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One-sided discrete square function

by

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Dedicated to Professor Carlos Segovia on his 65th birthday

Abstract. Let f be a measurable function defined on $\mathbb{R}$. For each $n \in \mathbb{Z}$ we consider the average $A_n f(x) = 2^{-n} \int_x^{x+2^n} f$. The square function is defined as

$$Sf(x) = \Big(\sum_{n=-\infty}^{\infty} |A_n f(x) - A_{n-1} f(x)|^2 \Big)^{1/2}.$$

The local version of this operator, namely the operator

$$S_1 f(x) = \Big(\sum_{n=-\infty}^{0} |A_n f(x) - A_{n-1} f(x)|^2 \Big)^{1/2},$$

is of interest in ergodic theory and it has been extensively studied. In particular it has been proved [3] that it is of weak type $(1,1)$, maps L^p into itself $(p > 1)$ and L^∞ into BMO. We prove that the operator S not only maps L^∞ into BMO but it also maps BMO into BMO. We also prove that the L^p boundedness still holds if one replaces Lebesgue measure by a measure of the form $w(x)dx$ if, and only if, the weight w belongs to the A_p^+ class introduced by E. Sawyer [8]. Finally we prove that the one-sided Hardy–Littlewood maximal function maps BMO into itself.

Introduction. Let f be a measurable function defined on $\mathbb{R}$. For each $n \in \mathbb{Z}$ define the operator A_n by

$$A_n f(x) = \frac{1}{2^n} \int_x^{x+2^n} f(y)\, dy.$$

It is a classical problem to study the different kinds of convergence of the sequence $\{A_n f\}_n$ when the function f belongs to $L^p(\mathbb{R}, dx)$, p being in the

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range $1 \le p < \infty$. A method of measuring the speed of convergence of this sequence is to analyze the boundedness of the square function

$$Sf(x) = \Big(\sum_{n=-\infty}^{\infty} |A_n f(x) - A_{n-1} f(x)|^2 \Big)^{1/2}. \tag{0.1}$$

Recently, among other operators, the local version of this operator, namely the operator

$$S_1 f(x) = \Big(\sum_{n=-\infty}^{0} |A_n f(x) - A_{n-1} f(x)|^2 \Big)^{1/2} \tag{0.2}$$

has been studied in [3] and [4]. It has been proved that S_1 maps $L^p(\mathbb{R}, dx)$ into itself for each p in the range $1 < p < \infty$ and that S_1 is of weak type $(1,1)$, that is,

$$|\{x : Sf(x) > \lambda\}| \le \frac{C}{\lambda} \int_{\mathbb{R}} |f(y)|\, dy,$$

where as usual we denote by $|E|$ the Lebesgue measure of a set $E \subset \mathbb{R}$.

The aim of this note is to characterize the weights ω (almost everywhere positive measurable functions) such that either, for each p in the range $1 < p < \infty$, the operator S maps $L^p(\mathbb{R}, \omega(x)dx)$ into itself, or the following weak type $(1,1)$ inequality is satisfied:

$$\omega(\{x : Sf(x) > \lambda\}) \le \frac{C}{\lambda} \int_{\mathbb{R}} |f(y)| \omega(y)\, dy, \tag{0.3}$$

where C is a positive constant.

In our opinion, the natural way of proving weighted results for this operator is to use the theory of vector-valued singular integrals. Therefore it would seem that the right class of weights were the A_p classes of Muckenhoupt, but this would overlook the fact that the operator S is *one-sided*, i.e. $Sf(x) = S(f(\cdot)\chi_{(x,\infty)}(\cdot))(x)$; clearly A_p is not a necessary condition. For one-sided operators the natural classes are the A_p^+ classes introduced by E. Sawyer [8] (see (2.1) and (2.2) in Section 2 for the corresponding definitions). In fact we shall prove the following result.

THEOREM A. *Given p in the range $1 \le p < \infty$, and a weight ω, the following are equivalent*:

(i) *There exists a constant C_p such that*

$$(\omega(\{x : Sf(x) > \lambda\}))^{1/p} \le \frac{C_p}{\lambda} \|f\|_{L^p(\mathbb{R}, \omega(x)dx)}, \tag{0.4}$$

(ii) $\omega \in A_p^+$.

Moreover, *in the case $1 < p < \infty$ they are also equivalent to the following statement*:

(iii) *There exists a constant* C_p *such that*

$$\|Sf\|_{L^p(\mathbb{R},\omega(x)dx)} \le C_p\|f\|_{L^p(\mathbb{R},\omega(x)dx)}. \tag{0.5}$$

In order to prove this theorem we shall introduce a one-sided vector-valued Calderón–Zygmund operator U (see Definition 1.1). We believe that our main contribution is a careful geometric analysis of the kernel of the operator U (see Lemma 1.2 and also (1.4)) that allows us to show that the kernel satisfies some one-sided Hörmander type conditions (see Definition 1.5). Conditions of this type suggest in general boundedness from L^∞ into BMO; however, the following example seems to forbid such a result.

EXAMPLE. *If* $f = \sum_{n=0}^{\infty} \chi_{[4^n,2\cdot 4^n]} = \sum_{i=0}^{\infty} \chi_{[2^{2i},2^{2i+1}]}$, *then* $Sf(x) = \infty$ *for every* x.

Proof. We shall see that $Sf(0) = \infty$; if $x \neq 0$ one can prove $Sf(x) = \infty$ in the same way. If $j = 2k+1$ then $A_jf(0) = (1/2^j)\sum_{i=0}^{k} 4^i$, while $A_{j+1}f(0) = \frac{1}{2}A_jf(0)$. It follows that $|A_{2k+2}f(0) - A_{2k+1}f(0)| = \frac{1}{2}A_{2k+1}f(0) > \frac{1}{4}$ and so $Sf(0) = \infty$. ∎

In fact as a byproduct of our study we shall obtain the following dichotomy results that we believe are of independent interest.

THEOREM B. (a) *Given a function* f *in* $L^\infty(\mathbb{R})$, *either* $Sf(x) = \infty$ *for a.e.* x *or* $Sf(x) < \infty$ *for a.e.* x. *Moreover in the second case* $Sf \in \mathrm{BMO}$ *and there exists a constant* C *such that* $\|Sf\|_{\mathrm{BMO}(\mathbb{R})} \le C\|f\|_{L^\infty(\mathbb{R})}$.

(b) *Given a function* f *in* BMO($\mathbb{R}$), *either* $Sf(x) = \infty$ *for a.e.* x *or* $Sf(x) < \infty$ *for a.e.* x. *Moreover in the second case* $Sf \in \mathrm{BMO}$ *and there exists a constant* C *such that* $\|Sf\|_{\mathrm{BMO}(\mathbb{R})} \le C\|f\|_{\mathrm{BMO}(\mathbb{R})}$.

We believe that the geometric analysis developed for the study of the square function can be of interest for other one-sided operators. In particular we apply these ideas to study the behaviour of the one-sided Hardy–Littlewood maximal operator acting on functions that belong to the BMO class, and again we get a dichotomy result of the type of Theorem B (see Theorem 3.10).

The organization of the paper is as follows. In Section 1 we develop the adapted one-sided Calderón–Zygmund theory that we need and as a quick consequence we prove Theorem B. Section 2 is devoted to the study of weighted inequalities, and in particular to the proof of Theorem A. Finally in Section 3 we analyze the one-sided Hardy–Littlewood maximal operator. We end this introduction with some notation. Given a measurable set E and a weight w, $w(E)$ will represent the integral of w on E. If I is an interval and f a locally integrable function, we will denote by f_I the average of f on I, i.e. $f_I = (1/|I|)\int_I f$.

1. Vector-valued analysis. Boundedness on BMO and L_∞

DEFINITION 1.1. Given a locally integrable function f we define the sequence-valued operator U as follows:

$$\begin{aligned}Uf(x) &= \{A_nf(x) - A_{n-1}f(x)\}_n\\ &= \Big\{\int_{\mathbb{R}}\Big(\frac{1}{2^n}\chi_{(-2^n,0)}(x-y) - \frac{1}{2^{n-1}}\chi_{(-2^{n-1},0)}(x-y)\Big)f(y)\,dy\Big\}_n\\ &= \int_{\mathbb{R}} K(x-y)f(y)\,dy,\end{aligned}$$

where K is the sequence-valued function

$$K(x) = \Big\{\frac{1}{2^n}\chi_{(-2^n,0)}(x) - \frac{1}{2^{n-1}}\chi_{(-2^{n-1},0)}(x)\Big\}_n.$$

Observe that $\|Uf(x)\|_{\ell^2} = Sf(x)$. Although the operator U is defined in terms of averages with nonsmooth kernels it satisfies a one-sided smoothness condition, which will play the role of the Hörmander condition in the classical theory of singular integrals.

LEMMA 1.2. *Given $x_0 \in \mathbb{R}$ and $i \in \mathbb{Z}$, consider x and y in $\mathbb{R}$ such that $x_0 < x \le x_0 + 2^i$ and $x_0 + 2^j < y \le x_0 + 2^{j+1}$ with $j > i$. Let $\chi_n(y) = \chi_{(-2^n,0)}(y)$. Then $\chi_n(x-y) - \chi_n(x_0-y) = 0$ unless $n = j$ in which case $\chi_j(x-y) - \chi_j(x_0-y) = \chi_{(x_0+2^j,x+2^j)}(y)$.*

Proof. It is clear that $\chi_n(x-y) = \chi_{(x,x+2^n)}(y)$. Now if $n < i$ then $x+2^n < x-x_0+x_0+2^i \le x_0+2^{i+1} \le x_0+2^j < y$. Therefore $\chi_n(x-y) = 0$. Obviously the same holds for $\chi_n(x_0-y)$. If $i \le n < j$ then $x+2^n \le x_0+2^i+2^n \le x_0+2\cdot 2^n \le x_0+2^j$, and $\chi_n(x-y) = \chi_n(x_0-y) = 0$. If $n > j$ then $x+2^n > x_0+2^n \ge x_0+2^{j+1} \ge y$, and since $y > x > x_0$ we have $\chi_n(x-y) - \chi_n(x_0-y) = 1-1 = 0$. Finally if $n = j$ then $\chi_j(x_0-y) = \chi_{(x_0,x_0+2^j)}(y) = 0$, while $\chi_j(x-y) = \chi_{(x,x+2^j)}(y) = 1$ whenever $x_0+2^j \le y \le x+2^j$. ∎

LEMMA 1.3 (Smoothness condition). *Assume x_0, x, y are as in the preceding lemma. Let K be the vector-valued kernel that appears in Definition* 1.1. *Then*

$$\|K(x-y) - K(x_0-y)\|_{\ell^2} = \frac{\sqrt{2}}{2^j}\chi_{(x_0+2^j,x+2^j)}(y). \tag{1.4}$$

Proof. We have

$$\begin{aligned}\|K(x-y) - K(x_0-y)\|_{\ell^2}^2 = \sum_n \Big|&\frac{1}{2^n}\chi_n(x-y) - \frac{1}{2^{n-1}}\chi_{n-1}(x-y)\\ &- \Big(\frac{1}{2^n}\chi_n(x_0-y) - \frac{1}{2^{n-1}}\chi_{n-1}(x_0-y)\Big)\Big|^2\end{aligned}$$

$$= \sum_n \left| \frac{1}{2^n} \chi_n(x-y) - \frac{1}{2^n} \chi_n(x_0-y) \right.$$
$$\left. - \left(\frac{1}{2^{n-1}} \chi_{n-1}(x-y) - \frac{1}{2^{n-1}} \chi_{n-1}(x_0-y) \right) \right|^2$$
$$= 2\left| \frac{1}{2^j} \chi_j(x-y) - \frac{1}{2^j} \chi_j(x_0-y) \right|^2 = 2\left| \frac{1}{2^j} \chi_{(x_0+2^j, x+2^j)}(y) \right|^2. \ \blacksquare$$

It follows from (1.4) that the kernel K does not satisfy the "gradient" condition

$$\|K(x-y) - K(x_0-y)\|_{\ell^2} \le C(x-x_0)(y-x_0)^{-2}$$

whenever $y - x_0 > 2(x-x_0)$. Nevertheless (1.4) will allow us to prove some kind of condition that implies Hörmander's.

Parallel to [7] we give the following

DEFINITION 1.5. We say that the kernel K satisfies one-sided condition D_r, for $1 \le r < \infty$, and write $K \in D_r$, if there exists a sequence $\{c_l\}_{l=1}^{\infty}$ of positive numbers such that $\sum_l c_l < \infty$ and for any $l \ge 2$ and $x > 0$,

$$\Big(\int\limits_{S_l(x)} \|K(x-y) - K(-y)\|_{\ell^2}^r \, dy \Big)^{1/r} \le c_l |S_l(x)|^{-1/r'},$$

where $S_l(x) = (2^l x, 2^{l+1} x)$.

It is easy to see that $D_s \subset D_r \subset D_1$ for $1 \le r < s$, where $K \in D_1$ means the following Hörmander's type condition:

$$\int\limits_{\{y>4x\}} \|K(x-y) - K(-y)\|_{\ell^2} \, dy \le C$$

where C is a positive constant.

THEOREM 1.6. *The kernel K introduced in Definition* 1.1 *satisfies D_r for any $r \ge 1$ with $c_l = C2^{-l/r}$.*

Proof. Given x, choose an integer i such that $2^{i-1} \le x < 2^i$. Lemma 1.3 and Hölder's inequality give us

$$\Big(\int\limits_{2^l x}^{2^{l+1} x} \|K(x-y) - K(-y)\|_{\ell^2}^r \, dy \Big)^{1/r}$$
$$\le \Big(\int\limits_{2^{l+i-1}}^{2^{l+i}} \|K(x-y) - K(-y)\|_{\ell^2}^r \, dy \Big)^{1/r}$$

$$+\Big(\int\limits_{2^{l+i}}^{2^{l+i+1}} \|K(x-y)-K(-y)\|_{\ell^2}^r\,dy\Big)^{1/r}$$

$$\le 2\,\frac{2^{i/r}}{2^{l+i}} = C2^{-l/r}|S_l(x)|^{-1/r'}. \ \blacksquare$$

Since we have Hörmander's condition and it is easy to check that the Fourier transform of the kernel of our vector-valued operator U is bounded we deduce that the operator S is bounded on L^p, $p>1$, and satisfies a weak type $(1,1)$ inequality. Now we shall use our smoothness condition in order to study the pointwise size of the operator. We start with the following technical lemma.

LEMMA 1.7. *Let f be a locally integrable function such that $Sf(x_0)<\infty$ for some $x_0\in\mathbb{R}$. Then $S(f(\cdot)\chi_{[x,\infty)}(\cdot))(x_0)<\infty$ for any $x>x_0$.*

Proof. Let i be an integer such that $2^i < x-x_0 \le 2^{i+1}$, consider $g = f\chi_{(x,\infty)}$ and $\mathcal{I}=\int_{x_0}^x f$. If $j\le i$, then $A_jg(x_0)=0$. If $j>i$ then

$$A_jg(x_0)=\frac{1}{2^j}\int\limits_{x_0}^{x_0+2^j} g = \frac{1}{2^j}\int\limits_{x_0}^{x_0+2^j} f - \frac{1}{2^j}\int\limits_{x_0}^{x} f = A_jf(x_0)-\frac{1}{2^j}\,\mathcal{I}.$$

Therefore

$$\begin{aligned}
Sg(x_0) &= \Big(\sum_j |A_jg(x_0)-A_{j-1}g(x_0)|^2\Big)^{1/2}\\
&= \Big|A_{i+1}f(x_0)-\frac{1}{2^{i+1}}\,\mathcal{I}\Big| + \Big(\sum_{j>i+2}|A_jf(x_0)-A_{j-1}f(x_0)|^2\Big)^{1/2}\\
&\quad + \Big(\sum_{j>i+2}\Big|\frac{1}{2^{j-1}}-\frac{1}{2^j}\Big|^2\Big)^{1/2}\mathcal{I}\\
&\le \Big|A_{i+1}f(x_0)-\frac{1}{2^{i+1}}\,\mathcal{I}\Big| + Sf(x_0) + \Big(\sum_{j>i+2}\Big|\frac{1}{2^{j-1}}-\frac{1}{2^j}\Big|^2\Big)^{1/2}\mathcal{I}\\
&< \infty. \ \blacksquare
\end{aligned}$$

PROPOSITION 1.8. *Let $f\in L^\infty(\mathbb{R})$ and let x_0 be such that $Sf(x_0)<\infty$. Then $Sf(x)<\infty$ for almost all $x>x_0$.*

Proof. We shall prove that the ℓ^2-valued operator U defined in 1.1 satisfies $\|Uf(x)\|_{\ell^2}<\infty$ for almost every $x>x_0$. Consider the interval $I_0=(x_0,x_0+4(x-x_0))$. Let $f_1=f\chi_{I_0}$ and $f_2=f-f_1$. Since the operator S is bounded on L^2, we have $\|Uf_1(y)\|_{\ell^2}=Sf_1(y)<\infty$. On the other hand, by using the one-sided nature of S and the last Lemma 1.7, we deduce that for $x > x_0$,

$$\|Uf_2(x_0)\|_{\ell^2} = Sf_2(x_0) = S(f(\cdot)\chi_{(x_0+4(x-x_0),\infty)}(\cdot))(x_0) < \infty.$$

Therefore it is enough to prove that $\|Uf_2(x) - Uf_2(x_0)\|_{\ell^2} < \infty$. By using again the one-sided nature and condition D_r with $r = 1$, we obtain

$$\|Uf_2(x) - Uf_2(x_0)\|_{\ell^2} = \Big\| \int_{x_0+4(x-x_0)}^{\infty} (K(x-y) - K(x_0-y))f(y)\,dy \Big\|_{\ell^2}$$

$$\le \int_{x_0+4(x-x_0)}^{\infty} \|K(x-y) - K(x_0-y)\|_{\ell^2}|f(y)|\,dy \le C\|f\|_\infty. \ \blacksquare$$

PROPOSITION 1.9. *Let $f \in L^\infty(\mathbb{R})$ and let x_0 be such that $Sf(x_0) < \infty$. Then $Sf(y) < \infty$ for almost all $y < x_0$.*

Proof. Following the proof of the last proposition, we shall see that $\|Uf(y)\|_{\ell^2} < \infty$ for almost every $y < x_0$. Given $y < x_0$. Set $I_0 = (y, y + 4(x_0 - y))$, $f_1 = f\chi_{I_0}$ and $f_2 = f - f_1$. Again $\|Uf_1(y)\|_{\ell^2} = Sf_1(y) < \infty$ for almost every y, because S is bounded on L^2. On the other hand, by using the one-sided nature of S and Lemma 1.7 (observe that $y + 4(x_0 - y) > x_0$), we see that for $y < x_0$,

$$\|Uf_2(x_0)\|_{\ell^2} = Sf_2(x_0) = S(f(\cdot)\chi_{(y+4(x_0-y),\infty)}(\cdot))(x_0) < \infty.$$

Therefore it is enough to prove that $\|Uf_2(y) - Uf_2(x_0)\|_{\ell^2} < \infty$. Now the proof ends as in the last proposition. $\blacksquare$

We have proved that for an L^∞ function f, Sf is either infinite a.e. or finite a.e. The same result can be proved, with minor modifications, for BMO functions. Therefore in order to prove Theorem B we need to prove that for functions f in L^∞ (respectively in BMO) with Sf finite almost everywhere, the function Sf is in BMO, and the BMO norm of Sf is controlled by the L^∞ norm (respectively the BMO norm) of f. We shall give only the proof in the case $f \in \mathrm{BMO}$. The case $f \in L^\infty$ is easier and we leave the details to the reader. We start with a technical lemma.

LEMMA 1.10. *Let C be a positive constant and let I_1 and I_2 be two intervals such that if J is the smallest interval that contains both then $|J| \le C|I_i|$, $i = 1, 2$. Then given a function $f \in \mathrm{BMO}$ we have*

$$|f_{I_1} - f_{I_2}| \le 2C\|f\|_{\mathrm{BMO}}.$$

Proof. It is clear that $|f_{I_1} - f_{I_2}| \le |f_{I_1} - f_J| + |f_J - f_{I_2}|$. Now

$$|f_{I_1} - f_J| \le \frac{C}{|J|}\int_J |f - f_J| \le C\|f\|_{\mathrm{BMO}}.$$

The other term is handled in the same way. $\blacksquare$

Corollary 1.11. *Let f be a* BMO *function*, $x_0 \in \mathbb{R}$, $h > 0$, i *an integer such that* $2^i \le h < 2^{i+1}$, j *any integer greater than* i, *and* $I = (x_0, x_0 + h)$. *Then*

$$\left(\frac{1}{2^j}\int_{x_0+2^j}^{x_0+2^{j+1}} |f(y) - f_I|^2\,dy\right)^{1/2} \le C(j - i + 1)\|f\|_{\mathrm{BMO}}.$$

Proof. For any integer l between i and j we denote by I_l the interval $(x_0 + 2^l, x_0 + 2^{l+1})$. Then

$$\left(\frac{1}{2^j}\int_{x_0+2^j}^{x_0+2^{j+1}} |f(y) - f_I|^2\,dy\right)^{1/2} \le \left(\frac{1}{2^j}\int_{x_0+2^j}^{x_0+2^{j+1}} |f(y) - f_{I_j}|^2\,dy\right)^{1/2} + \sum_{l=i+1}^{j} (|f_{I_l} - f_{I_{l-1}}| + |f_{I_{i+1}} - f_I|).$$

By John–Nirenberg,

$$\left(\frac{1}{2^j}\int_{x_0+2^j}^{x_0+2^{j+1}} |f(y) - f_{I_j}|^2\,dy\right)^{1/2} \le C\|f\|_{\mathrm{BMO}},$$

and by the preceding lemma, each of the other terms is dominated by $4\|f\|_{\mathrm{BMO}}$. ∎

Theorem 1.12. *Let f be a* BMO *function such that $Sf(x) < \infty$ a.e. Then $Sf \in$* BMO *and there exists C so that*

$$\|Sf\|_{\mathrm{BMO}} \le C\|f\|_{\mathrm{BMO}}.$$

Proof. Fix x_0 and $h > 0$. Consider the interval $I = (x_0, x_0 + h)$ and the average $f_I = (1/h)\int_I f$. Since $Sf(x)$ is finite a.e., it is enough to prove that there exists a positive constant C so that

$$\frac{1}{h}\int_{x_0}^{x_0+h} |Sf(x) - S((f(\cdot) - f_I)\chi_{(x_0+8h,\infty)}(\cdot))(x_0)|\,dx \le C\|f\|_{\mathrm{BMO}}.$$

We define $f_1 = (f - f_I)\chi_{(x_0,x_0+8h)}$ and $f_2 = (f - f_I)\chi_{(x_0+8h,\infty)}$; then $f = f_1 + f_2 + f_I$. By using the linear operator U defined in 1.1, we have

$$\begin{aligned}\frac{1}{h}\int_{x_0}^{x_0+h} |Sf(x) - Sf_2(x_0)|\,dx &= \frac{1}{h}\int_{x_0}^{x_0+h} \big|\|Uf(x)\|_{\ell^2} - \|Uf_2(x_0)\|_{\ell^2}\big|\,dx \\ &\le \frac{1}{h}\int_{x_0}^{x_0+h} \|Uf(x) - Uf_2(x_0)\|_{\ell^2}\,dx \\ &= \frac{1}{h}\int_{x_0}^{x_0+h} \|Uf_1(x) + Uf_2(x) - Uf_2(x_0)\|_{\ell^2}\,dx\end{aligned}$$

$$\leq \frac{1}{h}\int_{x_0}^{x_0+h} \|Uf_1(x)\|_{\ell^2}\,dx + \frac{1}{h}\int_{x_0}^{x_0+h} \|Uf_2(x)-Uf_2(x_0)\|_{\ell^2}\,dx = B_1+B_2.$$

The boundedness of S in L^2 and the John–Nirenberg inequality imply that

$$B_1 \leq \Big(\frac{1}{h}\int_{x_0}^{x_0+h} |Sf_1|^2\Big)^{1/2} \leq C\Big(\frac{1}{h}\int_{\mathbb{R}} |f_1|^2\Big)^{1/2} = C\Big(\frac{1}{h}\int_{x_0}^{x_0+8h} |f-f_I|^2\Big)^{1/2} \leq C\|f\|_{\mathrm{BMO}}.$$

For B_2 we just observe that if i is an integer such that $2^i \leq h < 2^{i+1}$, then using our smoothness condition and Corollary 1.11 we have

$$\begin{aligned} B_2 &\leq \int_{x_0+8h}^{\infty} \|(K(x-y)-K(x_0-y))(f(y)-f_I)\|_{\ell^2}\,dy \\ &\leq \sum_{j=i+3}^{\infty}\Big(\int_{x_0+2^j}^{x_0+2^{j+1}} \|K(x-y)-K(x_0-y)\|_{\ell^2}^2\,dy\Big)^{1/2} \\ &\quad\times\Big(\frac{1}{2^j}\int_{x_0+2^j}^{x_0+2^{j+1}} |f(y)-f_I|^2\,dy\Big)^{1/2} 2^{j/2} \\ &\leq C\sum_{j=i+3}^{\infty} 2^{i/2}(j-i)\,\frac{1}{2^j}\,2^{j/2}\|f\|_{\mathrm{BMO}} \leq C\|f\|_{\mathrm{BMO}}. \ \blacksquare \end{aligned}$$

2. Weights for the operator S. We recall that the A_p^+ classes were introduced by E. Sawyer [8] in the study of the one-sided Hardy–Littlewood maximal operators

$$M^+f(x) = \sup_{h>0}\int_x^{x+h} |f|, \qquad M^-f(x) = \sup_{h>0}\int_{x-h}^{x} |f|.$$

He proved the following.

THEOREM. *If $p>1$ then the inequality*

$$\int_{\mathbb{R}} M^+f(x)^p w(x)\,dx \leq C\int_{\mathbb{R}} |f(x)|^p w(x)\,dx$$

holds for all $f\in L^p(w)$ if and only if w satisfies the following condition:

(A_p^+) *There exists C such that for any three points $a < b < c$,*

$$\Big(\int_a^b w\Big)^{1/p}\Big(\int_b^c w^{1-p'}\Big)^{1/p'} \le C(c-a) \qquad (p+p'=pp'). \tag{2.1}$$

If $p = 1$ then the weak type inequality

$$\int_{\{x:M^+f(x)>\lambda\}} w \le \frac{C}{\lambda}\int |f(x)|w(x)\,dx$$

holds for all $f \in L^1(w)$ if and only if

(A_1^+) *There exists C such that for almost every x,*

$$M^-w(x) \le Cw(x). \tag{2.2}$$

REMARK 2.3. It is known (see [6]) that condition A_p^+ is equivalent to the following condition, called $\widetilde{A}_p^*$:

$$\int_{b-2h}^{b-h} w\Big(\int_b^{b+h} w^{1-p'}\Big)^{p-1} \le Ch^p.$$

Analogously in the case $p = 1$, condition A_1^+ is equivalent to the following condition: there exists C, which depends on the constant in (2.2), so that for almost every b, if $I = (a,b)$ then

$$\int_I w \le C|I|\inf\{w(x) : x \in (b, 2b-a)\}.$$

Of course there are similar results for the operator M^- reversing the orientation of $\mathbb{R}$. Since it is easy to see that any increasing function satisfies A_1^+, it is obvious that the A_p^+ classes are different from the Muckenhoupt A_p classes. It also follows that the A_p^+ weights do not satisfy the doubling condition nor the reverse Hölder inequality. Nevertheless there are nice substitutes for the doubling condition and the reverse Hölder inequality. Sawyer [8] proves that if $w \in A_1^+$, then $w^r \in A_1^+$ for some $r > 1$, and that $w \in A_p^+$ for some $p > 1$ implies $w \in A_s^+$ for some $1 < s < p$. On the other hand it is easy to see that any A_p^+ weight w satisfies the following one-sided doubling condition:

There exists C such that if $I = (a,b)$ and $I^+ = (b,c)$ with $b - a = c - b$ then $\int_I w \le C\int_{I^+} w$, which is clearly equivalent to $\int_{I\cup I^+} w \le C\int_{I^+} w$ for some constant C.

In this section we shall prove Theorem A. The result does not follow from [1] because although our operator can be considered as a one-sided (vector-valued) singular integral, it does not satisfy the gradient condition nor the cancellation conditions assumed in that article. Our main tool will be the following extrapolation theorem of Macías and Riveros [5]:

THEOREM (Extrapolation). *Let T be a sublinear operator with the following property: For every w such that $w^{-1} \in A_1^-$ there exists a constant $C(w)$, which may depend on w, such that*

$$\|w\chi_{[x-h,x]}\|_\infty \frac{1}{h} \int_x^{x+h} \left(|Tf|(y) - \frac{1}{h} \int_{x+h}^{x+2h} |Tf| \right)^+ dy \le C(w)\|fw\|_\infty$$

for every $h > 0$, $x \in \mathbb{R}$. Let $1 < p < \infty$ and $w \in A_p^+$. Then

$$\int |Tf|^p w \le C \int |f|^p w$$

provided the left hand side is finite.

Here is the estimate we get for the operator S.

THEOREM 2.4. *Assume that w is such that $w^{-1} \in A_1^-$. Then there exists a constant $C(w)$, which may depend on w, such that for every $h > 0$ and $x_0 \in \mathbb{R}$,*

$$\|w\chi_{[x_0-h,x_0]}\|_\infty \frac{1}{h} \int_{x_0}^{x_0+h} |Sf(x) - S(f\chi_{(x_0+8h,\infty)})(x_0)|\, dx \le C(w)\|fw\|_\infty.$$

Proof. Given a function f we define $f_1 = f\chi_{(x_0,x_0+8h)}$, $f_2 = f\chi_{(x_0+8h,\infty)}$. Then by using the linear operator introduced in Definition 1.1, we have

$$\begin{aligned}
\frac{1}{h} \int_{x_0}^{x_0+h} |Sf(x) - Sf_2(x_0)|\, dx &= \frac{1}{h} \int_{x_0}^{x_0+h} \big| \|Uf(x)\|_{\ell^2} - \|Uf_2(x_0)\|_{\ell^2} \big|\, dx \\
&\le \frac{1}{h} \int_{x_0}^{x_0+h} \|Uf(x) - Uf_2(x_0)\|_{\ell^2}\, dx \\
&\le \frac{1}{h} \int_{x_0}^{x_0+h} \|Uf_1(x)\|_{\ell^2}\, dx \\
&\quad + \frac{1}{h} \int_{x_0}^{x_0+h} \|Uf_2(x) - Uf_2(x_0)\|_{\ell^2}\, dx \\
&= B_1 + B_2.
\end{aligned}$$

We choose $t > 1$ sufficiently close to 1 such that:

(i) $(w^{-1})^t \in A_1^-$,

(ii) there exists s in the range $2 < s < \infty$ such that $1/2 + 1/s + 1/t = 1$.

Observe that in this case $1/s + 1/t = 1/2$. Then as the operator S is bounded in $L^p(\mathbb{R})$ for every p in the range $1 < p < \infty$, we have

$$\begin{aligned} B_1 &\le \left(\frac{1}{h}\int_{x_0}^{x_0+h} \|U(f_1)(x)\|_{\ell^2}^t\,dx\right)^{1/t} = \left(\frac{1}{h}\int_{x_0}^{x_0+h} |Sf_1|^t\right)^{1/t} \\ &\le C\left(\frac{1}{h}\int_{x_0}^{x_0+8h} |f|^t\,dx\right)^{1/t} \\ &\le C\|fw\|_\infty\left(\frac{1}{h}\int_{x_0}^{x_0+8h} (w^{-1}(x))^t\,dx\right)^{1/t}. \end{aligned}$$

Since $w^{-t} \in A_1^-$ we have

$$\|w\chi_{(x_0-h,x_0)}\|_\infty B_1 \le \|fw\|_\infty \|w\chi_{(x_0-h,x_0)}\|_\infty \left(\frac{1}{h}\int_{x_0}^{x_0+8h} (w^{-1})^t\right)^{1/t} \le C.$$

In order to bound B_2, we consider an integer i such that $2^i \le h < 2^{i+1}$. If we use Hölder's inequality and our smoothness condition, we get

$$\begin{aligned} &\|Uf_2(x) - Uf_2(x_0)\|_{\ell^2} \\ &\qquad \le \int_{x_0+8h}^{\infty} \|K(x-y) - K(x_0-y)\|_{\ell^2}|f(y)|\,dy \\ &\qquad \le \sum_{j=i+3}^{\infty} \left(\int_{x_0+2^j}^{x_0+2^{j+1}} \|K(x-y) - K(x_0-y)\|_{\ell^2}^2\,dy\right)^{1/2} \\ &\qquad\qquad \times \left(\int_{x_0+2^j}^{x_0+2^{j+1}} w^{-t}\right)^{1/t}\left(\int_{x_0+2^j}^{x_0+2^{j+1}} f^s w^s\right)^{1/s} \\ &\qquad \le C\|fw\|_\infty \sum_{j>i+3} 2^{i/2}2^{-j}2^{j/s}\left(\int_{x_0}^{x_0+2^{j+1}} w^{-t}\right)^{1/t}. \end{aligned}$$

Therefore as t was chosen in such a way that $w^{-t} \in A_1^-$ and $1/s+1/t = 1/2$, we have

$$\begin{aligned} \|w\chi_{(x_0-h,x_0)}\|_\infty B_2 &\le C\|fw\|_\infty \sum_{j>i+3} 2^{i/2}2^{-j}2^{j/s}2^{j/t} \\ &\le C\|fw\|_\infty \sum_{j>i+3} 2^{i/2}2^{-j/2} \le C\|fw\|_\infty. \ \blacksquare \end{aligned}$$

In order to check that the hypotheses of the extrapolation theorem are satisfied, we just look at the following inequalities:

$$\frac{1}{h}\int_x^{x+h}\left(Sf(y)-\frac{1}{h}\int_{x+h}^{x+2h}Sf\right)^+dy \le \frac{1}{h}\int_x^{x+h}|Sf(y)-S(f\chi_{(x+8h,\infty)})(x)|\,dy$$
$$+\frac{1}{h}\int_{x+h}^{x+2h}|Sf(y)-S(f\chi_{(x+8h,\infty)})(x))|\,dy$$
$$\le C\frac{1}{h}\int_x^{x+2h}|Sf(y)-S(f\chi_{(x+8h,\infty)})(x))|\,dy.$$

Therefore S maps $L^p(w)$ into itself if $w\in A_p^+$, provided $p>1$.

For the case $p=1$ we obtain weak type. The proof will use the following lemma:

LEMMA 2.5. *Let a be a function supported on $I=(x^*,x^*+h)$ such that $\int_I a(y)\,dy=0$. For any $w\in A_1^+$ there exists C depending only on w so that*

$$\int_{y<x^*-2h} Sa(y)w(y)\,dy \le C\int_I |a(y)|w(y)\,dy.$$

Proof. It is enough to prove the following

CLAIM. *Let a be a function supported on $I=(x^*,x^*+2^i)$ such that $\int_I a(y)\,dy=0$. For any $w\in A_1^+$ there exists C depending only on w so that*

$$\int_{y<x^*-2^i} Sa(y)w(y)\,dy \le C\int_I |a(y)|w(y)\,dy.$$

Observe that if the claim is true, given h we choose i such that $2^{i-1}\le h<2^i$; then $\int_{y<x^*-2^i} a(y)\,dy=0$ and $\{y<x^*-2h\}\subset\{y<x^*-2^i\}$.

Now we shall prove the claim. For $k=0,1,2\ldots$ let $x_k=x^*-2^{k+i}$. Then

$$\int_{y<x^*-2^i} Sa(y)w(y)\,dy=\sum_{k=1}^{\infty}\int_{x_k}^{x_{k-1}} Sa(y)w(y)\,dy.$$

Now if $x\in I_k=[x_k,x_{k-1}]$ and $j>k+i$ then

$$A_ja(x)=\frac{1}{2^j}\int_x^{x+2^j}a(y)\,dy=\frac{1}{2^j}\int_{x^*}^{x+2^j}a(y)\,dy=\frac{1}{2^j}\int_I a(y)\,dy=0,$$

because $x+2^j\ge x^*-2^{k+i}+2^{k+i+1}\ge x^*+2^i$. But if $j<k+i$ then $x+2^j\le x^*-2^{k+i-1}+2^{k+i-1}<x^*$ and again $A_ja(x)=0$. If $j=k+i$ and $x_k+2^i<x$ then $x+2^j\ge x^*-2^{k+i}+2^i+2^{k+i}=x^*+2^i$, and $A_j(x)=0$. In other words, on each I_k, Sf is zero except on the subinterval (x_k,x_k+2^i),

and on that interval it is less than or equal to $(C/2^{k+i})\int|a(y)|\,dy$. Now

$$\int_{y<x^*-2^i} Sa(y)w(y)\,dy \le C\sum_{k=1}^{\infty}\frac{1}{2^{k+i}}\int_{x_k}^{x_k+2^i} w(y)\int_I |a(z)|\,dz\,dy.$$

If, for each k, we use the fact that $w^r\in A_1^+$ for some $r>1$, we may write

$$\begin{aligned}\int_{x_k}^{x_k+2^i} w(y)\int_I|a(z)|\,dz\,dy &\le \Big(\int_{x_k}^{x_k+2^i} w(y)^r\,dy\Big)^{1/r}2^{i/r'}\int_I|a(z)|\,dz\\ &\le \Big(\int_{x_k}^{x^*} w(y)^r\,dy\Big)^{1/r}2^{i/r'}\int|a(z)|\,dz\\ &\le C2^{i/r'}2^{(k+i)/r}\int_I|a(z)|w(z)\,dz,\end{aligned}$$

where the constant C depends only on w. If we sum over k we have

$$\begin{aligned}\int_{y<x^*-2^i} Sa(y)w(y)\,dy &\le \sum_{k=1}^{\infty} C\,\frac{1}{2^{k+i}}\,2^{(k+i)/r}2^{i/r'}\int_I|a(y)|w(y)\,dy\\ &\le C\int_I|a(y)|w(y)\,dy. \ \blacksquare\end{aligned}$$

Theorem 2.6. *Let $w\in A_1^+$. Then there exists C, depending only on w, so that for any $\lambda>0$,*

$$\int_{\{x:Sf(x)>\lambda\}} w(x)\,dx \le \frac{C}{\lambda}\int|f(x)|w(x)\,dx.$$

Proof. Let $O_\lambda=\{x: M^+f(x)>\lambda\}$. It is well known [8] that if I_i are the connected components of O_λ, then $\lambda=(1/|I_i|)\int_{I_i}f=f_{I_i}$. We decompose f as

$$f=f\chi_{\mathbb{R}\setminus O_\lambda}+\sum f_{I_i}\chi_{I_i}+\sum(f-f_{I_i})\chi_{I_i}.$$

As usual $f\chi_{\mathbb{R}\setminus O_\lambda}+\sum f_{I_i}\chi_{I_i}$ will be denoted by g, and $\sum(f-f_{I_i})\chi_{I_i}=\sum b_i$ by b. Observe that each b_i has support on I_i and average zero. Now,

$$\begin{aligned}\int_{\mathbb{R}}|g(y)|w(y)\,dy &\le \int_{\mathbb{R}\setminus O_\lambda}|f(y)|w(y)\,dy+\sum w(I_i)f_{I_i}\\ &= \int_{\mathbb{R}\setminus O_\lambda}|f(y)|w(y)\,dy+\lambda\sum w(I_i)=\int_{\mathbb{R}\setminus O_\lambda}|f(y)|w(y)\,dy+\lambda w(O_\lambda)\\ &\le C\int|f|w,\end{aligned}$$

because the operator M^+f is of weak type $(1,1)$ with respect to w.

For each interval $I = (b,c)$, denote by $2I$ the interval $(b-2c, c)$. We also denote by $\widetilde{O}_\lambda$ the union of all the intervals $2I_i$, I_i being the connected components of O_λ. The one-sided doubling property of the weight (see the comments at the beginning of Section 2) gives

$$w(\widetilde{O}_\lambda) = w\Big(\bigcup_i 2I_i\Big) \le Cw(O_\lambda) \le \frac{C}{\lambda}\int |f(y)|w(y)\,dy.$$

Observe that

$$w\{x : Sf(x) > \lambda\} \le w\{x : Sg(x) > \lambda/2\} + w(\widetilde{O}_\lambda) + w\{x \notin \widetilde{O}_\lambda : Sb(x) > \lambda/2\}.$$

The second term is already known to be bounded by $(C/\lambda)\int |f(x)|w(x)\,dx$. Since S is a bounded operator in $L^2(w)$, and condition A_1^+ implies condition A_p^+ for any $p > 1$, we have

$$\begin{aligned} w\{x : Sg(x) > \lambda/2\} &\le \frac{C}{\lambda^2}\int (Sg(y))^2 w(y)\,dy \le \frac{C}{\lambda^2}\int |g(y)|^2 w(y)\,dy \\ &\le \frac{C}{\lambda}\int |g(y)|w(y)\,dy \le \frac{C}{\lambda}\int |f(y)|w(y)\,dy. \end{aligned}$$

In the last two inequalities we have used $|g| \le \lambda$ and $\int |g|w \le C\int |f|w$. Finally for the third term by using the preceding lemma and the one-sided nature of the operator S, we have

$$\begin{aligned} w\{x \notin \widetilde{O}_\lambda : Sb(x) > \lambda/2\} &\le \frac{C}{\lambda}\int_{\mathbb{R}\setminus\widetilde{O}_\lambda} Sb(x)w(x)\,dx \\ &\le \frac{C}{\lambda}\sum_j \int_{\mathbb{R}\setminus 2I_i} Sb_i(x)w(x)\,dx \\ &\le \sum_i \int_{I_i} |b_i(x)|w(x)\,dx. \end{aligned}$$

But since the I_i's are disjoint and $b(x) = b_i(x)$ on each I_i, the last term is bounded by

$$\int |b(x)|w(x)\,dx = \int |f(x) - g(x)|w(x)\,dx \le C\int |f(x)|w(x)\,dx. \ \blacksquare$$

Now we shall prove the converses of the last theorems.

THEOREM 2.7. *Assume that for some $p \ge 1$ there exists a constant C_p so that*

$$\int_{\{x:Sf(x)>\lambda\}} w \le \frac{C_p}{\lambda^p}\int |f|^p w.$$

Then w satisfies condition A_p^+, and the constant in the condition depends only on the constant C_p.

Proof. We shall prove that the weight w satisfies condition $\widetilde{A}_p^+$ (see Remark 2.3). Let $p > 1$. Let i be an integer such that $2^i < h < 2^{i+1}$. Let $f = w^{1-p'}\chi_{(b,b+h)}$. If $x \in (b-2h, b-h)$ then $A_i f(x) = 0$ and $A_{i+3}f(x) = (1/2^{i+3})\int_b^{b+h} w^{1-p'}$. If we choose $\lambda = (1/2^{i+3})\int_b^{b+h} w^{1-p'}$, we have

$$\begin{aligned}\lambda = A_{i+3}f(x) &\le |A_{i+3}f(x) - A_{i+2}f(x)| + |A_{i+2}f(x) - A_{i+1}f(x)| \\ &\quad + |A_{i+1}f(x) - A_i f(x)| \\ &\le CSf(x).\end{aligned}$$

This means that $(b-2h, b-h) \subset \{x : Sf(x) > \lambda\}$, and then

$$\int_{b-2h}^{b-h} w \le Ch^p\Big(\int_b^{b+h} w^{1-p'}\Big)^{1-p},$$

which is $\widetilde{A}_p^+$. We have thus proved that for $p > 1$ the operator S is bounded on $L^p(w)$ if and only if w satisfies A_p^+. If $p = 1$ and b is a Lebesgue point for w, we consider the interval $(b, b+2^i)$ and $f = \chi_{(b,b+h)}$, where $h < 2^i$. It is clear that if $x \in (b-2^{i+1}, b-2^i)$ then $A_i f(x) = 0$, while $A_{i+2}f(x) = (1/2^{i+2})h$. It follows that $Sf(x) > Ch/2^i$, and so $(b-2^{i+1}, b-2^i) \subset \{x : Sf(x) > h/2^i\}$. Therefore

$$w(b-2^{i+1}, b-2^i) \le C\,\frac{2^i}{h}\int_b^{b+h} w.$$

It follows that

$$\frac{1}{2^i}\,w(b-2^{i+1}, b-2^i) \le Cw(b).$$

If now $I = (a,b)$ is any interval of length 2^j we define $x_0 = a$ and for $k \ge 1$, $x_k = (x_{k-1}+b)/2$. We may then write

$$\int_I w = \sum \int_{x_k}^{x_{k+1}} w \le Cw(b)\sum(x_{k+1}-x_k) = Cw(b)(b-a),$$

which is A_1^+. ∎

REMARK 2.8. It is easy to see that the same methods prove that the operator S_1 defined in the introduction maps $L^p(w)$ into itself if and only if the weight w satisfies condition A_p^+, but restricted to intervals of length less than one.

3. The action of the one-sided maximal operator on BMO functions. It is easy to prove that for certain functions f in BMO the maximal operator M^+f is infinite at every point. Take for example $f(x) = \log^+ x$. In fact results similar to Propositions 1.8 and 1.9 can be proved in this case; we leave the details to the reader.

On the other hand it is extremely easy to prove that if a function f is in BMO and $M^+f(x) < \infty$ for a.e. x then $M^+f \in \mathrm{BMO}$. This fact is parallel to the corresponding result for the Hardy–Littlewood maximal operator in [2], but it does not follow from the fact that $M^+f(x) \le Mf(x)$, because $g \in \mathrm{BMO}$ and $0 \le f \le g$ do not imply $f \in \mathrm{BMO}$. We need the following lemma, whose detailed and easy proof is left to the reader.

LEMMA 3.9. *Let $I = (x_0, x_0 + h)$, $k > h$. Then*

$$\int_{x_0+k}^{x_0+k+h} |f(x) - f_I|\,dx \le Ck\|f\|_{\mathrm{BMO}}.$$

THEOREM 3.10. *If $f \in \mathrm{BMO}$ then either $M^+f(x) = \infty$ for a.e. x or $M^+f(x) < \infty$ for a.e. x. In the second case $M^+f \in \mathrm{BMO}$ and $\|M^+f\|_{\mathrm{BMO}} \le C\|f\|_{\mathrm{BMO}}$.*

Proof. Fix x_0 and $h > 0$. Let $I = (x_0, x_0 + h)$. We decompose f as $f = f_1 + f_2$, where $f_1(x) = (f(x) - f_I)\chi_{(x_0,x_0+2h)}(x)$ and $f_2(x) = f(x) - f_1(x)$. Since $M^+f(x)$ and $M^+f_2(x_0)$ are finite, we may write

$$\begin{aligned}\frac{1}{h}\int_{x_0}^{x_0+h} &|M^+f(x) - M^+f_2(x_0)|\,dx\\ &\le \frac{1}{h}\int_I M^+f_1(x)\,dx + \frac{1}{h}\int_I \sup_{k>0}\left|\frac{1}{k}\int_x^{x+k} f_2(y)\,dy - \frac{1}{k}\int_{x_0}^{x_0+k} f_2(y)\,dy\right|dx\\ &= B_1 + B_2.\end{aligned}$$

If we use Hölder's inequality, the fact that the operator M^+ is bounded in L^p for any $p > 1$, and the John–Nirenberg theorem, we get $B_1 \le C\|f\|_{\mathrm{BMO}}$.

Now we shall analyze B_2. Due to the one-sided nature of the operator M^+, we can substitute f_2 by $g_2(x) = (f(x) - f_I)\chi_{(x_0+2h,\infty)}(x)$. Now for each $k > 0$ it is clear that

$$\frac{1}{k}\int_x^{x+k} g_2(y)\,dy - \int_{x_0}^{x_0+k} g_2(y)\,dy$$

is 0 unless $k > h$ and in this case

$$\frac{1}{k}\left|\int_x^{x+k} g_2 - \int_{x_0}^{x_0+k} g_2\right| \le \frac{1}{k}\int_{x_0+k}^{x_0+h+k} |g_2(y)|\,dy.$$

But since $g_2(y) = (f(y) - f_I)\chi_{(x_0+2h,\infty)}(y)$, the last lemma tells us that

$$B_2 \le C\|f\|_{\mathrm{BMO}}.$$

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