from the inequalities (6) and (7) in each case that

$$\frac{f(\widetilde{R})}{f(\widetilde{r})f(r)} \leq \frac{c}{\sqrt{f(r)}},$$

which together with (9) implies that

$$\frac{1}{f(l)}\sum_{2\leq \text{char}\,(E_i(x))<r^{f(r)}}\|E_i(x)\| \leq \frac{4c^3}{\sqrt{f(r)}}\,\|x\|_r,$$

and, thus, that

$$\|x\|_r \leq \frac{d}{\sqrt{f(r)}}\,\|x\|_r + \frac{1}{f(l)}\sum_{i=1}^{l}\|E_i(x)\|_{r^{f(r)}},$$

yielding the lemma. ∎

REMARK. Note that in the proof of Lemma 2.6 the only properties of the function f which were needed were that it is increasing and that there is a $c>2$ so that the inequalities (4)–(7) hold. Thus if $l_0\in\mathbb{N}$ and $g:[l_0,\infty)\to(1,\infty)$ is an increasing function so that there is a $c>l_0$ for which (4) (whenever $\xi\geq l_0$), (5), (6) and (7) hold then the conclusion of Lemma 2.6 holds for the completion of c_{00} under the norm $\langle\!\langle\cdot\rangle\!\rangle$ defined implicitly by

$$\langle\!\langle x\rangle\!\rangle = \|x\|_{\ell_\infty} \vee \sup_{l\geq l_0,\,E_1<\ldots<E_l}\frac{1}{g(l)}\sum_{i=1}^{l}\langle\!\langle E_i(x)\rangle\!\rangle \quad \text{whenever } x\in c_{00}.$$

Proof of Theorem 2.4. Let $(y_n),(k_n)$ and (ε_n) be as in the statement of Theorem 2.4 and let $d\geq 1$ be as in Lemma 2.6. For $r\geq 1$ we put $r_0:=r$ and, assuming r_k was already defined, we let $r_{k+1}=r_k^{f(r_k)}$. From the properties

of the function f it follows that there is an $R > 1$ so that the value

$$(10) \qquad \beta(r) := \prod_{k=0}^{\infty} \left(\frac{1}{1 - d/\sqrt{f(r_k)}} \right) \frac{f(9r_k)}{f(r_k)}$$

is finite whenever $r \geq R$. By induction we will show that for every $m \in \mathbb{N}$ and every $(\alpha_i)_{i=1}^m \subset \mathbb{R}$,

$$(11) \qquad \left\| \left| \sum_{i=1}^m \alpha_i y_i \right| \right\|_r \leq \beta(r) \max_{i_0 \geq 1} \left[|\alpha_{i_0}| + \left\| \sum_{i > i_0} \alpha_i e_i \right\| + \sum_{i=1}^m |\alpha_i| \varepsilon_i \right].$$

Since $\| \cdot \|_r$ is equivalent to $\| \cdot \|$ for all $r \geq 1$, since $\sum \varepsilon_i < \infty$, and since $\| \sum \alpha_i y_i \| \geq \| \sum \alpha_i e_i \| \geq \max_{i \in \mathbb{N}} |\alpha_i|$ for $(\alpha_i) \in c_{00}$ this will prove the assertion of Theorem 2.4.

For $m = 1$ the claim is trivial. Assume it is true for all positive integers smaller than some $m > 1$ and let $r \geq R$ and $(\alpha_i)_{i=1}^m \in c_{00}$. Let $y = \sum_{i=1}^m \alpha_i y_i$. If $\|y\|_r = \|y\|_{\ell_\infty}$ the assertion follows easily since $\|y\|_{\ell_\infty} \leq \max_{i \leq m} |\alpha_i|$. Otherwise we can use Lemma 2.6 in order to find an $l \geq r$ and finite subsets $E_1 < \ldots < E_l$ of $\mathbb{N}$ so that (with $\gamma(r) = 1/(1 - d/\sqrt{f(r)})$)

$$(12) \qquad \|y\|_r \leq \gamma(r) \frac{1}{f(l)} \sum_{j=1}^l \|E_j(y)\|_{r^{f(r)}}.$$

We can assume that for all $j \leq l$, $E_j \subset \bigcup_{i=1}^m \operatorname{supp}(y_i)$. For $j = 1, \ldots, l$ we put $E_j^1 := E_j \cap \operatorname{supp}(y_{s(j)})$, $E_j^2 := E_j \cap \operatorname{supp}(y_{t(j)})$ and $E_j^3 = E_j \setminus (E_j^1 \cup E_j^2)$ where $s(j) := \min\{i : E_j \cap \operatorname{supp}(y_i) \neq \emptyset\}$ and $t(j) := \max\{i : E_j \cap \operatorname{supp}(y_i) \neq \emptyset\}$. We put $\widetilde{\mathcal{E}} := \{E_j^1, E_j^2, E_j^3 : j \leq l\} \setminus \{\emptyset\}$ and note that $\widetilde{\mathcal{E}}$ can be ordered into $\widetilde{\mathcal{E}} = \{\widetilde{E}_1, \ldots, \widetilde{E}_{\tilde{l}}\}$ with $l \leq \tilde{l} \leq 3l$ and $\widetilde{E}_1 < \ldots < \widetilde{E}_{\tilde{l}}$.

Secondly, we observe that $\widetilde{\mathcal{E}}$ can be partitioned into $m + 1$ sets $\widetilde{\mathcal{E}}_0, \widetilde{\mathcal{E}}_1, \ldots \ldots, \widetilde{\mathcal{E}}_m$ defined in the following way: $\widetilde{\mathcal{E}}_0 := \{E \in \widetilde{\mathcal{E}} : E \text{ fits with } (\operatorname{supp}(y_i))_{i \in \mathbb{N}}\}$ (where we say that E *fits* with a sequence (A_n) of disjoint subsets of $\mathbb{N}$ if for all n, $E \cap A_n \neq \emptyset$ implies that $A_n \subset E$) and for $1 \leq i \leq m$ we let $\widetilde{\mathcal{E}}_i := \{E : E \in \widetilde{\mathcal{E}} \text{ and } E \subsetneq \operatorname{supp}(y_i)\}$.

For $i = 1, \ldots, m$ we let $l_i := \#\widetilde{\mathcal{E}}_i$ (note that $\widetilde{\mathcal{E}}_i$ may be empty) and let $i_0 = 1$ if $l_i \leq k_{i-1}$ for all $i \leq m$, otherwise put $i_0 := \max\{i \leq m : l_i > k_{i-1}\}$.

From (12) we now deduce that (recall that $r_1 = r^{f(r)}$)

$$(13) \qquad \|y\|_r \leq \frac{\gamma(r)}{f(l)} \sum_{j=1}^{\tilde{l}} \|\widetilde{E}_j(y)\|_{r_1}$$

$$\leq \frac{\gamma(r)}{f(l)} \sum_{j=1}^{\tilde{l}} \left[\left\| \left| \widetilde{E}_j \left(\sum_{i < i_0} \alpha_i y_i \right) \right| \right\|_{r_1} + \|\widetilde{E}_j(\alpha_{i_0} y_{i_0})\|_{r_1} + \left\| \left| \widetilde{E}_j \left(\sum_{i > i_0} \alpha_i y_i \right) \right| \right\|_{r_1} \right]$$

$$\leq \frac{\gamma(r)}{f(l)} \sum_{i<i_0} \sum_{j=1}^{\widetilde{l}} |\alpha_i| \cdot \|E_j(y_i)\| + \frac{\gamma(r)f(\widetilde{l})}{f(l)} \frac{|\alpha_{i_0}|}{f(\widetilde{l})} \sum_{j=1}^{\widetilde{l}} \|\widetilde{E}_j(y_{i_0})\|$$

$$+ \frac{\gamma(r)}{f(l)} \sum_{i>i_0,\, \mathcal{E}_i \neq \emptyset} |\alpha_i| \sum_{E \in \mathcal{E}_i} \|E(y_i)\| + \frac{\gamma(r)}{f(l)} \sum_{E \in \mathcal{E}_0} \Big\|\Big|E\Big(\sum_{i>i_0} \alpha_i y_i\Big)\Big\|\Big|_{r_1}.$$

If $i_0 \neq 1$ we deduce that the first term in the above sum can be estimated as follows (we use condition (3) of the statement of Theorem 2.4 and note that from the choice of i_0 it follows that $l \geq \widetilde{l}/3 \geq l_{i_0}/3 \geq k_{i_0-1}/3 \geq k_i/3$ for $i < i_0$):

$$\frac{\gamma(r)}{f(l)} \sum_{i<i_0} \sum_{j=1}^{\widetilde{l}} |\alpha_i| \cdot \|E_j(y_i)\| \leq \gamma(r) \sum_{i<i_0} \frac{|\alpha_i|}{f(k_i/3)} \cdot \# \operatorname{supp}(y_i) \leq \gamma(r) \sum_{i=1}^{i_0-1} \varepsilon_i |\alpha_i| \,.$$

The second term can be estimated as follows:

$$\frac{\gamma(r)f(\widetilde{l})}{f(l)} |\alpha_{i_0}| \frac{1}{f(\widetilde{l})} \sum_{j=1}^{\widetilde{l}} \|\widetilde{E}_j(y_{i_0})\| \leq \frac{\gamma(r)f(3l)}{f(l)} |\alpha_{i_0}| \cdot \|y_{i_0}\| \leq \frac{\gamma(r)f(3r)}{f(r)} |\alpha_{i_0}|.$$

By condition (2) of the statement of Theorem 2.4 and the definition of i_0 we deduce that $\sum_{E \in \mathcal{E}_i} \|E(y_i)\| \leq 1 + \varepsilon_i$ if $i > i_0$ and $\mathcal{E}_i \neq \emptyset$. Thus, we observe for the third term that

$$\frac{\gamma(r)}{f(l)} \sum_{i>i_0,\, \mathcal{E}_i \neq \emptyset} |\alpha_i| \sum_{E \in \mathcal{E}_i} \|E(y_i)\| \leq \frac{\gamma(r)}{f(l)} \sum_{i>i_0,\, \mathcal{E}_i \neq \emptyset} (1+\varepsilon_i)|\alpha_i|.$$

For the last term we apply the induction hypothesis and find for each $E \in \mathcal{E}_0$ an $i_E \in \{i > i_0 : \operatorname{supp}(y_i) \subset E\} \cup \{0\}$ so that

$$\sum_{E \in \mathcal{E}_0} \Big\|\Big|E\Big(\sum_{i>i_0} \alpha_i y_i\Big)\Big\|\Big|_{r_1}$$

$$\leq \beta(r_1) \sum_{E \in \mathcal{E}_0} \Big[|\alpha_{i_E}| + \Big\|\sum_{\substack{i>i_E \\ \operatorname{supp}(y_i) \subset E}} \alpha_i e_i\Big\| + \sum_{\operatorname{supp}(y_i) \subset E} |\alpha_i|\varepsilon_i\Big].$$

Let $\mathcal{A} = \{\{i\} : i > i_0,\, \mathcal{E}_i \neq \emptyset\} \cup \{\{i_E\} : E \in \mathcal{E}_0\} \cup \{\{i > i_E : \operatorname{supp}(y_i) \subseteq E\} : E \in \mathcal{E}_0\}\setminus\{\emptyset\}$ and note that $\mathcal{A}$ consists of subsets of $\{i_0+1, i_0+2, \ldots\}$, has at most $3\widetilde{l} \leq 9l$ elements and is well ordered by $<$. Finally we deduce from (13) and the above estimates that

$$\|y\|_r \leq \gamma(r) \sum_{i<i_0} |\alpha_i|\varepsilon_i + \frac{\gamma(r)f(3r)}{f(r)} |\alpha_{i_0}| + \frac{\gamma(r)}{f(l)} \sum_{i>i_0,\, \mathcal{E}_i \neq \emptyset}^{m} |\alpha_i|$$

$$+ \frac{\gamma(r)}{f(l)} \sum_{i=1,\, \mathcal{E}_i \neq \emptyset}^{m} \varepsilon_i |\alpha_i| + \frac{\gamma(r)\beta(r_1)}{f(l)} \sum_{E \in \mathcal{E}_0} |\alpha_{i_E}|$$

$$+ \frac{\gamma(r)\beta(r_1)}{f(l)} \sum_{E \in \mathcal{E}_0} \Big\| \sum_{i > i_E, \, \mathrm{supp}(y_i) \subset E} \alpha_i e_i \Big\|$$

$$+ \frac{\gamma(r)\beta(r_1)}{f(l)} \sum_{\substack{\mathrm{supp}(y_i) \subset E \\ E \in \mathcal{E}_0}} |\alpha_i| \varepsilon_i$$

$$\leq \beta(r) \Big[|\alpha_{i_0}| + \frac{f(r)}{f(l)f(9r)} \sum_{A \in \mathcal{A}} \Big\| A\Big(\sum_{i=1}^{m} \alpha_i e_i\Big)\Big\| + \sum_{i=1}^{m} \varepsilon_i |\alpha_i| \Big]$$

$$\text{(note that } \beta(r) = \beta(r_1)\gamma(r)f(9r)/f(r))$$

$$\leq \beta(r) \Big[|\alpha_{i_0}| + \frac{f(r)}{f(9r)} \frac{f(9l)}{f(l)} \Big\| \sum_{i=1}^{m} \alpha_i e_i \Big\| + \sum_{i=i_0+1}^{m} \varepsilon_i |\alpha_i| \Big] \quad \text{(since } \#\mathcal{A} \leq 9l)$$

$$\leq \beta(r) \Big[|\alpha_{i_0}| + \Big\| \sum_{i=1}^{m} \alpha_i e_i \Big\| + \sum_{i=i_0+1}^{m} \varepsilon_i |\alpha_i| \Big] \quad \text{(since } l \geq r).$$

This proves the induction step and completes the proof of Theorem 2.4. ∎

3. The Banach space S is subsequentially prime. The main result of this section is the following theorem.

THEOREM 3.1. *The space S is subsequentially prime.*

Theorem 3.1 will essentially follow from A. Pełczyński's decomposition method and the following theorem.

THEOREM 3.2. *Let (x_i) be a normalized block sequence of (e_i) in S and let (k_i) be a subsequence of $\mathbb{N}$. There exists a subsequence (y_i) of (x_i) with the following property: If (s_i) and (t_i) are strictly increasing in $\mathbb{N}$, and $s_i, t_i \leq k_i$ for $i \in \mathbb{N}$, then (y_{s_i}) and (y_{t_i}) are equivalent.*

Before giving the proof of Theorem 3.2 we need a result, for which we introduce the following norm $\langle\!\langle \cdot \rangle\!\rangle$ on c_{00}. For $l \geq 3$ define $g(l) = \log_2(1+l/2)$ and let $\langle\!\langle \cdot \rangle\!\rangle$ be the norm which is implicitly defined by

$$\langle\!\langle x \rangle\!\rangle = \|x\|_{\ell_\infty} \vee \sup_{l \geq 3, \, E_1 < \ldots < E_l} \frac{1}{g(l)} \sum_{i=1}^{l} \langle\!\langle E_i(x) \rangle\!\rangle \quad \text{whenever } x \in c_{00}.$$

LEMMA 3.3. *The norm $\langle\!\langle \cdot \rangle\!\rangle$ is equivalent to the norm $\|\cdot\|$ on S.*

Proof. First note that since $f(l) \geq g(l+1)$ whenever $l \geq 2$ it follows that $\langle\!\langle \cdot \rangle\!\rangle \geq \|\cdot\|$. Thus, we only have to show that for some constant C it follows that $\langle\!\langle \cdot \rangle\!\rangle \leq C\|\cdot\|$. The proof will be similar to that of Theorem 2.4.

For $l, r \in [3, \infty)$ let $\langle\!\langle \cdot \rangle\!\rangle_l$ and $\langle\!\langle\!\langle \cdot \rangle\!\rangle\!\rangle_r$ be defined as $\|\cdot\|_l$ and $\|\!|\cdot\|\!|_r$ respectively. Let $\widetilde{d} > 1$ be chosen so that the statement of Lemma 2.6 holds (see

remark after the proof of Lemma 2.6), i.e. so that

$$(14) \qquad \langle\!\langle\!\langle x \rangle\!\rangle\!\rangle_r \leq \left[\frac{1}{1 - \widetilde{d}/\sqrt{g(r)}}\right] \sup_{\substack{l \geq r \\ E_1 < \ldots < E_l}} \frac{1}{g(l)} \sum_{i=1}^{l} \langle\!\langle\!\langle E_i(x) \rangle\!\rangle\!\rangle_{r^{g(r)}}$$

if $r \in \mathbb{R}_+$ with $g(r) > \widetilde{d}^2$, and $x \in c_{00}$ with $\langle\!\langle\!\langle x_r \rangle\!\rangle\!\rangle \neq \|x\|_{\ell_\infty}$.

For $r \in [3, \infty)$ we define $r_0 = r$ and, assuming that r_k had been defined, let $r_{k+1} = r_k^{g(r_k)}$. Then we deduce that there is an $R \geq 3$ so that for all $r \geq R$,

$$\widetilde{\beta}(r) = \prod_{k=0}^{\infty} \frac{1}{1 - \widetilde{d}/\sqrt{g(r_k)}} \frac{g(2r_k)}{g(r_k)}$$

is finite. By induction on $m \in \mathbb{N}$ we prove that for each $x \in c_{00}$ so that $\# \operatorname{supp}(x) \leq m$ and $r \geq R$ it follows that

$$(15) \qquad \langle\!\langle\!\langle x \rangle\!\rangle\!\rangle_r \leq \widetilde{\beta}(r)\|x\|.$$

Assume that (15) is true for all $z \in c_{00}$ for which $\# \operatorname{supp}(z) < m$ and assume that $x \in c_{00}$ with $\# \operatorname{supp}(x) = m$. If $\langle\!\langle\!\langle x \rangle\!\rangle\!\rangle_r = \|x\|_{\ell_\infty}$ the claim follows immediately. Otherwise it follows from (14) that for some $l \geq r$, $l \in \mathbb{N}$, and some choice of $E_1 < \ldots < E_l$ we have

$$\langle\!\langle\!\langle x \rangle\!\rangle\!\rangle_r \leq \left[\frac{1}{1 - \widetilde{d}/\sqrt{g(r)}}\right] \frac{1}{g(l)} \sum_{i=1}^{l} \langle\!\langle\!\langle E_i(x) \rangle\!\rangle\!\rangle_{r^{g(r)}}$$

$$\leq \widetilde{\beta}(r^{g(r)}) \left[\frac{1}{1 - \widetilde{d}/\sqrt{g(r)}}\right] \frac{1}{g(l)} \sum_{i=1}^{l} \|E_i(x)\|$$

$$\text{(by the induction hypothesis)}$$

$$= \widetilde{\beta}(r) \frac{g(r)}{g(2r)} \frac{1}{g(l)} \sum_{i=1}^{l} \|E_i(x)\| \leq \widetilde{\beta}(r)\|x\|$$

$$\left(\text{since } \frac{g(r)}{g(2r)} \leq \frac{g(l)}{g(2l)} = \frac{g(l)}{f(l)}\right),$$

which finishes the induction step and the proof of Lemma 3.3. ∎

Proof of Theorem 3.2. First we note that we can assume that $\lim_{n\to\infty} \|x_n\|_{\ell_\infty} = 0$. Indeed, from Theorem 2.4 it follows that there is a normalized block (z_k) in S which is equivalent to the unit vector basis (e_k) and has the property that $\lim_{n\to\infty} \|z_n\|_{\ell_\infty} = 0$. Thus, we could replace each x_n by the vector x'_n in the span of (z_k) whose coordinates with respect to the z_k's are the coordinates of x_n with respect to the e_k's.

Let $\varepsilon > 0$, and $y \in c_{00}$ with $\|y\| = 1$ and $\|y\|_\infty \leq \varepsilon/2$. We can write y as $y = \sum_{i=1}^{l(y,\varepsilon)} y(i,\varepsilon)$, where $l(y,\varepsilon) \in \mathbb{N}$ and $y(1,\varepsilon) < y(2,\varepsilon) < \ldots < y(l(y,\varepsilon),\varepsilon)$ such that $\|y(i,\varepsilon)\| \leq \varepsilon$ for $i = 1, \ldots, l(y,\varepsilon)$. Furthermore we could choose the $y(i,\varepsilon)$'s to have maximal support in the following sense. First we choose $y(i,\varepsilon) = [1,n_1](y)$ with $n_1 \in \mathbb{N}$ being the largest $n \in \mathbb{N}$, $n \leq \max \operatorname{supp}(y)$, so that $\|[1,n](y)\| \leq \varepsilon$. Then we choose $n_2 > n_1$ being the largest $n \in \mathbb{N}$, $n \leq \operatorname{supp}(y)$, so that $\|[n_1+1,n_2](y)\| \leq \varepsilon$. We can continue this way until we have exhausted the support of y. This defines $l(y,\varepsilon)$ and the vectors $(y(i,\varepsilon))_{i=1}^{l(y,\varepsilon)}$ uniquely and from the assumption that $\|y\|_\infty \leq \varepsilon/2$ it follows that $\varepsilon/2 \leq \|y(i,\varepsilon)\|$ for all $i \in \{1, \ldots, l(y,\varepsilon) - 1\}$. From the definition of the norm of S is follows that

$$\frac{\ell(y,\varepsilon) - 1}{f(\ell(y,\varepsilon))} \frac{\varepsilon}{2} \leq \frac{1}{f(l(y,\varepsilon))} \sum_{i=1}^{l(y,\varepsilon)} \|y(i,\varepsilon)\| \leq \|y\| = 1 \leq \sum_{i=1}^{l(y,\varepsilon)} \|y(i,\varepsilon)\| \leq l(y,\varepsilon)\varepsilon.$$

Thus for $\varepsilon > 0$ there are two numbers $H(\varepsilon) \geq h(\varepsilon)$ in $\mathbb{N}$ with $h(\varepsilon) \nearrow \infty$ if $\varepsilon \searrow 0$, and so that for any $y \in c_{00}$ with $\|y\| = 1$ and $\|y\|_\infty \leq \varepsilon/2$ it follows that $h(\varepsilon) \leq l(y,\varepsilon) \leq H(\varepsilon)$.

We now apply this "splitting procedure" to the elements of our sequence (x_n). By induction on $n \in \mathbb{N}$ we find an infinite subset M_n of $\mathbb{N}$ with $\mathbb{N} \supset M_1 \supset \ldots \supset M_n$ and $\min M_1 < \ldots < \min M_n$, and numbers $\varepsilon(n) > 0$ and $p(n) \in \mathbb{N}$ satisfying the following three properties:

(16)   For all $m \in M_n$ we have $l(\varepsilon(n), x_m) = p(n)$.

(17)   For any choice of $n \leq s_0 < s_1 < \ldots < s_{k_n+1}$ and $n \leq t_0 < t_1 < \ldots$
$< t_{k_n+1}$ in M_n it follows that $(x_{s_j}(i,\varepsilon(n)))_{0 \leq j \leq k_n+1, i \leq p(n)} \approx_{1+\varepsilon(n)}$
$(x_{t_j}(i,\varepsilon(n)))_{0 \leq j \leq k_n+1, i \leq p(n)}$.

(18)   If $n > 1$ it follows that $\sum_{1=i}^{n-1} \|x_{\min M_i}\|_{\ell_1} < f(h(\varepsilon(n)))2^{-n}$ and
$p(n-1)\varepsilon(n) < 2^{-n}$.

For $n = 1$ we simply choose $\varepsilon(1) = 1$ (thus $l(x_n, \varepsilon(1)) = 1$), $p(1) = 1$, and using compactness and the usual stabilization argument we can pass to a subsequence M_1 of $\mathbb{N}$ so that (17) holds. Assuming we made our choices of M_j, $\varepsilon(j)$, and $p(j)$ for all $j < n$, we first choose $\varepsilon(n)$ so that (18) is satisfied (recall that $h(\varepsilon) \nearrow \infty$ for $\varepsilon \searrow 0$), and then again using compactness and the usual stabilization argument we can pass to an $M_n \subset M_{n-1} \setminus \{\min M_{n-1}\}$ and find a $p(n) \in \mathbb{N}$ so that $\|x_m\| \leq \varepsilon(n)/2$ whenever $m \in M_n$ and so that (16) and (17) are satisfied.

For $n \in \mathbb{N}$ we now define $y_n = x_{\min(M_n)}$ and $y_n(i,j) = y_n(i,\varepsilon(j))$ if $i \leq n$ and $j \leq p(i)$, and prove the following claim by induction on $N \in \mathbb{N}$:

CLAIM. *For every $n \in \mathbb{N}$, every $K, L \in \{1, \ldots, p(n)\}$, every $(\alpha_i)_{i=0}^{n+N+1}$ $\subset \mathbb{R}$, and every choice of $n \leq s_0 < s_1 < \ldots < s_{n+N+1}$ and $n \leq t_0 < t_1 < \ldots$*

$< t_{n+N+1}$ *with* $s_i, t_i \leq k_{i+n}$ *for* $i = 0, 1 \ldots, n+N+1$, *it follows for*

$$(19) \qquad x = \alpha_0 \sum_{j=K}^{p(n)} y_{s_0}(n,j) + \sum_{i=1}^{N} \alpha_i y_{s_i} + \alpha_{N+n+1} \sum_{j=1}^{L} y_{s_{n+N+1}}(n,j),$$

$$(20) \qquad \widetilde{x} = \alpha_0 \sum_{j=K}^{p(n)} y_{t_0}(n,j) + \sum_{i=1}^{N} \alpha_i y_{t_i} + \alpha_{N+n+1} \sum_{j=1}^{L} y_{t_{n+N+1}}(n,j)$$

that

$$(21) \qquad \|x\| \leq c(n) \max_{E<F} [\langle\!\langle E(\widetilde{x})\rangle\!\rangle + \langle\!\langle F(\widetilde{x})\rangle\!\rangle]$$

where

$$(22) \qquad c(n) = \sum_{i=n}^{N} 2^{-i} + \varepsilon(i).$$

Since by Lemma 3.3, $\langle\!\langle \cdot \rangle\!\rangle$ is an equivalent norm on S (and therefore also the norm $c_{00} \ni x \mapsto \max_{E<F} \langle\!\langle E(x)\rangle\!\rangle + \langle\!\langle F(x)\rangle\!\rangle$ has this property) the claim implies the theorem.

For $N = 0$ the claim follows directly from (17). Assume the claim to be true for $\widetilde{N} < N$ and let x and $\widetilde{x}$ be given as in (19) and (20).

We choose $l \in \mathbb{N}$ so that $\|x\| = \|x\|_l$. If $l < p(k_n)$ we let $i_0 = 0$, and otherwise we choose $i_0 \in \mathbb{N} \cup \{0\}$ so that $p(k_{i_0+n}) \leq l < p(k_{i_0+n+1})$. We split x into three vectors $x^{(1)}$, $x^{(2)}$ and $x^{(3)}$ as follows. If $i_0 = 0$ we let $x^{(1)} = 0$, otherwise we put

$$(23) \qquad x^{(1)} = \alpha_0 \sum_{j=K}^{p(n)} y_{s_0}(n,j) + \sum_{i=1}^{(i_0-1)\wedge N} \alpha_i y_{s_i},$$

and $\widetilde{x}^{(1)}$ is defined as $x^{(1)}$, by replacing the s_i's by t_i's. From (18) and the choice of i_0 it follows that (note that $h(\varepsilon(s_{i_0-1}+1)) \leq h(\varepsilon(s_{i_0})) \leq p(s_{i_0}) \leq p(k_{n+i_0}) \leq l$)

$$(24) \qquad \|x^{(1)}\|_l \leq \max_{0 \leq i \leq n+N} |\alpha_i| \frac{1}{f(l)} \sum_{i=0}^{(i_0-1)\wedge N} \|y_{s_i}\|_{l_1} \leq \|\widetilde{x}\| \frac{f(h(\varepsilon(s_{i_0})))}{f(l)}$$

$$\leq \|\widetilde{x}\| 2^{-n-i_0}.$$

Secondly, if we let

$$(25) \qquad x^{(2)} = \begin{cases} \alpha_0 \displaystyle\sum_{j=K}^{p(n)} y_{s_0}(n,j) + \displaystyle\sum_{i=1}^{(k_n+1)\wedge N} \alpha_i y_{s_i} & \text{if } i_0 = 0, \\[2em] \displaystyle\sum_{i=(i_0-1)\wedge N+1}^{(k_{n+i_0}+1)\wedge N} \alpha_i y_{s_i} & \text{if } i_0 \neq 0, \end{cases}$$

and $\widetilde{x}^{(1)}$ is defined as $x^{(1)}$, by replacing the s_i's by t_i's, we deduce from (17) that

(26) $\qquad \|x^{(2)}\| \leq (1 + \varepsilon(s_{i_0}))\|\widetilde{x}^{(2)}\| \leq (1 + \varepsilon(n + i_0))\|\widetilde{x}^{(2)}\|.$

Finally we let

(27) $\qquad x^{(3)} = x - x^{(1)} - x^{(2)} = \sum_{i=k_{n+i_0}+2}^{N} \alpha_i y_{s_i} + \alpha_{N+n+1} \sum_{j=L}^{p(n)} y_{s_{n+N+1}}(n, j)$

and, again, define $\widetilde{x}^{(3)}$ as $x^{(3)}$, by replacing the s_i's by t_i's. Choose $E_1 < \ldots < E_l$ so that

$$\|x^{(3)}\|_l = \frac{1}{f(l)} \sum_{j=1}^{l} \|E_j(x^{(3)})\|.$$

Note that for any $i \geq k_{n+i_0}+2$ it follows that $s_i \geq k_{n+i_0}+2+n \geq k_{n+i_0}+2$ and by (18) it follows that $l\varepsilon(k_{n+i_0} + 2) \leq p(k_{n+i_0} + 1)\varepsilon(k_{n+i_0} + 2) < 2^{-k_{n+i_0}-1} \leq 2^{-n-i_0-1}$. Let $n' = k_{n+i_0} + 2 + n$. For each $j = 1, \ldots, l$ we can perturb E_j into a set F_j (by possibly taking some part of the support of some $y_{s_i}(n', u)$ away at the beginning of E_j and adding some part of the support of some $y_{s_j}(n', v)$ at the end of E_j) so that for each $j = 1, \ldots, l$, each $i \geq k_{n+i_0} + 2$, and each $u = 1, \ldots, p(n')$, the set F_j either contains $\text{supp}(y_{s_i}(n', u))$, or is disjoint from it, and so that

(28) $\qquad \|x^{(3)}\|_l = \frac{1}{f(l)} \sum_{j=1}^{l} \|F_j(x^{(3)})\| + l\varepsilon(n') \max_{0 \leq i \leq N+n+1} |\alpha_i|$

$$\leq \frac{1}{f(l)} \sum_{j=1}^{l} \|F_j(x^{(3)})\| + 2^{-n-i_0-1}\|x\|.$$

To each of the $F_j(x^{(3)})$ we can apply the induction hypothesis and obtain a splitting of each

$$\widetilde{F}_j = \bigcup \{\text{supp}(y_{t_i}(n', u)) : i \geq i_0 + 3, u \leq p(n') \text{ and } \text{supp}(y_{s_i}(n', u)) \subset F_j\}$$

($\widetilde{F}_j$ is the "$\widetilde{x}^{(3)}$ version of $x^{(3)}$") into $\widetilde{G}_j$ and $\widetilde{H}_j$, with $\widetilde{G}_j < \widetilde{H}_j$ so that

(29) $\quad \|x^{(3)}\|_l \leq 2^{-n-i_0-1}\|x\| + \frac{1}{f(l)}c(n') \sum_{j=1}^{l} (\langle\!\langle\widetilde{G}_j(\widetilde{x}^{(3)})\rangle\!\rangle + \langle\!\langle\widetilde{H}_j(\widetilde{x}^{(3)})\rangle\!\rangle)$

$$\leq 2^{-n-i_0-1}\|x\| + c(n')\langle\!\langle\widetilde{x}^{(3)}\rangle\!\rangle$$

(for the last inequality recall the definition of $\langle\!\langle \cdot \rangle\!\rangle$ and that $g(2l) = f(l)$).

Finally, putting (24), (26) and (29) together, observing that $\|\cdot\| \leq \langle\!\langle \cdot \rangle\!\rangle$, and that $c(n) \geq 2^{-n-i_0} + 2^{-n-i_0-1} + \varepsilon(n + 1) + c(n')$, we obtain

$$\|x\| = \|x\|_l \leq \|x^{(1)}\|_l + \|x^{(2)}\| + \|x^{(3)}\|_l$$

$$\leq \|\widetilde{x}\|2^{-n-i_0} + (1 + \varepsilon(n + i_0))\|\widetilde{x}^{(2)}\|$$

$$+ 2^{-n-i_0-1}\|\widetilde{x}\| + c(n')\langle\!\langle \widetilde{x}^{(3)}\rangle\!\rangle + c(n')\sup_{E<F}(\langle\!\langle E(\widetilde{x})\rangle\!\rangle + \langle\!\langle F(\widetilde{x})\rangle\!\rangle)$$

$$+ [2^{-n-i_0} + 2^{-n-i_0-1} + \varepsilon(n+1)]\|\widetilde{x}\|$$

$$\leq c(n)\sup_{E<F}(\langle\!\langle E(\widetilde{x})\rangle\!\rangle + \langle\!\langle F(\widetilde{x})\rangle\!\rangle),$$

which finishes the induction step and the proof of the theorem. ∎

Proof of Theorem 3.1. Let (x_n) be a seminormalized basic sequence in S whose closed linear span is complemented in S. Since S is reflexive we can assume, by passing to an appropriate subsequence, that for some $a \in S$, $x_n = a + z_n$ where z_n is weakly null. Since a is an element of the closed linear span of (x_n) and (x_n) is a seminormalized basic sequence it follows that $a = 0$, and thus (x_n) is seminormalized and weakly null. By applying the usual perturbation argument we can assume that (x_n) is a seminormalized block sequence and therefore apply Theorem 3.2 with $k_n = 2n + 1$ for $n \in \mathbb{N}$ to obtain a subsequence (y_n) so that (y_n), (y_{2n+1}) and (y_{2n}) are equivalent. In particular it follows for $Y = [y_n : n \in \mathbb{N}]$ that $Y \approx Y \oplus Y$ (the complemented sum of Y with itself). Since S has a subsymmetric basis, it also follows that $S \approx S \oplus S$. This means that we are in a position to apply Pełczyński's decomposition method [13], which is so elegant that we cannot restrain ourselves from repeating it here. We write $S = U \oplus Y$ (note that with $[x_i : i \in \mathbb{N}]$ also Y is complemented in S) and $Y = V \oplus S$. Then it follows that

$$S \approx U \oplus Y \approx U \oplus Y \oplus Y \approx S \oplus Y \approx S \oplus V \oplus S \approx S \oplus V \approx Y. \ \blacksquare$$

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