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On the vector-valued Fourier transform and compatibility of operators

by

IN SOOK PARK (Taejeon)

Abstract. Let $\mathbb{G}$ be a locally compact abelian group and let $1 < p \le 2$. $\mathbb{G}'$ is the dual group of $\mathbb{G}$, and p' the conjugate exponent of p. An operator T between Banach spaces X and Y is said to be compatible with the Fourier transform $F^{\mathbb{G}}$ if $F^{\mathbb{G}} \otimes T : L_p(\mathbb{G}) \otimes X \to L_{p'}(\mathbb{G}') \otimes Y$ admits a continuous extension $[F^{\mathbb{G}}, T] : [L_p(\mathbb{G}), X] \to [L_{p'}(\mathbb{G}'), Y]$. Let $\mathcal{FT}_p^{\mathbb{G}}$ denote the collection of such T's. We show that $\mathcal{FT}_p^{\mathbb{R}\times\mathbb{G}} = \mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}} = \mathcal{FT}_p^{\mathbb{Z}^n\times\mathbb{G}}$ for any $\mathbb{G}$ and positive integer n. Moreover, if the factor group of $\mathbb{G}$ by its identity component is a direct sum of a torsion-free group and a finite group with discrete topology then $\mathcal{FT}_p^{\mathbb{G}} = \mathcal{FT}_p^{\mathbb{Z}}$.

1. Introduction. A locally compact abelian group means a topological abelian group whose topology is locally compact Hausdorff and which is equipped with a Haar measure. The real line $\mathbb{R}$, the discrete group of integers $\mathbb{Z}$ and the circle group $\mathbb{T}$ are important examples. Further information can be found in [3], [5] and [13]. Let $\mathbb{G}$ be a locally compact abelian group and $\mathbb{G}'$ its dual (character) group. Then the Haar measure of $\mathbb{G}'$ can be determined so that Parseval's identity holds with constant 1. For $1 \le r < \infty$, we denote by $[L_r(\mathbb{G}, \mu_{\mathbb{G}}), X]$ the Banach space of all $\mu_{\mathbb{G}}$-measurable functions $\mathbf{f} : \mathbb{G} \to X$ such that $\|\mathbf{f}|L_r(\mathbb{G})\| := (\int_{\mathbb{G}} \|\mathbf{f}(s)\|^r \, d\mu_{\mathbb{G}}(s))^{1/r}$ is finite.

Let $\mathbb{G}$ be a fixed infinite locally compact abelian group and $1 < p \le 2$. Then $F^{\mathbb{G}}$ denotes the Fourier transform from $L_p(\mathbb{G})$ into $L_{p'}(\mathbb{G}')$. For a bounded linear operator T between Banach spaces X and Y,

$$F^{\mathbb{G}} \otimes T : \sum_{k=1}^{n} f_k \otimes \mathbf{x}_k \mapsto \sum_{k=1}^{n} F^{\mathbb{G}} f_k \otimes T\mathbf{x}_k$$

yields a well defined map from $L_p(\mathbb{G}) \otimes X$ into $L_{p'}(\mathbb{G}') \otimes Y$. The operator T is said to be *compatible with* $F^{\mathbb{G}}$, or have $\mathbb{G}$-*Fourier type* p, if the operator

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$F^{\mathbb{G}} \otimes T : L_p(\mathbb{G}) \otimes X \to L_{p'}(\mathbb{G}') \otimes Y$ admits a continuous extension

$$[F^{\mathbb{G}}, T] : [L_p(\mathbb{G}), X] \to [L_{p'}(\mathbb{G}'), Y].$$

For such T, we let

$$\|T|\mathcal{FT}_p^{\mathbb{G}}\| := \|[F^{\mathbb{G}}, T] : [L_p(\mathbb{G}), X] \to [L_{p'}(\mathbb{G}'), Y]\|.$$

The norm defined above is invariant under changing the Haar measure of $\mathbb{G}$. The class of these operators is a Banach ideal, denoted by $\mathcal{FT}_p^{\mathbb{G}}$. The definition and notation follow those of [11].

It is known that $\mathcal{FT}_p^{\mathbb{R}} = \mathcal{FT}_p^{\mathbb{Z}} = \mathcal{FT}_p^{\mathbb{T}}$ (see [11]), but the problem whether the operator ideal $\mathcal{FT}_p^{\mathbb{G}}$ depends on $\mathbb{G}$ or not is unsolved. There are several results about $\mathrm{FT}_p^{\mathbb{G}}$, the class of Banach spaces whose identity operators are compatible with $F^{\mathbb{G}}$. These are immediately extended to the case of $\mathcal{FT}_p^{\mathbb{G}}$ by replacing the identity operator on a Banach space X with T. Peetre [10] who introduced the concept of Banach space of Fourier type p proved that X belongs to $\mathrm{FT}_p^{\mathbb{R}}$ if and only if the dual space X' belongs to $\mathrm{FT}_p^{\mathbb{R}}$. In fact, T belongs to $\mathcal{FT}_p^{\mathbb{G}}$ if and only if the dual operator T' belongs to $\mathcal{FT}_p^{\mathbb{G}'}$, i.e. $\|T|\mathcal{FT}_p^{\mathbb{G}}\| = \|T'|\mathcal{FT}_p^{\mathbb{G}'}\|$ for any locally compact abelian group $\mathbb{G}$. Bourgain [2] showed that $\mathrm{FT}_p^{\mathbb{T}} \subset \mathrm{FT}_p^{\mathbb{R}}$ and König [8] modified Kwapień's argument [9] to show that $\mathrm{FT}_p^{\mathbb{R}} = \mathrm{FT}_p^{\mathbb{T}}$ and extended this to $\mathrm{FT}_p^{\mathbb{G}} = \mathrm{FT}_p^{\mathbb{T}}$ if $\mathbb{G}$ is one of $\mathbb{R}^m$ and $\mathbb{T}^m$, where m is a positive integer. García-Cuerva, Kazarian and Torrea [4] and Andersson [1] showed independently that $\mathrm{FT}_p^{\mathbb{G}} = \mathrm{FT}_p^{\mathbb{Z}}$ whenever $\mathbb{G}$ is one of $\mathbb{T}^m$, $\mathbb{T}^\infty$, $\mathbb{R}^m$, $\mathbb{Z}^m$ and $\mathbb{Z}^\infty$. Andersson [1] also proved that $\|I_X|\mathcal{FT}_p^{\mathbb{H}}\| \le \|I_X|\mathcal{FT}_p^{\mathbb{G}}\|$ when $\mathbb{H}$ is an open subgroup of $\mathbb{G}$, and that $\mathrm{FT}_p^{\mathbb{E}} = \mathrm{FT}_p^{\mathbb{Z}}$ if $\mathbb{E}$ is a nontrivial torsion free abelian group with the discrete topology.

In this paper we characterize $\mathcal{FT}_p^{\mathbb{G}}$ partly as follows. In Section 2, we show for every locally compact abelian group $\mathbb{G}$ that $\|T|\mathcal{FT}_p^{\mathbb{R}\times\mathbb{G}}\|$ is equivalent to $\|T|\mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}\|$, and $\|T|\mathcal{FT}_p^{\mathbb{Z}^n\times\mathbb{G}}\| = \|T|\mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}\|$. The cartesian product means the direct product. By applying these, we easily obtain some results in the above paragraph and the relation $\mathcal{FT}_p^{\mathbb{R}^k\times\mathbb{Z}^l\times\mathbb{T}^m\times\mathbb{G}} = \mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}$ for any nonnegative integers k, l, m with $k+l+m \ge 1$. In Section 3, we combine the results of Section 2 with the properties of locally compact abelian groups to show that $\mathcal{FT}_p^{\mathbb{R}^k\times\mathbb{Z}^l\times\mathbb{F}} = \mathcal{FT}_p^{\mathbb{Z}}$ for any compact abelian group $\mathbb{F}$ with finitely many components. Moreover, we show that if $\mathbb{G} \cong \mathbb{R}^k \times$[torsion-free group with discrete topology]$\times$[compact group with finitely many components] then $\mathcal{FT}_p^{\mathbb{G}} = \mathcal{FT}_p^{\mathbb{Z}}$. If the factor group of a locally compact abelian group $\mathbb{G}$ by the identity component is in the form of [torsion-free group]$\times$[finite group] with the discrete topology, then we have $\mathcal{FT}_p^{\mathbb{G}} = \mathcal{FT}_p^{\mathbb{Z}}$.

From now on, X and Y are Banach spaces and $T : X \to Y$ is a bounded linear operator. We denote the dual group of $\mathbb{G}$ by $\mathbb{G}'$. We use the fact that $\mathbb{R}' = \mathbb{R}$ and $\mathbb{Z}' = \mathbb{T}$. We use the abbreviation LCA for "locally compact abelian". The term "isomorphic" means "topologically and algebraically isomorphic". The integral of a vector-valued function is the Bochner integral.

2. Classifying $\mathcal{FT}_p^{\mathbb{G}}$ via direct product. First we observe that the proof in [4] of the fact $\mathrm{FT}_p^{\mathbb{R}} = \mathrm{FT}_p^{\mathbb{Z}}$ can be modified to yield the following:

PROPOSITION 1. *For any LCA group* $\mathbb{G}$, *we have the inequalities*

$$\|T|\mathcal{FT}_p^{\mathbb{R}\times\mathbb{G}}\| \le \|T|\mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}\| \le \frac{\pi}{2}\|T|\mathcal{FT}_p^{\mathbb{R}\times\mathbb{G}}\|,$$

and hence $\mathcal{FT}_p^{\mathbb{R}\times\mathbb{G}} = \mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}$.

Proof. For arbitrary $\delta > 0$, let

$$\mathbf{f}(s,t) = \sum_m \chi_{[\delta(m-1/2),\delta(m+1/2)]}(s)\mathbf{g}_m(t),$$

where the summation is over $\mathbb{Z}$, $\mathbf{g}_m$ is an X-valued simple function on $\mathbb{G}$ and $\mathbf{g}_m = 0$ except for finitely many m. Note that the set of all X-valued functions $\mathbf{f}$ as above is dense in $[L_p(\mathbb{R}\times\mathbb{G}), X]$. We compute

$$(1)\qquad \|\mathbf{f}|L_p(\mathbb{R}\times\mathbb{G})\| = \Big(\int_{\mathbb{R}\times\mathbb{G}} \Big\|\sum_m \chi_{[\delta(m-1/2),\delta(m+1/2)]}(s)\mathbf{g}_m(t)\Big\|^p ds\,dt\Big)^{1/p}$$
$$= \delta^{1/p}\Big(\int_{\mathbb{G}}\sum_m \|\mathbf{g}_m(t)\|^p\,dt\Big)^{1/p}.$$

Since $\int_{\delta(m-1/2)}^{\delta(m+1/2)} \exp(i\widetilde{s}s)\,ds = \exp(im\delta\widetilde{s})\,\frac{\sin(\delta\widetilde{s}/2)}{\widetilde{s}/2}$, we have

$$\widehat{T\mathbf{f}}(\widetilde{s},\widetilde{t}) = \int_{\mathbb{G}}\int_{\mathbb{R}}\sum_m T\mathbf{g}_m(t)\chi_{[\delta(m-1/2),\delta(m+1/2)]}(s)\exp(i\widetilde{s}s)\,(t,\widetilde{t})\,ds\,dt$$
$$= \int_{\mathbb{G}}\sum_m T\mathbf{g}_m(t)\,(t,\widetilde{t})\exp(i\delta m\widetilde{s})\,\frac{\sin(\delta\widetilde{s}/2)}{\widetilde{s}/2}\,dt.$$

Hence

$$\|[F^{\mathbb{R}\times\mathbb{G}}T]\mathbf{f}\|^{p'}$$
$$= \int_{\mathbb{G}'}\int_{\mathbb{R}}\Big\|\int_{\mathbb{G}}\sum_m T\mathbf{g}_m(t)\,(t,\widetilde{t})\exp(i\delta m\widetilde{s})\,\frac{\sin(\delta\widetilde{s}/2)}{\widetilde{s}/2}\,dt\Big\|^{p'}\frac{1}{2\pi}\,d\widetilde{s}\,d\widetilde{t}$$
$$= \delta^{p'}\int_{\mathbb{G}'}\int_{\mathbb{R}}\Big\|\int_{\mathbb{G}}\sum_m T\mathbf{g}_m(t)\,(t,\widetilde{t})\exp(i\delta m\widetilde{s})\,\frac{\sin(\delta\widetilde{s}/2)}{\delta\widetilde{s}/2}\,dt\Big\|^{p'}\frac{1}{2\pi}\,d\widetilde{s}\,d\widetilde{t}$$
$$= \delta^{p'-1}\int_{\mathbb{G}'}\int_{\mathbb{R}}\Big|\frac{\sin(\widetilde{s}/2)}{\widetilde{s}/2}\Big|^{p'}$$
$$\times\Big\|\int_{\mathbb{G}}\sum_k\Big(\sum_m T\mathbf{g}_m(t)\chi_{\{m\}}(k)\Big)\,(t,\widetilde{t})\exp(ik\widetilde{s})\,dt\Big\|^{p'}\frac{1}{2\pi}\,d\widetilde{s}\,d\widetilde{t}$$

$$= \delta^{p'-1} \int_{\mathbb{G}'} \int_{-\pi}^{\pi} \sum_n \left| \frac{\sin(\widetilde{s}/2)}{\widetilde{s}/2 - n\pi} \right|^{p'} \left\| [F^{\mathbb{Z}\times\mathbb{G}}, T]\Big(\sum_m \mathbf{g}_m \chi_{\{m\}}\Big) \right\|^{p'} \frac{1}{2\pi} \, d\widetilde{s} \, d\widetilde{t}$$

$$\le \delta^{p'-1} \|T|\mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}\|^{p'} \Big(\int_{\mathbb{G}} \sum_m \|\mathbf{g}_m(t)\|^p \, dt \Big)^{p'/p}$$

$$= \|T|\mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}\|^{p'} \|\mathbf{f}|L_p(\mathbb{R}\times\mathbb{G})\|^{p'} \quad \text{by (1)},$$

where we have used the inequality $\sum_n |\frac{\sin \widetilde{s}}{\widetilde{s}-n\pi}|^{p'} \le 1$ for any real $\widetilde{s} \ne n\pi$ and $2 \le p' < \infty$ (see [7]).

Therefore we have $\|T|\mathcal{FT}_p^{\mathbb{R}\times\mathbb{G}}\| \le \|T|\mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}\|$ and $\mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}} \subseteq \mathcal{FT}_p^{\mathbb{R}\times\mathbb{G}}$.

For the right inequality of the proposition let $\mathbf{f}(k,t) = \sum_m \mathbf{g}_m(t)\chi_{\{m\}}(k)$, where $\mathbf{g}_m$ is an X-valued simple function and $\mathbf{g}_m = 0$ except for finitely many m. By a density argument it is enough to consider $\mathbf{f}$ of the above form. Now we have

$$\text{(2)} \qquad \|\mathbf{f}|L_p(\mathbb{Z}\times\mathbb{G})\| = \Big(\int_{\mathbb{G}} \sum_k \Big\| \sum_m \mathbf{g}_m(t)\chi_{\{m\}}(k) \Big\|^p \, dt \Big)^{1/p} = \Big(\int_{\mathbb{G}} \sum_m \|\mathbf{g}_m(t)\|^p \, dt \Big)^{1/p};$$

and

$$\widehat{T\mathbf{f}}(\widetilde{s},\widetilde{t}) = \int_{\mathbb{G}} \sum_k \Big(\sum_m T\mathbf{g}_m(t)\chi_{\{m\}}(k) \Big) \exp(i\widetilde{s}k)\,(t,\widetilde{t})\, dt = \int_{\mathbb{G}} \sum_m T\mathbf{g}_m(t)\, \exp(i\widetilde{s}m)\,(t,\widetilde{t})\, dt.$$

We use the identity

$$\exp(im\widetilde{s}) = \int_{\mathbb{R}} \frac{\widetilde{s}/2}{\sin(\widetilde{s}/2)}\, \chi_{[m-1/2,m+1/2]}(s)\, \exp(is\widetilde{s})\, ds$$

to obtain the following inequality (we abbreviate $\chi_{[m-1/2,m+1/2]}$ to χ_m):

$$\|[F^{\mathbb{Z}\times\mathbb{G}}, T]\mathbf{f}\|^{p'}$$

$$= \frac{1}{2\pi} \int_{\mathbb{G}'} \int_{-\pi}^{\pi} \Big\| \int_{\mathbb{G}} \sum_m T\mathbf{g}_m(t)\,(t,\widetilde{t}) \int_{\mathbb{R}} \frac{\widetilde{s}/2}{\sin(\widetilde{s}/2)}\, \chi_m(s) \exp(i\widetilde{s}s)\, ds\, dt \Big\|^{p'} d\widetilde{s}\, d\widetilde{t}$$

$$\le \frac{1}{2\pi} \int_{\mathbb{G}'} \int_{-\pi}^{\pi} \left| \frac{\widetilde{s}/2}{\sin(\widetilde{s}/2)} \right|^{p'} \Big\| \int_{\mathbb{G}} \int_{\mathbb{R}} \sum_m T\mathbf{g}_m(t)\chi_m(s)\,(t,\widetilde{t}) \exp(i\widetilde{s}s)\, ds\, dt \Big\|^{p'} d\widetilde{s}\, d\widetilde{t}$$

(because $\left|\frac{\widetilde{s}/2}{\sin(\widetilde{s}/2)}\right| \le \frac{\pi}{2}$ for $-\pi \le \widetilde{s} \le \pi$)

$$\leq \frac{1}{2\pi}\left(\frac{\pi}{2}\right)^{p'} \int_{\mathbb{G}'}\int_{\mathbb{R}} \Big\| \int_{\mathbb{G}}\int_{\mathbb{R}} \sum_m T\mathbf{g}_m(t)\,\chi_m(s)\,(t,\widetilde{t})\,\exp(i\widetilde{s}s)\,ds\,dt\Big\|^{p'} d\widetilde{s}\,d\widetilde{t}$$

$$= \left(\frac{\pi}{2}\right)^{p'} \Big\|[F^{\mathbb{R}\times\mathbb{G}}, T]\Big(\sum_m \mathbf{g}_m\chi_m\Big)\Big\|^{p'}$$

$$\leq \left(\frac{\pi}{2}\right)^{p'} \|T|\mathcal{FT}_p^{\mathbb{R}\times\mathbb{G}}\|^{p'} \Big\|\sum_m \mathbf{g}_m\chi_m \,\Big|\, L_p(\mathbb{R}\times\mathbb{G})\Big\|^{p'}$$

$$\text{(since } \|\textstyle\sum_m \mathbf{g}_m\chi_m|L_p(\mathbb{R}\times\mathbb{G})\| = (\int_{\mathbb{G}}\sum_m \|\mathbf{g}_m(t)\|^p\,dt)^{1/p})$$

$$= \left(\frac{\pi}{2}\right)^{p'} \|T|\mathcal{FT}_p^{\mathbb{R}\times\mathbb{G}}\|^{p'} \|\mathbf{f}|L_p(\mathbb{Z}\times\mathbb{G})\|^{p'} \quad \text{by (2).}$$

Therefore we have $\|\,T\,|\mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}\| \leq \frac{\pi}{2}\|T|\mathcal{FT}_p^{\mathbb{R}\times\mathbb{G}}\|$ and $\mathcal{FT}_p^{\mathbb{R}\times\mathbb{G}} \subseteq \mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}$. ∎

According to Theorem 6.3 of [4], for any LCA group $\mathbb{G}$ and operator T, $\|T|\mathcal{FT}_p^{\mathbb{G}}\| = \|T'|\mathcal{FT}_p^{\mathbb{G}'}\|$. By applying this property, we have the following:

PROPOSITION 2. *For every LCA group* $\mathbb{G}$, $\mathcal{FT}_p^{\mathbb{R}\times\mathbb{G}} = \mathcal{FT}_p^{\mathbb{T}\times\mathbb{G}}$.

Proof. By Proposition 1 we have

$$\|T|\mathcal{FT}_p^{\mathbb{R}\times\mathbb{G}}\| = \|T'|\mathcal{FT}_p^{\mathbb{R}\times\mathbb{G}'}\| \leq \|T'|\mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}'}\| = \|T|\mathcal{FT}_p^{\mathbb{T}\times\mathbb{G}}\|.$$

Similarly

$$\|T|\mathcal{FT}_p^{\mathbb{T}\times\mathbb{G}}\| \leq \frac{\pi}{2}\,\|T|\mathcal{FT}_p^{\mathbb{R}\times\mathbb{G}}\|. \ ∎$$

From Propositions 1 and 2, we conclude that $\mathcal{FT}_p^{\mathbb{R}\times\mathbb{G}} = \mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}} = \mathcal{FT}_p^{\mathbb{T}\times\mathbb{G}}$ for every LCA group $\mathbb{G}$. We also have the following two corollaries.

COROLLARY 3. $\mathcal{FT}_p^{\mathbb{R}^n} = \mathcal{FT}_p^{\mathbb{Z}^n} = \mathcal{FT}_p^{\mathbb{T}^n}$ *for every integer* $n \geq 1$.

Proof. By Proposition 1 we have

$$\mathcal{FT}_p^{\mathbb{R}^n} = \mathcal{FT}_p^{\mathbb{R}\times\mathbb{R}^{n-1}} = \mathcal{FT}_p^{\mathbb{Z}\times\mathbb{R}^{n-1}}.$$

We continue this to obtain

$$\mathcal{FT}_p^{\mathbb{R}\times\mathbb{Z}\times\mathbb{R}^{n-2}} = \mathcal{FT}_p^{\mathbb{Z}\times\mathbb{Z}\times\mathbb{R}^{n-2}} = \cdots = \mathcal{FT}_p^{\mathbb{Z}^n}.$$

Similarly we apply Proposition 2 n times to obtain

$$\mathcal{FT}_p^{\mathbb{R}^n} = \mathcal{FT}_p^{\mathbb{T}^n}. \ ∎$$

COROLLARY 4. $\mathcal{FT}_p^{\mathbb{R}^n\times\mathbb{G}} = \mathcal{FT}_p^{\mathbb{Z}^n\times\mathbb{G}} = \mathcal{FT}_p^{\mathbb{T}^n\times\mathbb{G}}$ *for every LCA group* $\mathbb{G}$.

Proof. The proof is similar to that of Corollary 3. ∎

LEMMA 5. *For every LCA group* $\mathbb{G}$ *and every positive integer* n,

$$\|T|\mathcal{FT}_p^{\mathbb{Z}^n\times\mathbb{G}}\| = \|T|\mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}\|$$

and therefore $\mathcal{FT}_p^{\mathbb{Z}^n\times\mathbb{G}} = \mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}$.

Proof. It is enough to prove this statement for $n = 2$. Proposition 1.2 in [1] can be extended by replacing I_X by T to the following statement: For given LCA groups $\mathbb{G}_1$ and $\mathbb{G}_2$,

$$\|T|\mathcal{FT}_p^{\mathbb{G}_1}\| \, \|I_{\mathbb{C}}|\mathcal{FT}_p^{\mathbb{G}_2}\| \le \|T|\mathcal{FT}_p^{\mathbb{G}_1\times\mathbb{G}_2}\|,$$

where $I_{\mathbb{C}}$ is the identity operator on $\mathbb{C}$. We have $\|I_{\mathbb{C}}|\mathcal{FT}_p^{\mathbb{G}_2}\| = 1$ when $\mathbb{G}_2$ is compact or discrete. Hence if we let $\mathbb{G}_1 = \mathbb{Z}\times\mathbb{G}$ and $\mathbb{G}_2 = \mathbb{Z}$ then

$$\|T|\mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}\| \le \|T|\mathcal{FT}_p^{\mathbb{Z}^2\times\mathbb{G}}\|.$$

To show $\|T|\mathcal{FT}_p^{\mathbb{Z}^2\times\mathbb{G}}\| \le \|T|\mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}\|$ we consider

$$\mathbf{f}(k_1,k_2,t) = \sum_{l_1,l_2} \mathbf{g}_{l_1,l_2}(t)\chi_{\{l_1\}}(k_1)\chi_{\{l_2\}}(k_2),$$

where the summation is on a finite subset of $\mathbb{Z}\times\mathbb{Z}$, and $k_1,k_2\in\mathbb{Z}$, $\mathbf{g}_{l_1,l_2}(t)\in[L_p(\mathbb{G}),X]$. Then

$$\|\mathbf{f}|L_p(\mathbb{Z}^2\times\mathbb{G})\| = \Big(\int_{\mathbb{G}}\sum_{l_1,l_2}\|\mathbf{g}_{l_1,l_2}(t)\|^p\,dt\Big)^{1/p}. \tag{3}$$

Since

$$[F^{\mathbb{Z}^2\times\mathbb{G}},T]\mathbf{f}(\widetilde{s}_1,\widetilde{s}_2,\widetilde{t}) = \int_{\mathbb{G}}\sum_{l_1,l_2} T\mathbf{g}_{l_1,l_2}(t)\exp(i\widetilde{s}_1 l_1)\exp(i\widetilde{s}_2 l_2)\,(t,\widetilde{t})\,dt$$

we have

$$\begin{aligned}
&\|[F^{\mathbb{Z}^2\times\mathbb{G}},T]\mathbf{f}\|^{p'}\\
&= \int_{\mathbb{G}'}\int_{\mathbb{T}}\int_{\mathbb{T}}\Big\|\int_{\mathbb{G}}\sum_{l_1,l_2} T\mathbf{g}_{l_1,l_2}(t)\exp(i\widetilde{s}_1 l_1)\exp(i\widetilde{s}_2 l_2)\,(t,\widetilde{t})\,dt\Big\|^{p'} d\widetilde{s}_1\,d\widetilde{s}_2\,d\widetilde{t}\\
&= \int_{\mathbb{G}'}\int_{\mathbb{T}}\int_{\mathbb{T}}\Big\|\int_{\mathbb{G}}\sum_{l_1,l_2} T\mathbf{g}_{l_1,l_2}(t)\exp(i\widetilde{s}_1 l_1)\exp(i\widetilde{s}_2 A l_2)\,(t,\widetilde{t})\,dt\Big\|^{p'} d\widetilde{s}_1\,d\widetilde{s}_2\,d\widetilde{t},
\end{aligned}$$

where we have let $A := 2\max\{|l_1|\}+1$. Since the Haar measure is translation-invariant we obtain

$$\begin{aligned}
&\|[F^{\mathbb{Z}^2\times\mathbb{G}},T]\mathbf{f}\|^{p'}\\
&= \int_{\mathbb{G}'}\int_{\mathbb{T}}\int_{\mathbb{T}}\Big\|\int_{\mathbb{G}}\sum_{l_1,l_2} T\mathbf{g}_{l_1,l_2}(t)\exp(i\widetilde{s}_1 l_1)\exp(i(\widetilde{s}_1+\widetilde{s}_2) A l_2)\,(t,\widetilde{t})\,dt\Big\|^{p'} d\widetilde{s}_1\,d\widetilde{s}_2\,d\widetilde{t}\\
&= \int_{\mathbb{G}'}\int_{\mathbb{T}}\int_{\mathbb{T}}\Big\|\int_{\mathbb{G}}\sum_{l_1,l_2} T\mathbf{g}_{l_1,l_2}(t)\exp(i\widetilde{s}_1(l_1+Al_2))\exp(i\widetilde{s}_2 A l_2)\,(t,\widetilde{t})\,dt\Big\|^{p'} d\widetilde{s}_1\,d\widetilde{s}_2\,d\widetilde{t}\\
&= \int_{\mathbb{T}}\int_{\mathbb{G}'}\int_{\mathbb{T}}\Big\|\int_{\mathbb{G}}\sum_{l_1,l_2} T(\mathbf{g}_{l_1,l_2}(t)\exp(i\widetilde{s}_2 A l_2))\exp(i\widetilde{s}_1(l_1+Al_2))\,(t,\widetilde{t})\,dt\Big\|^{p'} d\widetilde{s}_1\,d\widetilde{t}\,d\widetilde{s}_2.
\end{aligned}$$

Since $(l_1, l_2) \mapsto l_1 + Al_2$ is a one-to-one correspondence, the above equals

$$\int_{\mathbb{T}} \Big\| [F^{\mathbb{Z}\times\mathbb{G}}, T]\Big(\sum_{l_1+Al_2} (\mathbf{g}_{l_1,l_2}(t)\exp(i\widetilde{s}_2 Al_2)\Big)\chi_{\{l_1+Al_2\}}\Big\|^{p'} d\widetilde{s}_2$$

$$\le \int_{\mathbb{T}} \Big[\|T|\mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}\|^{p'} \Big(\int_{\mathbb{G}} \sum_{l_1+Al_2} \|\mathbf{g}_{l_1,l_2}(t)\exp(i\widetilde{s}_2 Al_2)\|^p \, dt\Big)^{p'/p}\Big] d\widetilde{s}_2$$

$$= \|T|\mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}\|^{p'} \Big(\int_{\mathbb{G}} \sum_{l_1+Al_2} \|\mathbf{g}_{l_1,l_2}(t)\|^p \, dt\Big)^{p'/p}$$

$$= \|T|\mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}\|^{p'} \Big(\int_{\mathbb{G}} \sum_{l_1,l_2} \|\mathbf{g}_{l_1,l_2}(t)\|^p \, dt\Big)^{p'/p}$$

$$= \|T|\mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}\|^{p'} \|\mathbf{f}|L_p(\mathbb{Z}^2\times\mathbb{G}, X)\|^{p'} \quad \text{by (3)},$$

so it follows that $\|T|\mathcal{FT}_p^{\mathbb{Z}^2\times\mathbb{G}}\| \le \|T|\mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}\|$. ∎

Remark 1. In fact, every dissipative group $\mathbb{A}$ has the property that $\|T|\mathcal{FT}_p^{\mathbb{A}^n\times\mathbb{G}}\| = \|T|\mathcal{FT}_p^{\mathbb{A}\times\mathbb{G}}\|$ for any positive integer n and for any LCA group $\mathbb{G}$. The definition and properties of dissipative groups are found in [4].

Corollary 6. *$\|T|\mathcal{FT}_p^{\mathbb{T}^n\times\mathbb{G}}\| = \|T|\mathcal{FT}_p^{\mathbb{T}\times\mathbb{G}}\|$ and so $\mathcal{FT}_p^{\mathbb{T}^n\times\mathbb{G}} = \mathcal{FT}_p^{\mathbb{T}\times\mathbb{G}}$.*

Proof. This follows from Lemma 5 and a duality argument. ∎

Proposition 7. *Let $\mathbb{H}$ be $\mathbb{R}$, $\mathbb{Z}$ or $\mathbb{T}$, and $\mathbb{G}$ be an LCA group. Then $\mathcal{FT}_p^{\mathbb{H}^n\times\mathbb{G}} = \mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}$ for any integer $n \ge 1$. In particular, $\mathcal{FT}_p^{\mathbb{H}^n} = \mathcal{FT}_p^{\mathbb{Z}}$.*

Proof. It follows from Corollary 4 and Lemma 5 that $\mathcal{FT}_p^{\mathbb{H}^n\times\mathbb{G}} = \mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}$. And if $\mathbb{G}$ is the trivial group then we have $\mathcal{FT}_p^{\mathbb{H}^n} = \mathcal{FT}_p^{\mathbb{Z}}$. ∎

Notation: From now on, when $\mathbb{H}$ is $\mathbb{R}$, $\mathbb{Z}$ or $\mathbb{T}$ we denote $\mathcal{FT}_p^{\mathbb{H}}$ by $\mathcal{FT}_p$.

Corollary 8. *$\mathcal{FT}_p^{\mathbb{R}^a\times\mathbb{Z}^b\times\mathbb{T}^c} = \mathcal{FT}_p$ for any nonnegative integers a, b, c with $a + b + c \ge 1$.*

Proof. Without loss of generality we assume $a, b, c > 0$. By applying Proposition 7, we have

$$\mathcal{FT}_p^{\mathbb{R}^a\times\mathbb{Z}^b\times\mathbb{T}^c} = \mathcal{FT}_p^{\mathbb{Z}^{b+1}\times\mathbb{T}^c} = \mathcal{FT}_p^{\mathbb{Z}\times\mathbb{T}^c} = \mathcal{FT}_p^{\mathbb{Z}^2} = \mathcal{FT}_p^{\mathbb{Z}}. \ ∎$$

Theorem 9. *Let $\mathbb{G}$ be an LCA group. Then $\mathcal{FT}_p^{\mathbb{R}^a\times\mathbb{Z}^b\times\mathbb{T}^c\times\mathbb{G}} = \mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}$ for any nonnegative integers a, b, c with $a + b + c \ge 1$.*

Proof. By Proposition 7 we have

$$\mathcal{FT}_p^{\mathbb{R}^a\times\mathbb{Z}^b\times\mathbb{T}^c\times\mathbb{G}} = \mathcal{FT}_p^{\mathbb{Z}^{1+b}\times\mathbb{T}^c\times\mathbb{G}} = \mathcal{FT}_p^{\mathbb{Z}^{2+b}\times\mathbb{G}} = \mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}. \ ∎$$

3. Search for $\mathbb{G}$ satisfying $\mathcal{FT}_p^{\mathbb{G}} = \mathcal{FT}_p^{\mathbb{Z}}$ and further classification. We notice that to solve the problem of deciding whether $\mathcal{FT}_p^{\mathbb{G}}$ depends on $\mathbb{G}$, we should solve that problem for the compact abelian groups since any compactly generated LCA group is a product of a finite number of $\mathbb{R}$'s and $\mathbb{Z}$'s and a compact group [5]. The dual group of a compact group has the discrete topology [13]. Moreover if $\mathbb{G}$ is a compact abelian group then $\mathbb{G}$ is connected iff $\mathbb{G}'$ is torsion-free [5]. This fact gives a clue to the results in this section.

First, we introduce Weil's formula which factorizes integration on $\mathbb{G}$ into double integration on a closed subgroup and its factor group.

Theorem 10 ([12]). *Let $\mathbb{G}$ be an LCA group and $\mathbb{H}$ a closed subgroup. Then there are Haar measures $\mu_{\mathbb{G}}$, $\mu_{\mathbb{H}}$ and $\mu_{\mathbb{G}/\mathbb{H}}$ such that*

$$\int_{\mathbb{G}} \mathbf{f}(s)\, d_{\mu_{\mathbb{G}}}(s) = \int_{\mathbb{G}/\mathbb{H}} \Big(\int_{\mathbb{H}} \mathbf{f}(s+h)\, d_{\mu_{\mathbb{H}}}(h) \Big) d_{\mu_{\mathbb{G}/\mathbb{H}}}(s+\mathbb{H}) \tag{4}$$

whenever $\mathbf{f}$ is a compactly supported continuous Banach space-valued function or a nonnegative lower semicontinuous function on $\mathbb{G}$.

In Theorem 10 if any two of $\mu_{\mathbb{G}}$, $\mu_{\mathbb{H}}$ and $\mu_{\mathbb{G}/\mathbb{H}}$ are given then the third can be determined so that the statement holds.

Andersson [1] obtained the inequality $\|I_X\,|\mathcal{FT}_p^{\mathbb{H}}\| \le \|I_X|\mathcal{FT}_p^{\mathbb{G}}\|$, where $\mathbb{H}$ is an open subgroup of an LCA group $\mathbb{G}$ and I_X is the identity operator on a Banach space X. By replacing I_X with a bounded linear operator $T : X \to Y$ in the proof of [1], we have the following:

Proposition 11. *Let $\mathbb{H}$ be an open subgroup of an LCA group $\mathbb{G}$. Then*

$$\|T|\mathcal{FT}_p^{\mathbb{H}}\| \le \|T|\mathcal{FT}_p^{\mathbb{G}}\|. \tag{5}$$

Now we consider torsion-free LCA groups with the discrete topology.

Corollary 12. *For any nontrivial torsion-free LCA group $\mathbb{E}$ with the discrete topology,*

$$\|T|\mathcal{FT}_p^{\mathbb{E}}\| = \|T|\mathcal{FT}_p^{\mathbb{Z}}\|. \tag{6}$$

Proof. The proof is similar to that of $\mathrm{FT}_p^{\mathbb{E}} = \mathrm{FT}_p^{\mathbb{Z}}$ in [1]: Since $\mathbb{Z}$ is isomorphic to an open subgroup of $\mathbb{E}$, by applying (5) we have $\|T|\mathcal{FT}_p^{\mathbb{Z}}\| \le \|T|\mathcal{FT}_p^{\mathbb{E}}\|$. On the other hand, for any X-valued simple function $\mathbf{f}$ defined on $\mathbb{E}$ which has finite L_p-norm, $\mathbf{f}$ is nonzero only on a subset of an open subgroup which is isomorphic to $\mathbb{Z}^k$ for some positive integer k. Therefore we have

$$\frac{\|[F^{\mathbb{E}}, T]\mathbf{f}\|_{p'}}{\|\mathbf{f}|L_p(\mathbb{E})\|} = \frac{\|[F^{\mathbb{Z}^k}, T]\mathbf{f}\|_{p'}}{\|\mathbf{f}|L_p(\mathbb{Z}^k)\|} \le \|T|\mathcal{FT}_p^{\mathbb{Z}^k}\|$$

and hence

$$\|T|\mathcal{FT}_p^{\mathbb{E}}\| \le \sup_k \|T|\mathcal{FT}_p^{\mathbb{Z}^k}\| = \|T|\mathcal{FT}_p^{\mathbb{Z}}\|.$$

Thus equality (6) follows. ∎

LEMMA 13. *Let $\mathbb{H}$ be a closed subgroup of an LCA group $\mathbb{G}$ such that $\mathbb{G}/\mathbb{H}$ is finite. If n is the cardinality of $\mathbb{G}/\mathbb{H}$ then*

$$\|T|\mathcal{FT}_p^{\mathbb{H}}\| \le \|T|\mathcal{FT}_p^{\mathbb{G}}\| \le n^{1/p'}\|T|\mathcal{FT}_p^{\mathbb{H}}\|, \tag{7}$$

and hence $\mathcal{FT}_p^{\mathbb{G}} = \mathcal{FT}_p^{\mathbb{H}}$.

Proof. Since $\mathbb{G}/\mathbb{H}$ is finite, $\mathbb{H}$ is open and by (5) we have the left inequality. The proof of the right one is as follows. Choose $s_1, \ldots, s_n$ in $\mathbb{G}$ such that $\mathbb{G}/\mathbb{H} = \{s_1+\mathbb{H}, \ldots, s_n+\mathbb{H}\}$. The measure of $\mathbb{G}/\mathbb{H}$ is the Haar measure of unit mass. By applying Weil's formula, for any compactly supported continuous X-valued function $\mathbf{f}$ on $\mathbb{G}$ we have

$$\begin{aligned} \int_{\mathbb{G}} \|\mathbf{f}(s)\|^p \, d_{\mu_{\mathbb{G}}}(s) &= \int_{\mathbb{G}/\mathbb{H}} \Big(\int_{\mathbb{H}} \|\mathbf{f}(s+h)\|^p \, d_{\mu_{\mathbb{H}}}(h)\Big) \, d_{\mu_{\mathbb{G}/\mathbb{H}}}(s+\mathbb{H}) \\ &= \sum_{i=1}^{n} \frac{1}{n} \int_{\mathbb{H}} \|\mathbf{f}(s_i+h)\|^p \, d_{\mu_{\mathbb{H}}}(h) \end{aligned} \tag{8}$$

and

$$\begin{aligned} [F^{\mathbb{G}}, T]\mathbf{f}(\sigma) &= \sum_{i=1}^{n} \frac{1}{n}\Big(\int_{\mathbb{H}} T\mathbf{f}(s_i+h)\sigma(s_i+h) \, d_{\mu_{\mathbb{H}}}(h)\Big) \\ &= \sum_{i=1}^{n} \frac{\sigma(s_i)}{n}\Big(\int_{\mathbb{H}} T\mathbf{f}(s_i+h)\sigma(h) \, d_{\mu_{\mathbb{H}}}(h)\Big) \end{aligned} \tag{9}$$

for $\sigma \in \mathbb{G}'$. The dual group of $\mathbb{G}/\mathbb{H}$ is isomorphic to the closed subgroup $\mathbb{H}^{\perp} = \{\chi \in \mathbb{G}' \mid \chi(h) = 1 \text{ for all } h \in \mathbb{H}\}$, and $\mathbb{G}'/\mathbb{H}^{\perp}$ is isomorphic to $\mathbb{H}'$. Here the cardinality of $\mathbb{H}^{\perp}$ is n, the measure of $\mathbb{H}^{\perp}$ is the counting measure, and we write $\mathbb{H}^{\perp}$ as $\{\widetilde{\eta}_1, \ldots, \widetilde{\eta}_n\}$. Now $[F^{\mathbb{G}}, T](\mathbf{f})$ belongs to $[C_0(\mathbb{G}'), Y]$, so the Y-norm of $[F^{\mathbb{G}}, T](\mathbf{f})$ is continuous and by Weil's formula we have

$$\begin{aligned} &\|[F^{\mathbb{G}}, T]\mathbf{f}\| \\ &\quad = \Big(\int_{\mathbb{H}'} \sum_{j=1}^{n} \Big\| \frac{1}{n} \sum_{i=1}^{n} \sigma(s_i)\eta_j(s_i) \int_{\mathbb{H}} T\mathbf{f}(s_i+h)\sigma(h) \, d_{\mu_{\mathbb{H}}}(h)\Big\|^{p'} d_{\mu_{\mathbb{H}'}}(\sigma+\mathbb{H}^{\perp})\Big)^{1/p'} \\ &\quad \le \Big[\Big(\int_{\mathbb{H}'} \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} \frac{1}{n} \Big\| \int_{\mathbb{H}} T\mathbf{f}(s_i+h)\sigma(h) \, d_{\mu_{\mathbb{H}}}(h)\Big\|^{p}\Big)^{p'/p} d_{\mu_{\mathbb{H}'}}(\sigma+\mathbb{H}^{\perp})\Big)^{p/p'}\Big]^{1/p} \end{aligned}$$

$$\leq \Big[\sum_{i=1}^{n}\frac{1}{n}\Big(\int\limits_{\mathbb{H}'}\sum_{j=1}^{n}\Big\|\int\limits_{\mathbb{H}} T\mathbf{f}(s_i+h)\sigma(h)\,d_{\mu_{\mathbb{H}}}(h)\Big\|^{p'} d_{\mu_{\mathbb{H}'}}(\sigma+\mathbb{H}^{\perp})\Big)^{p/p'}\Big]^{1/p}$$

$$= \Big[\sum_{i=1}^{n}\frac{1}{n}\,n^{p/p'}\Big(\int\limits_{\mathbb{H}'}\Big\|\int\limits_{\mathbb{H}} T\mathbf{f}(s_i+h)\sigma(h)\,d_{\mu_{\mathbb{H}}}(h)\Big\|^{p'} d_{\mu_{\mathbb{H}'}}(\sigma+\mathbb{H}^{\perp})\Big)^{p/p'}\Big]^{1/p}$$

$$\leq \Big[\sum_{i=1}^{n}\frac{1}{n}\,n^{p/p'}\|T|\mathcal{FT}_p^{\mathbb{H}}\|^p\Big(\int\limits_{\mathbb{H}}\|\mathbf{f}(s_i+h)\|^p\,d_{\mu_{\mathbb{H}}}(h)\Big)\Big]^{1/p}$$

$$= n^{1/p'}\|T|\mathcal{FT}_p^{\mathbb{H}}\|\,\|\mathbf{f}|L_p(\mathbb{G})\|.$$

We have used Minkowski's inequality in the third inequality above. ∎

PROPOSITION 14. *If $\mathbb{F}$ is an infinite, compact and connected LCA group then for any nonnegative integers a, b, there exist $c(a,b), C(a,b) > 0$ such that*

$$c(a,b)\|T|\mathcal{FT}_p^{\mathbb{T}}\| \leq \|T|\mathcal{FT}_p^{\mathbb{R}^a\times\mathbb{Z}^b\times\mathbb{F}}\| \leq C(a,b)\|T|\mathcal{FT}_p^{\mathbb{T}}\|.$$

Therefore $\mathcal{FT}_p^{\mathbb{R}^a\times\mathbb{Z}^b\times\mathbb{F}} = \mathcal{FT}_p$.

Proof. If $a = b = 0$ then by applying (6) we have $\|T'|\mathcal{FT}_p^{\mathbb{F}'}\| = \|T'|\mathcal{FT}_p^{\mathbb{Z}}\|$, where $\mathbb{F}'$ is the dual group of $\mathbb{F}$ and hence a nontrivial torsion-free group with the discrete topology. It follows that

$$\|T|\mathcal{FT}_p^{\mathbb{F}}\| = \|T'|\mathcal{FT}_p^{\mathbb{Z}}\| = \|T|\mathcal{FT}_p^{\mathbb{T}}\| \quad \text{and} \quad \mathcal{FT}_p^{\mathbb{F}} = \mathcal{FT}_p. \tag{10}$$

If $a + b \geq 1$ then by applying Theorem 9 and Corollary 4, we have

$$\mathcal{FT}_p^{\mathbb{R}^a\times\mathbb{Z}^b\times\mathbb{F}} = \mathcal{FT}_p^{\mathbb{T}\times\mathbb{F}}$$

and in fact

$$c(a,b)\|T|\mathcal{FT}_p^{\mathbb{T}\times\mathbb{F}}\| \leq \|T|\mathcal{FT}_p^{\mathbb{R}^a\times\mathbb{Z}^b\times\mathbb{F}}\| \leq C(a,b)\|T|\mathcal{FT}_p^{\mathbb{T}\times\mathbb{F}}\|$$

for some positive reals $c(a,b), C(a,b)$. Here $c(a,b) = (2/\pi)^{ab}$, $C(a,b) = 1$. Note that $\mathbb{T}\times\mathbb{F}$ is connected and compact. Thus we again have

$$\|T|\mathcal{FT}_p^{\mathbb{T}\times\mathbb{F}}\| = \|T|\mathcal{FT}_p^{\mathbb{T}}\| \quad \text{by (10).}$$ ∎

If $\mathbb{F}$ has only finitely many components and $\mathbb{F}_0$ is the component (maximal connected set) of the identity element (briefly, the identity component), then $\mathbb{F}/\mathbb{F}_0$ is finite and we have the following result.

THEOREM 15. *Let $\mathbb{F}$ be an infinite compact LCA group with n components. Then for any nonnegative integers a, b, there exist $c(a,b), C(a,b) > 0$ such that*

$$c(a,b)\|T|\mathcal{FT}_p^{\mathbb{T}}\| \leq \|T|\mathcal{FT}_p^{\mathbb{R}^a\times\mathbb{Z}^b\times\mathbb{F}}\| \leq n^{1/p'}C(a,b)\|T|\mathcal{FT}_p^{\mathbb{T}}\|,$$

and therefore $\mathcal{FT}_p^{\mathbb{R}^a\times\mathbb{Z}^b\times\mathbb{F}} = \mathcal{FT}_p$.

Proof. Let $\mathbb{F}_0$ be the identity component of $\mathbb{F}$. Then the factor group $(\mathbb{R}^a \times \mathbb{Z}^b \times \mathbb{F})/(\mathbb{R}^a \times \mathbb{Z}^b \times \mathbb{F}_0)$ is a finite group with n elements. Thus by Lemma 13,

$$\|T|\mathcal{FT}_p^{\mathbb{R}^a\times\mathbb{Z}^b\times\mathbb{F}_0}\| \le \|T|\mathcal{FT}_p^{\mathbb{R}^a\times\mathbb{Z}^b\times\mathbb{F}_0}\| \le n^{1/p'}\|T|\mathcal{FT}_p^{\mathbb{R}^a\times\mathbb{Z}^b\times\mathbb{F}}\|$$

and

$$\mathcal{FT}_p^{\mathbb{R}^a\times\mathbb{Z}^b\times\mathbb{F}} = \mathcal{FT}_p^{\mathbb{R}^a\times\mathbb{Z}^b\times\mathbb{F}_0}.$$

Then the statement follows from Proposition 14. ∎

This theorem cannot say anything about the case when $\mathbb{F}$ has infinitely many components, because of the factor $n^{1/p'}$.

Theorem 15 can be extended beyond the scope of compactly generated LCA groups.

THEOREM 16. *Let $\mathbb{E}$ be a nontrivial torsion-free group with the discrete topology and $\mathbb{F}$ an infinite, compact and connected LCA group. Then for any LCA group $\mathbb{G}$,*

$$\|T|\mathcal{FT}_p^{\mathbb{E}\times\mathbb{G}}\| = \|T|\mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}\|, \qquad \|T|\mathcal{FT}_p^{\mathbb{F}\times\mathbb{G}}\| = \|T|\mathcal{FT}_p^{\mathbb{T}\times\mathbb{G}}\|.$$

If $\widetilde{\mathbb{F}}$ is a compact LCA group with n components then for any nonnegative integer k there exist $c(k), C(k)$ such that

$$c(k)\|T|\mathcal{FT}_p^{\mathbb{T}\times\mathbb{G}}\| \le \|T|\mathcal{FT}_p^{\mathbb{R}^k\times\mathbb{E}^l\times\widetilde{\mathbb{F}}^m\times\mathbb{G}}\| \le n^{m/p'}C(k)\|T|\mathcal{FT}_p^{\mathbb{T}\times\mathbb{G}}\|,$$

and hence

$$\mathcal{FT}_p^{\mathbb{R}^k\times\mathbb{E}^l\times\widetilde{\mathbb{F}}^m\times\mathbb{G}} = \mathcal{FT}_p^{\mathbb{T}\times\mathbb{G}},$$

where $l, m = 0$ or 1 and $k+l+m \ge 1$. In particular if $\mathbb{G}$ is the trivial group then

$$\mathcal{FT}_p^{\mathbb{R}^k\times\mathbb{E}^l\times\widetilde{\mathbb{F}}^m} = \mathcal{FT}_p.$$

Proof. First, $\mathbb{Z}$ is isomorphic to an open subgroup of $\mathbb{E}$, and so $\mathbb{Z}\times\mathbb{G}$ is an open subgroup of $\mathbb{E}\times\mathbb{G}$. By applying (5) we have $\|T|\mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}\| \le \|T|\mathcal{FT}_p^{\mathbb{E}\times\mathbb{G}}\|$. Conversely, for any simple X-valued function $\mathbf{f}$ which is defined on $\mathbb{E}\times\mathbb{G}$ and has finite L_p-norm, the support of $\mathbf{f}$ is a subset of $\mathbb{Z}^a\times\mathbb{G}$ for some positive integer a. Therefore Lemma 5 yields $\|T|\mathcal{FT}_p^{\mathbb{E}\times\mathbb{G}}\| \le \sup_a \|T|\mathcal{FT}_p^{\mathbb{Z}^a\times\mathbb{G}}\| = \|T|\mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}}\|$.

Second, from the duality argument it follows that

$$\|T|\mathcal{FT}_p^{\mathbb{F}\times\mathbb{G}}\| = \|T'|\mathcal{FT}_p^{\mathbb{F}'\times\mathbb{G}'}\| = \|T'|\mathcal{FT}_p^{\mathbb{Z}\times\mathbb{G}'}\| = \|T|\mathcal{FT}_p^{\mathbb{T}\times\mathbb{G}}\|.$$

The rest follows by applying the above two results, Propositions 1, 2 and Lemma 13. ∎

REMARK 2. If $\mathbb{G}$ is an LCA group and $\mathbb{G}_0$ is the identity component of $\mathbb{G}$ then $\mathbb{G}_0$ is a closed normal subgroup of $\mathbb{G}$ and the factor group $\mathbb{G}/\mathbb{G}_0$ is totally disconnected and Hausdorff. From (24.45) of [5], if $\mathbb{G}_0$ is open then

$\mathbb{G}$ is isomorphic to $\mathbb{G}_0 \times (\mathbb{G}/\mathbb{G}_0)$. In particular $\mathbb{G}_0$ is open when $\mathbb{G}$ is locally connected. Hence to answer the question, denoted by (P), whether $\|T|\mathcal{FT}_p^{\mathbb{G}}\|$ is equivalent to $\|T|\mathcal{FT}_p^{\mathbb{T}}\|$ or not, it is useful to find $\mathbb{G}_0$ and $\mathbb{G}/\mathbb{G}_0$. Since $\mathbb{G}_0$ is isomorphic to $\mathbb{R}^n \times \mathbb{K}$, where n is a nonnegative integer and $\mathbb{K}$ is a compact connected group (see Theorem 9.14 of [5]), the answer to (P) is affirmative when $\mathbb{G}/\mathbb{G}_0$ is good.

THEOREM 17. *Let $\mathbb{G}$ be an LCA group with n components. Then there are positive real numbers c and C such that*

$$c\|T|\mathcal{FT}_p^{\mathbb{T}}\| \le \|T|\mathcal{FT}_p^{\mathbb{G}}\| \le n^{1/p'}C\|T|\mathcal{FT}_p^{\mathbb{T}}\|, \tag{11}$$

and hence $\mathcal{FT}_p^{\mathbb{G}} = \mathcal{FT}_p$.

Proof. Let $\mathbb{G}_0$ be the identity component of $\mathbb{G}$. Then $\mathbb{G}/\mathbb{G}_0$ is a finite LCA group. Hence by Lemma 13 we have $\|T|\mathcal{FT}_p^{\mathbb{G}_0}\| \le \|T|\mathcal{FT}_p^{\mathbb{G}}\| \le n^{1/p'}\|T|\mathcal{FT}_p^{\mathbb{G}_0}\|$. Moreover, $\mathbb{G}_0$ is isomorphic to $\mathbb{R}^k \times \mathbb{K}$, where k is a nonnegative integer and $\mathbb{K}$ is a compact connected group. Now, $\mathbb{K}$ is trivial or infinite. If $\mathbb{K}$ is trivial then from Proposition 2 it follows that

$$c\|T|\mathcal{FT}_p^{\mathbb{T}^k}\| \le \|T|\mathcal{FT}_p^{\mathbb{G}_0}\| \le C\|T|\mathcal{FT}_p^{\mathbb{T}^k}\|$$

for some $c, C > 0$. Moreover, by (10), $\|T|\mathcal{FT}_p^{\mathbb{T}^k}\| = \|T|\mathcal{FT}_p^{\mathbb{T}}\|$. Thus we have

$$c\|T|\mathcal{FT}_p^{\mathbb{T}}\| \le \|T|\mathcal{FT}_p^{\mathbb{G}_0}\| \le C\|T|\mathcal{FT}_p^{\mathbb{T}}\|. \tag{12}$$

If $\mathbb{K}$ is infinite, we also have (12) by Proposition 14. Therefore the inequality (11) follows, and $\mathcal{FT}_p^{\mathbb{G}} = \mathcal{FT}_p$. ∎

THEOREM 18. *If $\mathbb{G}_0$ is the identity component of an LCA group $\mathbb{G}$ and $\mathbb{G}/\mathbb{G}_0$ is the direct sum of a nontrivial discrete torsion-free group and a finite group with cardinality n, then $\|T|\mathcal{FT}_p^{\mathbb{G}_0}\| \le \|T|\mathcal{FT}_p^{\mathbb{G}}\| \le n^{1/p'}\|T|\mathcal{FT}_p^{\mathbb{G}_0\times\mathbb{Z}}\|$ and $\mathcal{FT}_p^{\mathbb{G}} = \mathcal{FT}_p$.*

Proof. $\mathbb{G}$ is isomorphic to $\mathbb{G}_0 \times \mathbb{G}/\mathbb{G}_0$ because $\mathbb{G}_0$ is open. The direct sum of a nontrivial discrete torsion-free group and a finite group is isomorphic to the direct product of the two. We can apply Proposition 11, Lemma 13 and Theorem 16 to obtain the assertion. ∎

As for the class of Banach spaces which have Fourier type p for an infinite LCA group, we have the following:

COROLLARY 19. (i) *Under the assumptions of Theorem* 16,

$$\mathrm{FT}_p^{\mathbb{R}^k\times\mathbb{E}^l\times\widetilde{\mathbb{F}}^m} = \mathrm{FT}_p.$$

(ii) *Under the assumptions of Theorem* 18,

$$\mathrm{FT}_p^{\mathbb{G}} = \mathrm{FT}_p.$$

Hinrichs and Lee in [6], by combining the results of Propositions 1, 2 and Lemma 5 of this paper with the structure of LCA groups, prove that $\mathcal{FT}_p^{\mathbb{G}} = \mathcal{FT}_p^{\mathbb{G}'}$ for any LCA group $\mathbb{G}$ and $1 < p \le 2$. They derive the general statement:

> Let $\mathbb{G}_1$, $\mathbb{G}_2$ and $\mathbb{H}$ be LCA groups and $1 < p \le 2$. If there exists a constant c such that $\|T|\mathcal{FT}_p^{\mathbb{G}_1}\| \le c\|T|\mathcal{FT}_p^{\mathbb{G}_2}\|$ for all operators $T \in \mathcal{FT}_p^{\mathbb{G}_2}$ then also $\|T|\mathcal{FT}_p^{\mathbb{G}_1\times\mathbb{H}}\| \le c\|T|\mathcal{FT}_p^{\mathbb{G}_2\times\mathbb{H}}\|$ for all operators $T \in \mathcal{FT}_p^{\mathbb{G}_2\times\mathbb{H}}$.

This statement, combined with the inequalities

$$\|T|\mathcal{FT}_p^{\mathbb{R}}\| \le \|T|\mathcal{FT}_p^{\mathbb{F}}\| = \|T|\mathcal{FT}_p^{\mathbb{Z}}\| \le \frac{\pi}{2}\,\|T|\mathcal{FT}_p^{\mathbb{R}}\|,$$

where $\mathbb{F}$ is a torsion-free group, implies Propositions 1, 2 and Lemma 5.

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Division of Applied Mathematics
Korea Advanced Institute of Science and Technology
373-1 Kuseong-dong, Yuseong-gu
Taejeon 305-701, Republic of Korea
E-mail: ispark@amath.kaist.ac.kr

Current address:
Next generation radio transmission research team
Mobile telecommunication group
Electronics and Telecommunications Research Institute
161 Gajeong-dong, Yuseong-gu
Daejeon 305-350, Republic of Korea

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