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Maximal regularity of delay equations in Banach spaces

by

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Abstract. We characterize existence and uniqueness of solutions for an inhomogeneous abstract delay equation in Hölder spaces. The main tool is the theory of operator-valued Fourier multipliers.

1. Introduction. Partial differential equations with delay have been extensively studied in the last years. In an abstract way they can be written as

$$u'(t) = Au(t) + Fu_t + f(t), \qquad t \in \mathbb{R}, \tag{1.1}$$

where $(A, D(A))$ is an (unbounded) linear operator on a Banach space X, $u_t(\cdot) = u(t + \cdot)$ on $[-r, 0]$, $r > 0$, and the delay operator F is supposed to belong to $\mathcal{B}(C([-r, 0], X), X)$.

First studies on equation (1.1) go back to J. Hale [8] and G. Webb [12]. A general and systematic study of linear delay equations with emphasis on the qualitative behavior and asymptotic properties can be found in the recent monograph by Bátkai and Piazzera [5]. See also [13]. The problem to find conditions for all solutions of (1.1) to be in the same space as f arises naturally from recent studies on maximal regularity and their application to nonlinear problems in the theory of evolution equations; see the recent monograph by Denk–Hieber–Prüss [7] and references therein.

Recently, a significant progress has been made in finding sufficient conditions for operator-valued functions to be C^α-Fourier multipliers (see [3]). In particular, in [4] the theory of operator-valued Fourier multipliers is applied to obtain results on the hyperbolicity of delay equations and in [9] to obtain stability of linear control systems in Banach spaces. Also in [10] existence and uniqueness of periodic solutions for equation (1.1) via L^p-Fourier multiplier theorems has recently been obtained.

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In this paper we obtain necessary and sufficient conditions of well-posedness of the delay equation (1.1) in the Hölder spaces $C^\alpha(\mathbb{R}, X)$ $(0 < \alpha < 1)$, under the condition that X is a B-convex space. We stress that here A is not necessarily the generator of a C_0-semigroup.

The Fourier multiplier approach allows us to give a direct treatment of the equation, in contrast with the approach using the correspondence between (1.1) and the solutions of the abstract Cauchy problem

$$\mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t) + \mathcal{F}(t), \quad t \geq 0,$$

where $\mathcal{A} = \left(\begin{smallmatrix} A & F \\ 0 & d/d\sigma \end{smallmatrix}\right)$. In the latter case the question of well-posedness of the delay equation reduces to the question whether or not the operator $(\mathcal{A}, D(\mathcal{A}))$ generates a C_0-semigroup; see [5, 6, 11] and references therein.

2. Preliminaries. Let X, Y be Banach spaces and let $0 < \alpha < 1$. We consider the spaces

$$\dot{C}^\alpha(\mathbb{R}, X) = \{f : \mathbb{R} \to X : f(0) = 0,\ \|f\|_\alpha < \infty\}$$

normed by

$$\|f\|_\alpha = \sup_{t \neq s} \frac{\|f(t) - f(s)\|}{|t - s|^\alpha}.$$

Let $\Omega \subset \mathbb{R}$ be an open set. By $C_c^\infty(\Omega)$ we denote the space of all C^∞-functions in $\Omega \subseteq \mathbb{R}$ having compact support in Ω.

We denote by $\mathcal{F}f$ or $\widetilde{f}$ the Fourier transform, i.e.

$$(\mathcal{F}f)(s) := \int_{\mathbb{R}} e^{-ist} f(t)\, dt \quad (s \in \mathbb{R},\ f \in L^1(\mathbb{R}, X)).$$

Definition 2.1. Let $M : \mathbb{R}\setminus\{0\} \to \mathcal{B}(X, Y)$ be continuous. We say that M is a $\dot{C}^\alpha$-*multiplier* if there exists a mapping $L : \dot{C}^\alpha(\mathbb{R}, X) \to \dot{C}^\alpha(\mathbb{R}, Y)$ such that

$$\int_{\mathbb{R}} (Lf)(s)(\mathcal{F}\phi)(s)\, ds = \int_{\mathbb{R}} (\mathcal{F}(\phi \cdot M))(s) f(s)\, ds \tag{2.1}$$

for all $f \in C^\alpha(\mathbb{R}, X)$ and $\phi \in C_c^\infty(\mathbb{R} \setminus \{0\})$.

Here $(\mathcal{F}(\phi \cdot M))(s) = \int_{\mathbb{R}} e^{-ist} \phi(t) M(t)\, dt \in \mathcal{B}(X, Y)$. Note that L is well defined, linear and continuous (cf. [3, Definition 5.2]).

Define

$$C^\alpha(\mathbb{R}, X) = \{f : \mathbb{R} \to X : \|f\|_{C^\alpha} < \infty\}$$

with the norm

$$\|f\|_{C^\alpha} = \|f\|_\alpha + \|f(0)\|.$$

Let $C^{\alpha+1}(\mathbb{R}, X)$ be the Banach space of all $u \in C^1(\mathbb{R}, X)$ such that $u' \in C^\alpha(\mathbb{R}, X)$, equipped with the norm

$$\|u\|_{C^{\alpha+1}} = \|u'\|_{C^\alpha} + \|u(0)\|.$$

By Definition 2.1 and since

$$\int_{\mathbb{R}} (\mathcal{F}(\phi M)(s))(s)\,ds = 2\pi(\phi M)(0) = 0,$$

it follows that $f \in C^\alpha(\mathbb{R}, X)$ implies $Lf \in C^\alpha(\mathbb{R}, X)$. Moreover, if $f \in C^\alpha(\mathbb{R}, X)$ is bounded then Lf is bounded as well (see [3, Remark 6.3]).

The following multiplier theorem is due to Arendt–Batty and Bu [3, Theorem 5.3].

THEOREM 2.2. *Let $M \in C^2(\mathbb{R} \setminus \{0\}, \mathcal{B}(X, Y))$ be such that*

$$\sup_{t\neq 0} \|M(t)\| + \sup_{t\neq 0} \|tM'(t)\| + \sup_{t\neq 0} \|t^2 M''(t)\| < \infty. \tag{2.2}$$

Then M is a $\dot{C}^\alpha$-multiplier.

REMARK 2.3. If X is B-convex, in particular if X is a UMD space, Theorem 2.2 remains valid if condition (2.2) is replaced by the weaker condition

$$\sup_{t\neq 0} \|M(t)\| + \sup_{t\neq 0} \|tM'(t)\| < \infty, \tag{2.3}$$

where $M \in C^1(\mathbb{R} \setminus \{0\}, \mathcal{B}(X, Y))$ (cf. [3, Remark 5.5]).

We use the symbol $\widehat{f}(\lambda)$ for the Carleman transform:

$$\widehat{f}(\lambda) = \begin{cases} \int_0^\infty e^{-\lambda t} f(t)\,dt, & \operatorname{Re}\lambda > 0, \\ -\int_{-\infty}^0 e^{-\lambda t} f(t)\,dt, & \operatorname{Re}\lambda < 0, \end{cases}$$

where $f \in L^1_{\mathrm{loc}}(\mathbb{R}, X)$ is of subexponential growth; by this we mean

$$\int_{-\infty}^{\infty} e^{-\varepsilon|t|}\|f(t)\|\,dt < \infty \quad \text{for each } \varepsilon > 0.$$

We remark that if $u' \in L^1_{\mathrm{loc}}(\mathbb{R}, X)$ is of subexponential growth, then

$$\widehat{u'}(\lambda) = \lambda\widehat{u}(\lambda) - u(0), \quad \operatorname{Re}\lambda \neq 0.$$

3. A characterization. In this section we consider the equation

$$u'(t) = Au(t) + Fu_t + f(t), \quad t \in \mathbb{R}, \tag{3.1}$$

where $A : D(A) \subseteq X \to X$ is a closed linear operator, $f \in C^\alpha(\mathbb{R}, X)$, and, for some $r > 0$, $F : C([-r, 0], X) \to X$ is a bounded linear operator. Moreover u_t is an element of $C([-r, 0], X)$ defined by $u_t(\theta) = u(t + \theta)$ for $-r \le \theta \le 0$.

Example 3.1. Let $\mu : [-r,0] \to \mathcal{B}(X)$ be of bounded variation. Let $F : C([-r,0],X) \to X$ be the bounded operator given by the Riemann–Stieltjes integral

$$F(\phi) = \int_{-r}^{0} \phi\, d\mu \quad \text{for all } \phi \in C([-r,0],X).$$

An important special case involves operators F defined by

$$F(\phi) = \sum_{k=0}^{n} C_k \phi(\tau_k), \qquad \phi \in C([-r,0],X),$$

where $C_k \in \mathcal{B}(X)$ and $\tau_k \in [-r,0]$ for $k = 0,1,\dots,n$. For concrete equations with the above classes of delay operators see the monograph of Bátkai and Piazzera [5, Chapter 3].

Definition 3.2. We say that (1.1) is C^α*-well posed* if for each $f \in C^\alpha(\mathbb{R},X)$ there is a unique function $u \in C^{\alpha+1}(\mathbb{R},X) \cap C^\alpha(\mathbb{R},[D(A)])$ such that (1.1) is satisfied.

Set $e_\lambda(t) := e^{i\lambda t}$ for all $\lambda \in \mathbb{R}$, and define the operators $\{F_\lambda\}_{\lambda\in\mathbb{R}} \subseteq \mathcal{B}(X)$ by

$$F_\lambda x = F(e_\lambda x) \quad \text{for all } \lambda \in \mathbb{R} \text{ and } x \in X. \tag{3.2}$$

We define the *real spectrum* of (3.1) by

$$\sigma(\Delta) = \{s \in \mathbb{R} : isI - F_s - A \in \mathcal{B}([D(A)],X) \text{ is not invertible}\}.$$

Proposition 3.3. *Let X be a Banach space and let $A : D(A) \subset X \to X$ be a closed linear operator. Suppose that* (1.1) *is C^α-well posed. Then*

(i) $\sigma(\Delta) = \emptyset$,
(ii) $\{i\eta(i\eta I - A - F_\eta)^{-1}\}_{\eta\in\mathbb{R}}$ *is bounded.*

Proof. Let $x \in D(A)$ and let $u(t) = e^{i\eta t}x$ for $\eta \in \mathbb{R}$. Then $u_t(s) = e^{it\eta}e^{is\eta}x$. Thus

$$F(u_t) = e^{it\eta}F(e_\eta x) = e^{it\eta}F_\eta x. \tag{3.3}$$

Now if $(i\eta - A - F_\eta)x = 0$, then $u(t)$ is a solution of equation (1.1) when $f \equiv 0$. Hence by uniqueness $x = 0$. Now let $L : C^\alpha(\mathbb{R},X) \to C^{\alpha+1}(\mathbb{R},X)$ be the bounded operator which takes each $f \in C^\alpha(\mathbb{R},X)$ to the unique solution $u \in C^{\alpha+1}(\mathbb{R},X)$ of (1.1). Fix $y \in X$ and $s_0 \in \mathbb{R}$, and define $f(t) = e^{it\eta}y$, $t \in \mathbb{R}$. Let $u(t)$ be the unique solution of (1.1) such that $L(f) = u$.

We claim that $v(t) := u(t+s_0)$ and $w(t) := e^{i\eta s_0}u(t)$ both satisfy (1.1) when f is replaced by $e^{is_0\eta}f(t)$. First we notice that

$$v_t(s) = u(t+s_0+s) = u_{t+s_0}(s).$$

Hence $F(v_t) = F(u_{t+s_0})$. Then an easy computation shows that $v(t)$ satisfies (1.1). On the other hand,

$$w_t(s) = w(t+s) = e^{i\eta s_0}u(t+s) = e^{i\eta s_0}u_t(s).$$

Hence $F(w_t) = e^{is_0\eta}F(u_t)$. Thus

$$e^{i\eta s_0}u'(t) = e^{i\eta s_0}(Au(t) + F(u_t) + f(t)) = Aw(t) + F(w_t) + e^{i\eta s_0}f(t),$$

that is, $w(t)$ satisfies (1.1). By uniqueness we again have

$$u(t+s) = e^{i\eta s}u(t)$$

for all $t, s \in \mathbb{R}$. In particular, when $t = 0$ we obtain

$$u(s) = e^{i\eta s}u(0), \qquad s \in \mathbb{R}.$$

Now let $x = u(0) \in D(A)$. Then $u(t) = e^{i\eta t}x$ satisfies (1.1), that is, by (3.3),

$$i\eta u(t) = Au(t) + F(u_t) + e^{i\eta t}y = Au(t) + e^{i\eta t}F_\eta x + e^{i\eta t}y.$$

In particular, if $t = 0$ we obtain

$$i\eta x = Ax + F_\eta x + y,$$

since $x = u(0)$. Thus

$$(i\eta I - A - F_\eta)x = y \tag{3.4}$$

and hence $i\eta I - A - F_\eta$ is bijective. This shows assertion (i) of the proposition.

Next we notice that $u(t) = (i\eta - A - F_\eta)^{-1}y$ by (3.4). Since $\|e_\eta \otimes x\|_\alpha = K_\alpha|\eta|^\alpha\|x\|$, we have

$$\begin{aligned} K_\alpha|\eta|^\alpha\|i\eta(i\eta - A - F_\eta)^{-1}y\| &= \|e_\eta \otimes i\eta(i\eta - A - F_\eta)^{-1}y\|_\alpha = \|u'\|_\alpha \\ &\le \|u\|_{1+\alpha} = \|Lf\|_{1+\alpha} \le \|L\|\,\|f\|_\alpha \le \|L\|(\|f\|_\alpha + \|f(0)\|) \\ &= \|L\|(\|e_\eta \otimes y\|_\alpha + \|y\|) \le \|L\|(K_\alpha|\eta|^\alpha + 1)\|y\|. \end{aligned}$$

Hence for $\varepsilon > 0$ it follows that

$$\sup_{|\eta|>\varepsilon}\|i\eta(i\eta - A - F_\eta)^{-1}y\| \le \|L\| \sup_{|\eta|>\varepsilon}\left(1 + \frac{1}{K_\alpha|\eta|^\alpha}\right) < \infty. \ \blacksquare$$

Recall that a Banach space X has *Fourier type* p, where $1 \le p \le 2$, if the Fourier transform defines a bounded linear operator from $L^p(\mathbb{R}, X)$ to $L^q(\mathbb{R}, X)$, where q is the conjugate index of p. For example, the space $L^p(\Omega)$, where $1 \le p \le 2$, has Fourier type p; X has Fourier type 2 if and only if X is a Hilbert space; X has Fourier type p if and only if X^* has Fourier type p. Every Banach space has Fourier type 1; X is B-convex if it has Fourier type p for some $p > 1$. Every uniformly convex space is B-convex.

Our main result in this paper establishes that the converse of Proposition 3.3 is true.

THEOREM 3.4. *Let A be a closed linear operator defined on a B-convex space X. Then the following assertions are equivalent*:

(i) *Equation* (1.1) *is C^α-well posed.*
(ii) $\sigma(\Delta) = \emptyset$ *and* $\sup_{\eta\in\mathbb{R}} \|i\eta(i\eta I - A - F_\eta)^{-1}\| < \infty$.

Proof. (ii)⇒(i). Define the operator $M(t) = (B_t - A)^{-1}$, with $B_t = itI - F_t$. Note that by hypothesis $M \in C^1(\mathbb{R}, \mathcal{B}(X, [D(A)]))$.

We claim that M is a C^α-multiplier. In fact, by hypothesis it is clear that $\sup_{t\in\mathbb{R}} \|M(t)\| < \infty$. On the other hand, we have

$$M'(t) = -M(t)B_t'M(t)$$

with $B_t' = iI - F_t'$ and $F_t'(x) = F(e_t'x)$ where $e_t'(s) = ise^{ist}$. Note that for each $x \in X$,

$$\|F_t x\|_X \le \|F(e_t x)\|_X \le \|F\|\,\|e_t x\|_\infty \le \|F\|\,\|x\|_X, \tag{3.5}$$

and

$$\|F_t' x\|_X \le \|F(e_t' x)\|_X \le \|F\|\,\|e_t' x\|_\infty \le r\|F\|\,\|x\|_X. \tag{3.6}$$

Hence B_t' is uniformly bounded with respect to $t \in \mathbb{R}$ and we conclude from the hypothesis that

$$\sup_{t\in\mathbb{R}} \|tM'(t)\| = \sup_{t\in\mathbb{R}} \|[tM(t)]B_t'M(t)\| < \infty, \tag{3.7}$$

and hence the claim follows from Theorem 2.2 and Remark 2.3.

Now, define $N \in C^1(\mathbb{R}, \mathcal{B}(X))$ by $N(t) = (id \cdot M)(t)$, where $id(t) := it$ for all $t \in \mathbb{R}$. We will prove that N is a C^α-multiplier. In fact, with a direct calculation, we have

$$\begin{aligned} tN'(t) &= itM(t) + it^2M'(t) = itM(t) + i[itM(t)]B_t'[itM(t)] \\ &= N(t) + iN(t)B_t'N(t). \end{aligned}$$

By hypothesis and (3.6) it follows that

$$\sup_{t\in\mathbb{R}} \|tN'(t)\| \le \sup_{t\in\mathbb{R}} \|N(t)\| + \sup_{t\in\mathbb{R}} \|N(t)B_t'N(t)\| < \infty,$$

hence from Theorem 2.2 and Remark 2.3 the claim is proved.

A similar calculation proves that $P \in C^1(\mathbb{R} \setminus \{0\}, \mathcal{B}(X))$ defined by $P(t) = F_tM(t)$ is a C^α-multiplier. In fact, we have $tP'(t) = F_t'N(t) + F_ttM'(t)$, and hence from (3.5), (3.6) and (3.7) we see that $\sup_{t\in\mathbb{R}} \|P(t)\| + \sup_{t\in\mathbb{R}} \|tP'(t)\| < \infty$.

Let $f \in C^\alpha(\mathbb{R}, X)$. Since M, N and P are C^α-multipliers, there exist $\overline{u} \in C^\alpha(\mathbb{R}, [D(A)])$, $v \in C^\alpha(\mathbb{R}, X)$ and $w \in C^\alpha(\mathbb{R}, X)$ such that

$$\int_{\mathbb{R}} \overline{u}(s)(\mathcal{F}\phi)(s)\,ds = \int_{\mathbb{R}} \mathcal{F}(\phi \cdot M)(s) f(s)\,ds, \tag{3.8}$$

$$\int_{\mathbb{R}} v(s)(\mathcal{F}\psi)(s)\,ds = \int_{\mathbb{R}} \mathcal{F}(\psi \cdot id \cdot M)(s) f(s)\,ds, \tag{3.9}$$

$$\int_{\mathbb{R}} w(s)(\mathcal{F}\varphi)(s)\,ds = \int_{\mathbb{R}} \mathcal{F}(\varphi \cdot F_{\cdot} M)(s) f(s)\,ds, \tag{3.10}$$

for all $\phi, \psi, \varphi \in C_c^\infty(\mathbb{R})$.

Note that for $x \in X$ and $\phi \in C_c^\infty(\mathbb{R})$ we have

$$\mathcal{F}(\phi F_{\cdot} M)(s)x = \int_{\mathbb{R}} e^{-ist}\phi(t) F_t M(t) x\,dt = \int_{\mathbb{R}} e^{-ist}\phi(t) F(e_t M(t)x)\,dt, \tag{3.11}$$

where $\int_{\mathbb{R}} e^{-ist}\phi(t) e_t M(t) x\,dt \in C([-r,0],X)$. Now, for all $\theta \in [-r,0]$ we have

$$\Big\| \int_{\mathbb{R}} e^{-ist}\phi(t) e_t(\theta) M(t) x\,dt \Big\|_X \le \int_{\mathbb{R}} |\phi(t)|\, \|M(t)x\|_X\,dt.$$

Since F is bounded, we deduce that

$$\mathcal{F}(\phi \cdot F_{\cdot} M)(s)x = F(\mathcal{F}(\phi \cdot e_{\cdot} M)(s)x). \tag{3.12}$$

Furthermore, observe that for $\theta \in [-r,0]$ fixed we have $e_{\cdot}(\theta)\phi \in C_c^\infty(\mathbb{R})$. Using (3.8) we obtain

$$\begin{aligned}
\int_{\mathbb{R}} \overline{u}(s+\theta)(\mathcal{F}\phi)(s)ds &= \int_{\mathbb{R}} \overline{u}(s+\theta) \int_{\mathbb{R}} e^{-ist}\phi(t)\,dt\,ds \\
&= \int_{\mathbb{R}} \overline{u}(s+\theta) \int_{\mathbb{R}} e^{-i(s+\theta)t} e_t(\theta)\phi(t)\,dt\,ds \\
&= \int_{\mathbb{R}} \overline{u}(s+\theta)(\mathcal{F}e_{\cdot}(\theta)\phi)(s+\theta)\,ds \\
&= \int_{\mathbb{R}} \overline{u}(s)(\mathcal{F}e_{\cdot}(\theta)\phi)(s)\,ds \\
&= \int_{\mathbb{R}} \mathcal{F}(e_{\cdot}(\theta)\phi \cdot M)(s) f(s)\,ds,
\end{aligned}$$

hence $\int_{\mathbb{R}} \overline{u}_s(\mathcal{F}\phi)(s)\,ds = \int_{\mathbb{R}} \mathcal{F}(e_{\cdot}\phi \cdot M)(s) f(s)\,ds$.

Since $\theta \mapsto \int_{\mathbb{R}} \overline{u}_s(\theta)(\mathcal{F}\phi)(s)\,ds \in C([-r,0],X)$ (see [3, p. 25]), from the boundedness of F and (3.12) it follows that

$$\begin{aligned}
\int_{\mathbb{R}} \mathcal{F}(\phi \cdot F_{\cdot} M)(s) f(s)\,ds &= \int_{\mathbb{R}} F\mathcal{F}(\phi \cdot e_{\cdot} M)(s) f(s)\,ds \\
&= \int_{\mathbb{R}} F\overline{u}_s(\mathcal{F}\phi)(s)\,ds
\end{aligned} \tag{3.13}$$

for all $\phi \in C_c^\infty(\mathbb{R})$. Since $F_\cdot M$ is a C^α-multiplier, from (3.10) we obtain

$$\int_{\mathbb{R}} w(s)(\mathcal{F}\phi)(s)\,ds = \int_{\mathbb{R}} F\overline{u}_s(\mathcal{F}\phi)(s)\,ds$$

for all $\phi \in C_c^\infty(\mathbb{R})$. We conclude that there exists $y_1 \in X$ satisfying $w(t) = F\overline{u}_t + y_1$, proving that $F\overline{u}_\cdot \in C^\alpha(\mathbb{R}, X)$.

Choosing $\phi = id \cdot \psi$ in (3.8) we deduce from (3.9) that

$$\int_{\mathbb{R}} \overline{u}(s)\mathcal{F}(id \cdot \psi)(s)\,ds = \int_{\mathbb{R}} v(s)(\mathcal{F}\psi)(s)\,ds, \tag{3.14}$$

and it follows from Lemma 6.2 in [3] that $\overline{u} \in C^{\alpha+1}(\mathbb{R}, X)$ and $\overline{u}' = v + y_2$ for some $y_2 \in X$.

Since $(id\, I - F_\cdot - A)M = I$ we have $id \cdot M = I + F_\cdot M + AM$ and replacing in (3.9) gives

$$\begin{aligned} \int_{\mathbb{R}} v(s)(\mathcal{F}\phi)(s)\,ds &= \int_{\mathbb{R}} \mathcal{F}(\phi \cdot (I + F_\cdot M + AM))(s) f(s)\,ds \\ &= \int_{\mathbb{R}} (\mathcal{F}\phi)(s) f(s)\,ds + \int_{\mathbb{R}} \mathcal{F}(\phi \cdot F_\cdot M)(s) f(s)\,ds \\ &\quad + \int_{\mathbb{R}} \mathcal{F}(\phi \cdot AM)(s) f(s)\,ds \end{aligned} \tag{3.15}$$

for all $\phi \in C_c^\infty(\mathbb{R})$.

Since $\overline{u}(t) \in D(A)$ and $\mathcal{F}(\phi \cdot M)(s)x \in D(A)$ for all $x \in X$, using the fact that A is closed and inserting (3.8) and (3.13) in (3.15) we obtain

$$\begin{aligned} \int_{\mathbb{R}} v(s)(\mathcal{F}\phi)(s)\,ds &= \int_{\mathbb{R}} F\overline{u}_s(\mathcal{F}\phi)(s)\,ds + \int_{\mathbb{R}} A\overline{u}(s)(\mathcal{F}\phi)(s) f(s)\,ds \\ &\quad + \int_{\mathbb{R}} f(s)(\mathcal{F}\phi)(s)\,ds \end{aligned} \tag{3.16}$$

for all $\phi \in C_c^\infty(\mathbb{R})$. By Lemma 5.1 in [3] this implies that for some $y_3 \in X$ one has

$$v(t) = F\overline{u}_t + A\overline{u}(t) + f(t) + y_3, \qquad t \in \mathbb{R}.$$

Consequently, $\overline{u}'(t) = v(t) + y_2 = F\overline{u}_t + A\overline{u}(t) + f(t) + y$ where $y = y_2 + y_3$. In particular $A\overline{u} \in C^\alpha(\mathbb{R}, X)$. Now, by hypothesis we can define $x = (A + F)^{-1}y \in D(A)$, and then it is clear that $u(t) := \overline{u}(t) + x$ is in $C^{\alpha+1}(\mathbb{R}, X) \cap C^\alpha(\mathbb{R}, [D(A)])$ and satisfies (1.1). We have shown that a solution of (1.1) exists.

In order to prove uniqueness, suppose that

$$u'(t) = Au(t) + Fu_t, \qquad t \in \mathbb{R}, \tag{3.17}$$

where $u \in C^{\alpha+1}(\mathbb{R}, X) \cap C^\alpha(\mathbb{R}, [D(A)])$ and, as shown, $Au, Fu_\cdot \in C^\alpha(\mathbb{R}, X)$.

We claim that $\widehat{u_\cdot}(\lambda) \in C([-r,0],X)$ for $\operatorname{Re}\lambda \neq 0$. In fact, let $\operatorname{Re}\lambda > 0$. Then

$$\|e^{-\lambda t}u_t\|_\infty = \sup_{\theta\in[-r,0]} \|e^{-\lambda t}u(t+\theta)\|_X \le \sup_{\theta\in[-r,0]} e^{-\operatorname{Re}\lambda t}(1+|t+\theta|^\alpha)$$
$$\le e^{-\operatorname{Re}\lambda t}(1+(|t|+r)^\alpha).$$

Since $e^{-\operatorname{Re}\lambda t}(1+(|t|+r)^\alpha) \in L^1(\mathbb{R}_+)$, applying the dominated convergence theorem we obtain the claim. Analogously we argue for $\operatorname{Re}\lambda < 0$.

Now, note that for $\operatorname{Re}\lambda > 0$ and $\theta \in [-r,0]$,

$$\int_0^\infty e^{-\lambda t}u_t(\theta)\,dt = \int_0^\infty e^{-\lambda t}u(t+\theta)\,dt = \int_\theta^\infty e^{-\lambda(t-\theta)}u(t)\,dt$$
$$= e^{\lambda\theta}\int_\theta^\infty e^{-\lambda t}u(t)\,dt = e^{\lambda\theta}\Big(\int_0^\infty e^{-\lambda t}u(t)\,dt + \int_\theta^0 e^{-\lambda t}u(t)\,dt\Big)$$
$$= e^{\lambda\theta}\widehat{u}(\lambda) + e^{\lambda\theta}\int_\theta^0 e^{-\lambda t}u(t)\,dt.$$

Analogously if $\operatorname{Re}\lambda < 0$ and $\theta \in [-r,0]$, then

$$-\int_{-\infty}^0 e^{-\lambda t}u_t(\theta)\,dt = -\int_{-\infty}^0 e^{-\lambda t}u(t+\theta)\,dt = -\int_{-\infty}^\theta e^{-\lambda(t-\theta)}u(t)\,dt$$
$$= -e^{\lambda\theta}\Big(\int_{-\infty}^0 e^{-\lambda t}u(t)\,dt - \int_\theta^0 e^{-\lambda t}u(t)\,dt\Big)$$
$$= e^{\lambda\theta}\widehat{u}(\lambda) + e^{\lambda\theta}\int_\theta^0 e^{-\lambda t}u(t)\,dt.$$

Since F is bounded, we obtain

$$\widehat{Fu_\cdot}(\lambda) = F\widehat{u_\cdot}(\lambda) = Fg\widehat{u}(\lambda) + Fgh \quad \text{for } \operatorname{Re}\lambda \neq 0 \tag{3.18}$$

where $g(\theta) = e^{\lambda\theta}$ and $h(\theta) = \int_\theta^0 e^{-\lambda t}u(t)\,dt$. Note that $gh \in C([-r,0],X)$.

Since $\widehat{u'}(\lambda) = \lambda\widehat{u}(\lambda) - u(0)$ for $\operatorname{Re}\lambda \neq 0$, one has $\widehat{u}(\lambda) \in D(A)$ and

$$\widehat{u'}(\lambda) = \widehat{Au}(\lambda) + \widehat{Fu_\cdot}(\lambda) \quad \text{for } \operatorname{Re}\lambda \neq 0. \tag{3.19}$$

Using the fact that A is closed, from (3.18) and (3.19) we get

$$(\lambda I - Fg - A)\widehat{u}(\lambda) = u(0) + Fgh \quad \text{for all } \lambda \in \mathbb{C}\setminus i\mathbb{R}.$$

Since $i\mathbb{R} \subset \varrho(A)$, it follows that the Carleman spectrum $\operatorname{sp}_{\mathrm{C}}(u)$ of u is empty. Hence $u \equiv 0$ by [2, Theorem 4.8.2]. ∎

We denote by $\mathcal{K}_F(X)$ the class of operators in X satisfying (ii) in the above theorem. If $A \in \mathcal{K}_F(X)$ we have $u', Au, Fu_\cdot \in C^\alpha(\mathbb{R},X)$, and hence we deduce the following result.

COROLLARY 3.5. *Let X be B-convex and $A \in \mathcal{K}_F(X)$. Then*

(i) (1.1) *has a unique solution in $Z := C^{\alpha+1}(\mathbb{R}, X) \cap C^{\alpha}(\mathbb{R}, [D(A)])$ if and only if $f \in C^{\alpha}(\mathbb{R}, X)$.*

(ii) *There exists a constant $M > 0$ independent of $f \in C^{\alpha}(\mathbb{R}, X)$ such that*

$$\|u'\|_{C^{\alpha}(\mathbb{R},X)} + \|Au\|_{C^{\alpha}(\mathbb{R},X)} + \|Fu.\|_{C^{\alpha}(\mathbb{R},X)} \leq M\|f\|_{C^{\alpha}(\mathbb{R},X)}. \tag{3.20}$$

REMARK 3.6. The inequality (3.20) is a consequence of the closed graph theorem and known as the *maximal regularity property* for equation (1.1). From it we deduce that the operator L defined by

$$D(L) = Z, \qquad (Lu)(t) = u'(t) - Au(t) - Fu_t,$$

is an isomorphism onto. In fact, since A is closed, the space Z becomes a Banach space under the norm

$$\|u\|_Z := \|u\|_{C^{\alpha}(\mathbb{R},X)} + \|u'\|_{C^{\alpha}(\mathbb{R},X)} + \|Au\|_{C^{\alpha}(\mathbb{R},X)}.$$

Such isomorphisms are crucial for the treatment of nonlinear versions of (1.1).

Assume X is B-convex and $A \in \mathcal{K}_F(X)$ and consider the semilinear problem

$$u'(t) = Au(t) + Fu_t + f(t, u(t)), \quad t \geq 0. \tag{3.21}$$

Define the Nemytskiĭ superposition operator $N : Z \to C^{\alpha}(\mathbb{R}, X)$ by $N(v)(t) = f(t, v(t))$, and the bounded linear operator

$$S : C^{\alpha}(\mathbb{R}, X) \to Z$$

by $S(g) = u$ where u is the unique solution of the linear problem

$$u'(t) = Au(t) + Fu_t + g(t).$$

Then to solve (3.21) we have to show that the operator $H : Z \to Z$ defined by $H = SN$ has a fixed point.

For related information we refer to Amann [1] where results on quasilinear delay equations involving the method of maximal regularity are presented.

We finish this paper with the following result which gives us a useful criterion to verify condition (ii) in the above theorem.

THEOREM 3.7. *Let X be a B-convex space and let $A : D(A) \subset X \to X$ be a closed linear operator such that $i\mathbb{R} \subset \varrho(A)$ and $\sup_{s\in\mathbb{R}} \|A(isI - A)^{-1}\| =: M < \infty$. Suppose that*

$$\|F\| < \frac{1}{\|A^{-1}\|M}. \tag{3.22}$$

Then for each $f \in C^{\alpha}(\mathbb{R}, X)$ there is a unique function $u \in C^{\alpha+1}(\mathbb{R}, X) \cap C^{\alpha}(\mathbb{R}, [D(A)])$ such that (1.1) *is satisfied.*

Proof. From the identity

$$isI - A - F_s = (isI - A)(I - F_s(isI - A)^{-1}), \qquad s \in \mathbb{R},$$

it follows that $isI - A - F_s$ is invertible whenever $\|F_s(isI - A)^{-1}\| < 1$. Next observe that

$$\|F_s\| \le \|F\|, \tag{3.23}$$

and hence

$$\|F_s(isI - A)^{-1}\| = \|F_s A^{-1} A(isI - A)^{-1}\| \le \|F\|\,\|A^{-1}\|M =: \alpha.$$

Therefore, under the condition (3.22) we obtain $\sigma(\Delta) = \emptyset$ and the identity

$$\begin{aligned}(isI - A - F_s)^{-1} &= (isI - A)^{-1}(I - F_s(isI - A)^{-1}) \\ &= (isI - A)^{-1} \sum_{n=0}^{\infty} [F_s(isI - A)^{-1}]^n.\end{aligned} \tag{3.24}$$

For all $n \in \mathbb{N}$ we have

$$\begin{aligned}&\|is(isI - A)^{-1}[F_s(isI - A)^{-1}]^n\| \\ &\qquad \le \|is(isI - A)^{-1}\|\,\|F_s A^{-1} A(isI - A)^{-1}\|^n \\ &\qquad \le \|is(isI - A)^{-1}\|\|F_s A^{-1}\|^n \|A(isI - A)^{-1}\|^n \\ &\qquad \le \|is(isI - A)^{-1}\|\,\|A^{-1}\|^n \|F_s\|^n \|A(isI - A)^{-1}\|^n.\end{aligned}$$

By (3.23) we obtain

$$\begin{aligned}\|is(isI - A)^{-1}[F_s(isI - A)^{-1}]^n\| &\le \|is(isI - A)^{-1}\|\,\|A^{-1}\|^n \|F\|^n M^n \\ &= \|is(isI - A)^{-1}\|\alpha^n.\end{aligned}$$

Finally, by (3.24), one has

$$\|is(isI - A - F_s)^{-1}\| \le \|is(isI - A)^{-1}\| \frac{1}{1 - \alpha} \le \frac{M + 1}{1 - \alpha}.$$

This proves that $\{is(isI - A - F_s)^{-1}\}$ is bounded and the conclusion follows from Theorem 3.4. ∎

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