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On locally convex extension of H^∞ in the unit ball and continuity of the Bergman projection

by

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Abstract. We define locally convex spaces LW and HW consisting of measurable and holomorphic functions in the unit ball, respectively, with the topology given by a family of weighted-sup seminorms. We prove that the Bergman projection is a continuous map from LW onto HW. These are the smallest spaces having this property. We investigate the topological and algebraic properties of HW.

1. Introduction. The Bergman projection in the unit disc does not map the space of bounded measurable functions L^∞ into the space of bounded holomorphic functions H^∞. It can be shown that the image of L^∞ is the Bloch space. In [13] locally convex spaces L_V^∞, H_V^∞, h_V^∞ were defined consisting of measurable, holomorphic, and harmonic functions in the unit disc, respectively, with the topology given by the seminorms

$$\|f\|_v = \sup_{z\in\mathbb{D}} |f(z)|v(|z|), \tag{1}$$

where v: $[0,1) \to \mathbb{R}$ is a continuous function and $v(r)|\log(1-r)|^k$ is bounded for each $k \in \mathbb{N}$. From now on we call seminorms of the form (1) *weighted-sup seminorms.* It was shown that the Bergman projection B is a continuous operator from the space L_V^∞ of measurable functions onto the space H_V^∞ of holomorphic functions. Moreover it was proved that harmonic conjugation is a continuous operator from h_V^∞ onto h_V^∞, and the Szegő projection is continuous from h_V^∞ onto H_V^∞. The spaces L_V^∞, H_V^∞, h_V^∞ are the smallest extensions of L^∞, h^∞, and H^∞ having these properties.

This paper is devoted to the case of the unit ball. We concentrate on the Bergman projection and extend the results of [13] concerning locally convex space structure and algebraic properties of the space of holomorphic functions with the same weight family. We believe that investigating the properties of extensions of the algebra of bounded holomorphic functions may also throw some light on H^∞.

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In the unit ball $\mathbb{B}$ of $\mathbb{C}^d$ $(d > 1)$ the behaviour of the Bergman projection is, to some extent, similar to the one-dimensional case. It does not map the space of bounded measurable functions into the space of bounded analytic functions. Therefore it is natural to expect that the same extension can also be constructed. In [6] a precise description of the behaviour of the operator B on L^p spaces was given. Among other results, it was shown that the integral of the Bergman kernel behaves asymptotically like $|\log(1 - |z|)|$. The description of the behaviour of the Bergman kernel in the unit ball is crucial to the determination of a weight family defining seminorms. It is worth mentioning that the asymptotic behaviour of the Bergman projection was also described in the case of general strongly pseudoconvex domains in $\mathbb{C}^d$. This makes us believe that one may construct a similar extension and prove its minimality in a more general setting. To show that the extension is minimal we use some modification of the Bell operator [2], [3]. Our approach differs from that in [13], where integral formulas were used.

In Section 3 we study the topological properties of the space HW. Using the results of [5] and [4] we show that HW is the inductive limit of a compact inductive system of Banach spaces. We also establish the dual space. From the inductive description of HW and Grothendieck's factorization theorem we obtain at once interesting information about the behaviour of the Bergman projection.

There is a growing interest in topological algebras as natural generalizations of Banach algebras (see e.g. [10]). It is easy to see that HW is a topological algebra. We show that the only continuous characters on HW are evaluations at points of $\mathbb{B}$. Their kernels are the only closed maximal ideals in HW. An easy argument shows that there must also exist in HW dense ideals of infinite codimension. We give examples of such ideals. It is a basic fact in Banach algebra theory that the set of invertible elements is open. We show that the constant function 1 is not an interior point of the set of invertible elements in HW. It is also of interest to know whether maximal closed ideals in HW are finitely generated. Using ideas of [1] we solve the Gleason problem for the algebra HW.

Let Ω be a bounded domain in $\mathbb{C}^d$. We denote by $H(\Omega)$ or briefly H the set of holomorphic functions in Ω. The space of all holomorphic functions which are square integrable with respect to a volume measure V, normalized so that $V(\mathbb{B}) = 1$, is called the *Bergman space* and denoted by $H^2(\Omega)$ or H^2 if no confusion can occur. It is a closed subspace of $L^2(\Omega)$. Thus there exists a projection B: $L^2 \to H^2$ called the *Bergman projection* (for details see [8], [12]). For the unit ball $\mathbb{B}$ in $\mathbb{C}^d$ this operator can be written explicitly:

$$Bf(z) = \int_{\mathbb{B}} \frac{f(\zeta)}{(1 - \langle z, \zeta \rangle)^{d+1}} \, dV(\zeta).$$

The function

$$K(z,\zeta)=\frac{1}{(1-\langle z,\zeta\rangle)^{d+1}}$$

is called the *Bergman kernel*.

Let $r(z)=1-|z|^2$ be the defining function for the unit ball. As stated before, the following fact [6] is crucial.

PROPOSITION 1. *If* $0<\alpha<1$, *then*

$$\sup_{z\in\mathbb{B}} r^{\alpha}(z)\int_{\mathbb{B}}|K(z,\zeta)|r^{-\alpha}(\zeta)\,dV(\zeta)=\frac{n!\Gamma(1-\alpha)\Gamma(\alpha)}{|\Gamma((n+1)/2)|^2},$$

$$\sup_{z\in\mathbb{B}}\frac{1}{1+|\log r(z)|}\int_{\mathbb{B}}|K(z,\zeta)|\,dV(\zeta)=\frac{n!}{|\Gamma((n+1)/2)|^2}.$$

For the reader's convenience we sketch the proofs of those theorems from [13] that we use.

2. Definitions and minimality properties. We define the weight family and the corresponding spaces.

DEFINITION 1. Let W be the set of all continuous functions $v\colon[0,1)\to\mathbb{R}_+$ such that

$$\sup_{r\in[0,1)} v(r)|\log(1-r)|^n<\infty \quad \text{for each } n\in\mathbb{N}. \tag{2}$$

DEFINITION 2. Set

$$\begin{aligned}LW&=LW(\mathbb{B})\\&=\{f\colon\mathbb{B}\to\mathbb{C}: f \text{ measurable}, \|f\|_v=\operatorname*{ess\,sup}_{z\in\mathbb{B}}|f(z)|v(|z|)<\infty\}.\end{aligned}$$

We denote by $HW(\mathbb{B})$ or briefly HW the set of holomorphic functions belonging to LW. The HW space for functions defined on the unit disc $\mathbb{D}$ in the complex plane will be denoted by $HW(\mathbb{D})$.

Our notation differs from that in [13] but it is justified by [5]. We intend to show that the Bergman projection is a continuous operator from LW onto HW. The fact that each function in W can be majorized by a more regular function still belonging to the weight family is important in the proof of the continuity. We formulate it in the following lemma.

LEMMA 2. *Let* $v\in W$ *and* $\alpha>0$.

(i) *The function* $w(r)=\sup_{s\ge r}v(s)$ *belongs to* W.

(ii) *If* ϕ *is a biholomorphic mapping of the unit ball*, *then the function* $w(r)=\sup_{|z|=r}v\circ|\phi(z)|$ *belongs to* W.

(iii) *Define*

$$v_\delta(r)=\begin{cases} v(r) & \text{if } r\le 1-\delta,\\ v(1-\delta)\delta^{-\alpha}(1-r)^\alpha & \text{if } r>1-\delta.\end{cases}$$

Then the function $w(r)=\max\{\sup_{0<\delta<1/2}v_\delta(r),v(r)\}$ *belongs to* W.

(iv) *The function* v^α *for* $\alpha>0$ *belongs to* W.

Proof. We omit the easy proofs of (i), (iv) and sketch the proofs of (ii) and (iii). All of them are similar to those in [13].

(ii) Let $v\in W$ and suppose that the function $w(r)=\sup_{|z|=r}v\circ|\phi(z)|$ does not belong to W. Then there exists $n\in\mathbb{N}$ and a sequence of points r_j in the unit ball with $|r_j|\to 1$ such that

$$v(|r_j|)|\log(1-|\phi(r_j)|^2)|^n\to\infty$$

because $\log(1-x)\sim\log(1-x^2)$. Making use of the well-known description of automorphisms of the unit ball (see [12]) it is easy to calculate that

$$1-|\phi(r_j)|^2=\frac{(1-|a|^2)(1-|r_j|^2)}{|1-\langle r_j,a\rangle|^2}$$

for some $a\in\mathbb{B}$. From obvious estimates it follows that

$$v(|r_j|)|\log(1-|r_j|)|^k\to\infty$$

for some $0\le k\le n$. This is impossible because $v\in W$.

(iii) Suppose that $w\notin W$. Thus there exists $n\in\mathbb{N}$, a sequence r_j converging to 1 and a sequence δ_j satisfying $0<\delta_j<1/2$ such that

$$|\log(1-r_j)|^n v_{\delta_j}(r_j)\to\infty$$

as $j\to\infty$. Observe that we can assume that $r_j>1-\delta_j$ since otherwise v would not belong to W. From the monotonicity of the function $|\log r|^n r^\alpha$ it follows that

$$|\log(1-r_j)|^n v(1-\delta_j){\delta_j}^{-\alpha}(1-r_j)^\alpha\le|\log\delta_j|^n v(1-\delta_j).$$

Thus if $w\notin W$, then $v\notin W$. ∎

For $f\in LW$ we can write $|gh(z)|v(|z|)=|g(z)|v^{1/2}(|z|)|h(z)|v^{1/2}(|z|)$. From Lemma 2(iv) it follows that LW and HW are topological algebras under pointwise multiplication. Observe also that if $f\in HW$ then $f\circ\phi\in HW$ for any holomorphic automorphism ϕ of the unit ball. This is a consequence of Lemma 2(ii).

THEOREM 3. *The Bergman projection is a continuous operator* $B\colon LW\to HW$.

Proof. Let $v\in W$. We may assume that v is a decreasing function. Let $\varrho=|z|$ and let w be the function defined for $v^{1/2}$ as in Lemma 2(iii) with

$\alpha = 1/2$. From Proposition 1 it follows that

$$\begin{aligned}
|Bf(z)| &\le \int_{\mathbb{B}} |f(\zeta)K(z,\zeta)|\, dV(\zeta) \\
&\le \int_{\varrho\mathbb{B}} \frac{\|f\|_w}{w(|\zeta|)}\, |K(z,\zeta)|\, dV(\zeta) + \int_{\mathbb{B}\setminus\varrho\mathbb{B}} \frac{\|f\|_w}{w(|\zeta|)}\, |K(z,\zeta)|\, dV(\zeta) \\
&\le \frac{\|f\|_w}{v^{1/2}(\varrho)} \int_{\varrho\mathbb{B}} |K(z,\zeta)|\, dV(\zeta) \\
&\quad + \frac{(1-\varrho)^{1/2}}{v^{1/2}(\varrho)} \int_{\mathbb{B}\setminus\varrho\mathbb{B}} \frac{\|f\|_w}{(1-|\zeta|)^{1/2}}\, |K(z,\zeta)|\, dV(\zeta) \\
&\le C\biggl(\frac{1}{v^{1/2}(\varrho)}\, |\log(1-|z|)| + \frac{(1-\varrho)^{1/2}}{v^{1/2}(\varrho)}\, \frac{1}{(1-|z|)^{1/2}}\biggr)\|f\|_w.
\end{aligned}$$

We have just shown that

$$|B(f)(z)|v(|z|) \le C(v^{1/2}(|z|)|\log(1-|z|)| + v^{1/2}(|z|))\|f\|_w. \ \blacksquare$$

The same arguments, easier than in [13], work in the one-dimensional case as well.

The next lemma is stated in a more general form than is in fact needed. The reason for this is our belief that similar extensions can be constructed not only for the ball but for bounded strongly pseudoconvex domains. We intend to deal with this problem in another paper. Let $\Omega = \{z \in \mathbb{C}^d\colon r(z) < 0\}$ be a bounded domain in $\mathbb{C}^d$. Assume that r is a smooth function defined on some neighbourhood of Ω and satisfies $|\operatorname{grad} r| \ge \delta > 0$ on $\partial\Omega$. Define

$$H^2_{1,r}(\Omega) = \{h \in H^2(\Omega)\colon |\partial_i hr| \in L^2,\ i = 1,\ldots,d\}.$$

(The symbol ∂_i denotes the partial derivative with respect to the ith coordinate.) Set $U_\delta = \{z \in \mathbb{C}^d : |\operatorname{grad} r| > \delta/2\}$. Thus we have $\partial\Omega \subset U_\delta$. Choose an open covering $\{U_i\}_{i=1}^d$ of U_δ such that $|\partial_i r| > \delta(2d)^{-1/2}$ on U_i. Let $\{\phi_i\}_{i=1}^d$ be a smooth partition of unity subordinate to $\{U_i\}$. The following lemma is a modification of a construction given by Bell (see [2], [3]). Its proof is also modified. We do not assume pseudoconvexity of the domain Ω. By $C_{0,r}(\Omega)$ we denote the space

$$\{h \in C(\Omega) : hr \text{ vanishes at infinity}\}.$$

LEMMA 4. *Define the operator Φ: $H^2_{1,r}(\Omega) \cap C_{0,r}(\Omega) \to L^2(\Omega)$ by the formula*

$$\Phi h = h - \sum_{i=1}^d \partial_i\biggl(\frac{\phi_i}{\partial_i r}\, hr\biggr).$$

Then $B\Phi$ is equal to the identity operator on $H^2_{1,r}(\Omega) \cap C_{0,r}(\Omega)$.

Proof. To prove the lemma it is enough to show that for each i,

$$\partial_i\left(\frac{\phi_i}{\partial_i r}\,hr\right) \perp H^2.$$

Since the polynomials are dense in H^2, it suffices to check the orthogonality only on polynomials. Take a complex polynomial p. Observe that there exists $\varrho < 0$ such that if we set $K_\varrho = \{z \in \Omega\colon r(z) < \varrho\}$ then

$$\int\limits_{\Omega\setminus K_\varrho} \left|p\,\overline{\partial_i\left(\frac{\phi_i}{\partial_i r}\,hr\right)}\right| dV < \varepsilon, \qquad \left|p\,\frac{\phi_i}{\partial_i r}\,hr\right| < \varepsilon \quad \text{in } \Omega\setminus K_\varrho.$$

Thus we can estimate

$$\begin{aligned}\left|\left\langle p, \partial_i\left(\frac{\phi_i}{\partial_i r} hr\right)\right\rangle\right| &= \left|\left\{\int\limits_{K_\varrho} + \int\limits_{\Omega\setminus K_\varrho}\right\} p\,\overline{\partial_i\left(\frac{\phi_i}{\partial_i r}\,hr\right)}\,dV\right| \\ &\le C\left|\int\limits_{\partial K_\varrho\cap U_i} p\,\overline{\frac{\phi_i}{\partial_i r}}\,hr\,dz_1\wedge\ldots\wedge dz_n\wedge d\overline{z}_1\wedge\ldots\wedge\widehat{d\overline{z}_i}\wedge\ldots\wedge d\overline{z}_n\right| + \varepsilon.\end{aligned}$$

For simplicity we now assume that $i = 1$. Using the standard "partition of unity" argument we can assume that $|\partial r/\partial x_1| > c$ on $\partial K_\varrho \cap U_1$. This estimate does not depend on ϱ. Denote by $\mathcal{U}_1$ the projection of $\partial K_\varrho \cap U_1$ onto the last $2d-1$ real variables. From the implicit function theorem it follows that for some $a, b, C \in \mathbb{C}$,

$$\begin{aligned}&\left|\int\limits_{\partial K_\varrho\cap U_1} p\,\overline{\frac{\phi_1}{\partial_1 r}\,hr}\,dz_1\wedge\ldots\wedge dz_n\wedge\widehat{d\overline{z}_1}\wedge d\overline{z}_2\wedge\ldots\wedge d\overline{z}_n\right| \\ &\quad = \left|\int\limits_{\mathcal{U}_1} p\,\overline{\frac{\phi_1}{\partial_1 r}\,hr}\left(a + b\,\frac{\partial r}{\partial y_1}\left(\frac{\partial r}{\partial x_1}\right)^{-1}\right) dy_1\wedge dx_2\wedge\ldots\wedge dy_n\right| \le C\varepsilon. \ ■\end{aligned}$$

THEOREM 5. *Let $L^\infty(\mathbb{B}) \subset E \subset L^2(\mathbb{B})$ be a locally convex space with a topology defined by weighted-sup seminorms. Let F be the subspace of E consisting of analytic functions in the unit ball. Assume that the Bergman projection is a continuous map from E into F. Then $LW \subset E$ and $HW \subset F$.*

In other words, LW and HW are the smallest locally convex extensions of L^∞ and H^∞ with the properties described in the theorem.

Proof. We show that if the Bergman projection is a continuous operator on the space of measurable functions with a topology given by some weighted-sup seminorms, then the functions defining these seminorms must belong to the set W.

Observe that in $U_\delta \cap \mathbb{B}$ we have

$$\Phi h = \sum_{i=1}^{d} \biggl(\partial_i \biggl(\frac{\phi_i}{\partial_i r} \biggr) hr + \frac{\phi_i}{\partial_i r} \partial_i (h) r \biggr).$$

Take a function $h_k(z) = (\log(1 - z_1))^k \in H^2_{1,r}$. Then for $z \in U_\delta \cap \mathbb{B}$ sufficiently close to $\partial\mathbb{B}$,

$$|\Phi h_k(z)| \le C|h_k r(z)| + C|h_{k-1}(z)| \le \widetilde{C}|h_{k-1}(z)|$$

because $|h_k r(z)|$ is bounded. Assume that the Bergman projection is a continuous operator from E into F. Then for each $v \in W$ there exist $w \in W$ and $C, \widetilde{C} > 0$ such that

$$\begin{aligned} |h_k(z)|v(|z|) = |B\Phi(h_k)(z)|v(|z|) &\le C \sup_{z\in\mathbb{B}} |\Phi(h_k)(z)|w(|z|) \\ &\le \widetilde{C} \sup_{z\in\mathbb{B}} |h_{k-1}(z)|w(|z|). \end{aligned}$$

Thus for each $v \in W$, if $\sup |\log(1 - |z|)|^{k-1} v(|z|) < \infty$, then we also have $\sup |\log(1 - |z|)|^{k} v(|z|) < \infty$. Obviously taking the largest family of continuous functions satisfying condition (2) one obtains the smallest space. ∎

Remark. It is worth mentioning that taking upper semicontinuous functions satisfying the condition from the definition of the weight family instead of continuous functions does not change the space HW. This will follow from the inductive description of the space HW given in the next section.

It is known [9] that Bloch functions in the unit ball satisfy $|f(z)| \le C|\log(1-|z|)|$. The space of Bloch functions is not an algebra with pointwise multiplication. We can formulate the following

Corollary 6. *The space HW is the smallest algebra with pointwise multiplication with a topology given by weighted-sup seminorms containing the Bloch space.*

3. Locally convex space properties. The space HW is locally convex. One cannot expect that there exists a countable family of seminorms giving the same topology. Thus HW is not a Fréchet space. Nevertheless some interesting facts concerning the topological structure of this space can be proved. It is obvious that the topology in HW is stronger than the topology of uniform covergence on compact sets.

Proposition 7. *The polynomials are dense in HW.*

Proof. The proof is exactly the same as in the one-dimensional case in [13]. Therefore we only sketch it. It is enough to show that if we define $T_\varrho f(z) = f(\varrho z)$ for $\varrho > 1$ then $T_\varrho f \to f$ in HW because a function holomorphic in some open neighbourhood of the closed unit ball can be approximated in $\mathbb{B}$ by its Taylor polynomials. The convergence $T_\varrho f \to f$ in HW follows from the fact that $T_\varrho f$ converges uniformly on compact subsets of $\mathbb{B}$ and that for each $f \in HW$ and $v \in W$ the function $|f(z)|v(|z|)$ vanishes at infinity. ∎

Define continuous functions

$$v_k(r) = \begin{cases} 1 & \text{if } r \leq 1 - e^{-1}, \\ |\log(1-r)|^{-k} & \text{if } r > 1 - e^{-1}, \end{cases}$$

and spaces

$$Hv_k = \{f \in H\colon \sup_{\mathbb{B}} |f(z)| v_k(|z|) < \infty\}.$$

The spaces Hv_k with the natural injections $\iota_k\colon Hv_k \hookrightarrow Hv_{k+1}$ form an injective inductive system of Banach spaces.

THEOREM 8. *The mappings ι_k are compact. The space HW is the inductive limit of the system (Hv_n, ι_n).*

We recall that a locally convex space which is the inductive limit of a countable, compact, injective system of Banach spaces is called a *Silva space* (in the literature, Silva spaces are also called (DFS)-spaces or (LS)-spaces).

Proof. Compactness of the injections ι_n follows from the Montel theorem and the estimate

$$\sup_{\mathbb{B}} |f_n - f_m| v_{k+1} \leq C_\varrho \sup_{\varrho\overline{\mathbb{B}}} |f_n - f_m| + \frac{2}{|\log(1-\varrho)|}$$

for functions f_n, f_m belonging to the unit ball of Hv_k.

Observe that the functions v_n are strictly positive and $v_l/v_k \to 0$ if $l > k$. Therefore according to [5, 1.6],

$$\operatorname{ind}_{\to} Hv_k = \operatorname{ind}_{\to} (Hv_k)_0 = H\overline{W}_0$$

as topological vector spaces. The symbol $(Hv_k)_0$ denotes the space of functions f from Hv_k for which fv_k vanishes at infinity. The meaning of $H\overline{W}_0$ is analogous. The family $\overline{W}$ is the largest family of upper semicontinuous functions which can be pointwise majorized by some function of the form

$$v(r) = \inf\{\alpha_n v_n(r)\colon n \in \mathbb{N}\},$$

where α_n is a sequence of positive numbers. Because v_n is positive and $v_{n+1} \leq v_n$ it follows from [5, 0.2] that $\overline{W}$ is equivalent to

$$\{v \in C[0,1)\colon \sup_{r\in[0,1)} vv_n^{-1}(r) < \infty \text{ for } n \in \mathbb{N}\}.$$

To prove the theorem it remains to notice that for a continuous function v we have $\sup_{[0,1)} v(r)v_k^{-1}(r) < \infty$ if and only if $\sup_{[0,1)} v(r)|\log(1-r)|^k < \infty$. ∎

REMARK. From Grothendieck's factorization theorem it follows that for each $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $B\colon Lv_k \to Hv_n$. In other words if $|f(z)| \le C|\log(1-|z|)|^k$ then there exists $n \in \mathbb{N}$ such that $|Bf(z)| \le C|\log(1-|z|)|^n$.

COROLLARY 9. *HW is a complete reflexive* (LB)*-space. It is also a separable Schwartz space and hence Montel. Moreover HW carries the finest topology which makes all injections ι_k continuous. The dual projective sequence $((Hv_n)_b)'$ is again compact and its limit is a reflexive Fréchet space.*

For the proofs of these facts see [11]. From Corollary 9 it follows for example that such fundamental tools of functional analysis as the open mapping or the closed graph theorems can be used for the space HW.

Let E, F be locally convex spaces. Assume that F is a subspace of the set of all linear functionals on E separating points of E. A set $M \subset E$ is called $\sigma(E,F)$-*bounded* or briefly *weakly bounded* if $\sup_{x\in M}|y(x)| < \infty$ for each $y \in F$. *The strong topology* $b(F,E)$ on F is the topology defined by the seminorms

$$p_M(y) = \sup_{x\in M}|y(x)|.$$

If E is a Banach space then from the Banach–Steinhaus theorem it follows that a set M is $\sigma(E,E')$-bounded if and only if it is bounded in E. Therefore in this case $b(E',E)$ is just the norm topology of E'. From now on by the dual space E' of a locally convex space E we mean the space of all continuous linear functionals on E with the strong topology.

Define

$$H_v^1 = \Big\{ f \in H\colon \int_{\mathbb{B}} |f| v_n^{-1}\, dV < \infty \text{ for } n = 1,2,\ldots \Big\}.$$

This space is the projective limit of the projective system of spaces

$$H_{v_n}^1 = \Big\{ f \in H\colon \int_{\mathbb{B}} |f| v_n^{-1}\, dV < \infty \Big\}.$$

LEMMA 10. *H_v^1 is a Fréchet–Schwartz space.*

Proof. The proof is the same as in the one-dimensional case in [13]. To show that H_v^1 is a Fréchet space it suffices to prove the completeness of this space. This amounts to showing that the topology of H_v^1 is finer than the topology of uniform convergence on compact subsets. To prove the Schwartz property, let $B_n = \{f \in H_v^1 : \int_{\mathbb{B}} |f| v_n^{-1}\, dV \le 1\}$. Fix $r \in (0,1)$ such that $1/|\log(1-r)| < \varepsilon/4$. From the Montel theorem it follows that there exist functions $g_1,\ldots,g_m \in B_{n+1}$ such that for each $f \in B_{n+1}$,

$$\int_{|z|\le r} |f-g_k|\, dV < \frac{\varepsilon}{2|\log(1-r)|^n}$$

for some $1 \le k \le m$. The choice of r implies that

$$\int_{\mathbb{B}} |f - g_k| v_n^{-1}\, dV \le |\log(1-r)|^n \int_{|z|\le r} |f - g_k|\, dV + \frac{2}{|\log(1-r)|} \le \varepsilon.$$

Therefore $B_{n+1} \subset \bigcup_{i=1}^{m} (g_k + \varepsilon B_n)$. ■

Thus there exists a compact inductive system (E_n, j_n) of Banach spaces with $(H_v^1)' = \operatorname{ind}_{n\to} E_n$ (see [11, 25.20]). We show that in fact $HW = \operatorname{ind}_{n\to} Hv_n$ is isomorphic to the dual space of H_v^1. From the reflexivity of HW it will follow at once that H_v^1 is isomorphic to the dual space of HW. Our approach is different from that in [13] because we do not use Köthe spaces.

Proposition 11. *The space HW is isomorphic to the strong dual of H_v^1.*

Proof. Define the map Ψ: $HW \to (H_v^1)'$ by

$$\Psi(g)f = \int_{\mathbb{B}} f\overline{g}\, dV.$$

The map Ψ is injective because H_v^1 contains the polynomials. Take $y \in (H_v^1)'$. From the Hahn–Banach theorem it follows that y can be regarded as a continuous functional on $H_{v_n}^1$ for some $n \in \mathbb{N}$. Thus there exists a bounded measurable function h such that

$$y(f) = \int_{\mathbb{B}} f\bar{h} v_n^{-1}\, dV.$$

From the Fubini theorem it follows that

$$y(f) = \int_{\mathbb{B}} B(f)\bar{h} v_n^{-1}\, dV = \int_{\mathbb{B}} f\, \overline{B(hv_n^{-1})}\, dV.$$

Therefore Ψ is surjective because $B(hv_n^{-1})$ belongs to HW.

It remains to prove that Ψ is a homeomorphism. HW has a web as an inductive limit of Banach spaces. We know from the remarks before the proposition that the dual space of H_v^1 is an ultrabornological space. Therefore from the open mapping theorem (see [11, 24.30]) it follows that to show that Ψ is a homeomorphism it is enough to prove that it is continuous. Because HW is the limit of a compact inductive system of Banach spaces it suffices to prove that if B_n is the unit ball of Hv_n then $\Psi(B_n)$ is bounded (see [11, 25.19]). Let M be a weakly bounded set in H_v^1. Then for each $h \in HW$,

$$\sup_{f\in M} |\Psi(h)f| < \infty.$$

Observe that for each $f \in M$ the functional $h \mapsto \Psi(h)f$ is continuous on

Hv_n. From the Banach–Steinhaus theorem it follows that

$$\sup_{h\in B_n} p_M(\Psi h) = \sup_{h\in B_n}\sup_{f\in M} |\Psi(h)f| < \infty. \ \blacksquare$$

4. Algebraic properties. From Lemma 1(iv) it follows that the space HW is a locally convex algebra. By *a character* on an algebra we mean a nonzero linear multiplicative functional. It is well known that on a Banach algebra each character is automatically continuous, and there is a one-to-one correspondence between the maximal ideals of a commutative unital Banach algebra and the kernels of characters. None of these facts is true in the general case of topological algebras (see [10]). The set of continuous characters on HW, denoted by $\mathfrak{M}$, as a subset of $(HW)'$ can be equipped with the weak-$*$ topology (for details see [10]). In general $\mathfrak{M}$ is not compact.

PROPOSITION 12. *Evaluations at points of $\mathbb{B}$ are the only continuous characters on the algebra HW.*

Proof. If m is a continuous character on HW then it is a character on the ball algebra (of all functions holomorphic in the unit ball and continuous on its closure). Thus there exists a point $z \in \overline{\mathbb{B}}$ such that on the ball algebra, m is the evaluation m_z at the point z. From the density of the polynomials it follows that the functional m has a unique extension from the ball algebra to HW. Observe that the Gelfand transform is the identity on $\mathbb{B}$. Assume that $z \in \partial\mathbb{B}$. Since the composition of a function belonging to HW with a holomorphic automorphism of the unit ball also belongs to HW, we may assume that $z = (1, 0, \ldots, 0)$. Define $f(\zeta) = \log(1-\zeta_1)$. From the continuity of the functional m (see Proposition 7) it follows that

$$m(f) = \lim_{r\to 1-} f(rz) = -\infty.$$

Thus z must belong to $\mathbb{B}$. $\blacksquare$

Take $\lambda \in \mathbb{C}$ such that $\lambda - f$ is not invertible in HW. Let $I = (\lambda - f)$ be the principal ideal generated by $\lambda - f$. There exists a maximal ideal $\mathfrak{M}$ containing I. If the only maximal ideals in HW were the kernels of evaluations there would exist a point $z \in \mathbb{B}$ such that $f(z) = \lambda$. Therefore we would have

$$\sigma(f) = \{\lambda \in \mathbb{C}\colon \lambda - f \text{ not invertible in } HV\} = f(\mathbb{B}).$$

This is not true because it is easy to see that the function $1 - z_1$ is not invertible in HW. Thus there must exist other maximal ideals in HW. These ideals are certainly dense and of infinite codimension. To describe some examples of dense ideals we need the following lemma.

LEMMA 13. *Let p be a complex polynomial having no zeros in $\mathbb{B}$. Then for every $1/2 < r < 1$ and $z \in \mathbb{B}$,*

$$\left|\frac{p(z_1,\dots,z_d)}{p(rz_1,\dots,rz_d)}\right| \le 2^{\deg p}.$$

For a simple proof see [7].

PROPOSITION 14. *Let $p_1,\dots,p_m$ be complex polynomials. The ideal $I = (p_1,\dots,p_m)$ is proper if and only if the intersection of the vanishing set $\mathcal{V}(p_1,\dots,p_m) = \{z \in \mathbb{C}^d\colon p_1(z) = \dots = p_m(z) = 0\}$ with the closed unit ball in $\mathbb{C}^d$ is nonempty.*

The ideal of $HW(\mathbb{D})$ generated by polynomials $p_1,\dots,p_m \in \mathbb{C}[Z]$ is dense if and only if $\mathcal{V}(p_1,\dots,p_m) \cap \overline{\mathbb{D}} \subset \partial\mathbb{D}$.

Proof. If $\mathcal{V}(p_1,\dots,p_m)\cap\overline{\mathbb{B}}$ is empty then there exist functions $q_1,\dots,q_m$ in the ball algebra such that

$$\sum_{i=1}^{m} q_i p_i = 1. \tag{3}$$

Suppose now that $\mathcal{V}(p_1,\dots,p_m)\cap\overline{\mathbb{B}}$ is not empty and that the ideal I is not proper. We may assume that $\zeta \in \mathcal{V}(p_1,\dots,p_m)\cap\partial\mathbb{B}$. Observe that if $f \in HW(\mathbb{B})$ then the function $T_\zeta f(z) = f(\zeta z)$ belongs to $HW(\mathbb{D})$. This follows easily from the decription of HW as an inductive limit of normed spaces. Because I is not proper there exist $q_1,\dots,q_n \in HW(\mathbb{B})$ such that (3) holds. Therefore for some polynomials $\widetilde{p}_1,\dots,\widetilde{p}_m$ of one variable,

$$1 = \sum_{i=1}^{m} T_\zeta(q_i)T_\zeta(p_i) = (z-1)\sum_{i=1}^{m} T_\zeta(q_i)\widetilde{p}_i.$$

Thus the function $z-1$ is invertible in $HW(\mathbb{D})$, which is a contradiction.

If $\mathcal{V}(p_1,\dots,p_m)\cap\overline{\mathbb{D}}$ is not contained in $\partial\,\mathbb{D}$ then there exists a point $z \in \mathbb{D}\cap\mathcal{V}(p_1,\dots,p_m)$. In other words the ideal I is contained in the kernel of a continuous character. Hence I is not dense in HW.

Assume that $\mathcal{V}(p_1,\dots,p_m)\cap\overline{\mathbb{D}} \subset \partial\,\mathbb{D}$. Since $\mathbb{C}[Z]$ is a principal ideal domain there exists a polynomial $q \in \mathbb{C}[Z]$ such that the ideals $(p_1,\dots,p_m)$ and (q) are equal as ideals of the algebra $\mathbb{C}[Z]$. Consequently, $(q) \subset (p_1,\dots,p_m)$ in HW and $\mathcal{V}(q) = \mathcal{V}(p_1,\dots,p_m)$. Define $p^{(r)}(z) = p(rz)$. Observe that $q/q^{(r)}$ converges to 1 on compact subsets of $\mathbb{D}$. Take a decreasing function $v \in W$. From the previous lemma it follows that

$$\sup_{z\in\mathbb{B}}\left|\frac{q}{q^{(r)}}(z) - 1\right| v(|z|) \le \sup_{z\in\varrho\mathbb{B}}\left|\frac{q}{q^{(r)}}(z) - 1\right| v(|z|) + (2^{\deg q}+1)v(\varrho). \ \blacksquare$$

It is a fundamental property of unital Banach algebras that the set of invertible elements is open. One can prove it using the fact that the series

$\sum_{i=0}^{\infty} a^i$ with $\|a\| < 1$ converges to $(1-a)^{-1}$. The series $\sum_{i=0}^{\infty} f^i$ converges in HW if and only if $\sup|f| < 1$ because evaluations are continuous functionals on HW. Thus the following proposition is no surprise.

PROPOSITION 15. *The constant function* 1 *is not an interior point of the set of invertible elements of the algebra HW. Thus the set of invertible elements of this algebra is not open.*

Proof. Assume that the function $f(z) \equiv 1$ is an interior point of the set $\mathrm{Inv}(HW)$ of invertible elements of HW. Then there exists a function $v \in W$ and a positive number ε such that for $\varepsilon' \le \varepsilon$,

$$\{1 - f : \|f\|_v < \varepsilon'\} \subset \mathrm{Inv}(HW).$$

We know that $|\log(1 - z_1)|v(|z|) \le C_v$. Thus the function

$$f(z) = \frac{\varepsilon'}{2C_v} \log(1 - z_1)$$

satisfies $\|f\|_v < \varepsilon$. This leads to a contradiction because one can choose $\varepsilon' < \varepsilon$ such that the equation

$$1 + \frac{\varepsilon'}{2C_v} \log(1 - z_1) = 0$$

has a solution in the unit ball. ∎

The properties of HW described above are rather negative. In particular they show that algebraically this algebra is not similar to the Banach algebra of bounded holomorphic functions. Moreover the topological algebra HW differs from the algebra H of holomorphic functions with the topology of uniform convergence on compact sets. However, there is some similarity. For both H^∞ and H, the kernels of evaluations are finitely generated ideals. To show this in the case of HW we make use of an idea of [1] with the Bergman projection instead of the Szegő projection.

THEOREM 16. *Assume that for some $f \in HW$ and $z \in \mathbb{B}$ we have $f(z) = 0$. Then there exist $g_1, \ldots, g_d \in HW$ such that*

$$f(\zeta) = \sum_{i=1}^{d} (\zeta_i - z_i) g_i(\zeta).$$

In other words, each closed maximal ideal in HW is generated by the functions $\zeta_1 - z_1, \ldots, \zeta_d - z_d$ for some $(z_1, \ldots, z_d) \in \mathbb{B}$.

Proof. We use the fact that the Bergman projection maps LW onto HW. We have

$$
\begin{aligned}
f(\zeta)-f(z) &= Bf(\zeta)-Bf(z)\\
&= \sum_{k=1}^{d}(\zeta_k-z_k)\int_{\mathbb{B}} \frac{1}{\langle \zeta - z, \eta\rangle}\,(K(\zeta,\eta)-K(z,\eta))\overline{\eta}_k f(\eta)\,dV(\eta)\\
&= \sum_{k=1}^{d}(\zeta_k-z_k)\int_{\mathbb{B}} \frac{\overline{\eta}_k \sum_{i=1}^{d}(1-\langle\zeta,\eta\rangle)^i(1-\langle z,\eta\rangle)^{d-i}}{(1-\langle z,\eta\rangle)^{d+1}(1-\langle\zeta,\eta\rangle)^{d+1}}\,f(\eta)\,dV(\eta)\\
&= \sum_{k=1}^{d}(\zeta_k-z_k)B\bigg(\frac{\overline{\eta}_k \sum_{i=1}^{d}(1-\langle\zeta,\eta\rangle)^i(1-\langle z,\eta\rangle)^{d-i}}{(1-\langle z,\eta\rangle)^{d+1}}\,f\bigg).
\end{aligned}
$$

The functions

$$
g_k(\eta)=\frac{\overline{\eta}_k \sum_{i=1}^{d}(1-\langle\zeta,\eta\rangle)^i(1-\langle z,\eta\rangle)^{d-i}}{(1-\langle z,\eta\rangle)^{d+1}}\,f(\eta)
$$

belong to LW, which proves the theorem. ■

COROLLARY 17. *Let $\mathfrak{M}$ be a closed maximal ideal of HW. Then for each $n\in\mathbb{N}$ the ideal $\mathfrak{M}^n$ is closed in HW.*

Proof. This follows from the previous proposition and the continuity of the Bergman projection. ■

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